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VANISHING VISCOSITY LIMITS FOR AXISYMMETRIC FLOWS WITH BOUNDARY

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ABSTRACT. We construct global weak solutions of the Euler equations in an infinite cylinder $\Pi = \{x \in \mathbb{R}^3 \mid x_h = (x_1, x_2), r = |x_h| < 1\}$ for axisymmetric initial data without swirl when initial vorticity $\omega_0 = \omega_0^\theta e_\theta$ satisfies $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 3)$. The solutions constructed are Hölder continuous for spatial variables in $\bar{\Pi}$ if in addition that $\omega_0^\theta/r \in L^s$ for $s \in (3, \infty)$ and unique if $s = \infty$. The proof is by a vanishing viscosity method. We show that the Navier-Stokes equations subject to the Neumann boundary condition is globally well-posed for axisymmetric data without swirl in L^p for all $p \in [3, \infty)$. It is also shown that the energy dissipation tends to zero if $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 2]$, and Navier-Stokes flows converge to Euler flow in L^2 locally uniformly for $t \in [0, \infty)$ if additionally $\omega_0^\theta/r \in L^\infty$. The L^2 -convergence in particular implies the energy equality for weak solutions.

Résumé. Nous développons des solutions faibles globales des équations d'Euler dans un cylindre infini $\Pi = \{x \in \mathbb{R}^3 \mid x_h = (x_1, x_2), r = |x_h| < 1\}$ pour les données initiales axisymétriques sans tourbillon lorsque la vortacité initiale $\omega_0 = \omega_0^\theta e_\theta$ satisfait à $\omega_0^\theta/r \in L^q$ pour $q \in [3/2, 3)$. Les solutions développées sont la continuité de Hölder pour les variables en $\bar{\Pi}$ si, par ailleurs, $\omega_0^\theta/r \in L^s$ pour $s \in (3, \infty)$, et sont uniques si $s = \infty$. La preuve est établie par une méthode de disparition de la viscosité. Nous démontrons que les équations de Navier-Stokes soumises à la condition aux limites de Neumann sont globalement bien posées pour les données axisymétriques sans tourbillon dans L^p pour tout $p \in [3, \infty)$. Nous démontrons également que la dissipation d'énergie tend vers zéro si $\omega_0^\theta/r \in L^q$ pour $q \in [3/2, 2]$ et que les flux de Navier-Stokes convergent vers le flux d'Euler dans L^2 localement de manière uniforme pour $t \in [0, \infty)$ si, par ailleurs, $\omega_0^\theta/r \in L^\infty$. La convergence L^2 implique notamment l'égalité énergétique pour les solutions faibles.

1. INTRODUCTION

We consider the Navier-Stokes equations:

$$(1.1) \quad \begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= 0, & \operatorname{div} u &= 0 & \text{in } \Pi \times (0, \infty), \\ \nabla \times u \times n &= 0, & u \cdot n &= 0 & \text{on } \partial\Pi \times (0, \infty), \\ u &= u_0 & \text{on } \Pi \times \{t = 0\}, \end{aligned}$$

for the infinite cylinder

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$$\Pi = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_h = (x_1, x_2), |x_h| < 1\}.$$

Here, n denotes the unit outward normal vector field on $\partial\Pi$ and $\nu > 0$ is the kinematic viscosity.

We study the problem (1.1) for axisymmetric initial data. We say that a vector field u is axisymmetric if $u(x) = {}^tRu(Rx)$ for $x \in \Pi$, $\eta \in [0, 2\pi]$, $R = (e_r, e_\theta, e_z)$ and $e_r = {}^t(\cos \eta, \sin \eta, 0)$, $e_\theta = {}^t(-\sin \eta, \cos \eta, 0)$, $e_z = {}^t(0, 0, 1)$. By the cylindrical coordinate (r, θ, z) , an axisymmetric vector field is decomposed into three terms $u = u^r e_r + u^\theta e_\theta + u^z e_z$ and the azimuthal component u^θ is called swirl velocity (e.g., [67]). It is known that the Cauchy problem is globally well-posed for axisymmetric initial data without swirl in H^2 [58], [81], [60]. See also [3] for $H^{1/2}$.

The purpose of this paper is to study axisymmetric solutions in L^p . It is well known that the Cauchy problem is locally well-posed in L^p for all $p \geq 3$ [53]. However, global well-posedness results are unknown even if initial data is axisymmetric without swirl. For the two-dimensional case, the problem is globally well-posed in L^p for all $p \geq 2$ (including $p = \infty$ [41]), since vorticity of two-dimensional flows are uniformly bounded. On the other hand, for axisymmetric flows without swirl, vorticity estimates are more involved due to the vortex stretching as $r \rightarrow \infty$.

Recently, global-in-time solutions of the Cauchy problem are constructed in [32] for axisymmetric data without swirl when initial vorticity $\omega_0 = \omega_0^\theta e_\theta$ is a vortex ring, i.e., $\omega_0^\theta = \kappa \delta_{r_0, z_0}$ for $\kappa \in \mathbb{R}$ and a Dirac measure δ_{r_0, z_0} in the (r, z) -plane. See [37] for the uniqueness. For such initial data, initial velocity belong to L^p for $p \in (1, 2)$ and BMO^{-1} by the Biot-Savart law. For small data in BMO^{-1} , a global well-posedness result is known [57].

In this paper, we study axisymmetric solutions in the infinite cylinder $\Pi = \{r < 1\}$, subject to the Neumann boundary condition. Since the cylinder is horizontally bounded, vorticity estimates are simpler than those in the whole space. We prove that vorticity of axisymmetric solutions without swirl to (1.1) is uniformly bounded in the infinite cylinder Π , and unique global-in-time solutions exist for large axisymmetric data without swirl in L^p for all $p \in [3, \infty)$.

An important application of our well-posedness result is a vanishing viscosity limit as $\nu \rightarrow 0$. We apply our global well-posedness result to (1.1) and construct global weak solutions of the Euler equations. Although local well-posedness results are well known for the Euler equations with boundary (see below Theorem 1.1), a few results are known on existence of global weak solutions. The well-posedness result to (1.1) in L^p for $p \in [3, \infty)$ enable us to study weak solutions of the Euler equations when initial vorticity is in L^q for $q \in [3/2, 3)$ by the Biot-Savart law $1/p = 1/q - 1/3$.

To state a result, let L_σ^p denote the L^p -closure of $C_{c,\sigma}^\infty$, the space of all smooth solenoidal vector fields with compact support in Π . The space L_σ^p agrees with the space of all divergence-free vector fields whose normal trace is vanishing on $\partial\Pi$ [75]. By a local well-posedness result in the companion paper [1], unique local-in-time solutions to (1.1) exist for $u_0 \in L_\sigma^p$ and $p \in [3, \infty)$. Our first result is:

Theorem 1.1. *Let $u_0 \in L^p_\sigma$ be an axisymmetric vector field without swirl for $p \in [3, \infty)$. Then, there exists a unique axisymmetric solution without swirl $u \in C([0, \infty); L^p) \cap C^\infty(\bar{\Pi} \times (0, \infty))$ of (1.1) with some associated pressure $p \in C^\infty(\bar{\Pi} \times (0, \infty))$.*

We apply Theorem 1.1 to construct global weak solutions of the Euler equations:

$$(1.2) \quad \begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0, & \operatorname{div} u &= 0 & \text{in } \Pi \times (0, \infty), \\ u \cdot n &= 0 & \text{on } \partial\Pi \times (0, \infty), \\ u &= u_0 & \text{on } \Pi \times \{t = 0\}. \end{aligned}$$

Unique existence of local-in-time solutions of the Euler equations $u \in C([0, T]; W^{k,q}) \cap C^1([0, T]; W^{k-1,q})$ is known for sufficiently smooth initial data $u_0 \in W^{k,q}$ with integers $k > 1 + n/q$ and $q \in (1, \infty)$, when Π is smoothly bounded in \mathbb{R}^n for $n \geq 2$ [27], [12], [79], [54]. For axisymmetric data without swirl, it is known that local-in-time solutions for $u_0 \in H^s$ ($s \geq 3$) are continued for all time [74]. Observe that for $k = 2$ and $n = 3$, the condition $q \in (3, \infty)$ is required in order to construct local-in-time unique solutions. We construct global weak solutions under the lower regularity condition $q \in [3/2, 3)$; see below.

When Π is a two-dimensional bounded and simply-connected domain (e.g., a unit disk), global weak solutions of the Euler equations are constructed in [70] by a vanishing viscosity method for initial vorticity satisfying $\omega_0 \in L^q$ and $q \in (1, 2)$ (see [25], [43] for \mathbb{R}^2). For the two-dimensional case, the Neumann boundary condition in (1.1) is reduced to the condition $\omega = 0$ and $u \cdot n = 0$ on $\partial\Pi$, called the free condition [62] ([63, p.129]). The vanishing viscosity method subject to the free condition is studied in [7], [82] for $\omega_0 \in L^2$. The condition $q \in (1, 2)$ implies that initial velocity belongs to L^p for some $p \in (2, \infty)$ by the Biot-Savart law $u_0 = \nabla^\perp (-\Delta_D)^{-1} \omega_0$ for $1/p = 1/q - 1/2$. Here, $\nabla^\perp = (\partial_2, -\partial_1)$ and $-\Delta_D$ denotes the Laplace operator subject to the Dirichlet boundary condition.

Our goal is to construct three-dimensional weak solutions in the infinite cylinder Π for axisymmetric data without swirl when initial vorticity $\omega_0 = \omega_0^\theta e_\theta$ satisfies $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 3)$. The assumption for ω_0^θ/r is stronger than that for vorticity itself and implies that the initial velocity is in L^p for some $p \in [3, \infty)$ by the Biot-Savart law $u_0 = \nabla \times (-\Delta_D)^{-1} \omega_0$ and $1/p = 1/q - 1/3$. For such initial data, unique global-in-time solutions to (1.1) exist by Theorem 1.1. Note that the condition $\omega_0^\theta/r \in L^q$ is weaker than $u_0 \in W^{2,q}$ for $q \in [3/2, 3)$ since $\omega_0^\theta/r = -e_r \cdot (e_\theta \cdot \nabla \omega_0)$.

Let $BC_w([0, \infty); L^p)$ denote the space of bounded and weakly continuous (resp. weakly-star continuous) functions from $[0, \infty)$ to L^p for $p \in (1, \infty)$ (resp. for $p = \infty$). Let \mathbb{P} denote the Helmholtz projection on L^p [75]. We construct global weak solutions for $\omega_0^\theta/r \in L^q$ and $q \in [3/2, 3)$, which are L^p -integrable and may not be continuous. Under the additional regularity assumptions $\omega_0^\theta/r \in L^s$ for $s \in (3, \infty)$ and $s = \infty$, the weak solutions are Hölder continuous and unique. The main result of this paper is the following:

Theorem 1.2. *Let $u_0 \in L^p_\sigma$ be an axisymmetric vector field without swirl for $p \in [3, \infty)$ such that $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 3)$ and $1/p = 1/q - 1/3$.*

(i) (Existence) *There exists a weak solution $u \in BC_w([0, \infty); L^p)$ of (1.2) in the sense that $\nabla u \in BC_w([0, \infty); L^q)$ and*

$$(1.3) \quad \int_0^\infty \int_\Pi (u \cdot \partial_t \varphi + uu : \nabla \varphi) dx dt = - \int_\Pi u_0 \cdot \varphi_0 dx$$

for all $\varphi \in C_c^1(\overline{\Pi} \times [0, \infty))$ such that $\operatorname{div} \varphi = 0$ in Π and $\varphi \cdot n = 0$ on $\partial\Pi$ for $t \geq 0$, where $\varphi_0(x) = \varphi(x, 0)$.

(ii) (Hölder continuity) If $\omega_0^\theta/r \in L^s$ for $s \in (3, \infty)$, then $u \in BC([0, \infty); L^s)$ satisfies $\nabla u \in BC_w([0, \infty); L^s)$, $\partial_t u \in L^\infty(0, \infty; L^s)$ and

$$(1.4) \quad \partial_t u + \mathbb{P}u \cdot \nabla u = 0 \quad \text{on } L^s \quad \text{for a.e. } t > 0.$$

In particular, $u(\cdot, t)$ is bounded and Hölder continuous in $\overline{\Pi}$ of exponent $1 - 3/s$ for each $t \geq 0$.

(iii) (Uniqueness) If in addition that $\omega_0^\theta/r \in L^\infty$, then $\nabla \times u \in BC_w([0, \infty); L^\infty)$ and the weak solution is unique.

It is an interesting question whether the weak solutions constructed in Theorem 1.2 conserve the energy. Since the Poincaré inequality holds for the infinite cylinder (see Remarks 4.4 (ii)), the condition $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 2]$ implies the finite energy $u_0 \in L_\sigma^p \cap L^2$ and the energy equality holds for global-in-time solutions to (1.1); see below (1.5). In the sequel, we consider the case $q \in [3/2, 2]$.

In the Kolmogorov's theory of turbulence, it is a basic hypothesis that the energy dissipation tends to a positive constant at large Reynolds numbers. See, e.g., [33]. If the energy dissipation converges to a positive constant for global-in-time solutions u_ν to (1.1) as $\nu \rightarrow 0$, we would obtain a weak solution strictly decreasing the energy as a vanishing viscosity limit. Unfortunately, due to a regularizing effect, the energy dissipation converges to zero at least under the initial condition $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 2]$. However, it is still non-trivial whether vanishing viscosity limits conserve the energy since they are no longer continuous.

In the sequel, we prove that global-in-time solutions to (1.1) converge to a limit in L^2 locally uniformly for $t \in [0, \infty)$ under the additional assumption $\omega_0^\theta/r \in L^\infty$. If the limit is a C^1 -solution, the L^2 -convergence to a limit is equivalent to the convergence of the energy dissipation [52]. The equivalence may not always hold if a limit is a weak solution. The assumption $\omega_0^\theta/r \in L^\infty$ does not imply that ∇u is bounded for the limit and at present is optimal in order to obtain the L^2 -convergence. Once we know the L^2 -convergence, the energy conservation immediately follows as a consequence.

Theorem 1.3. *Let $u_0 \in L_\sigma^p \cap L^2$ be an axisymmetric vector field without swirl such that $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 2]$ and $1/p = 1/q - 1/3$. Let u_ν be a solution of (1.1) as in Theorem 1.1.*

(i) (Energy dissipation) *The solution $u_\nu \in BC([0, \infty); L^2)$ satisfies the energy equality*

$$(1.5) \quad \int_\Pi |u_\nu|^2 dx + 2\nu \int_0^t \int_\Pi |\nabla u_\nu|^2 dx ds = \int_\Pi |u_0|^2 dx, \quad t \geq 0,$$

and

$$(1.6) \quad \nu \int_0^T \int_{\Pi} |\nabla u_\nu|^2 dx ds \leq C \nu^{5/2-3/q} \rightarrow 0 \quad \text{as } \nu \rightarrow 0 \quad \text{for each } T > 0,$$

with some constant $C = C(q, T)$.

(ii) (L^2 -convergence) Assume in addition that $\omega_0^\theta/r \in L^\infty$. Then,

$$(1.7) \quad \lim_{\nu, \mu \rightarrow 0} \sup_{0 \leq t \leq T} \|u_\nu - u_\mu\|_{L^2(\Pi)} = 0.$$

In particular, the limit $u \in BC([0, \infty); L^2)$ satisfies the energy equality of (1.2):

$$(1.8) \quad \int_{\Pi} |u|^2 dx = \int_{\Pi} |u_0|^2 dx, \quad t \geq 0.$$

It is noted that there is a possibility that the energy equality (1.8) holds under a weaker assumption than $\omega_0^\theta/r \in L^\infty$ although we assumed it in order to prove the L^2 -convergence (1.7). In fact, it is known as a celebrated Onsager's conjecture [71] that Hölder continuous weak solutions to the Euler equations of exponent $\alpha > 1/3$ conserve the energy (but not necessarily if $\alpha \leq 1/3$). The conjecture is studied in [31] and the energy conservation is proved for weak solutions in the whole space under a stronger assumption. A simple proof is given in [20] under a weaker and natural assumption in the Besov space $u \in L^3(0, T; B_3^{\alpha, \infty})$ for $\alpha > 1/3$. See [26], [16] for further developments and [30] for a review. Recently, the energy conservation is proved in [8] for weak solutions in a bounded domain in the Hölder space $u \in L^3(0, T; C^\alpha(\bar{\Pi}))$ for $\alpha > 1/3$. The weak solutions constructed in Theorem 1.2 are indeed Hölder continuous of exponent $\alpha = 1 - 3/s > 1/3$ if in addition that $\omega_0^\theta/r \in L^s$ for $s \in (9/2, \infty]$. If the result of [8] holds also for the infinite cylinder, the weak solutions in Theorem 1.2 satisfy (1.8) even for $s \in (9/2, \infty]$.

We outline the proofs of Theorems 1.1-1.3. By a local well-posedness result of (1.1) in [1], there exist local-in-time smooth axisymmetric solutions without swirl $u \in C([0, T]; L^p) \cap C^\infty(\bar{\Pi} \times (0, T])$ for $u_0 \in L_\sigma^p$ and $p \in [3, \infty)$ satisfying the integral equation

$$u = e^{-t\nu A} u_0 - \int_0^t e^{-(t-s)\nu A} \mathbb{P} (u \cdot \nabla u)(s) ds.$$

Here, A denotes the Stokes operator subject to the Neumann boundary condition. We establish an a priori estimate in L^p based on the vorticity equation. Since the vorticity $\omega = \omega^\theta e_\theta$ vanishes on the boundary subject to the Neumann boundary condition, ω^θ/r satisfies the drift-diffusion equation with the homogeneous Dirichlet boundary condition:

$$(1.9) \quad \begin{aligned} \partial_t \left(\frac{\omega^\theta}{r} \right) + u \cdot \nabla \left(\frac{\omega^\theta}{r} \right) - \nu \left(\Delta + \frac{2}{r} \partial_r \right) \left(\frac{\omega^\theta}{r} \right) &= 0 \quad \text{in } \Pi \times (0, T), \\ \frac{\omega^\theta}{r} &= 0 \quad \text{on } \partial\Pi \times (0, T). \end{aligned}$$

We prove the a priori estimate

$$(1.10) \quad \left\| \frac{\omega^\theta}{r} \right\|_{L^p(\Pi)} \leq \frac{C}{(\nu t)^{\frac{3}{2}(\frac{1}{q} - \frac{1}{\rho})}} \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)}, \quad t > 0, \nu > 0,$$

for $1 \leq q \leq \rho \leq \infty$, with $C = C(q)$, independent of ρ and ν . The estimate (1.10) for $\rho = q$ is proved in [81] for $\Pi = \mathbb{R}^3$. Moreover, the decay estimate for $\rho \in [1, \infty]$ and $q = 1$ is established in [32]. We prove (1.10) for the infinite cylinder Π . Since local-in-time solutions of (1.1) are smooth for $t > 0$, we may assume that ω_0^θ/r is bounded. Then, the a priori estimate (1.10) for $\rho = q = \infty$ implies that vorticity is uniformly bounded in the infinite cylinder $\Pi = \{r < 1\}$. Since $\mathbb{P}u \cdot \nabla u = \mathbb{P}\omega \times u$, by the Gronwall's inequality we obtain an exponential bound of the form

$$\|u\|_{L^p(\Pi)} \leq C \|u_0\|_{L^p(\Pi)} \exp \left(C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\infty(\Pi)} t \right), \quad t \geq 0,$$

with some constant $C = C(p)$, independent of ν . This estimate implies that the L^p -norm does not blow-up. Hence the local-in-time solutions are continued for all time. (Moreover, the solutions converge to zero in L^p for $\rho \in (p, \infty)$ as time goes to infinity; see Remarks 2.6 (ii).)

The proof of Theorem 1.2 is based on the Biot-Savart law in the infinite cylinder. We show that axisymmetric vector fields without swirl satisfy

$$(1.11) \quad u = \nabla \times (-\Delta_D)^{-1} (\nabla \times u),$$

$$(1.12) \quad \|u\|_{L^p} + \|\nabla u\|_{L^q} \leq C \|\nabla \times u\|_{L^q}, \quad 1/p = 1/q - 1/3,$$

with $C = C(q)$. The existence of global weak solutions (i) and regularity properties (ii) follow from the a priori estimate (1.10) for $\rho = q$ and (1.12) by taking a vanishing viscosity limit and applying an abstract compactness theorem. The uniqueness in Theorem 1.2 (iii) is based on the growth estimate of the L^ρ -norm

$$(1.13) \quad \|\nabla u\|_{L^p(\Pi)} \leq C\rho \|\nabla \times u\|_{L^p \cap L^{\rho_0}(\Pi)},$$

for $3 < \rho_0 < \rho < \infty$ with some absolute constant C , independent of ρ . The estimate (1.13) is proved in [84] for bounded domains. We extend it for the infinite cylinder and adjust the Yudovich's energy method of uniqueness [85] for solutions with infinite energy by a cut-off function argument.

The convergence of the energy dissipation (1.6) follows from the vorticity estimate (1.10) for $\rho = 2$ and $q \in [3/2, 2]$. The L^2 -convergence (1.7) is based on the estimate (1.13). Since the condition $\omega_0^\theta/r \in L^\infty$ implies that $\nabla \times u_\nu$ is uniformly bounded for $\nu > 0$ and $\rho > 3$, we estimate the energy norm of $u_\nu - u_\mu$ for two solutions of (1.1) by using the estimate (1.13).

Let us remark a novelty from the perspective of technique. Although the a priori estimate (1.10) for $\rho = q = \infty$ is well known, the exponential bound on the L^p -norm of u is a new estimate. Since a bound for ω^θ/r does not imply that for ω^θ in \mathbb{R}^3 , the exponential bound can not be obtained for \mathbb{R}^3 in the same way as the cylinder Π . We developed the L^p -theory of (1.1) for the cylinder with the Neumann boundary condition which may be non-standard compared with \mathbb{R}^3 or bounded domains with the Dirichlet boundary condition. The technical difficulty is the construction of the local-in-time solutions in this setting [1], which relies on linear tools developed in [75], [6], [39]. Once the local well-posedness is established, global-in-time solutions for axisymmetric data without swirl are constructed by a simple approach as explained above.

During the revision of this paper, the author learned existence results on global weak solutions to the Euler equation (1.2) in \mathbb{R}^3 for axisymmetric data without swirl [14], [13]. In the papers, global weak solutions are constructed under a weak regularity condition for ω_0^θ/r , motivated by two-dimensional weak solutions for vortex-sheet initial data [23], [29], [66] (see [67]). The constructions in [14], [13] are based on approximation of initial data and a global well-posedness result of the Euler equations. More recently, global weak solutions are constructed in [48] for axisymmetric data $u_0 \in L_\sigma^2(\mathbb{R}^3)$ satisfying $\omega_0^\theta/r \in L^q \cap L^1(\mathbb{R}^3)$ for $q \in (1, \infty)$, by a vanishing viscosity method based on the global well-posedness result to (1.1) in $H^2(\mathbb{R}^3)$ [58], [81], [60]. Our construction is based on the L^p -global wellposedness result to (1.1) (Theorem 1.1) which enables us to study vanishing viscosity limits under the condition $u_0 \in L_\sigma^p(\Pi)$ for $p \in [3, \infty)$ without approximation of initial data or a finite energy condition.

The L^p -theory for axisymmetric solutions to (1.1) is a new tool to study Leray-Hopf weak solutions [61], [44]. The Leray-Hopf weak solutions are known to exist for general $u_0 \in L_\sigma^2(\mathbb{R}^3)$ and their regularity and uniqueness are questions. If the problem (1.1) is globally well-posed on L^p for $p \in [3, \infty)$, one can deduce that they are smooth for all $t > 0$. See [61] for Leray's structure theorem (also [40, Theorem 5]). In the present setting, we may construct axisymmetric Leray-Hopf weak solutions without swirl for $u_0 \in L_\sigma^2$ by approximation of initial data and using Theorem 1.1. Since (1.1) is globally well-posed on L^p for axisymmetric data without swirl by Theorem 1.1, such solutions will be smooth for all $t > 0$.

On the other hand, uniqueness is a question even if Leray-Hopf weak solutions are axisymmetric without swirl. See [59], [47]. It is also a question whether there exists a Leray-Hopf weak solution for $u_0 \in L_\sigma^2$ satisfying the energy equality (1.5). If $u_0 \in L_\sigma^p \cap L^2$ for $p \in [3, \infty)$, (1.5) holds as in Theorem 1.3. The energy equality (1.5) is necessary to study the energy conservation for vanishing viscosity limits (1.8).

It is an interesting question whether the energy dissipation tends to a positive constant as $\nu \rightarrow 0$. If $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 2]$, u_0 belongs to L_σ^p for some $p \in [3, 6]$ by the Biot-Savart law and the energy dissipation converges to zero as in (1.6). The case $q \in (1, 3/2)$ is a question. We may study this case for strong solutions assuming that $u_0 \in L_\sigma^p$ for $p \in [3, \infty)$ or for Leray-Hopf weak solutions for $u_0 \in L_\sigma^2$. The case $q \in (1, 6/5]$ seems particularly

important since the decay estimate (1.10) does not imply the convergence (1.6). We give a detailed comment for $q = 1$ in Remarks 6.4 (i).

This paper is organized as follows. In Section 2, we prove the vorticity estimate (1.10) for local-in-time solutions to (1.1). Since we only use the estimate (1.10) for $\rho = q = \infty$ in order to prove Theorem 1.1, we give a proof for the case $\rho \neq q$ in Appendix A. In Section 3, we prove the Biot-Savart law (1.11) and the estimate (1.13). In Section 4, we prove Theorem 1.2 (i) and (ii) by applying a vanishing viscosity method. In Section 5, we prove Theorem 1.2 (iii). In Section 6, we prove Theorem 1.3.

2. GLOBAL SMOOTH SOLUTIONS WITH VISCOSITY

We prove Theorem 1.1. We first observe unique existence of local-in-time axisymmetric solutions without swirl for $u_0 \in L_\sigma^p$ and $p \in [3, \infty)$. The vorticity estimate (1.10) for $\rho = q$ is obtained by integration by parts for $q \in [1, \infty]$. Throughout this section, we denote solutions of (1.1) by $u = u_\nu$ and suppressing $\nu > 0$.

2.1. Local-in-time solutions. We set the Laplace operator subject to the Neumann boundary condition

$$\begin{aligned} Bu &= -\Delta u, \quad \text{for } u \in D(B), \\ D(B) &= \{u \in W^{2,p}(\Pi) \mid \nabla \times u \times n = 0, u \cdot n = 0 \text{ on } \partial\Pi\}, \end{aligned}$$

It is proved in [1, Lemma B.1] that the operator $-B$ generates a bounded C_0 -analytic semigroup on L^p ($1 < p < \infty$) for the infinite cylinder Π . We set the Stokes operator

$$\begin{aligned} Au &= Bu, \quad \text{for } u \in D(A), \\ D(A) &= L_\sigma^p \cap D(B). \end{aligned}$$

Since $Au \in L_\sigma^p$ by the Neumann boundary condition, the operator $-A$ generates a bounded C_0 -analytic semigroup on the solenoidal vector space L_σ^p . By the analyticity of the semigroup, we are able to construct local-in-time solutions satisfying the integral form

$$(2.1) \quad u = e^{-tvA}u_0 - \int_0^t e^{-(t-s)\nu A} \mathbb{P}(u \cdot \nabla u)(s) ds, \quad 0 \leq t \leq T,$$

for some $T > 0$, depending on $\nu > 0$. By a standard argument using a fractional power of the Stokes operator, it is not difficult to see that all derivatives of the mild solution belong to the Hölder space $C^\mu((0, T]; L^s)$ for $\mu \in (0, 1/2)$ and $s \in (p, \infty)$. Hence the mild solution is smooth for $t > 0$ and satisfies (1.1).

Lemma 2.1. *For an axisymmetric vector field without swirl $u_0 \in L^p_\sigma$ and $p \in [3, \infty)$, there exists $T > 0$ and a unique axisymmetric mild solution without swirl $u \in C([0, T]; L^p) \cap C^\infty(\overline{\Pi} \times (0, T])$ of (1.1). All derivatives of u belong to $C^\mu((0, T]; L^s)$ for $\mu \in (0, 1/2)$ and $s \in (p, \infty)$.*

Proof. The unique existence of local-in-time smooth mild solutions is proved in [1, Theorems 1.1, 6.3]. The axial symmetry follows from the uniqueness. We consider a rotation operator $U : f \mapsto {}^t R f(Rx)$ for $R = (e_r(\eta), e_\theta(\eta), e_z)$ and $\eta \in [0, 2\pi]$. Since the Stokes semigroup $e^{-t\nu A}$ and the Helmholtz projection \mathbb{P} are commutable with the operator U (see [2, Proposition 2.6]), by multiplying U by (2.1) we see that Uu is a mild solution for the same axisymmetric initial data u_0 . By the uniqueness of mild solutions, the function Uu agrees with u . Hence $u(x, t) = {}^t R u(Rx, t)$ for $\eta \in [0, 2\pi]$ and u is axisymmetric.

It is not difficult to see that u is without swirl. By the Neumann boundary condition in (1.1), we see that an axisymmetric solution $u = u^r e_r + u^\theta e_\theta + u^z e_z$ satisfies

$$(2.2) \quad u^r = 0, \quad \partial_r u^\theta + u^\theta = 0, \quad \partial_r u^z = 0 \quad \text{on } \{r = 1\}.$$

Since u is smooth for $t > 0$ and $u_0^\theta = 0$, by a fundamental calculation, we see that $\varphi = u^\theta e_\theta \in C([0, T]; L^p) \cap C^\infty(\overline{\Pi} \times (0, T))$ satisfies

$$\begin{aligned} \partial_t \varphi - \Delta \varphi + u \cdot \nabla \varphi - (\partial_r u^r + \partial_z u^z) \varphi &= 0 \quad \text{in } \Pi \times (0, T), \\ \partial_n \varphi + \varphi &= 0 \quad \text{on } \partial \Pi \times (0, T), \\ \varphi &= 0 \quad \text{on } \Pi \times \{t = 0\}. \end{aligned}$$

Since the Laplace operator $-\Delta_R$ with the Robin boundary condition generates a C_0 -analytic semigroup on L^p [5] ([65, Theorem 3.1.3]), by the uniqueness of the inhomogeneous heat equation, the function φ satisfies the integral form

$$\varphi = - \int_0^t e^{(t-s)\nu \Delta_R} (u \cdot \nabla \varphi - (\partial_r u^r + \partial_z u^z) \varphi) ds.$$

Since $u \in C([0, T]; L^p)$ and $t^{1/2} \nabla u \in C([0, T]; L^p)$, it is not difficult to show that $\varphi \equiv 0$ by estimating L^p -norms of φ . Hence the local-in-time solution u is axisymmetric without swirl. \square

In order to prove Theorem 1.3 later in Section 6, we show that local-in-time solutions satisfy the energy equality (1.5) for initial data with finite energy.

Proposition 2.2. *For axisymmetric initial data without swirl $u_0 \in L^p_\sigma \cap L^2$ for $p \in [3, \infty)$, the local-in-time solution u satisfies*

$$(2.3) \quad u, t^{1/2} \nabla u \in C([0, T]; L^p \cap L^2),$$

and the energy equality (1.5) for $t \geq 0$.

Proof. We prove (2.3). The energy equality (1.5) follows from (2.2) and integration by parts. We give a proof for the case $p = 3$ since we are able to prove the case $p \in (3, \infty)$ in a similar way. We may assume that $\nu = 1$. We invoke an iterative argument in [1, Theorem 5.2]. We use regularizing estimates of the Stokes semigroup [1, Lemma 5.1],

$$(2.4) \quad \|\partial_x^k e^{-tA} f\|_{L^2} \leq \frac{C}{t^{\frac{3}{2}(\frac{1}{\rho}-\frac{1}{2})+\frac{|k|}{2}}} \|f\|_{L^\rho}$$

for $t \leq T_0$, $|k| \leq 1$ and $\rho \in [6/5, 2]$ with $C = C(\rho, k, T_0)$. We set a sequence $\{u_j\}$ as usual by $u_1 = e^{-tA} u_0$,

$$u_{j+1} = e^{-tA} u_0 - \int_0^t e^{-(t-s)A} \mathbb{P} u_j \cdot \nabla u_j ds, \quad j \geq 1,$$

and the constants

$$K_j = \sup_{0 \leq t \leq T} t^\gamma \|u_j\|_{L^q}(t),$$

$$M_j = \sup_{0 \leq t \leq T} (\|u_j\|_{L^2} + t^{1/2} \|\nabla u_j\|_{L^2}),$$

for $\gamma = 3/2(1/3 - 1/q)$ and $q \in (3, \infty)$. Then, we have $K_j \leq K_1$ for all $j \geq 1$ for sufficiently small $T > 0$. We set $1/\rho = 1/2 + 1/q$. Since $\rho \in (6/5, 2]$, applying (2.4) and the Hölder inequality imply that

$$\begin{aligned} \|u_{j+1}\|_{L^2} &\leq \|e^{-tA} u_0\|_{L^2} + \int_0^t \frac{C}{(t-s)^{\frac{3}{2}(\frac{1}{\rho}-\frac{1}{2})}} \|u_j \cdot \nabla u_j\|_{L^\rho} ds \\ &\leq \|e^{-tA} u_0\|_{L^2} + C' K_j M_j. \end{aligned}$$

We estimate the L^2 -norm of ∇u_{j+1} in a similar way and obtain

$$M_{j+1} \leq M_1 + CK_1 M_j,$$

with $C = C(q)$. We take $T > 0$ sufficiently small so that $CK_1 \leq 1/2$ and obtain the uniform bound $M_{j+1} \leq 2M_1$ for all $j \geq 1$. Since the sequence $\{u_j\}$ converges to a limit $u \in C([0, T]; L^3)$ such that $t^{1/2} \nabla u \in C([0, T]; L^3)$, in a similar way, the uniform estimate for M_j is inherited by the limit. We obtained (2.3). \square

2.2. Vorticity estimates. We shall prove the vorticity estimate (1.10) for $\rho = q \in [1, \infty]$.

Lemma 2.3. *Let u be an axisymmetric solution as in Lemma 2.1. Assume that $\omega_0^\theta/r \in L^q$ for $q \in [1, \infty]$. Then, the estimate*

$$(2.5) \quad \left\| \frac{\omega^\theta}{r} \right\|_{L^q(\Pi)} \leq \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)}, \quad t > 0$$

holds.

Proof. We prove (2.5) for $q \in [1, \infty)$. The case $q = \infty$ follows by sending $q \rightarrow \infty$. We observe that ω^θ/r is smooth for $t > 0$ and satisfies

$$(2.6) \quad \begin{aligned} \partial_t \left(\frac{\omega^\theta}{r} \right) + u \cdot \nabla \left(\frac{\omega^\theta}{r} \right) - \nu \left(\Delta + \frac{2}{r} \partial_r \right) \left(\frac{\omega^\theta}{r} \right) &= 0 \quad \text{in } \Pi \times (0, T), \\ \left(\frac{\omega^\theta}{r} \right) &= 0 \quad \text{on } \partial\Pi \times (0, T). \end{aligned}$$

In order to differentiate the L^q -norm of $\Omega = \omega^\theta/r$, we approximate the absolute value function $\psi(s) = |s|$. For an arbitrary $\varepsilon > 0$, we set a smooth non-negative convex function $\psi_\varepsilon(s) = (s^2 + \varepsilon^2)^{1/2} - \varepsilon$ for $s \in \mathbb{R}$, i.e., $0 \leq \psi_\varepsilon \leq |s|$, $\dot{\psi}_\varepsilon > 0$. The function ψ_ε satisfies $\psi_\varepsilon(0) = \dot{\psi}_\varepsilon(0) = 0$. We differentiate $\psi_\varepsilon^q(\Omega)$ to see that

$$\begin{aligned} \partial_t(\psi_\varepsilon^q(\Omega)) &= q(\partial_t \Omega) \dot{\psi}_\varepsilon(\Omega) \psi_\varepsilon^{q-1}(\Omega), \\ \nabla(\psi_\varepsilon^q(\Omega)) &= q(\nabla \Omega) \dot{\psi}_\varepsilon(\Omega) \psi_\varepsilon^{q-1}(\Omega). \end{aligned}$$

Since $\psi_\varepsilon(\Omega) = \dot{\psi}_\varepsilon(\Omega) = 0$ on $\partial\Pi$ by the boundary condition, integration by parts yields

$$\begin{aligned} \frac{d}{dt} \int_\Pi \psi_\varepsilon^q(\Omega) dx &= q \int_\Pi \left(-u \cdot \nabla \Omega + \nu \Delta \Omega + 2\nu \frac{1}{r} \partial_r \Omega \right) \dot{\psi}_\varepsilon(\Omega) \psi_\varepsilon^{q-1}(\Omega) dx \\ &= - \int_\Pi u \cdot \nabla(\psi_\varepsilon^q(\Omega)) dx + \nu q \int_\Pi \Delta \Omega \dot{\psi}_\varepsilon(\Omega) \psi_\varepsilon^{q-1}(\Omega) dx + 2\nu \int_\Pi \frac{1}{r} \partial_r(\psi_\varepsilon^q(\Omega)) dx. \end{aligned}$$

The first-term vanishes by the divergence-free condition. Since ψ_ε is non-negative and convex, we see that

$$\begin{aligned} \int_\Pi \Delta \Omega \dot{\psi}_\varepsilon(\Omega) \psi_\varepsilon^{q-1}(\Omega) dx &= - \int_\Pi |\nabla \Omega|^2 ((q-1) \psi_\varepsilon^{q-2}(\Omega) |\dot{\psi}_\varepsilon(\Omega)|^2 + \psi_\varepsilon^{q-1}(\Omega) \ddot{\psi}_\varepsilon(\Omega)) dx \\ &\leq -\frac{4}{q} \left(1 - \frac{1}{q}\right) \int_\Pi |\nabla(\psi_\varepsilon^{q/2}(\Omega))|^2 dx, \\ \int_\Pi \frac{1}{r} \partial_r(\psi_\varepsilon^q(\Omega)) dx &= 2\pi \int_{\mathbb{R}} dz \int_0^1 \partial_r(\psi_\varepsilon^q(\Omega)) dr = -2\pi \int_{\mathbb{R}} \psi_\varepsilon^q(\Omega(0, z, t)) dz \leq 0, \end{aligned}$$

for $\Omega = \Omega(r, z, t)$. Hence we have

$$(2.7) \quad \frac{d}{dt} \int_{\Pi} \psi_{\varepsilon}^q(\Omega) dx + 4\nu \left(1 - \frac{1}{q}\right) \int_{\Pi} |\nabla(\psi_{\varepsilon}^{q/2}(\Omega))|^2 dx \leq 0.$$

We integrate in $[0, t]$ and estimate

$$\int_{\Pi} \psi_{\varepsilon}^q(\Omega) dx \leq \int_{\Pi} \psi_{\varepsilon}^q(\Omega_0) dx, \quad t \geq 0.$$

Since $\psi_{\varepsilon}(s)$ monotonically converges to $\psi(s) = |s|$, sending $\varepsilon \rightarrow 0$ implies the desired estimate for $q \in [1, \infty)$. \square

Remark 2.4. Since the operator $\Delta_x + 2r^{-1}\partial_r$ can be regarded as the Laplace operator in \mathbb{R}^5 , it is also possible to prove (2.5) for $q = \infty$ by a maximum principle. See [56], [32].

We further deduce the decay estimate of vorticity from the inequality (2.7). We give a proof for the following Lemma 2.5 in Appendix A.

Lemma 2.5. *Under the same assumption of Lemma 2.3, the estimate*

$$(2.8) \quad \left\| \frac{\omega^{\theta}}{r} \right\|_{L^p(\Pi)} \leq \frac{C}{(\nu t)^{\frac{3}{2}(\frac{1}{q} - \frac{1}{\rho})}} \left\| \frac{\omega_0^{\theta}}{r} \right\|_{L^q(\Pi)}, \quad t > 0, \nu > 0,$$

holds for $1 \leq q \leq \rho \leq \infty$ with some constant $C = C(q)$, independent of ρ and ν .

2.3. An exponential bound. We now complete:

Proof of Theorem 1.1. Let $u \in C([0, T]; L^p) \cap C^{\infty}(\overline{\Pi} \times (0, T])$ be a local-in-time axisymmetric solution as in Lemma 2.1 for $T = T_{\nu} > 0$. Since all derivatives of u belong to $C^{\mu}((0, T]; L^s)$ for $\mu \in (0, 1/2)$ and $s \in (p, \infty)$, particularly we have $\nabla^2 u \in C^{\mu}((0, T]; L^{\infty})$ and $\omega^{\theta}/r = -e_r \cdot (e_{\theta} \cdot \nabla \omega) \in C^{\mu}((0, T]; L^{\infty})$.

We take an arbitrary $t_0 \in (0, T)$. We show that the L^p -norm of u is globally bounded for $t \geq t_0$. By translation, we set

$$\tilde{u}(\cdot, t) = u(\cdot, t + t_0), \quad 0 \leq t \leq T - t_0.$$

The function \tilde{u} is a local-in-time solution to (1.1) satisfying $\tilde{u}_0 = \tilde{u}(\cdot, 0) \in L^p_{\sigma}$, $\tilde{\omega}_0/r = (\tilde{\omega}_0/r)(\cdot, 0) \in L^{\infty}$ and

$$(2.9) \quad \tilde{u} = e^{-t\nu A} \tilde{u}_0 - \int_0^t e^{-(t-s)\nu A} \mathbb{P}(\tilde{u} \cdot \nabla \tilde{u})(s) ds, \quad 0 \leq t \leq T - t_0.$$

We apply Lemma 2.3 for \tilde{u} to estimate

$$\left\| \frac{\tilde{\omega}^\theta}{r} \right\|_{L^\infty(\Pi)} \leq \left\| \frac{\tilde{\omega}_0^\theta}{r} \right\|_{L^\infty(\Pi)}, \quad t \geq 0.$$

Since $r < 1$, this yields a uniform estimate of vorticity

$$\|\tilde{\omega}^\theta\|_{L^\infty(\Pi)} \leq \left\| \frac{\tilde{\omega}_0^\theta}{r} \right\|_{L^\infty(\Pi)}, \quad t \geq 0.$$

Since $\mathbb{P}\tilde{u} \cdot \nabla \tilde{u} = \mathbb{P}\tilde{\omega} \times \tilde{u}$ and the Stokes semigroup is a bounded semigroup on L^p , it follows from (2.9) that

$$\|\tilde{u}\|_{L^p(\Pi)} \leq C_1 \|\tilde{u}_0\|_{L^p(\Pi)} + C_2 \left\| \frac{\tilde{\omega}_0^\theta}{r} \right\|_{L^\infty(\Pi)} \int_0^t \|\tilde{u}\|_{L^p(\Pi)} ds, \quad t \geq 0.$$

with some constants C_1 and C_2 , independent of $\nu > 0$. Applying the Gronwall's inequality yields

$$(2.10) \quad \|\tilde{u}\|_{L^p(\Pi)} \leq C_1 \|\tilde{u}_0\|_{L^p(\Pi)} \exp\left(C_2 \left\| \frac{\tilde{\omega}_0^\theta}{r} \right\|_{L^\infty(\Pi)} t\right), \quad t \geq 0.$$

Since the L^p -norm of \tilde{u} is globally bounded, so is u . Hence the local-in-time solution u is continued for all $t \geq T$. The proof is complete. \square

Remarks 2.6. (i) ($p = \infty$) It is unknown whether the assertion of Theorem 1.1 holds for $p = \infty$. For the two-dimensional Cauchy problem, unique existence of global-in-time solutions is known for bounded and non-decaying initial data $u_0 \in L^\infty_\sigma$ [41]. Moreover, global-in-time solutions satisfy a single exponential bound of the form

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C_1 \|u_0\|_{L^\infty(\mathbb{R}^2)} \exp(C_2 \|\omega_0\|_{L^\infty(\mathbb{R}^2)} t), \quad t \geq 0,$$

with some constants C_1 and C_2 , independent of viscosity [72]. The single exponential bound is further improved to a linear growth estimate as $t \rightarrow \infty$ by using viscosity. See [86], [34]. For a two-dimensional layer, unique global-in-time solutions exist for bounded initial data subject to the Neumann boundary condition. Moreover, the L^∞ -norm of solutions are uniformly bounded for all time [36], [35].

(ii) (Large time behavior) Global-in-time solutions in Theorem 1.1 are uniformly bounded for all time, i.e., $u \in BC([0, \infty); L^p)$. In fact, we are able to assume that $\omega_0^\theta/r \in L^p$ by

replacing the initial time since local-in-time solutions $u(\cdot, t)$ belong to $W^{2,p}$ for $p \in [3, \infty)$. As proved later in Lemma 3.5 and Proposition 4.3, since axisymmetric solutions of (1.1) are uniquely determined by the Biot-Savart law and the Poincaré inequality holds in the cylinder, we have

$$\|u\|_{L^p(\Pi)} \leq C\|\nabla u\|_{L^p(\Pi)} \leq C'\|\nabla \times u\|_{L^p(\Pi)} \leq C'\left\|\frac{\omega^\theta}{r}\right\|_{L^p(\Pi)},$$

with some constants $C = C(p)$ and $C' = C'(p)$. By the vorticity estimate (2.8), the solutions are uniformly bounded in L^p and tend to zero in L^p for $p \in (p, \infty)$ as $t \rightarrow \infty$.

3. THE BIOT-SAVART LAW

In this section, we give a Biot-Savart law in the infinite cylinder (Lemma 3.5). Since stream functions exist for axisymmetric vector fields without swirl and satisfy the Dirichlet boundary condition, we are able to represent axisymmetric vector fields without swirl by the Laplace operator with the Dirichlet boundary condition $u = \nabla \times (-\Delta_D)^{-1} \nabla \times u$. We first prepare L^p -estimates for the Dirichlet problem of the Poisson equation and apply them to axisymmetric vector fields without swirl.

3.1. L^p -estimates for the Poisson equation. We consider the Poisson equation in the infinite cylinder:

$$(3.1) \quad -\Delta\phi = f \quad \text{in } \Pi, \quad \phi = 0 \quad \text{on } \partial\Pi.$$

Lemma 3.1. (i) Let $q \in (1, \infty)$. For $f \in L^q$, there exists a unique solution $\phi \in W^{2,q}$ of (3.1) satisfying

$$(3.2) \quad \|\phi\|_{W^{2,q}} \leq C\|f\|_{L^q},$$

with some constant $C = C(q)$.

(ii) For $q \in (1, 3)$ and $p \in (3/2, \infty)$ satisfying $1/p = 1/q - 1/3$, there exists a constant C' such that

$$(3.3) \quad \|\nabla\phi\|_{L^p} \leq C'\|f\|_{L^q}.$$

with some constant $C' = C'(q)$.

We prove Lemma 3.1 by using the heat semigroup $e^{t\Delta_D}$.

Proposition 3.2. (i) *There exists a constant M such that*

$$(3.4) \quad \|e^{t\Delta_D} f\|_{L^q} \leq e^{-\mu_q t} \|f\|_{L^q}, \quad t > 0,$$

for $f \in L^q$ with the constant $\mu_q = M/(qq')$, where q' is the conjugate exponent to $q \in (1, \infty)$.

(ii) *The heat kernel $K(x, y, t)$ of $e^{t\Delta_D}$ satisfies the Gaussian upper bound,*

$$(3.5) \quad 0 \leq K(x, y, t) \leq \frac{1}{(4\pi t)^{3/2}} e^{-|x-y|^2/4t}, \quad x, y \in \Pi, t > 0.$$

Proof. The pointwise upper bound (3.5) is known for an arbitrary domain. See [22, Example 2.1.8]. We prove the assertion (i). It suffices to show (3.4) for $f \in C_c^\infty$. Suppose that $f \geq 0$. Then, $u = e^{t\Delta_D} f$ is non-negative by a maximum principle. By multiplying qu^{q-1} by the heat equation and integration by parts, we see that $\varphi = u^{q/2}$ satisfies

$$\frac{d}{dt} \int_{\Pi} |\varphi|^2 dx + \frac{4}{q'} \int_{\Pi} |\nabla \varphi|^2 dx = 0.$$

Since the function φ vanishes on $\partial\Pi$, we apply the Poincaré inequality in the cylinder $\|\varphi\|_{L^2} \leq C\|\nabla\varphi\|_{L^2}$ [4, 6.30 THEOREM] to estimate

$$\frac{d}{dt} \int_{\Pi} |\varphi|^2 dx \leq -\frac{4}{C^2 q'} \int_{\Pi} |\varphi|^2 dx.$$

By the Gronwall's inequality,

$$\|\varphi\|_2^2 \leq \|\varphi(\cdot, 0)\|_2^2 e^{-4t/C^2 q'}.$$

Since $\|\varphi\|_2^2 = \|u\|_q^q$, the estimate (3.4) holds with the constant $M = 4/C^2$. For general $f \in C_c^\infty$, we approximate the absolute value function as in the proof of Lemma 2.3 and obtain (3.4). \square

By (3.4), we are able to define the inverse operator of $-\Delta_D$ by using the Laplace transform.

Proof of Lemma 3.1. We prove (i). We set

$$\phi = \int_0^\infty e^{t\Delta_D} f dt \quad \text{for } f \in L^q.$$

Since the heat semigroup is an analytic semigroup on L^q , it follows from (3.4) that

$$(3.6) \quad \|\phi\|_{W^{1,q}} \leq C \left(1 + \frac{1}{\mu_q}\right) \|f\|_{L^q}$$

with some constant $C = C(q)$, independent of q . Since $-\Delta\phi = f$, by the elliptic regularity estimate [5], it follows that

$$(3.7) \quad \|\nabla^2\phi\|_{L^q} \leq C(\|f\|_{L^q} + \|\phi\|_{W^{1,q}}),$$

with $C = C(q)$. We obtained (3.2). The uniqueness follows from a maximum principle.

We prove (ii). Since $\phi = (-\Delta_D)^{-1}f = (-\Delta_D)^{-1/2}(-\Delta_D)^{-1/2}f$, we use a fractional power of the operator $-\Delta_D$. We set the domain $D(-\Delta_D)$ by a space of all functions in $W^{2,q}$, vanishing on $\partial\Pi$. By estimates of pure imaginary powers of the operator [73], the domain of the fractional power $D((-\Delta_D)^{1/2})$ is continuously embedded to the Sobolev space $W^{1,q}$. Hence the operator $\partial(-\Delta_D)^{-1/2}$ acts as a bounded operator on L^q .

It suffices to show that the fractional power $(-\Delta_D)^{-1/2}$ acts as a bounded operator from L^q to L^p . We see that

$$((-\Delta_D)^{-1/2}f)(x) = \int_0^\infty t^{-1/2} e^{t\Delta_D} f dt = \int_\Pi f(y) dy \int_0^\infty t^{-1/2} K(x, y, t) dt.$$

By (3.5), we have

$$\int_0^\infty t^{-1/2} K(x, y, t) dt \leq \frac{C}{|x - y|^2},$$

with some absolute constant C . Since the operator $f \mapsto |x|^{-2} * f$ acts as a bounded operator from L^q to L^p for $1/p = 1/q - 1/3$ by the Hardy-Littlewood-Sobolev inequality [77, p.354], so is $(-\Delta_D)^{-1/2}$. The proof is complete. \square

3.2. Dependence of a constant. The growth rate of the constant in (3.2) is at most linear as $q \rightarrow \infty$.

Lemma 3.3. *Let $q_0 \in (3, \infty)$. There exists a constant C such that*

$$(3.8) \quad \|\phi\|_{W^{2,q}(\Pi)} \leq Cq \|f\|_{L^q \cap L^{q_0}(\Pi)}$$

holds for solutions of (3.1) for $f \in L^q \cap L^{q_0}(\Pi)$ and $q \in [q_0, \infty)$, where

$$\|f\|_{L^q \cap L^{q_0}} = \max\{\|f\|_{L^q}, \|f\|_{L^{q_0}}\}.$$

The constant C is independent of q .

We prove Lemma 3.3 by a cut-off function argument.

Proposition 3.4. *Let G be a smoothly bounded domain in \mathbb{R}^3 . Let $q_0 \in (3, \infty)$. There exists a constant C such that*

$$(3.9) \quad \|\phi\|_{W^{2,q}(G)} \leq Cq\|f\|_{L^q(G)}$$

holds for solutions of (3.1) for $f \in L^q(G)$ and $q \in [q_0, \infty)$. The constant C is independent of q .

Proof. The assertion is proved in [84, Corollary 1] for general elliptic operators and n -dimensional bounded domains. \square

Proof of Lemma 3.3. Let $\{\varphi_j\}_{j=-\infty}^{\infty} \subset C_c^\infty(\mathbb{R})$ be a partition of the unity such that $0 \leq \varphi_j \leq 1$, $\text{spt } \varphi_j \subset [j-1, j+1]$ and $\sum_{j=-\infty}^{\infty} \varphi_j(x_3) = 1$, $x_3 \in \mathbb{R}$. Let $\phi \in W^{2,q}(\Pi)$ be a solution of (3.1) for $f \in L^q \cap L^{q_0}(\Pi)$. We set $\phi_j = \phi\varphi_j$ and observe that

$$\begin{aligned} -\Delta\phi_j &= f_j & \text{in } G_j, \\ \phi_j &= 0 & \text{on } \partial G_j, \end{aligned}$$

for $G_j = D \times (j-1, j+1)$ and $f_j = f\varphi_j - 2\nabla\phi \cdot \nabla\varphi_j - \phi\Delta\varphi_j$. We take a smooth bounded domain \tilde{G}_j such that $G_j \subset \tilde{G}_j \subset D \times [j-2, j+2]$ and apply (3.9) to estimate

$$\|\phi_j\|_{W^{2,q}(\tilde{G}_j)} \leq Cq\|f_j\|_{L^q(\tilde{G}_j)}$$

for $q \in [q_0, \infty)$ with some constant C , independent of j and q . It follows that

$$\|\nabla^2\phi\varphi_j\|_{L^q(\Pi)} \leq Cq(\|f\|_{L^q(G_j)} + \|\phi\|_{W^{1,q}(G_j)}).$$

By summing over j , we obtain

$$(3.10) \quad \|\nabla^2\phi\|_{L^q(\Pi)} \leq Cq(\|f\|_{L^q(\Pi)} + \|\phi\|_{W^{1,q}(\Pi)}).$$

We estimate the lower order term of ϕ . By Lemma 3.1(i), we have $\|\phi\|_{W^{2,q_0}} \leq C\|f\|_{L^{q_0}}$ with $C = C(q_0)$. In particular, $\|\phi\|_{W^{1,\infty}} \leq C\|f\|_{L^{q_0}}$ by the Sobolev inequality. Applying the Hölder inequality implies that

$$(3.11) \quad \|\phi\|_{W^{1,q}(\Pi)} \leq C\|f\|_{L^{q_0}(\Pi)}$$

for $q \in [q_0, \infty)$ with some constant C , independent of q . The estimate (3.8) follows from (3.10) and (3.11). \square

3.3. Stream functions. We shall give a Biot-Savart law for axisymmetric vector fields without swirl. We see that a smooth axisymmetric solenoidal vector field without swirl $u = u^r e_r + u^z e_z$ in Π satisfies

$$\begin{aligned} \partial_z(ru^z) + \partial_r(ru^r) &= 0 \quad (z, r) \in \mathbb{R} \times (0, 1), \\ ru^r &= 0 \quad \text{on } \{r = 0, 1\}. \end{aligned}$$

Since (ru^z, ru^r) is regarded as a solenoidal vector field in the two-dimensional layer $\mathbb{R} \times (0, 1)$, there exists a stream function $\psi(r, z)$ such that

$$ru^z = \frac{\partial \psi}{\partial r}, \quad ru^r = -\frac{\partial \psi}{\partial z}.$$

Since ψ is constant on the boundary, we may assume that $\psi = 0$ on $\{r = 1\}$. Since $\phi = (\psi/r)e_\theta$ satisfies

$$\operatorname{div} \phi = 0, \quad \nabla \times \phi = u \quad \text{in } \Pi, \quad \phi = 0 \quad \text{on } \partial\Pi,$$

we see that $-\Delta \phi = \nabla \times u$. Thus the stream function is represented by $\phi = (-\Delta_D)^{-1} \nabla \times u$.

Lemma 3.5. (i) *Let u be an axisymmetric vector field without swirl in L^p_σ such that $\nabla \times u \in L^q$ for $q \in (1, 3)$ and $1/p = 1/q - 1/3$. Then,*

$$(3.12) \quad u = \nabla \times (-\Delta_D)^{-1} (\nabla \times u).$$

(ii) *The estimates*

$$(3.13) \quad \|u\|_{L^p} + \|\nabla u\|_{L^q} \leq C_1 \|\nabla \times u\|_{L^q},$$

$$(3.14) \quad \|\nabla u\|_{L^p} \leq C_2 \|\nabla \times u\|_{L^p}, \quad 1 < p < \infty,$$

$$(3.15) \quad \|\nabla u\|_{L^p} \leq C_3 \rho \|\nabla \times u\|_{L^p \cap L^{\rho_0}}, \quad 3 < \rho_0 < \rho < \infty,$$

hold with some constants $C_1 = C_1(q)$ and $C_2 = C_2(\rho)$. The constant C_3 is independent of ρ .

It suffices to show (3.12). The assertion (ii) follows from Lemmas 3.1 and 3.3.

Proposition 3.6. *Let w be an axisymmetric vector field without swirl in L^p for $p \in (1, \infty)$. Assume that*

$$\operatorname{div} w = 0, \nabla \times w = 0 \quad \text{in } \Pi, \quad w \cdot n = 0 \quad \text{on } \partial\Pi.$$

Then, $w \equiv 0$.

Proof. Since $w = w^r e_r + w^z e_z$ is a harmonic vector field in Π and $\Delta = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_\theta^2 + \partial_z^2$ by the cylindrical coordinate, we see that

$$\begin{aligned} 0 = \Delta w &= \Delta(w^r e_r) + \Delta(w^z e_z) \\ &= \left\{ (\partial_r^2 + r^{-1} \partial_r - r^{-2} + \partial_z^2) w^r \right\} e_r + (\Delta w^z) e_z \\ &= \left\{ (\Delta - r^{-2}) w^r \right\} e_r + (\Delta w^z) e_z. \end{aligned}$$

Hence, $(\Delta - r^{-2})w^r = 0$ and $\Delta w^z = 0$. By $\Delta(w^r e_r) = \{(\Delta - r^{-2})w^r\} e_r = 0$, $w^r e_r$ is harmonic in Π . Since w^r vanishes on the boundary and the operator $-\Delta_D$ is invertible on L^p , we see that $w^r \equiv 0$. By the divergence-free condition $\partial_r w^r + w^r/r + \partial_z w^z = 0$ and a decay condition $w^z \in L^p$, we have $w^z \equiv 0$. \square

Proof of Lemma 3.5. We set

$$\tilde{\phi} = (-\Delta_D)^{-1}(\nabla \times u), \quad \tilde{u} = \nabla \times \tilde{\phi}.$$

Since u is axisymmetric without swirl, $\tilde{\phi}$ is axisymmetric and $\tilde{\phi} = \tilde{\phi}^\theta e_\theta$. Since $\tilde{\phi}$ satisfies

$$\operatorname{div} \tilde{\phi} = 0, \quad -\Delta \tilde{\phi} = \nabla \times u \quad \text{in } \Pi, \quad \tilde{\phi} = 0 \quad \text{on } \partial\Pi,$$

it follows that

$$\operatorname{div} \tilde{u} = 0, \quad \nabla \times \tilde{u} = \nabla \times u \quad \text{in } \Pi, \quad \tilde{u} \cdot n = 0 \quad \text{on } \partial\Pi.$$

Applying Proposition 3.6 for $w = u - \tilde{u}$ implies $u \equiv \tilde{u}$. We proved (3.12). \square

4. VANISHING VISCOSITY LIMITS

We prove Theorem 1.2 (i) and (ii). When the initial vorticity satisfies $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 3)$, the initial velocity belongs to L^p for $p \in [3, \infty)$ by the Biot-Savart law and a global-in-time unique solution $u = u_\nu$ of (1.1) exists by Theorem 1.1. We use the vorticity estimate (2.3) and construct global weak solutions of the Euler equations by sending $\nu \rightarrow 0$. In the subsequent section, we prove Hölder continuity of weak solutions.

4.1. Convergence to a limit. We first derive a priori estimates independent of the viscosity $\nu > 0$.

Lemma 4.1. (i) Let $u_0 \in L_\sigma^p$ be an axisymmetric vector field without swirl such that $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 3)$ and $1/p = 1/q - 1/3$. Let $u_\nu \in C([0, \infty); L^p) \cap C^\infty(\bar{\Pi} \times (0, \infty))$ be a solution of (1.1) for u_0 as in Theorem 1.1. There exists a constant $C = C(q)$ such that

$$(4.1) \quad \|u_\nu\|_{L^p(\Pi)} + \|\nabla u_\nu\|_{L^q(\Pi)} \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)}, \quad t \geq 0, \nu > 0.$$

Moreover, for each bounded domain $G \subset \Pi$, there exists a constant $C' = C'(q)$ such that

$$(4.2) \quad \|\partial_t u_\nu\|_{W^{-1,q}(G)} \leq C' \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)} \left(\nu + \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)} \right), \quad t \geq 0, \nu > 0,$$

where $W^{-1,q}$ denotes the dual space of $W_0^{1,q'}$ and q' is the conjugate exponent to q .

(ii) If $\omega_0^\theta/r \in L^s$ for $s \in (3, \infty)$, then

$$(4.3) \quad \|\nabla u_\nu\|_{L^s(\Pi)} \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^s(\Pi)}, \quad t \geq 0, \nu > 0.$$

with some constant $C = C(s)$.

(iii) If $\omega_0^\theta/r \in L^\infty$, then

$$(4.4) \quad \|\nabla \times u_\nu\|_{L^\infty(\Pi)} \leq \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\infty(\Pi)}, \quad t \geq 0, \nu > 0.$$

Proof. Since $\omega_0^\theta/r \in L^q$, applying Lemma 2.3 implies the vorticity estimate

$$\left\| \frac{\omega_\nu^\theta}{r} \right\|_{L^q} \leq \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q}.$$

Since $r < 1$, the L^q -norm of the vorticity $\nabla \times u_\nu$ is bounded. By the estimate of the Biot-Savart law (3.13), we obtain (4.1). The estimates (4.3) and (4.4) follow in the same way.

We prove (4.2). We take an arbitrary $\varphi \in C_c^\infty(G)$ and consider its zero extension to $\Pi \setminus \overline{G}$ (denoted by $\tilde{\varphi}$). We set $f = \mathbb{P}\varphi$ by the Helmholtz projection operator \mathbb{P} . By a higher regularity estimate of the Helmholtz projection operator [75, Theorem 6], we see that $f \in C^\infty(\overline{\Pi})$ and

$$\|f\|_{W^{1,s}} = \|\mathbb{P}\varphi\|_{W^{1,s}} \leq C\|\varphi\|_{W^{1,s}}$$

with some constant $C = C(s)$ and $s \in (1, \infty)$. By multiplying f by (1.1) and integration by parts, we see that

$$\begin{aligned} \int_{\Pi} \partial_t u_\nu \cdot f dx &= \nu \int_{\Pi} \Delta u_\nu \cdot f dx - \int_{\Pi} (u_\nu \cdot \nabla u_\nu) \cdot f dx \\ &= -\nu \int_{\Pi} \nabla \times u_\nu \cdot \nabla \times f dx + \int_{\Pi} u_\nu u_\nu : \nabla f dx. \end{aligned}$$

By $\operatorname{div} u = 0$, the left-hand side equals to the integral of $\partial_t u_\nu \cdot \varphi$ in G . By applying the estimate of the Helmholtz projection, we obtain

$$\left| \int_G \partial_t u_\nu \cdot \varphi dx \right| \leq C \left(\nu \|\nabla u_\nu\|_{L^q(\Pi)} \|\varphi\|_{W^{1,q'}(G)} + \|u_\nu\|_{L^p(\Pi)}^2 \|\varphi\|_{W^{1,p/(p-2)}(G)} \right),$$

with $C = C(q)$. Since $p/(p-2) \leq q'$, the norms of φ are estimated by the $W^{1,q'}$ -norm of φ in G . By (4.1), we obtain (4.2). \square

We apply the estimates (4.1) and (4.2) in order to extract a subsequence of $\{u_\nu\}$. We recall an abstract compactness theorem in [80, Chapter III, Theorem 2.1].

Proposition 4.2. (i) *Let X_0 , X and X_1 be Banach spaces such that $X_0 \subset X \subset X_1$ with continuous injections, X_0 and X_1 are reflexive and the injection $X_0 \subset X$ is compact. For $T \in (0, \infty)$ and $s \in (1, \infty)$, set the Banach space*

$$Y = \{u \in L^s(0, T; X_0) \mid \partial_t u \in L^s(0, T; X_1)\},$$

equipped with the norm $\|u\|_Y = \|u\|_{L^s(0, T; X_0)} + \|\partial_t u\|_{L^s(0, T; X_1)}$. Then, the injection $Y \subset L^s(0, T; X)$ is compact.

Proof of Theorem 1.2 (i). For an arbitrary bounded domain $G \subset \Pi$, we set $X_0 = W^{1,q}(G)$, $X = L^q(G)$ and $X_1 = W^{-1,q}(G)$. Since $\{u_\nu\}$ is a bounded sequence in Y by (4.1) and (4.2), we apply Proposition 4.2 to get a subsequence (still denoted by u_ν) that converges to a limit u in $L^s(0, T; L^q(G))$. By choosing a subsequence, we may assume that

$$u_\nu \rightarrow u \quad \text{in } L^s(0, T; L^q(G)),$$

for arbitrary $G \subset \Pi$ and $T > 0$.

We take an arbitrary $\varphi \in C_c^1(\overline{\Pi} \times [0, \infty))$ such that $\operatorname{div} \varphi = 0$ in Π and $\varphi \cdot n = 0$ on $\partial\Pi$. By multiplying φ by (1.1) and integration by parts, we see that

$$(4.5) \quad \int_0^\infty \int_\Pi (u_\nu \cdot \partial_t \varphi - \nu \nabla u_\nu \cdot \nabla \varphi + u_\nu u_\nu : \nabla \varphi) dx dt = - \int_\Pi u_0 \cdot \varphi_0 dx$$

Note that the integral of $\partial_n u_\nu \cdot \varphi$ on $\partial\Pi$ vanishes since $\partial_r u_\nu^z = 0$ and $\varphi \cdot n = 0$ on $\partial\Pi$. The first term converges to the integral of $u \cdot \partial_t \varphi$ and the second term vanishes by (4.1). We take a bounded domain $G \subset \Pi$ and $T > 0$ such that $\operatorname{spt} \varphi \subset \overline{G} \times [0, T]$. Since u_ν is uniformly bounded in $L^\infty(0, \infty; L^p(\Pi))$, by (4.1) and $p \geq q'$, u_ν is uniformly bounded in $L^{s'}(0, T; L^{q'}(G))$. Hence

$$\begin{aligned} & \left| \int_0^\infty \int_\Pi (u_\nu u_\nu : \nabla \varphi - uu : \nabla \varphi) dx dt \right| \\ & \leq \left| \int_0^T \int_G u_\nu (u_\nu - u) : \nabla \varphi dx dt \right| + \left| \int_0^T \int_G (u_\nu - u) u : \nabla \varphi dx dt \right| \\ & \leq \|\nabla \varphi\|_{L^\infty(G \times (0, T))} \left(\|u_\nu\|_{L^{s'}(0, T; L^{q'}(G))} + \|u\|_{L^{s'}(0, T; L^{q'}(G))} \right) \|u_\nu - u\|_{L^s(0, T; L^q(G))} \rightarrow 0 \quad \text{as } \nu \rightarrow 0. \end{aligned}$$

Thus sending $\nu \rightarrow 0$ to (4.5) yields (1.3). Since the estimate (4.1) is inherited to the limit u , we see that $u \in L^\infty(0, \infty; L^p)$ and $\nabla u \in L^\infty(0, \infty; L^q)$.

We show the weak continuity $u \in BC_w([0, \infty); L^p)$. We take an arbitrary $\varphi \in C_c^\infty(\Pi)$ and $\eta \in C_c^\infty(0, \infty)$. By multiplying $\varphi \eta$ by (1.1) and integration by parts as we did in the proof of Lemma 4.1, we obtain the estimate

$$\begin{aligned} & \left| \int_0^\infty \left(\int_\Pi u_\nu(x, t) \cdot \varphi(x) dx \right) \dot{\eta}(t) dt \right| \\ & \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)} \left(\nu \|\varphi\|_{W^{1, q'}(\Pi)} + \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)} \|\varphi\|_{W^{1, p/(p-2)}(\Pi)} \right) \left(\int_0^\infty \eta(t) dt \right) \end{aligned}$$

with $C = C(q)$, independent of ν . Sending $\nu \rightarrow 0$ yields

$$\left| \int_0^\infty \left(\int_\Pi u(x, t) \cdot \varphi(x) dx \right) \dot{\eta}(t) dt \right| \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)}^2 \|\varphi\|_{W^{1, p/(p-2)}(\Pi)} \left(\int_0^\infty \eta(t) dt \right).$$

Thus

$$h(t) = \int_\Pi u(x, t) \cdot \varphi(x) dx$$

is differentiable in the sense of distribution and

$$\left| \int_0^\infty \dot{h}(t)\eta(t)dt \right| = \left| \int_0^\infty h(t)\dot{\eta}(t)dt \right| \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)}^2 \|\varphi\|_{W^{1,p/(p-2)}(\Pi)} \left(\int_0^\infty \eta(t)dt \right).$$

By duality, $\dot{h} \in L^\infty(0, \infty)$ and

$$\left| \frac{d}{dt} \int_\Pi u(x, t) \cdot \varphi(x)dx \right| \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q(\Pi)}^2 \|\varphi\|_{W^{1,p/(p-2)}(\Pi)}, \quad \text{a.e. } t > 0.$$

Hence for $s \in [0, \infty)$, we have

$$\int_\Pi u(x, t) \cdot \varphi(x)dx \rightarrow \int_\Pi u(x, s) \cdot \varphi(x)dx \quad \text{as } t \rightarrow s.$$

By $u \in L^\infty(0, \infty; L^p)$ and the density, the above convergence holds for all $\varphi \in L^{p'}$. Thus $u \in BC_w([0, \infty); L^p)$. The weak continuity of ∇u on L^q follows from that of u on L^p . We proved the assertion (i). \square

4.2. Regularity of weak solutions. We prove Theorem 1.2 (ii). We use the Poincaré inequality.

Proposition 4.3. *Let $u \in C(\bar{\Pi})$ be an axisymmetric vector field without swirl such that $\operatorname{div} u = 0$ in Π , $u \cdot n = 0$ on $\partial\Pi$ and $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Assume that $\nabla u \in L^s(\Pi)$ for $s \in (1, \infty)$. Then, the estimate*

$$(4.6) \quad \|u\|_{L^s(\Pi)} \leq C \|\nabla u\|_{L^s(\Pi)}$$

holds with some constant $C = C(s)$.

Proof. Since the radial component u^r vanishes on $\partial\Pi$ by $u \cdot n = 0$, we apply the Poincaré inequality [4] to estimate

$$\|u^r\|_{L^s(\Pi)} \leq C \|\nabla u^r\|_{L^s(\Pi)},$$

with $C = C(s)$. We estimate u^z . For arbitrary $z_1, z_2 \in \mathbb{R}$, we set $G = D \times (z_1, z_2)$ by $D = \{x_h = (x_1, x_2) \mid |x_h| < 1\}$. Since $\operatorname{div} u = 0$, it follows that

$$0 = \int_G \operatorname{div} u dx = \int_D u^z(r, z_2) d\mathcal{H} - \int_D u^z(r, z_1) d\mathcal{H}.$$

Since u decays as $|z_2| \rightarrow \infty$, we see that the flux on D is zero, i.e.,

$$\int_D u^z(r, z_1) d\mathcal{H} = 0.$$

We apply the Poincaré inequality [28] to estimate

$$\|u^z\|_{L^s(D)}(z_1) \leq C \|\nabla_h u^z\|_{L^s(D)}(z_1),$$

with $C = C(s)$, where ∇_h denotes the gradient for the horizontal variable $x_h = (x_1, x_2)$. By integrating for $z_1 \in \mathbb{R}$, we obtain (4.6). \square

Proof of Theorem 1.2 (ii). If $\omega_0^\theta/r \in L^s$ for $s \in (3, \infty)$, the limit $u \in BC_w([0, \infty); L^p)$ satisfies $\nabla u \in L^\infty(0, \infty; L^s)$ by (4.3). Thus $u(\cdot, t)$ is Hölder continuous in $\bar{\Pi}$ and decaying as $|x| \rightarrow \infty$. We apply the Poincaré inequality (4.6) to see that $u \in L^\infty(0, \infty; W^{1,s})$ and $u \cdot \nabla u \in L^\infty(0, \infty; L^s)$ by the Sobolev inequality. By integration by parts, it follows from (1.3) that

$$(4.7) \quad \int_0^\infty \int_\Pi u \cdot \varphi \eta dx dt = \int_0^\infty \int_\Pi (u \cdot \nabla u) \cdot \varphi \eta dx dt$$

for all $\varphi \in L_\sigma^{s'}$ and $\eta \in C_c^\infty(0, \infty)$, where s' is the conjugate exponent to s . By the boundedness of the Helmholtz projection on $L^{s'}$ and a duality, we see that $\partial_t u \in L^\infty(0, \infty; L^s)$ and $u \in BC([0, \infty); L^s)$. The equation (1.4) follows from (4.7) by integration by parts. Since $\nabla u \in L^\infty(0, \infty; L^s)$ and $u \in BC([0, \infty); L^s)$, we see that for $\varphi \in C_c^\infty(\Pi)$,

$$\begin{aligned} \int_\Pi \partial u(x, t) \cdot \varphi(x) dx &= - \int_\Pi u(x, t) \cdot \partial \varphi(x) dx \rightarrow - \int_\Pi u_0(x) \cdot \partial \varphi(x) dx \\ &= \int_\Pi \partial u_0(x) \cdot \varphi(x) dx \quad \text{as } t \rightarrow 0, \end{aligned}$$

where $\partial = \partial_x^k$ for $|k| = 1$. By density of $C_c^\infty(\Pi)$ in $L^{s'}(\Pi)$, we have for $\varphi \in L^{s'}(\Pi)$

$$\int_\Pi \partial u(x, t) \cdot \varphi(x) dx \rightarrow \int_\Pi \partial u_0(x) \cdot \varphi(x) dx \quad \text{as } t \rightarrow 0.$$

Thus ∇u is weakly continuous at $t = 0$, i.e., $\nabla u \in BC_w([0, \infty); L^s)$.

If in addition that $\omega_0^\theta/r \in L^\infty$, the limit satisfies $\nabla \times u \in BC_w([0, \infty); L^\infty)$ by (4.4). \square

Remarks 4.4. (i) The equation (1.4) is written as

$$\partial_t u + u \cdot \nabla u + \nabla p = 0 \quad \text{on } L^s \quad \text{for a.e. } t > 0,$$

by the associated pressure $\nabla p = -(I - \mathbb{P})u \cdot \nabla u \in L^\infty(0, \infty; L^s)$.

(ii) The weak solutions in Theorem 1.2 are with finite energy for $q \in [3/2, 2]$. In fact, by the global estimate (4.1) and applying the Poincaré inequality (4.6) for global-in-time solutions $u = u_\nu$ of (1.1) for $\omega_0^\theta/r \in L^q$ in Theorem 1.1, we see that

$$\|u_\nu\|_{L^p} + \|u_\nu\|_{L^q} \leq C \|\nabla u_\nu\|_{L^q} \leq C' \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q}, \quad t \geq 0, \nu > 0,$$

with $C' = C'(q)$ for $p \in [3, 6]$ satisfying $1/p = 1/q - 1/3$. By the Hölder inequality,

$$\|u_\nu\|_{L^p} \leq \|u_\nu\|_{L^q}^\theta \|u_\nu\|_{L^p}^{1-\theta} \leq C' \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q}, \quad t > 0, \nu > 0,$$

for $1/\rho = \theta/q + (1 - \theta)/p$. Hence u_ν is uniformly bounded in L^ρ for all $\rho \in [q, p]$. Since $[2, 3] \subset [q, p]$, in particular $u_\nu \in L^\infty(0, \infty; L^2)$ and the limit as $\nu \rightarrow 0$ belongs to the same space.

5. UNIQUENESS

We prove Theorem 1.2 (iii). It remains to show the uniqueness. Since the weak solutions are with infinite energy for $q \in (2, 3)$, we estimate a local energy of two weak solutions in the cylinder by using a cut-off function θ_R . We then send $R \rightarrow \infty$ and prove the uniqueness by using the growth bound of the L^ρ -norm (3.15). To this end, we show decay properties of weak solutions.

5.1. Decay properties of weak solutions. We use the Poincaré inequality (4.6) and deduce decay properties of velocity as $|x_3| \rightarrow \infty$.

Proposition 5.1. *The weak solutions (u, p) as in Theorem 1.2 (iii) satisfy*

$$(5.1) \quad u, \nabla u, \partial_t u, \nabla p \in L^\infty(0, \infty; L^q).$$

Proof. Since $u(\cdot, t)$ is bounded and Hölder continuous in $\overline{\Pi}$ and $\nabla u \in BC_w([0, \infty); L^q)$, we see that $u \cdot \nabla u \in L^\infty(0, \infty; L^q)$. Thus, $\partial_t u$ and $\nabla p = -(I - \mathbb{P})u \cdot \nabla u$ belong to $L^\infty(0, \infty; L^q)$ by (1.4). By the Poincaré inequality (4.6), $u \in L^\infty(0, \infty; L^q)$ follows. \square

We estimate the pressure p as $|x_3| \rightarrow \infty$. We set

$$(5.2) \quad \begin{aligned} \tilde{p}(x_h, x_3, t) &= p(x_h, x_3, t) - \hat{p}(x_3, t), \\ \hat{p}(x_3, t) &= \frac{1}{|D|} \int_D p(x_h, x_3, t) dx_h. \end{aligned}$$

Proposition 5.2.

$$(5.3) \quad \tilde{p} \in L^\infty(0, \infty; L^q),$$

$$(5.4) \quad |\hat{p}(x_3, t)| \leq C(1 + |x_3|)^{1/3}, \quad x_3 \in \mathbb{R}, t > 0,$$

with some constant C , independent of $x_3 \in \mathbb{R}$ and $t > 0$.

Proof. The property (5.3) follows from (5.1) by applying the Poincaré inequality on D . We show (5.4). We integrate the vertical component of (1.2) on D to see that

$$\frac{\partial}{\partial t} \int_D u^z dx_h + \int_D u \cdot \nabla u^z dx_h + \frac{\partial}{\partial z} \int_D p dx_h = 0.$$

Since the flux of u on D is zero as we have seen in the proof of Proposition 4.3, the first term vanishes. We integrate the equation by the vertical variable between $(0, z)$ to get

$$\int_D p(r, z, t) dx_h = \int_D p(r, 0, t) dx_h - \int_0^z \int_D u \cdot \nabla u^z dx.$$

We observe that $u \in L^\infty(0, \infty; W^{1,\rho})$ for $\rho \in [q, 3)$ by (5.1) and Theorem 1.2 (ii). Since $u \cdot \nabla u \in L^\infty(0, \infty; L^{\rho/2})$ for $\rho \geq 2$, we apply the Hölder inequality to estimate

$$\left| \int_0^z \int_D u \cdot \nabla u^z dx \right| \leq |D \times (0, z)|^{1-2/\rho} \|u \cdot \nabla u\|_{L^{\rho/2}} \leq C|z|^{1-2/\rho}, \quad t > 0.$$

Since $\rho \in [2, 3)$ and $1 - 2/\rho < 1/3$, we obtain (5.4). \square

We use the growth bound for the L^ρ -norm of ∇u as $\rho \rightarrow \infty$.

Proposition 5.3.

$$\nabla u \in BC_w([0, \infty); L^\rho), \quad \rho \in (3, \infty),$$

$$\nabla \times u \in BC_w([0, \infty); L^\infty),$$

$$\|\nabla u\|_{L^\rho} \leq C\rho, \quad \rho > 3, t \geq 0,$$

with some constant C , independent of ρ .

Proof. By construction, the weak solution u satisfies

$$\|\nabla \times u\|_{L^\rho} \leq \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\rho}, \quad t \geq 0.$$

for all $\rho \in (3, \infty)$. We fix $\rho \in (3, \infty)$ and take $\rho_0 \in (3, \rho)$. It follows from (3.15) that

$$\|\nabla u\|_{L^\rho} \leq C\rho \|\nabla \times u\|_{L^\rho \cap L^{\rho_0}} \leq C\rho \left\| \frac{\omega_0^\theta}{r} \right\|_{L^\rho \cap L^{\rho_0}},$$

with the constant C , independent of ρ . Since the L^ρ -norm of ω_0^θ/r is uniformly bounded for all $\rho > 3$ by $\omega_0^\theta/r \in L^\infty$, we obtain the desired estimate for ∇u . \square

5.2. Local energy estimates. We now prove the uniqueness. Let (u_1, p_1) and (u_2, p_2) be two weak solutions to (1.2) in Theorem 1.2 (iii) for the same initial data. Then, $w = u_1 - u_2$ and $\pi = p_1 - p_2$ satisfy

$$(5.5) \quad \begin{aligned} \partial_t w + u_1 \cdot \nabla w + w \cdot \nabla u_2 + \nabla \pi &= 0, & \operatorname{div} w &= 0 & \text{in } \Pi \times (0, \infty), \\ w \cdot n &= 0 & & & \text{on } \partial\Pi \times (0, \infty), \\ w &= 0 & & & \text{on } \Pi \times \{t = 0\}. \end{aligned}$$

Let $\theta \in C_c^\infty[0, \infty)$ be a smooth monotone non-increasing function such that $\theta \equiv 1$ in $[0, 1]$ and $\theta \equiv 0$ in $[2, \infty)$. We set $\theta_R(x_3) = \theta(|x_3|/R)$ for $R \geq 1$ so that $\theta_R \equiv 1$ in $[0, R]$, $\theta_R \equiv 0$ in $[2R, \infty)$, $\|\partial_{x_3} \theta_R\|_\infty \leq C/R$ and $\operatorname{spt} \partial_{x_3} \theta_R \subset I_R$ for $I_R = [R, 2R]$. By multiplying $2w\theta_R$ by (1.2) and integration by parts, we see that

$$(5.6) \quad \frac{d}{dt} \int_\Pi |w|^2 \theta_R dx + 2 \int_\Pi (w \cdot \nabla u_2) \cdot w \theta_R dx - \int_\Pi u_1 |w|^2 \cdot \nabla \theta_R dx - 2 \int_\Pi \pi w \cdot \nabla \theta_R dx = 0.$$

We set

$$\phi_R(t) = \int_\Pi |w|^2(x, t) \theta_R(x_3) dx.$$

By Theorem 1.2 (ii), the function $\phi_R \in C[0, \infty)$ is differentiable for a.e. $t > 0$ and satisfies $\phi_R(0) = 0$. We estimate errors in the cut-off procedure.

Proposition 5.4. *There exists a constant $C = C(R)$ such that*

$$(5.7) \quad \left| \int_\Pi u_1 |w|^2 \cdot \nabla \theta_R dx \right| + \left| 2 \int_\Pi \pi w \cdot \nabla \theta_R dx \right| \leq C, \quad t > 0.$$

The constant $C(R)$ converges to zero as $R \rightarrow \infty$ for each $t > 0$.

Proof. Since $u_1 \in L^\infty(0, \infty; L^q)$ and $w \in L^\infty(\Pi \times (0, \infty))$ by Proposition 5.1 and Theorem 1.2 (ii), applying the Hölder inequality yields

$$\left| \int_{\Pi} u_1 |w|^2 \cdot \nabla \theta_R dx \right| \leq \frac{C}{R} \int_{D \times I_R} |u_1| dx \leq \frac{C'}{R} |D \times I_R|^{1/q'} \|u_1\|_{L^q} \leq \frac{C''}{R^{1/q}},$$

with some constant C'' , independent of R . We next estimate the second term of (5.7). We set $\pi = \tilde{\pi} + \hat{\pi}$ by (5.2). Since $\tilde{\pi} \in L^\infty(0, \infty; L^q)$ by (5.3), it follows that

$$\left| \int_{\Pi} \tilde{\pi} w \cdot \nabla \theta_R dx \right| \leq \frac{C}{R} \int_{D \times I_R} |\tilde{\pi}| dx \leq \frac{C'}{R^{1/q}}.$$

It follows from (5.4) that

$$\left| \int_{\Pi} \hat{\pi} w \cdot \nabla \theta_R dx \right| \leq \frac{C}{R^{2/3}} \int_{D \times I_R} |w| dx \leq \frac{C}{R^{2/3}} |D \times I_R|^{1/q'} \|w\|_{L^q} \leq \frac{C'}{R^{2/3-1/q'}}.$$

with some constant C' , independent of R . Since $2/3 - 1/q' > 0$ for $q \in [3/2, 3)$, the right-hand side converges to zero as $R \rightarrow \infty$. \square

Proof of Theorem 1.2 (iii). By Proposition 5.3, there exist constants M_1 and M_2 such that

$$\begin{aligned} \|w\|_{L^\infty} &\leq M_1, \\ \|\nabla u_2\|_{L^\rho} &\leq M_2 \rho, \quad \rho > 3, t \geq 0. \end{aligned}$$

For an arbitrary $\delta \in (0, 2/3)$, we set $\rho = 2/\delta$. We apply the Hölder inequality with the conjugate exponent $\rho' = 2/(2 - \delta)$ to see that

$$\begin{aligned} \left| 2 \int_{\Pi} (w \cdot \nabla u_2) \cdot w \theta_R dx \right| &\leq 2 \int_{\Pi} |\nabla u_2| (|w| \theta_R^{1/2})^2 dx \\ &\leq 2M_1^\delta \int_{\Pi} |\nabla u_2| (|w| \theta_R^{1/2})^{2-\delta} dx \\ &\leq 2M_1^\delta \|\nabla u_2\|_{L^\rho} \left(\int_{\Pi} |w|^2 \theta_R dx \right)^{1/\rho'} \\ &\leq 2M_1^\delta M_2 \rho \phi_R^{1/\rho'}. \end{aligned}$$

Thus, ϕ_R satisfies the differential inequality

$$\begin{aligned}\dot{\phi}_R(t) &\leq a\phi_R(t)^{1/\rho'} + b, \quad t > 0, \\ \phi_R(0) &= 0,\end{aligned}$$

with the constants $a = 2M_1^\delta M_2 \rho$ and $b = C(R)$ by (5.6) and (5.7). Hence we have

$$(5.8) \quad \int_0^{\phi_R(t)} \frac{ds}{as^{1/\rho'} + b} \leq t.$$

We prove that

$$(5.9) \quad \overline{\lim}_{R \rightarrow \infty} \phi_R(t) < \infty \quad \text{for each } t > 0.$$

Suppose on the contrary that (5.9) were false for some $t_0 > 0$. Then, there exists a sequence $\{R_j\}$ such that $\lim_{j \rightarrow \infty} \phi_{R_j}(t_0) = \infty$. For an arbitrary $K > 0$, we take a constant $N \geq 1$ such that $\phi_{R_j}(t_0) \geq K$ for $j \geq N$. It follows from (5.8) that

$$\int_0^K \frac{ds}{as^{1/\rho'} + b} \leq t_0.$$

Since the constant $b = C(R_j)$ converges to zero as $R_j \rightarrow \infty$, sending $j \rightarrow \infty$ yields $(r/a)K^{1/r} \leq t_0$. Since $K > 0$ is arbitrary, this yields a contradiction. Thus (5.9) holds.

Since $|w|^2 \theta_R$ monotonically converges to $|w|^2$ in Π , it follows from (5.9) that

$$\phi(t) := \int_{\Pi} |w|^2(x, t) dx = \lim_{R \rightarrow \infty} \int_{\Pi} |w|^2(x, t) \theta_R(x_3) dx < \infty.$$

Sending $R \rightarrow \infty$ to (5.8) implies $(\rho/a)\phi^{1/\rho}(t) \leq t$. We thus obtain

$$\int_{\Pi} |w(x, t)|^2 dx \leq M_1^2 (2M_2 t)^{2/\delta}.$$

Since the right-hand side converges to zero as $\delta \rightarrow 0$ for $t \in [0, T]$ and $T = (4M_2)^{-1}$, we see that $w \equiv 0$ in $[0, T]$. Applying the same argument for $t \geq T$ implies $u_1 \equiv u_2$ for all $t \geq 0$. The proof is now complete. \square

Remark 5.5. By a similar cut-off function argument, uniqueness of weak solutions of the Euler equations with infinite energy is proved in [15, Theorem 5.1.1] for the whole space under different assumptions from Theorem 1.2 (iii). See also [21, Theorem 2]. We proved uniqueness of weak solutions in the infinite cylinder based on the Yudovich's estimate (Lemma 3.3).

6. SOLUTIONS WITH FINITE ENERGY

It remains to prove Theorem 1.3. The proof of the L^2 -convergence (1.7) is simpler than that of uniqueness of weak solutions since solutions are with finite energy.

6.1. Energy dissipation.

Proposition 6.1. *The assertion of Theorem 1.3 (i) holds.*

Proof. Let $u_0 \in L_\sigma^p \cap L^2$ be an axisymmetric vector field without swirl such that $\omega_0^\theta/r \in L^q$ for $q \in [3/2, 2]$ and $1/p = 1/q - 1/3$. By Theorem 1.1 and Proposition 2.2, there exists a unique global-in-time solution $u_\nu \in BC([0, \infty); L^p \cap L^2)$ of (1.1) satisfying the energy equality (1.5). Since $\omega_0^\theta/r \in L^q$, applying Lemma 2.5 yields

$$\left\| \frac{\omega^\theta}{r} \right\|_{L^2} \leq \frac{C}{(\nu t)^{\frac{3}{2}(1-\frac{1}{q})}} \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q}, \quad t \geq 0, \nu > 0,$$

with $C = C(q)$, independent of ν . It follows that

$$\nu \int_0^T \|\nabla u_\nu\|_{L^2}^2 dt = \nu \int_0^T \|\omega^\theta\|_{L^2}^2 dt \leq \nu \int_0^T \left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2 dt \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^q}^2 (\nu T)^{\frac{5}{2}-\frac{3}{q}}.$$

Thus, (1.6) holds. \square

6.2. L^2 -convergence. We prove Theorem 1.3 (ii). In order to apply the Yudovich's energy method, we prepare the following estimate (6.2) for $\|\nabla u_\nu\|_{L^p}$, which is uniform for $\nu > 0$, $t > 0$, and linearly growing as $\rho \rightarrow \infty$.

Proposition 6.2. *Let $u_0 \in L_\sigma^p \cap L^2$ be an axisymmetric vector field without swirl such that $\omega_0^\theta/r \in L^q \cap L^\infty$ for $q \in [3/2, 2]$ and $1/p = 1/q - 1/3$. Let $u_\nu \in BC([0, \infty); L^p \cap L^2) \cap C^\infty(\bar{\Pi} \times (0, \infty))$ be a solution of (1.1). Then, the estimates*

$$(6.1) \quad \|u_\nu\|_{L^\infty} \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^{p_0}},$$

$$(6.2) \quad \|\nabla u_\nu\|_{L^p} \leq C' \rho \left\| \frac{\omega_0^\theta}{r} \right\|_{L^p \cap L^{p_0}},$$

$$(6.3) \quad \|\nabla u_\nu\|_{L^2} \leq \left\| \frac{\omega_0^\theta}{r} \right\|_{L^2},$$

hold for $t > 0$ and $3 < \rho_0 < \rho < \infty$ with some constants $C = C(\rho_0)$ and C' , independent of ρ and ν .

Proof. We take $\rho_0 \in (3, \infty)$. It follows from (4.3) and (4.6) that

$$\|u_\nu\|_{W^{1,\rho_0}} \leq C \left\| \frac{\omega_0^\theta}{r} \right\|_{L^{\rho_0}}, \quad t \geq 0, \nu > 0.$$

By the Sobolev inequality, the estimate (6.1) follows. The estimates (6.2) and (6.3) follow from (2.5) and (3.15). \square

Let (u_ν, p_ν) and (u_μ, p_μ) be two solutions of (1.1) for the same initial data u_0 . We may assume that $\nu \geq \mu$. Then, $w = u_\nu - u_\mu$ and $\pi = p_\nu - p_\mu$ satisfy

$$\begin{aligned} \partial_t w - \nu \Delta w - (\nu - \mu) \Delta u_\mu + u_\nu \cdot \nabla w + w \cdot \nabla u_\mu + \nabla \pi &= 0 & \operatorname{div} w &= 0 & \text{in } \Pi \times (0, \infty), \\ \nabla \times w \times n = 0, w \cdot n &= 0 & & & \text{on } \partial \Pi \times (0, \infty), \\ w &= 0 & & & \text{on } \Pi \times \{t = 0\}. \end{aligned}$$

By multiplying $2w$ by the equation and integration by parts, we see that

$$\frac{d}{dt} \int_{\Pi} |w|^2 dx + 2\nu \int_{\Pi} |\nabla w|^2 dx + 2(\nu - \mu) \int_{\Pi} \nabla u_\mu \cdot \nabla w dx + 2 \int_{\Pi} (w \cdot \nabla u_\mu) \cdot w dx = 0.$$

We set

$$\phi_\nu(t) = \int_{\Pi} |w(x, t)|^2 dx.$$

We show that $K_\nu(T) = \sup_{0 \leq t \leq T} \phi_\nu(t)$ converges to zero as $\nu \rightarrow 0$ for each $T > 0$.

Proposition 6.3. *There exist constants $M_1 - M_3$, independent of $\nu, \mu > 0$ such that*

$$\begin{aligned} \|w\|_{L^\infty} &\leq M_1, \\ \|\nabla u_\mu\|_{L^\rho} &\leq M_2 \rho, \\ \|\nabla u_\mu\|_{L^2} + \|\nabla w\|_{L^2} &\leq M_3, \end{aligned}$$

hold for $t > 0$ and $\rho > 3$.

Proof. The assertion follows from Proposition 6.2. \square

Proof of Theorem 1.3 (ii). By Proposition 6.3, we estimate

$$\left| 2(\nu - \mu) \int_{\Pi} \nabla u_{\mu} \cdot \nabla w dx \right| \leq 2\nu \|\nabla u_{\mu}\|_{L^2} \|\nabla w\|_{L^2} \leq 2\nu M_3^2.$$

For an arbitrary $\delta \in (0, 2/3)$, we set $\rho = 2/\delta$. Since $\rho' = 2/(2 - \delta)$, in a similar way as in the proof of Theorem 1.2 (iii), we estimate

$$\left| 2 \int_{\Pi} (w \cdot \nabla u_{\mu}) \cdot w dx \right| \leq 2M_1^{\delta} M_2 \rho \phi_{\nu}^{1/\rho'}.$$

Thus, ϕ_{ν} satisfies

$$\begin{aligned} \dot{\phi}_{\nu}(t) &\leq a \phi_{\nu}^{1/\rho'}(t) + b, \\ \phi_{\nu}(0) &= 0, \end{aligned}$$

for $a = 2M_1^{\delta} M_2 \rho$ and $b = 2\nu M_3^2$. We take an arbitrary $T > 0$. We integrate the differential inequality between $(0, t)$ and take a supremum for $t \in [0, T]$ to estimate

$$\int_0^{K_{\nu}(T)} \frac{ds}{as^{1/\rho'} + b} \leq T.$$

Since $b = b_{\nu}$ converges to zero as $\nu \rightarrow 0$, by the same way as in the proof of Theorem 1.2 (iii), we see that the limit superior of $K_{\nu}(T)$ is finite for each $T > 0$. We set

$$K(T) := \overline{\lim}_{\nu \rightarrow 0} K_{\nu}(T) < \infty.$$

Sending $\nu \rightarrow 0$ to the above inequality implies $(\rho/a)K^{1/\rho'}(T) \leq T$. Since $a = 2M_1^{\delta} M_2 \rho$ and $\rho = 2/\delta$, it follows that

$$K(T) \leq M_1^2 (2M_2 T)^{2/\delta}.$$

Since the right-hand side converges to zero as $\delta \rightarrow 0$ for $T \leq T_0$ and $T_0 = (4M_2)^{-1}$, we see that $K(T) \equiv 0$. Thus the convergence (1.7) holds. By replacing the initial time and applying the same argument for $T \geq T_0$, we are able to show the convergence (1.7) for an arbitrary $T > 0$. The proof is now complete. \square

Remarks 6.4. (i) ($q = 1$) We constructed global weak solutions to the Euler equations for axisymmetric data without swirl by a vanishing viscosity method. As explained in the introduction, the condition $\omega_0^{\theta}/r \in L^q$ for $q \in [3/2, 3)$ in Theorem 1.2 (i) is satisfied if $u_0 \in W^{2,q}$. This condition is weaker than $u_0 \in W^{2,q}$ for $q \in (3, \infty)$, required for the local well-posedness of the Euler equations.

An interesting case is $q = 1$ for which the condition is invariant under the scaling of (1.1) in \mathbb{R}^3 . In this case, vorticity can be a measure and we need to consider a space larger than $L^3(\mathbb{R}^3)$ such as the Morrey space $M^3(\mathbb{R}^3)$ [42] or $\text{BMO}^{-1}(\mathbb{R}^3)$ [57]. See [38]. It is known [37, Lemma 2.8] that the condition $\omega_0^\theta/r \in L^1(\mathbb{R}^3)$ implies that $u_0 \in (L^\infty)^{-1}(\mathbb{R}^3) \subset \text{BMO}^{-1}(\mathbb{R}^3)$. We are able to write $\omega_0^\theta/r \in L^1(\mathbb{R}^3)$ as $\omega_0^\theta \in L^1(H)$ for $H = \{(r, z) \mid r > 0, z \in \mathbb{R}\}$, or more generally $\omega_0^\theta \in M(\overline{H})$, the space of measures on \overline{H} . The existence of global-in-time solutions to (1.1) in \mathbb{R}^3 for a measure with compact support is proved in [32]. See [37] for uniqueness. Moreover, global well-posedness for $\omega_0^\theta \in M(\overline{H})$ with a small pure point part is proved in [38].

It seems unknown whether global weak solutions to (1.2) exist for vortex-sheet initial data $\omega_0^\theta \in M(\overline{H})$ (e.g., $\omega_0^\theta \geq 0$, with compact support and $u_0 \in L^2_{\text{loc}}(H)$). See [24], [49] for convergence results. For such initial data global-in-time unique solutions to (1.1) are known to exist [32], [38].

(ii) (Convergence in Sobolev space) Our approach is based on the a priori estimate (1.10) which is a special property of axisymmetric solutions and is not available at the broad level. On the other hand, there is another approach to study vanishing viscosity limits when the Euler equation is locally well-posed. When $\Pi = \mathbb{R}^3$, unique local-in-time solutions of the Euler equations are constructed in [78], [50], [51] by a vanishing viscosity method. See also [18]. In particular, for a local-in-time solution $u \in C([0, T]; H^s)$ of the Euler equations and $u_0 \in H^s$, $s > 5/2$, the convergence

$$u_\nu \rightarrow u \quad \text{in } L^\infty(0, T; H^s),$$

is known to hold [68]. The case with boundary is a difficult question related to analysis of boundary layer. See [19] for a survey. However, convergence results are known subject to the Neumann boundary condition (1.1). See [83], [9], [10] for the case with flat boundaries and [11], [76] for curved boundaries.

(iii) (Navier boundary condition) The Neumann boundary condition in (1.1) may be viewed as a special case of the Navier boundary condition,

$$(6.4) \quad (D(u)n + \alpha u)_{\text{tan}} = 0, \quad u \cdot n = 0 \quad \text{on } \partial\Pi,$$

where $D(u) = (\nabla u + \nabla^T u)/2$ is the deformation tensor and $f_{\text{tan}} = f - n(f \cdot n)$ for a vector field f . Indeed, for the two-dimensional case, the Neumann boundary condition is reduced to the free condition $\omega = 0$ and $u \cdot n = 0$ on $\partial\Pi$. The free condition is a special case of (6.4), which is written as $\omega + 2(\alpha - \kappa)u \cdot n^\perp = 0$ and $u \cdot n = 0$ on $\partial\Pi$, with the curvature $\kappa(x)$ and $n^\perp = (-n^2, n^1)$. For a two-dimensional bounded domain, vanishing viscosity limits subject to (6.4) are studied in [17], [64], [55]. For the three-dimensional case, it is shown in [45] that a Leray-Hopf weak solution u_ν subject to (6.4) converges to the local-in-time solution $u \in C([0, T]; H^s)$ of the Euler equations for $u_0 \in H^3$ in the sense that

$$u_\nu \rightarrow u \quad \text{in } L^\infty(0, T; L^2).$$

For the Dirichlet boundary condition, the same convergence seems unknown. See [46] for a boundary layer expansion subject to (6.4) and [69] for a stronger convergence result.

APPENDIX A. DECAY ESTIMATES OF VORTICITY

We prove the decay estimate (2.8) (Lemma 2.5). It suffices to show:

Lemma A.1. *There exists a constant $C = C(q)$ such that the estimate (2.8) holds for $\rho = 2^m q$, $q \in [1, \infty)$ and non-negative integers $m \geq 0$.*

Proof of Lemma 2.5. We apply Lemma A.1. Since

$$\lim_{\rho \rightarrow \infty} \left\| \frac{\omega^\theta}{r} \right\|_{L^\rho} (t) = \left\| \frac{\omega^\theta}{r} \right\|_{L^\infty} (t),$$

sending $m \rightarrow \infty$ implies (2.8) for $\rho = \infty$ and $q \in [1, \infty)$. Since (2.8) holds for $\rho = q \in [1, \infty)$, we obtain (2.8) for all $1 \leq q \leq \rho \leq \infty$ by the Hölder inequality. \square

Let $\psi_\varepsilon(s)$ be a non-negative convex function as in the proof of Lemma 2.3. We prove the estimate (2.8) for $\psi_\varepsilon(\Omega)$ and $\Omega = \omega^\theta/r$. The assertion of Lemma A.1 follows by sending $\varepsilon \rightarrow 0$.

Proposition A.2. *There exists a constant $C = C(q)$ such that the estimate*

$$(A.1) \quad \left\| \psi_\varepsilon(\Omega) \right\|_{L^\rho(\Pi)} \leq \frac{C}{(\nu t)^{\frac{3}{2}(\frac{1}{q} - \frac{1}{\rho})}} \left\| \psi_\varepsilon(\Omega_0) \right\|_{L^q(\Pi)}, \quad t > 0, \nu > 0,$$

holds for all $\varepsilon > 0$, $\rho = 2^m q$ and $m \geq 0$. The constant C is independent of m , ν and ε .

We consider differential inequalities for L^ρ -norms of $\psi_\varepsilon(\Omega)$.

Proposition A.3. *The function*

$$\phi_\rho(t) = \int_{\Pi} \psi_\varepsilon^\rho(\Omega) dx$$

satisfies

$$(A.2) \quad \dot{\phi}_\rho(t) \leq -\kappa v \left(1 - \frac{1}{\rho}\right) \frac{\phi_\rho^{5/3}}{\phi_{\rho/2}^{4/3}}, \quad t > 0$$

with some absolute constant κ , independent of ρ and v .

Proof. We apply the interpolation inequality

$$\|\varphi\|_{L^2} \leq C_0 \|\varphi\|_{L^1}^{2/5} \|\nabla \varphi\|_{L^2}^{3/5}$$

for $\varphi \in H_0^1$ with some absolute constant C_0 . Since $\psi_\varepsilon(\Omega)$ satisfies

$$\frac{d}{dt} \int_{\Pi} \psi_\varepsilon^\rho(\Omega) dx + 4v \left(1 - \frac{1}{\rho}\right) \int_{\Pi} \left| \nabla(\psi_\varepsilon^{\rho/2}(\Omega)) \right|^2 dx \leq 0,$$

by (2.7), applying the interpolation inequality for $\varphi = \psi_\varepsilon^{\rho/2}$ yields

$$\int_{\Pi} \left| \nabla(\psi_\varepsilon^{\rho/2}(\Omega)) \right|^2 dx \geq \frac{1}{C_0^{10/3}} \frac{\left(\int_{\Pi} \psi_\varepsilon^\rho dx \right)^{5/3}}{\left(\int_{\Pi} \psi_\varepsilon^{\rho/2} dx \right)^{4/3}} = \frac{\phi_\rho^{5/3}}{C_0^{10/3} \phi_{\rho/2}^{4/3}}.$$

The differential inequality (A.2) follows from the above two inequalities with $\kappa = 4C_0^{-10/3}$. \square

Proof of Proposition A.2. We set $\lambda = \|\psi_\varepsilon(\Omega_0)\|_{L^q}$. The estimate (A.1) is written as

$$(A.3) \quad \phi_\rho^{1/\rho}(t) \leq \frac{C}{(vt)^{\frac{3}{2}\left(\frac{1}{q}-\frac{1}{\rho}\right)}} \lambda, \quad t > 0,$$

for $\rho = 2^m q$ and $m \geq 0$. We prove (A.3) by induction for $m \geq 0$. For $m = 0$, the estimate (A.3) holds with $C = 1$ by Lemma 2.3.

Suppose that (A.3) holds for $m = k$ with some constant $C = C_k$. We set $s = 2\rho$ for $\rho = 2^m q$. By the assumption of our induction, we see that

$$\frac{1}{\phi_\rho(t)} \geq \frac{(vt)^{\frac{3}{2}(2^k-1)}}{C_k^r \lambda^\rho}.$$

It follows from (A.2) that

$$\begin{aligned}\phi_s^{-5/3} \dot{\phi}_s &\leq -\kappa \nu \left(1 - \frac{1}{s}\right) \phi_\rho^{-4/3} \\ &\leq -\kappa \nu \left(1 - \frac{1}{s}\right) \frac{(\nu t)^{2^{k+1}-2}}{C_k^{4\rho/3} \lambda^{4\rho/3}}.\end{aligned}$$

We integrate the both sides between $[t_1, t]$ and estimate

$$\frac{3}{2} \left(\frac{1}{\phi_s^{2/3}(t)} - \frac{1}{\phi_s^{2/3}(t_1)} \right) \geq \kappa \left(1 - \frac{1}{s}\right) \frac{(\nu t)^{2^{k+1}-1} - (\nu t_1)^{2^{k+1}-1}}{C_k^{4\rho/3} \lambda^{4\rho/3}} \left(\frac{1}{2^{k+1}-1} \right).$$

Since the left-hand side is smaller than $3/2\phi_s^{-2/3}$, sending $t_1 \rightarrow 0$ yields

$$\phi_s^{2/3}(t) \leq \frac{(C_k \lambda)^{4\rho/3}}{(\nu t)^{2^{k+1}-1}} \frac{2^{k+2}}{\kappa(1-1/s)}.$$

Since

$$\frac{3}{2s} \frac{4\rho}{3} = 1, \quad \frac{3}{2s} (2^{k+1} - 1) = \frac{3}{2} \left(\frac{1}{q} - \frac{1}{s} \right),$$

and $1 - 1/s \geq 1/2$, it follows that

$$\phi_s^{1/s}(t) \leq \frac{C_k \lambda}{(\nu t)^{\frac{3}{2} \left(\frac{1}{q} - \frac{1}{s} \right)}} \left(\frac{2^{k+3}}{\kappa} \right)^{\frac{3}{2^{k+2}q}}.$$

We proved that (A.3) holds for $m = k + 1$ with the constant $C_{k+1} = a_k C_k$ for $a_k = b^{\frac{1}{2^{k+2}}} d^{\frac{k+3}{2^{k+2}}}$ and

$$b = \kappa^{-3/q}, \quad d = 2^{3/q}.$$

Thus (A.3) holds for all $m \geq 0$. Since

$$C_{k+1} = a_k C_k = \prod_{j=1}^k a_j = b^{\sum_{j=1}^k 2^{-j-2}} d^{\sum_{j=1}^k (j+3)2^{-j-2}},$$

and the right-hand side converges as $k \rightarrow \infty$, we are able to take a uniform constant $C = C(q)$ in (A.3) for all $m \geq 0$. The proof is now complete. \square

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