SHARP HARDY-LERAY AND RELLICH-LERAY INEQUALITIES FOR CURL-FREE VECTOR FIELDS

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Abstract. In this paper, we prove Hardy-Leray and Rellich-Leray inequalities for curl-free vector fields with sharp constants. This complements the former work by Costin-Maz'ya [2] on the sharp Hardy-Leray inequality for axisymmetric divergence-free vector fields.

1. Introduction

In this paper, we concern the classical functional inequalities called Hardy-Leray and Rellich-Leray inequalities for smooth vector fields and study how the best constants will change according to the pointwise constraints on their differentials.

Let $N \in \mathbb{N}$ be an integer with $N \geq 2$ and put $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$. In the following, $C_c^\infty(\mathbb{R}^N)^N$ denotes the set of smooth vector fields

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_N) : \mathbb{R}^N \ni \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}) \in \mathbb{R}^N$$

having compact supports on $\mathbb{R}^N$. Let $\gamma \neq 1 - N/2$. Then it is well known that

$$\left(\gamma + \frac{N}{2} - 1\right)^2 \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \tag{1}$$

holds for any vector field $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ with $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ if $\gamma < 1 - N/2$. This is a higher dimensional extension of the 1-dimensional inequality by G. H. Hardy, see [8], also [12], and was first proved by J. Leray [10] in 1933 when the weight $\gamma = 0$, see also the book by Ladyzhenskaya [9]. It is also known that the constant $\left(\gamma + \frac{N}{2} - 1\right)^2$ is sharp and never attained. In [2], Costin and Maz'ya proved that if the smooth vector fields are axisymmetric and subject to the divergence-free constraint $\operatorname{div} \boldsymbol{u} = 0$, then the constant $\left(\gamma + \frac{N}{2} - 1\right)^2$ in (1) can be improved and replaced by a larger one. More precisely, they proved the following:

Theorem A. (Costin-Maz'ya [2]) *Let $N \geq 3$. Let $\gamma \neq 1 - N/2$ be a real number and $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ be an axisymmetric divergence-free vector field. Assume that $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ if $\gamma < 1 - N/2$. Then*

$$C_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx$$

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holds with the optimal constant $C_{N,\gamma}$ *given by*

$$C_{N,\gamma}=\begin{cases}\left(\gamma+\frac{N}{2}-1\right)^2\dfrac{N+1+\left(\gamma-\frac{N}{2}\right)^2}{N-1+\left(\gamma-\frac{N}{2}\right)^2} & (\gamma\le 1),\\ \left(\gamma+\frac{N}{2}-1\right)^2+2+\min\limits_{\kappa\ge 0}\left(\kappa+\frac{4(N-1)(\gamma-1)}{\kappa+N-1+\left(\gamma-\frac{N}{2}\right)^2}\right) & (N\ge 4,\gamma>1),\\ \left(\gamma+\frac{1}{2}\right)^2+2 & (N=3,\gamma>1),\end{cases}$$

Note that the expression of the best constant $C_{N,\gamma}$ is slightly different from that in [2] when $N\ge 4$, but a careful checking the proof in [2] leads to the above formula in Theorem A. Choosing $\gamma=0$ in Theorem A, we see that the best constant in (1) is actually improved for axisymmetric divergence-free vector fields in the sense that

$$C_{N,0}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}dx\le\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2dx$$

holds with the optimal constant $C_{N,0}=\left(\frac{N}{2}-1\right)^2\frac{N^2+4N+4}{N^2+4N-4}>\left(\frac{N-2}{2}\right)^2$.

In 2-dimensional case, the result in [2] reads as follows:

Theorem B. (Costin-Maz'ya [2]) *Let* $\gamma\ne 0$ *be a real number and* $\boldsymbol{u}\in C_c^\infty(\mathbb{R}^2)^2$ *be a divergence-free vector field. We assume that* $\boldsymbol{u}(\boldsymbol{0})=\boldsymbol{0}$ *if* $\gamma<0$ *. Then*

$$C_{2,\gamma}\int_{\mathbb{R}^2}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx\le\int_{\mathbb{R}^2}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx$$

holds with the optimal constant $C_{2,\gamma}$ *given by*

$$C_{2,\gamma}=\begin{cases}\gamma^2\dfrac{3+(\gamma-1)^2}{1+(\gamma-1)^2} & \text{for}\quad |\gamma+1|\le\sqrt{3},\\ \gamma^2+1 & \text{otherwise.}\end{cases}$$

When $N=2$, the divergence-free field $\boldsymbol{u}$ in Theorem B need not be axisymmetric. Furthermore if we consider $\boldsymbol{u}^\perp=(-u_2,u_1)$ for $\boldsymbol{u}=(u_1,u_2)$ in Theorem B, then the condition $\operatorname{div}\boldsymbol{u}=0$ is replaced by $\operatorname{curl}\boldsymbol{u}^\perp=0$ and also $|\nabla\boldsymbol{u}|^2=|\nabla\boldsymbol{u}^\perp|^2$. Thus the above inequality in Theorem B holds also for curl-free vector fields with the same constant.

Motivated by this observation, our aim in this paper is to generalize Costin-Maz'ya's result for curl-free vector fields when $N=2$ to higher-dimensional cases. In addition, we also consider the Rellich type inequality involving the higher-order derivative, $\Delta\boldsymbol{u}$, for curl-free vector fields. We refer to [5] for the Rellich-Leray inequality for divergence-free vector fields. See also [6], [7] for other improvements of [2].

Now, main results of this paper are as follows:

Theorem 1. *Let* $\gamma\ne 1-N/2$ *be a real number and let* $\boldsymbol{u}\in C_c^\infty(\mathbb{R}^N)^N$ *be a curl-free vector field. We assume that* $\boldsymbol{u}(\boldsymbol{0})=\boldsymbol{0}$ *if* $\gamma<1-N/2$. *Then*

$$H_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx\le\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \tag{2}$$

with the optimal constant $H_{N,\gamma}$ *given by*

$$H_{N,\gamma} = \begin{cases} \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{3(N-1)+\left(\gamma+\frac{N}{2}-2\right)^2}{N-1+\left(\gamma+\frac{N}{2}-2\right)^2} & \text{if } \left|\gamma + \frac{N}{2}\right| \leq \sqrt{N+1}, \\ \left(\gamma + \frac{N}{2} - 1\right)^2 + N - 1 & \text{otherwise.} \end{cases} \tag{3}$$

We remark that no symmetry assumption for $\boldsymbol{u}$ is needed. Theorem 1 corresponds to the higher-dimensional analogue of Theorem B in the sense that $C_{2,\gamma} = H_{2,\gamma}$.

For curl-free vector fields $\boldsymbol{u}$, Poincaré's lemma implies that there exists a smooth scalar potential ϕ such that $\boldsymbol{u} = \nabla\phi$. Thus by using the potential function ϕ, Theorem 1 is equivalent to the following corollary.

Corollary 2. *Let* $\gamma \neq 1 - N/2$ *be a real number and let* $\phi \in C_c^\infty(\mathbb{R}^N)$*. We assume that* $\nabla\phi(\mathbf{0}) = \mathbf{0}$ *if* $\gamma < 1 - N/2$*. Then*

$$H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\nabla\phi|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |D^2\phi|^2 |\boldsymbol{x}|^{2\gamma} dx$$

with the optimal constant $H_{N,\gamma}$ *given in* (3)*. Here* $D^2\phi(\boldsymbol{x}) = \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(\boldsymbol{x})\right)_{1\leq i,j\leq N}$ *denotes the Hessian matrix of* ϕ*.*

By similar arguments, we prove the following Rellich-Leray inequality for curl-free vector fields.

Theorem 3. *Let* $\gamma \neq 2-N/2$ *be a real number and let* $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ *be a curl-free vector field. We assume that* $\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}|^2 dx < \infty$*. Then*

$$R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\Delta\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \tag{4}$$

with the optimal constant $R_{N,\gamma}$ *given by*

$$R_{N,\gamma} = \min_{\nu\in\mathbb{N}\cup\{0\}} \frac{(1 - \frac{N}{2} - \gamma)^2 + \alpha_\nu}{\left(3 - \frac{N}{2} - \gamma\right)^2 + \alpha_\nu} (\alpha_{3-\frac{N}{2}-\gamma} - \alpha_\nu)^2 \tag{5}$$

for $\gamma \neq 3 - N/2$*, where we put*

$$\alpha_s = s(s + N - 2), \quad s \in \mathbb{R},$$

and

$$R_{N,3-N/2} = \begin{cases} 4(N-2)^2 & \text{for } N = 2,3,4, \\ (N+3)(N-1) & \text{for } N \geq 5. \end{cases} \tag{6}$$

Corollary 4. *Let* $\gamma \neq 2-N/2$ *be a real number and let* $\phi \in C_c^\infty(\mathbb{R}^N)$ *be a potential function such that* $\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\nabla\phi|^2 dx < \infty$*. Then*

$$R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\nabla\phi|^2}{|\boldsymbol{x}|^4} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla\Delta\phi|^2 |\boldsymbol{x}|^{2\gamma} dx$$

holds with the optimal constant $R_{N,\gamma}$ *as in* (5) *and* (6)*.*

Remark 5. *The best constants* $H_{N,\gamma}$ *and* $R_{N,\gamma}$ *in Theorem 1 and 3 are respectively unchanged even if we additionally assume the axisymmetry condition on the curl-free fields* $\boldsymbol{u}$*. Indeed,* $\psi_\nu(\boldsymbol{\sigma}) = P_\nu(-\cos\theta_1)$*, where* P_ν *is a Legendre polynomial of* ν*-th order (see Appendix in* [5]*), is the axisymmetric eigenfunction of the Laplace-Beltrami operator on the sphere* $\mathbb{S}^{N-1}$ *associated with the eigenvalue* $\alpha_\nu = \nu(\nu +$

$N-2$). Therefore, in the proof of the optimality for $H_{N,\gamma}$ or $R_{N,\gamma}$, we may use the axisymmetric curl-free test fields by applying (35) to $\psi_{\nu_0}(\boldsymbol{\sigma}) = P_{\nu_0}(-\cos\theta_1)$. This implies the claim.

Remark 6. *We do not know that the optimal constants $H_{N,\gamma}$ and $R_{N,\gamma}$ are attained or not in the class of vector fields in Theorem 1 and Theorem 3.*

Also in Theorem 1 and Theorem 3, if we restrict ourselves only on vector fields in $C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$, then the additional assumptions $\boldsymbol{u}(\mathbf{0}) = \mathbf{0}$ if $\gamma < 1 - N/2$, or $\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}|^2 dx < \infty$ are not needed and the same conclusions hold true.

Concerning Corollary 2 which is equivalent to Theorem 1, we should remark that the similar results already exist by [13], [3]; see also [4] Chapter 6.5. More precisely, improving the work by Tertikas and Zographopoulos [13], Ghoussoub and Moradifam ([3]: Appendix B) proved the following: Let $C_c^\infty(B_R)$ denote the set of smooth functions having compact supports on a ball $B_R \subset \mathbb{R}^N$ with radius R. Define

$$A_{N,\gamma}(R) = \inf \left\{ \frac{\int_{B_R} |\Delta\phi|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{B_R} \frac{|\nabla\phi|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx} \; : \; \phi \in C_c^\infty(B_R) \right\}.$$

Assume $\gamma \geq 1 - N/2$. Then $A_{N,\gamma}(R)$ is independent of R, and is equal to

$$A_{N,\gamma} = \min_{\nu \in \mathbb{N}\cup\{0\}} \left\{ \frac{\left(\frac{(N-4+2\gamma)(N-2\gamma)}{4} + \alpha_\nu \right)^2}{\left(\frac{N-4+2\gamma}{2} \right)^2 + \alpha_\nu} \right\},$$

where $\alpha_\nu = \nu(N+\nu-2)$ ($\nu \in \mathbb{N}\cup\{0\}$) is the ν-th eigenvalue of the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Note that by the simple formula

$$|D^2\phi|^2 = \sum_{i,j=1}^N \left(\frac{\partial^2\phi}{\partial x_i \partial x_j} \right)^2 = \mathrm{div} \left(\frac{1}{2}\nabla|\nabla\phi|^2 - (\Delta\phi)\nabla\phi \right) + (\Delta\phi)^2,$$

for $\phi \in C_c^\infty(B_R)$, we have $\int_{B_R} |D^2\phi|^2 dx = \int_{B_R} |\Delta\phi|^2 dx$ which implies $H_{N,0} = A_{N,0}$. However, in weighted cases, it holds $\int_{B_R} |D^2\phi|^2 |\boldsymbol{x}|^{2\gamma} dx \neq \int_{B_R} |\Delta\phi|^2 |\boldsymbol{x}|^{2\gamma} dx$, and in general we have $H_{N,\gamma} \neq A_{N,\gamma}$. Also the inequality in Corollary 4 seems new.

The organization of this paper is as follows: In §2, we recall the method by Costin-Maz'ya in [2] and derive the equivalent curl-free condition in polar coordinates. In §3, we prove Theorem 1 and the sharpness of the constant (3). In §4, we prove Theorem 3 and the sharpness of the constants (5) and (6). Since the test vector fields introduced in [2] may not have compact supports, we will use different test vector fields for the proof of the sharpness of the constants.

2. Preparation: Costin-Maz'ya's setting

In this section, we recall the method of Costin-Maz'ya [2] and derive the polar coordinate representation of the curl-free condition.

Spherical polar coordinate. We introduce the spherical polar coordinates

$$(\rho, \theta_1, \theta_2, \cdots, \theta_{N-2}, \theta_{N-1}) \in (0,\infty) \times (0,\pi)^{N-2} \times [0, 2\pi)$$

whose relation to the standard Euclidean coordinates $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$ is given by

$$\boldsymbol{x} = \rho(\cos\theta_1, \pi_1\cos\theta_2, \pi_2\cos\theta_3, \cdots, \pi_{k-1}\cos\theta_k, \cdots, \pi_{N-2}\cos\theta_{N-1}, \pi_{N-1}),$$

hereafter we use the notation

$$\pi_0 = 1, \quad \pi_k = \prod_{j=1}^{k} \sin\theta_j, \quad (k = 1, 2, \cdots, N-1)$$

for simplicity. Also we use the notation

$$\partial_\rho = \frac{\partial}{\partial\rho}, \quad \partial_{\theta_k} = \frac{\partial}{\partial\theta_k}, \quad (k = 1, 2, \cdots, N-1)$$

for the partial derivatives, and

$$dx = \prod_{k=1}^{N} dx_k = dx_1 dx_2 \cdots dx_N, \quad d\sigma = \prod_{k=1}^{N-1} (\sin\theta_k)^{N-k-1} d\theta_k$$

for the volume elements on $\mathbb{R}^N$ and $\mathbb{S}^{N-1}$.

The orthonormal basis vector fields $\boldsymbol{e}_\rho, \boldsymbol{e}_{\theta_1}, \boldsymbol{e}_{\theta_2}, \cdots, \boldsymbol{e}_{\theta_{N-1}}$ along the polar coordinates are given by

$$(7) \quad \begin{cases} \boldsymbol{e}_\rho = \frac{\partial_\rho \boldsymbol{x}}{|\partial_\rho \boldsymbol{x}|} = (\cos\theta_1, \pi_1\cos\theta_2, \pi_2\cos\theta_3, \cdots \pi_{N-2}\cos\theta_{N-1}, \pi_{N-1}), \\ \boldsymbol{e}_{\theta_k} = \frac{\partial_{\theta_k} \boldsymbol{x}}{|\partial_{\theta_k} \boldsymbol{x}|} = \dfrac{1}{\pi_{k-1}} \partial_{\theta_k} \boldsymbol{e}_\rho, \quad (k = 1, 2, \cdots, N-1) \end{cases}$$

that are clearly independent of the radius ρ. Note that we can rewrite them as

$$\begin{aligned} \boldsymbol{e}_\rho &= (\cos\theta_1,\ \pi_1\cos\theta_2\ ,\ \pi_2\cos\theta_3\ , \cdots, \pi_{k-1}\cos\theta_k\ , \pi_k\boldsymbol{\varphi}_k)\ , \\ \boldsymbol{e}_{\theta_k} &= \big(\underbrace{0, 0, \cdots, 0}_{k-1}, -\sin\theta_k,\ \cos\theta_k\boldsymbol{\varphi}_k\big)\ , \end{aligned}$$

where

$$\boldsymbol{\varphi}_k = \left(\cos\theta_{k+1}, \frac{\pi_{k+1}}{\pi_k}\cos\theta_{k+2}, \frac{\pi_{k+2}}{\pi_k}\cos\theta_{k+3}, \cdots, \frac{\pi_{N-2}}{\pi_k}\cos\theta_{N-1}, \frac{\pi_{N-1}}{\pi_k}\right) \in \mathbb{S}^{N-k-1}$$

is a $(N-k)$-vector, which depends only on $\theta_{k+1}, \cdots, \theta_{N-1}$. From these expressions, we can easily check the orthonormality of $\boldsymbol{e}_\rho, \boldsymbol{e}_{\theta_1}, \boldsymbol{e}_{\theta_2}, \cdots, \boldsymbol{e}_{\theta_{N-1}}$.

For any smooth vector field $\boldsymbol{u} = (u_1, u_2, \cdots, u_N) : \mathbb{R}^N \to \mathbb{R}^N$, its polar components u_ρ, u_{θ_1}, u_{θ_2}, $\cdots$, $u_{\theta_{N-1}}$ as $\mathbb{R}$-valued smooth functions are defined by

$$\boldsymbol{u} = u_\rho \boldsymbol{e}_\rho + \sum_{k=1}^{N-1} u_{\theta_k} \boldsymbol{e}_{\theta_k}\ .$$

The second term of the right-hand side is denoted by

$$\boldsymbol{u}_{\boldsymbol{\sigma}} = \sum_{k=1}^{N-1} u_{\theta_k} \boldsymbol{e}_{\theta_k}$$

and we call this the spherical component of $\boldsymbol{u}$. Thus we have the polar decomposition of $\boldsymbol{u}$:

$$(8) \qquad \boldsymbol{u} = u_\rho \boldsymbol{e}_\rho + \boldsymbol{u}_{\boldsymbol{\sigma}}$$

which gives the decomposition of $\boldsymbol{u}$ into the radial and the spherical parts. Also by using the chain rules together with (7), we have

$$\partial_\rho = \boldsymbol{e}_\rho \cdot \nabla, \quad \text{and} \quad \frac{1}{\rho}\partial_{\theta_k} = \pi_{k-1}\boldsymbol{e}_{\theta_k} \cdot \nabla, \quad (k = 1, \cdots, N-1),$$

which give the polar decomposition of the gradient operator ∇:

$$\nabla = \boldsymbol{e}_\rho \partial_\rho + \frac{1}{\rho}\nabla_{\boldsymbol{\sigma}} \ , \tag{9}$$

where

$$\nabla_{\boldsymbol{\sigma}} = \sum_{k=1}^{N-1} \frac{\boldsymbol{e}_{\theta_k}}{\pi_{k-1}} \partial_{\theta_k} \tag{10}$$

is the gradient operator on $\mathbb{S}^{N-1}$.

Moreover, it is well-known that the polar representation of the Laplace operator $\Delta = \sum_{k=1}^{N} \partial^2/\partial x_k^2$ is given by

$$\Delta = \frac{1}{\rho^{N-1}}\partial_\rho \left(\rho^{N-1}\partial_\rho\right) + \frac{1}{\rho^2}\Delta_{\boldsymbol{\sigma}} \ , \tag{11}$$

where

$$\Delta_{\boldsymbol{\sigma}} = \sum_{k=1}^{N-1} \frac{(\sin\theta_k)^{k+1-N}}{\pi_{k-1}^2} \partial_{\theta_k} \left((\sin\theta_k)^{N-k-1}\partial_{\theta_k}\right) = \sum_{k=1}^{N-1} \frac{1}{\pi_{k-1}^2} D_{\theta_k}\partial_{\theta_k} \tag{12}$$

is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$ and for every $k = 1, \cdots, N-1$

$$D_{\theta_k} = \partial_{\theta_k} + (N-k-1)\cot\theta_k$$

is the adjoint operator of $-\partial_{\theta_k}$ in $L^2(d\sigma, \mathbb{S}^{N-1})$: it holds that

$$-\int_{\mathbb{S}^{N-1}} f\left(\partial_{\theta_k} g\right) d\sigma = \int_{\mathbb{S}^{N-1}} \left(D_{\theta_k} f\right) g d\sigma$$

for any smooth functions f, g on $\mathbb{S}^{N-1}$.

We also introduce the deformed radial coordinate $t \in \mathbb{R}$ by the Emden transformation

$$t = \log\rho. \tag{13}$$

Note that (13) leads to the transformation law of the differential operators $\rho\partial_\rho = \partial_t$. By this transformation, it is easy to check that the polar decomposition of ∇ , Δ in (9) , (11) are also given by

$$\rho\nabla = \boldsymbol{e}_\rho\partial_t + \nabla_{\boldsymbol{\sigma}}, \tag{14}$$

$$\rho^2\Delta = \partial_t^2 + (N-2)\partial_t + \Delta_{\boldsymbol{\sigma}}. \tag{15}$$

For the later use, we prove the following lemma.

Lemma 7. *Let $\nabla_{\boldsymbol{\sigma}}$ and $\Delta_{\boldsymbol{\sigma}}$ are given by (10) and (12) respectively. Then for any $f \in C^\infty(\mathbb{S}^{N-1})$, $\boldsymbol{\sigma} = \boldsymbol{e}_\rho \in \mathbb{S}^{N-1}$ and $\alpha \in \mathbb{C}$, there holds that*

$$\Delta_{\boldsymbol{\sigma}}(\boldsymbol{e}_\rho f) - \boldsymbol{e}_\rho\Delta_{\boldsymbol{\sigma}} f = \big(2\nabla_{\boldsymbol{\sigma}} - (N-1)\boldsymbol{e}_\rho\big) f, \tag{16}$$

$$\Delta_{\boldsymbol{\sigma}}\nabla_{\boldsymbol{\sigma}} f - \nabla_{\boldsymbol{\sigma}}\Delta_{\boldsymbol{\sigma}} f = \big((N-3)\nabla_{\boldsymbol{\sigma}} - 2\boldsymbol{e}_\rho\Delta_{\boldsymbol{\sigma}}\big) f, \tag{17}$$

$$\begin{aligned} \Delta_{\boldsymbol{\sigma}}\left(f\boldsymbol{e}_\rho + \alpha\nabla_{\boldsymbol{\sigma}} f\right) &= \boldsymbol{e}_\rho\big((1-2\alpha)\Delta_{\boldsymbol{\sigma}} f - (N-1)f\big) \\ &\quad + \big(2 + (N-3)\alpha\big)\nabla_{\boldsymbol{\sigma}} f + \alpha\nabla_{\boldsymbol{\sigma}}\Delta_{\boldsymbol{\sigma}} f. \end{aligned} \tag{18}$$

Proof. Take any $f \in C^\infty(\mathbb{S}^{N-1})$. We identify f with the function $\widetilde{f} \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$ defined by $\widetilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}/|\boldsymbol{x}|)$. Since $f = \widetilde{f}$ does not depend on the radius ρ, we have $\nabla_{\boldsymbol{\sigma}} f = \rho \nabla f$ by (9) and $\Delta_{\boldsymbol{\sigma}} f = \rho^2 \Delta f$ by (11). Thus we compute

$$\begin{aligned}
\Delta_{\boldsymbol{\sigma}}(\boldsymbol{e}_\rho f) - \boldsymbol{e}_\rho \Delta_{\boldsymbol{\sigma}} f &= \rho^2 \Delta \left(\frac{\boldsymbol{x} f}{\rho} \right) - \frac{\boldsymbol{x}}{\rho} \rho^2 \Delta f \\
&= \rho^2 \left(\frac{\Delta(\boldsymbol{x} f)}{\rho} + 2 \left(\left(\nabla \rho^{-1} \right) \cdot \nabla \right) (\boldsymbol{x} f) + (\Delta \rho^{-1}) \boldsymbol{x} f \right) - \rho \boldsymbol{x} \Delta f \\
&= 2\rho (\nabla f \cdot \nabla) \boldsymbol{x} - 2 \left(\nabla \rho \cdot \nabla \right) (\boldsymbol{x} f) + \rho^3 (\Delta \rho^{-1}) \boldsymbol{e}_\rho f \\
&= 2\rho \nabla f - 2\partial_\rho (\rho \boldsymbol{e}_\rho f) - (N-3) \boldsymbol{e}_\rho f \\
&= \big(2\nabla_{\boldsymbol{\sigma}} - (N-1) \boldsymbol{e}_\rho \big) f,
\end{aligned}$$

here we have used $\nabla \rho \cdot \nabla = \partial_\rho$ and $\Delta \rho^{-1} = -(N-3)\rho^{-3}$. This proves (16). Similarly, also noting the commutativity $\Delta \nabla = \nabla \Delta$ and using $\Delta \rho = (N-1)\rho^{-1}$, we have

$$\begin{aligned}
(\Delta_{\boldsymbol{\sigma}} \nabla_{\boldsymbol{\sigma}} - \nabla_{\boldsymbol{\sigma}} \Delta_{\boldsymbol{\sigma}}) f &= \rho^2 \Delta \nabla_{\boldsymbol{\sigma}} f - \rho \nabla \Delta_{\boldsymbol{\sigma}} f \\
&= \rho^2 \Delta \big(\rho \nabla f \big) - \rho \nabla \big(\rho^2 \Delta f \big) \\
&= \rho^2 \big((\Delta \rho) \nabla f + 2 (\nabla \rho \cdot \nabla) \nabla f \big) - \rho \big(\nabla \rho^2 \big) \Delta f \\
&= (N-1) \rho \nabla f + 2\rho^2 \partial_\rho \rho^{-1} \nabla_{\boldsymbol{\sigma}} f - 2\rho^2 \boldsymbol{e}_\rho \Delta f \\
&= (N-3) \nabla_{\boldsymbol{\sigma}} f - 2 \boldsymbol{e}_\rho \Delta_{\boldsymbol{\sigma}} f \ .
\end{aligned}$$

This proves (17). Finally, by (16) and (17), we see

$$\begin{aligned}
\Delta_{\boldsymbol{\sigma}} \left(f \boldsymbol{e}_\rho + \alpha \nabla_{\boldsymbol{\sigma}} f \right) &= \Delta_{\boldsymbol{\sigma}} (\boldsymbol{e}_\rho f) + \alpha \Delta_{\boldsymbol{\sigma}} \nabla_{\boldsymbol{\sigma}} f \\
&= \left(\boldsymbol{e}_\rho \Delta_{\boldsymbol{\sigma}} + 2\nabla_{\boldsymbol{\sigma}} - (N-1) \boldsymbol{e}_\rho \right) f + \alpha \left(\nabla_{\boldsymbol{\sigma}} \Delta_{\boldsymbol{\sigma}} + (N-3) \nabla_{\boldsymbol{\sigma}} - 2 \boldsymbol{e}_\rho \Delta_{\boldsymbol{\sigma}} \right) f \\
&= \boldsymbol{e}_\rho \left((1-2\alpha) \Delta_{\boldsymbol{\sigma}} f - (N-1) f \right) + \left(2 + (N-3)\alpha \right) \nabla_{\boldsymbol{\sigma}} f + \alpha \nabla_{\boldsymbol{\sigma}} \Delta_{\boldsymbol{\sigma}} f.
\end{aligned}$$

This proves (18). □

Representing the curl-free condition in polar coordinates. In the following, let "$\cdot$" denote the standard inner product in $\mathbb{R}^N$, "$\wedge$" the wedge product for differential forms and "d" the exterior derivative operator. For a vector field $\boldsymbol{a} = (a_1, a_2, \cdots, a_N) : \mathbb{R}^N \to \mathbb{R}^N$, we define the vector valued 1-form $d\boldsymbol{a} = (da_1, da_2, \cdots, da_N)$. If $\boldsymbol{u} = (u_1, u_2, \cdots, u_N)$ is a vector field, then $\boldsymbol{u} \cdot d\boldsymbol{a}$ denotes the 1-form $\sum_{k=1}^N u_k da_k$. Now, for any C^∞ vector field $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^N$ with variable $\boldsymbol{x} = (x_1, \cdots, x_N)$, we define its curl as the differential 2-form

$$\operatorname{curl} \boldsymbol{u} = d(\boldsymbol{u} \cdot d\boldsymbol{x}).$$

This can be expressed in terms of the standard Euclidean coordinates, according to the usual manipulations for the exterior derivative d and the wedge product $\wedge$:

$$d(\boldsymbol{u} \cdot d\boldsymbol{x}) = \sum_{k=1}^N du_k \wedge dx_k = \sum_{k=1}^N \sum_{j=1}^N \frac{\partial u_k}{\partial x_j} dx_j \wedge dx_k = \sum_{j<k} \sum \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right) dx_j \wedge dx_k.$$

As well as the standard representation, we want to find a representation of $d(\boldsymbol{u} \cdot d\boldsymbol{x})$ in terms of the polar coordinates $(\rho, \theta_1, \cdots, \theta_{N-1})$. For this purpose, first we differentiate the unit vector field $\boldsymbol{e}_\rho$ given by (7) and expand it in the spherical-coordinate basis:

$$d\boldsymbol{e}_\rho = \sum_{k=1}^{N-1} \frac{\partial \boldsymbol{e}_\rho}{\partial \theta_k} d\theta_k = \sum_{k=1}^{N-1} \boldsymbol{e}_{\theta_k} \pi_{k-1} d\theta_k \ .$$

Then, taking the inner product with the vector field $\boldsymbol{u} = u_\rho \boldsymbol{e}_\rho + \sum_{k=1}^{N-1} u_{\theta_k} \boldsymbol{e}_{\theta_k}$ and also taking its exterior derivative, we see that

$$\boldsymbol{u} \cdot d\boldsymbol{e}_\rho = \sum_{k=1}^{N-1} u_{\theta_k} \pi_{k-1} d\theta_k \ ,$$

$$d(\boldsymbol{u} \cdot d\boldsymbol{e}_\rho) = d\rho \wedge \sum_{k=1}^{N-1} (\partial_\rho u_{\theta_k}) \pi_{k-1} d\theta_k + \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \partial_{\theta_j} (\pi_{k-1} u_{\theta_k}) d\theta_j \wedge d\theta_k \ .$$

Also we have

$$\boldsymbol{u} \cdot d\boldsymbol{x} = \boldsymbol{u} \cdot \Big((d\rho) \boldsymbol{e}_\rho + \rho \, d\boldsymbol{e}_\rho \Big) = u_\rho d\rho + \rho \boldsymbol{u} \cdot d\boldsymbol{e}_\rho .$$

From these relations, we obtain the polar representation of the curl of $\boldsymbol{u}$:

$$\begin{aligned} d(\boldsymbol{u} \cdot d\boldsymbol{x}) &= d(u_\rho d\rho + \rho \boldsymbol{u} \cdot d\boldsymbol{e}_\rho) \\ &= du_\rho \wedge d\rho + d\rho \wedge (\boldsymbol{u} \cdot d\boldsymbol{e}_\rho) + \rho d(\boldsymbol{u} \cdot d\boldsymbol{e}_\rho) \\ &= d\rho \wedge \Big(- du_\rho + \sum_k u_{\theta_k} \pi_{k-1} d\theta_k + \sum_k \rho \partial_\rho u_{\theta_k} \pi_{k-1} d\theta_k \Big) \\ &\quad + \rho \sum_j \sum_k \partial_{\theta_j} \big(\pi_{k-1} u_{\theta_k} \big) d\theta_j \wedge d\theta_k \\ &= d\rho \wedge \sum_k \Big(\pi_{k-1} \partial_\rho (\rho u_{\theta_k}) - \partial_{\theta_k} u_\rho \Big) d\theta_k \\ &\quad + \rho \sum_{j<k}\sum \Big(\partial_{\theta_j} \big(\pi_{k-1} u_{\theta_k} \big) - \partial_{\theta_k} \big(\pi_{j-1} u_{\theta_j} \big) \Big) d\theta_j \wedge d\theta_k \ . \end{aligned}$$

Therefore, the curl-free condition $d(\boldsymbol{u} \cdot \boldsymbol{x}) = 0$ for the vector field $\boldsymbol{u}$ is represented by

$$\left\{ \begin{array}{l} \partial_\rho \big(\rho \pi_{k-1} u_{\theta_k} \big) = \partial_{\theta_k} u_\rho \\ \partial_{\theta_j} \big(\pi_{k-1} u_{\theta_k} \big) = \partial_{\theta_k} \big(\pi_{j-1} u_{\theta_j} \big) \end{array} \right. , \quad \big(j, k = 1, 2, \cdots, N-1 \ \big). \tag{19}$$

We claim that the second relation in (19) is actually a consequence of the first. Indeed, by integrating the first equation in (19) on any interval $(0, r] \subset \mathbb{R}$ with respect to the measure $d\rho$, we have $r \pi_{k-1} u_{\theta_k} = \partial_{\theta_k} \int_0^r u_\rho d\rho$ for every k. Thus the function $\phi \in C^\infty \big(\mathbb{R}^N \backslash \{ \boldsymbol{0} \} \big)$ defined by $\phi(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|} \int_0^{|\boldsymbol{x}|} u_\rho \big(\rho \boldsymbol{x} / |\boldsymbol{x}| \big) d\rho$ satisfies $\pi_{k-1} u_{\theta_k} = \partial_{\theta_k} \phi$ for all k. Then the second relation in (19) is equivalent to $\partial_{\theta_j} \partial_{\theta_k} \phi = \partial_{\theta_k} \partial_{\theta_j} \phi$, which holds trivially. This proves the claim.

Consequently we have proved that a vector field $\boldsymbol{u} \in C^\infty(\mathbb{R}^N)^N$ is curl-free if and only if

$$\partial_\rho (\rho u_{\theta_k}) = \frac{1}{\pi_{k-1}} \partial_{\theta_k} u_\rho \ , \qquad k = 1, \cdots, N-1 \ .$$

That is, using the same vector notation as in (8) and (9), we have

$$\partial_\rho (\rho \boldsymbol{u}_{\boldsymbol{\sigma}}) = \nabla_{\boldsymbol{\sigma}} u_\rho \qquad (\rho, \boldsymbol{\sigma}) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}. \tag{20}$$

In what follows we also call (20) the curl-free condition for $\boldsymbol{u}$.

Brezis-Vazquez, Maz'ya transformation. Let $\varepsilon \neq 1$ be a real number. As in [1], [11], we introduce a new vector field $\boldsymbol{v}$ by the formula

$$\boldsymbol{v}(\boldsymbol{x}) = \rho^{1-\varepsilon}\boldsymbol{u}(\boldsymbol{x}). \tag{21}$$

Then the curl-free condition (20) is transformed into

$$\nabla_{\boldsymbol{\sigma}}\left(\rho^{\varepsilon-1}v_\rho\right) = \partial_\rho(\rho^\varepsilon \boldsymbol{v}_{\boldsymbol{\sigma}}) \ ,$$

that is,

$$(\varepsilon + \rho\partial_\rho)\boldsymbol{v}_{\boldsymbol{\sigma}} = \nabla_{\boldsymbol{\sigma}} v_\rho \ . \tag{22}$$

Fourier transformation in radial direction. In the following, let us use the abbreviation $\boldsymbol{v}(t,\boldsymbol{\sigma}) = \boldsymbol{v}(e^t\boldsymbol{\sigma})$ for a vector field $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}(\rho\boldsymbol{\sigma})$, where $t = \log\rho$ is the Emden transformation given in (13). As in [2], we apply the one-dimensional Fourier transformation

$$\boldsymbol{v}(t,\boldsymbol{\sigma}) \mapsto \widehat{\boldsymbol{v}}(\lambda,\boldsymbol{\sigma}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t}\boldsymbol{v}(t,\boldsymbol{\sigma})dt$$

with respect to the variable t. By the transformation law between the derivative operators

$$\rho\partial_\rho = \partial_t \ \mapsto \ \widehat{\partial_t} = i\lambda\cdot \ ,$$

the curl-free condition (22) is changed into the equation

$$(\varepsilon + i\lambda)\widehat{\boldsymbol{v}_{\boldsymbol{\sigma}}} = \nabla_{\boldsymbol{\sigma}}\widehat{v_\rho} \ ,$$

that is,

$$\widehat{\boldsymbol{v}_{\boldsymbol{\sigma}}} = \frac{1}{\varepsilon + i\lambda}\nabla_{\boldsymbol{\sigma}} f \qquad \text{where} \quad f = \widehat{v_\rho} \ .$$

Thus we see that $\widehat{\boldsymbol{v}}_{\boldsymbol{\sigma}}$ is expressed by the spherical gradient of some function $f = \widehat{v_\rho}$. In this sense, we may consider f as a kind of scalar potential of $\widehat{\boldsymbol{v}}$, corresponding to the fact that the curl-free vector field $\boldsymbol{u}$ has a scalar potential.

Now we have proved the following proposition:

Proposition 8. *Let $\varepsilon \neq 1$ and let $\boldsymbol{u}$ be a smooth vector field on $\mathbb{R}^N$. Then $\boldsymbol{u}$ is curl-free if and only if its Brezis-Vázquez, Maz'ya transformation $\boldsymbol{v}(t,\boldsymbol{\sigma}) = e^{t(1-\varepsilon)}\boldsymbol{u}(e^t\boldsymbol{\sigma})$ satisfies*

$$(\varepsilon + \partial_t)\boldsymbol{v}_{\boldsymbol{\sigma}} = \nabla_{\boldsymbol{\sigma}} v_\rho \ . \tag{23}$$

In particular, if $\boldsymbol{u}$ is curl-free and has a compact support on $\mathbb{R}^N$, then the Fourier transformation of $\boldsymbol{v}$ satisfies

$$\widehat{\boldsymbol{v}}(\lambda,\boldsymbol{\sigma}) = f\boldsymbol{e}_\rho + \frac{1}{\varepsilon + i\lambda}\nabla_{\boldsymbol{\sigma}} f \tag{24}$$

for some complex-valued scalar function $f = f(\lambda,\boldsymbol{\sigma}) \in C^\infty(\mathbb{R}\times\mathbb{S}^{N-1})$.

We list up some formulae for $\widehat{\boldsymbol{v}}$ and its differentials. The square length of $\widehat{\boldsymbol{v}}$ is

$$|\widehat{\boldsymbol{v}}|^2 = |f|^2 + \frac{1}{\varepsilon^2+\lambda^2}|\nabla_{\boldsymbol{\sigma}} f|^2.$$

By using Lemma 7, we also see that

$$-\Delta_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}} = \boldsymbol{e}_\rho\left((N-1)f + \left(\frac{2}{\varepsilon+i\lambda}-1\right)\Delta_{\boldsymbol{\sigma}} f\right) - \left(\frac{N-3}{\varepsilon+i\lambda}+2\right)\nabla_{\boldsymbol{\sigma}} f - \frac{1}{\varepsilon+i\lambda}\nabla_{\boldsymbol{\sigma}}\Delta_{\boldsymbol{\sigma}} f\,.$$

Then integrating $|\widehat{\boldsymbol{v}}|^2$, $-\overline{\widehat{\boldsymbol{v}}}\cdot\Delta_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}$ and $|\Delta_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2$ over $\mathbb{S}^{N-1}$, we find that

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |\widehat{\boldsymbol{v}}|^2 d\sigma &= \int_{\mathbb{S}^{N-1}} \overline{f}\left(1+\frac{1}{\varepsilon^2+\lambda^2}(-\Delta_{\boldsymbol{\sigma}})\right) f d\sigma,\\
\int_{\mathbb{S}^{N-1}} |\nabla_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2 d\sigma &= \int_{\mathbb{S}^{N-1}} \overline{f}\left(N-1+\left(1+\frac{3-4\varepsilon-N}{\varepsilon^2+\lambda^2}\right)(-\Delta_{\boldsymbol{\sigma}}) + \frac{1}{\varepsilon^2+\lambda^2}(-\Delta_{\boldsymbol{\sigma}})^2\right) f d\sigma,\\
\int_{\mathbb{S}^{N-1}} |\Delta_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2 d\sigma &= \int_{\mathbb{S}^{N-1}} \overline{f}\Bigg((N-1)^2 + \left(2N+2+\frac{(N-3)^2-8\varepsilon}{\varepsilon^2+\lambda^2}\right)(-\Delta_{\boldsymbol{\sigma}})\\
&\qquad + \left(1+\frac{10-8\varepsilon-2N}{\varepsilon^2+\lambda^2}\right)(-\Delta_{\boldsymbol{\sigma}})^2 + \frac{1}{\varepsilon^2+\lambda^2}(-\Delta_{\boldsymbol{\sigma}})^3\Bigg) f d\sigma.
\end{aligned}$$

Thus, we have proved the following lemma.

Lemma 9. *Let* $\widehat{\boldsymbol{v}} = f\boldsymbol{e}_\rho + \frac{1}{\varepsilon+i\lambda}\nabla_{\boldsymbol{\sigma}} f$ *as in* (24)*. Then*

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |\widehat{\boldsymbol{v}}|^2 d\sigma &= \int_{\mathbb{S}^{N-1}} \overline{f} P_1(\lambda,-\Delta_{\boldsymbol{\sigma}}) f d\sigma,\\
\int_{\mathbb{S}^{N-1}} |\nabla_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2 d\sigma &= \int_{\mathbb{S}^{N-1}} \overline{f} P_2(\lambda,-\Delta_{\boldsymbol{\sigma}}) f d\sigma,\\
\int_{\mathbb{S}^{N-1}} |\Delta_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2 d\sigma &= \int_{\mathbb{S}^{N-1}} \overline{f} P_3(\lambda,-\Delta_{\boldsymbol{\sigma}}) f d\sigma
\end{aligned}$$

where the three polynomials $\alpha \mapsto P_k(\lambda,\alpha)$ $(k=1,2,3)$ *are given by*

$$\begin{aligned}
P_1(\lambda,\alpha) &= 1 + \frac{1}{\varepsilon^2+\lambda^2}\,\alpha,\\
P_2(\lambda,\alpha) &= N-1+\left(1+\frac{3-4\varepsilon-N}{\varepsilon^2+\lambda^2}\right)\alpha + \frac{1}{\varepsilon^2+\lambda^2}\,\alpha^2,\\
P_3(\lambda,\alpha) &= (N-1)^2 + \left(2N+2+\frac{(N-3)^2-8\varepsilon}{\varepsilon^2+\lambda^2}\right)\alpha\\
&\qquad + \left(1+\frac{10-8\varepsilon-2N}{\varepsilon^2+\lambda^2}\right)\alpha^2 + \frac{1}{\varepsilon^2+\lambda^2}\,\alpha^3.
\end{aligned}$$

3. Proof of Theorem 1

Let $\gamma \neq 1-N/2$ be a real number and put $\varepsilon = 2-N/2-\gamma \neq 1$. If the right-hand side of (2) diverges, there is nothing to prove. When the right-hand side of (2) is finite, the smoothness of $\boldsymbol{u}$ implies the existence of an integer $m > \varepsilon - 2$ such that $\nabla\boldsymbol{u}(\boldsymbol{x}) = O(|\boldsymbol{x}|^m)$ as $|\boldsymbol{x}| \to +0$. If $\varepsilon < 1$, then the vector field $\boldsymbol{v}(\boldsymbol{x})$ in (21) is Hölder continuous at $\boldsymbol{x} = \boldsymbol{0}$ and satisfies $\boldsymbol{v}(\boldsymbol{0}) = \boldsymbol{0}$. When $\varepsilon > 1$, again the assumption $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ implies $\boldsymbol{u}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{m+1})$ and $\boldsymbol{v}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{m+2-\varepsilon})$, ~~thus the same properties hold for $\boldsymbol{v}$.~~ Also since

$$\begin{aligned}\nabla\boldsymbol{u} = \nabla(\rho^{\varepsilon-1}\boldsymbol{v}) &= \rho^{\varepsilon-2}\left((\varepsilon-1)\boldsymbol{e}_\rho\otimes\boldsymbol{v}+\rho\nabla\boldsymbol{v}\right)\\ &= \rho^{\varepsilon-2}\left(\boldsymbol{e}_\rho\otimes(\varepsilon-1+\partial_t)\,\boldsymbol{v}+\nabla_{\boldsymbol{\sigma}}\boldsymbol{v}\right)\end{aligned}$$

by (14), and since $\iint_{\mathbb{S}^{n-1}\times\mathbb{R}}\partial_t\boldsymbol{v}\cdot\boldsymbol{v}d\sigma dt$ vanishes, we calculate

(25)

$$\begin{aligned}\int_{\mathbb{R}^N}|\boldsymbol{x}|^{2\gamma}|\nabla\boldsymbol{u}|^2dx &= \int_{\mathbb{R}^N}|\boldsymbol{x}|^{4-2\varepsilon-N}|\nabla\boldsymbol{u}|^2dx\\ &= \int_{\mathbb{S}^{N-1}}d\sigma\int_{\mathbb{R}}\left|\boldsymbol{e}_\rho\otimes(\varepsilon-1+\partial_t)\,\boldsymbol{v}+\nabla_{\boldsymbol{\sigma}}\boldsymbol{v}\right|^2dt\\ &= \iint_{\mathbb{S}^{N-1}\times\mathbb{R}}\left((\varepsilon-1)^2|\boldsymbol{v}|^2+|\partial_t\boldsymbol{v}|^2+|\nabla_{\boldsymbol{\sigma}}\boldsymbol{v}|^2\right)d\sigma dt\\ &= \iint_{\mathbb{S}^{N-1}\times\mathbb{R}}\left(\left((\varepsilon-1)^2+\lambda^2\right)|\widehat{\boldsymbol{v}}|^2+|\nabla_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2\right)d\sigma d\lambda\\ &= \iint_{\mathbb{S}^{N-1}\times\mathbb{R}}\overline{f}\left(\left((\varepsilon-1)^2+\lambda^2\right)P_1(\lambda,-\Delta_{\boldsymbol{\sigma}})+P_2(\lambda,-\Delta_{\boldsymbol{\sigma}})\right)fd\sigma d\lambda\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}|\boldsymbol{x}|^{2\gamma-2}|\boldsymbol{u}|^2dx &= \int_{\mathbb{R}^N}|\boldsymbol{x}|^{2-2\varepsilon-N}|\boldsymbol{u}|^2dx\\ &= \int_{\mathbb{S}^{N-1}}d\sigma\int_0^\infty|\boldsymbol{v}|^2\frac{d\rho}{\rho} = \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}|^2dtd\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\widehat{\boldsymbol{v}}|^2d\lambda d\sigma = \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\overline{f}P_1(\lambda,-\Delta_{\boldsymbol{\sigma}})fd\lambda d\sigma\end{aligned} \tag{26}$$

by Lemma 9. Therefore, by (25) and (26), the optimal constant in (2) can be expressed as

(27)

$$H_{N,\gamma} = \inf_{\boldsymbol{u}\neq 0,\operatorname{curl}\boldsymbol{u}=\boldsymbol{0}}\frac{\int_{\mathbb{R}^N}|\boldsymbol{x}|^{2\gamma}|\nabla\boldsymbol{u}|^2dx}{\int_{\mathbb{R}^N}|\boldsymbol{x}|^{2\gamma-2}|\boldsymbol{u}|^2dx} = \inf_{f\neq 0}\frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\overline{f}Q_1(\lambda,-\Delta_{\boldsymbol{\sigma}})fd\lambda d\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\overline{f}P_1(\lambda,-\Delta_{\boldsymbol{\sigma}})fd\lambda d\sigma},$$

where $Q_1(\lambda,\,\cdot\,)$ is the polynomial defined by

$$Q_1(\lambda,\alpha) = \left((\varepsilon-1)^2+\lambda^2\right)P_1(\lambda,\alpha)+P_2(\lambda,\alpha). \tag{28}$$

Calculation of a lower bound. In the same manner as Costin-Maz'ya [2], we expand f in $L^2(\mathbb{S}^{N-1})$ by eigenfunctions $\{\psi_\nu\}_{\nu\in\{0\}\cup\mathbb{N}}$ of $-\Delta_{\boldsymbol{\sigma}}$ as

$$f(\lambda,\boldsymbol{\sigma}) = \sum_{\nu=0}^{\infty}f_\nu(\lambda)\psi_\nu(\boldsymbol{\sigma}),\qquad \begin{cases}-\Delta_{\boldsymbol{\sigma}}\psi_\nu = \alpha_\nu\psi_\nu,\\ \alpha_\nu = \nu(\nu+N-2)\quad(\nu = 0,1,2,\cdots).\end{cases} \tag{29}$$

Then we find that (27) is estimated from below by

$$H_{N,\gamma} = \inf_{f\neq 0}\frac{\sum_{\nu\in\mathbb{N}\cup\{0\}}\int_{\mathbb{R}}Q_1(\lambda,\alpha_\nu)|f_\nu(\lambda)|^2d\lambda}{\sum_{\nu\in\mathbb{N}\cup\{0\}}\int_{\mathbb{R}}P_1(\lambda,\alpha_\nu)|f_\nu(\lambda)|^2d\lambda} \geq \inf_{\lambda\in\mathbb{R}\setminus\{0\}}\inf_{\nu\in\mathbb{N}\cup\{0\}}\frac{Q_1(\lambda,\alpha_\nu)}{P_1(\lambda,\alpha_\nu)},$$

where P_1, Q_1 are the same as in Lemma 9, (28) and where in the last inequality we have used Lemma 10 in Appendix, applied to $X = \{(\nu,\lambda) \in (\mathbb{N}\cup\{0\}) \times \mathbb{R}\}$, $\mu = \left(\sum_{\nu\in\mathbb{N}\cup\{0\}} \delta_\nu\right) \times d\lambda$ and $g(\nu,\lambda) = |f_\nu(\lambda)|^2$. Therefore, we have

$$H_{N,\gamma} \geq \inf_{\kappa>0} \inf_{\nu\in\mathbb{N}\cup\{0\}} F(\kappa,\alpha_\nu) \tag{30}$$

with $F(\kappa,\,\cdot\,)$ defined by

$$F(\kappa,\alpha) = \frac{Q_1(\sqrt{\kappa},\alpha)}{P_1(\sqrt{\kappa},\alpha)} = (\varepsilon-1)^2 + N - 1 + \kappa + \alpha - 2\alpha\frac{2\varepsilon+N-2}{\varepsilon^2+\kappa+\alpha} \tag{31}$$

for $\kappa > 0$ and $\alpha \geq 0$. Here we also define $F(0,\alpha)$ by

$$\begin{aligned} F(0,\alpha) &= \lim_{|\lambda|\searrow+0} \frac{Q_1(\lambda,\alpha)}{P_1(\lambda,\alpha)} = \lim_{\kappa\searrow+0} F(\kappa,\alpha) \\ &= \begin{cases} (\varepsilon-1)^2 + N - 1 + \alpha - 2\alpha\dfrac{2\varepsilon+N-2}{\varepsilon^2+\alpha} & \text{for} \quad \alpha > 0 \\ (\varepsilon-1)^2 + N - 1 & \text{for} \quad \alpha = 0\,. \end{cases} \end{aligned} \tag{32}$$

In this setting, we calculate the right-hand side of (30). In the case $\varepsilon < 1 - N/2$, by differentiating (31) directly with respect to α, we see that

$$\frac{\partial}{\partial\alpha}F(\kappa,\alpha) = 1 - 2(2\varepsilon+N-2)\frac{\varepsilon^2+\kappa}{(\varepsilon^2+\kappa+\alpha)^2} > 0\,.$$

Thus $0 \leq \alpha \mapsto F(\kappa,\alpha)$ is monotone increasing for each $\kappa > 0$, and

$$F(\kappa,\alpha) \geq F(\kappa,0) = (\varepsilon-1)^2 + N - 1 + \kappa > F(0,0) = F(0,\alpha_0)\,,$$

that implies

$$\inf_{\kappa>0} \inf_{\nu\in\mathbb{N}\cup\{0\}} F(\kappa,\alpha_\nu) = F(0,\alpha_0) \qquad \text{when} \quad \varepsilon < 1 - N/2.$$

In the case $\varepsilon \geq 1 - N/2$, by (31) we see that $F(\kappa,\alpha)$ is increasing in $\kappa > 0$ for each $\alpha \geq 0$. Thus $F(\kappa,\alpha) \geq F(0,\alpha)$ and

$$\inf_{\kappa>0} \inf_{\nu\in\mathbb{N}\cup\{0\}} F(\kappa,\alpha_\nu) = \inf_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu).$$

To evaluate the right-hand side, we compute

$$\begin{aligned} \frac{\partial}{\partial\alpha}F(0,\alpha) &= 1 - 2(2\varepsilon+N-2)\frac{\varepsilon^2}{(\varepsilon^2+\alpha)^2} = \frac{\varepsilon^4 - 4\varepsilon^3 + 2(\alpha-(N-2))\varepsilon^2 + \alpha^2}{(\varepsilon^2+\alpha)^2} \\ &\geq \frac{\varepsilon^2(\varepsilon+2)^2 + \alpha^2}{(\varepsilon^2+\alpha)^2} > 0 \qquad \text{if} \quad \alpha \geq N\,. \end{aligned}$$

Thus we have $F(0,\alpha) > F(0,N)$ for any $\alpha \geq N$, which implies $F(0,\alpha_\nu) \geq F(0,\alpha_2) = F(0,2N)$ for all $\nu \geq 2$. This in turn implies

$$\inf_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu) = \min_{\nu\in\{0,1,2\}} F(0,\alpha_\nu)\,.$$

Moreover, by computing

$$\begin{aligned} F(0,\alpha_2) - F(0,\alpha_1) &= F(0,2N) - F(0,N-1) \\ &= \frac{(N+1)\varepsilon^2\big((\varepsilon-2)^2 + N - 1\big) + 2N(N-1)}{(\varepsilon^2+N-1)(\varepsilon^2+2N)} > 0\,, \end{aligned}$$

we see that

$$\inf_{\nu\in\{0,1,2\}} F(0,\alpha_\nu) = \min_{\nu\in\{0,1\}} F(0,\alpha_\nu).$$

Therefore, by calculating

$$F(0,\alpha_1) - F(0,\alpha_0) = F(0,N-1) - F(0,0) = (N-1)\frac{(\varepsilon-2)^2-(N+1)}{\varepsilon^2+N-1},$$

it turns out that

$$\begin{aligned}\inf_{\kappa>0}\inf_{\nu\in\mathbb{N}\cup\{0\}} F(\kappa,\alpha_\nu) &= \min_{\nu\in\{0,1\}} F(0,\alpha_\nu)\\ &= \begin{cases} F(0,\alpha_1) & \text{for} \quad (\varepsilon-2)^2 \le N+1\ , \\ F(0,\alpha_0) & \text{for} \quad (\varepsilon-2)^2 > N+1 \end{cases}\end{aligned} \tag{33}$$

when $\varepsilon \ge 1 - N/2$. The expression (33) holds true even for $\varepsilon < 1 - N/2$ since $\varepsilon < 1 - N/2$ implies $(\varepsilon-2)^2 > N+1$.

Inserting this result into (30), we have

$$\begin{aligned} H_{N,\gamma} &\ge \min_{\nu\in\{0,1\}} F(0,\alpha_\nu)\\ &= \begin{cases} F(0,\alpha_1) = (\varepsilon-1)^2\frac{\varepsilon^2+3(N-1)}{\varepsilon^2+N-1} & \text{for} \quad |\varepsilon-2| \le \sqrt{N+1}\ , \\ F(0,\alpha_0) = (\varepsilon-1)^2 + N - 1 & \text{otherwise.} \end{cases}\end{aligned}$$

Returning to $\varepsilon = 2 - \frac{N}{2} - \gamma$, we arrive at the desired infimum value in Theorem 1.

Optimality for $H_{N,\gamma}$. In this subsection, we prove that the former lower bound of $H_{N,\gamma}$ is indeed realized as an equality:

$$H_{N,\gamma} = \min_{\nu\in\{0,1\}} F(0,\alpha_\nu) = \min_{\nu\in\{0,1\}} \lim_{|\lambda|\searrow+0} \frac{Q_1(\lambda,\alpha_\nu)}{P_1(\lambda,\alpha_\nu)}.$$

For that purpose, let $\nu_0 \in \{0,1\}$ be such that

$$\min_{\nu\in\{0,1\}} F(0,\alpha_\nu) = F(0,\alpha_{\nu_0}).$$

By the argument in the last subsection, it is enough to prove that there exists a sequence of curl-free vector fields $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)^N$ such that

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\nabla\boldsymbol{u}_n|^2 dx}{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-2}|\boldsymbol{u}_n|^2 dx} = \lim_{|\lambda|\searrow+0} \frac{Q_1(\lambda,\alpha_{\nu_0})}{P_1(\lambda,\alpha_{\nu_0})}. \tag{34}$$

For the construction of $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$, take any nonnegative $h \in C_c^\infty(\mathbb{R})$, $h \not\equiv 0$ and put $h_n(t) = h(t/n)$ for $n \in \mathbb{N}$. Set

$$\boldsymbol{v}_n(\rho,\boldsymbol{\sigma}) = \boldsymbol{e}_\rho\left(\varepsilon h_n(t) + h_n'(t)\right)\psi_{\nu_0}(\boldsymbol{\sigma}) + h_n(t)\nabla_{\boldsymbol{\sigma}}\psi_{\nu_0}(\boldsymbol{\sigma}) \tag{35}$$

where $\rho = e^t$ and ψ_{ν_0} denotes an eigenfunction of $-\Delta_{\boldsymbol{\sigma}}$ associated with the eigenvalue $\alpha_{\nu_0} = \nu_0(\nu_0+N-2)$. Then it is clear that $\boldsymbol{v}_n$ satisfies (23). Define

$$\boldsymbol{u}_n(\rho,\boldsymbol{\sigma}) = \rho^{\varepsilon-1}\boldsymbol{v}_n(\rho,\boldsymbol{\sigma}) \tag{36}$$

for $\varepsilon = 2 - N/2 - \gamma$. Then $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is a sequence of curl-free vector fields having compact supports on $\mathbb{R}^N \setminus \{\boldsymbol{0}\}$. Put

$$f_n(\lambda,\boldsymbol{\sigma}) = \widehat{(v_n)_\rho}(\lambda,\boldsymbol{\sigma}) = (\varepsilon + i\lambda)\,\widehat{h_n}(\lambda)\psi_{\nu_0}(\boldsymbol{\sigma})$$

and compute the Hardy-Leray quotient for $\boldsymbol{u}_n$ by using (25) and (26). We see that

$$\frac{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\nabla \boldsymbol{u}_n|^2dx}{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-2}|\boldsymbol{u}_n|^2dx} = \frac{\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \overline{f_n}Q_1(\lambda,-\Delta_{\boldsymbol{\sigma}})f_n d\sigma d\lambda}{\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \overline{f_n}P_1(\lambda,-\Delta_{\boldsymbol{\sigma}})f_n d\sigma d\lambda}$$
$$= \frac{\int_{\mathbb{R}}(\varepsilon^2+\lambda^2)Q_1(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2d\lambda}{\int_{\mathbb{R}}(\varepsilon^2+\lambda^2)P_1(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2d\lambda}$$
$$= \frac{\int_{\mathbb{R}} Q_{01}(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2d\lambda}{\int_{\mathbb{R}} P_{01}(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2d\lambda},$$

here

$$\begin{aligned} P_{01}(\lambda,\alpha) &= (\varepsilon^2+\lambda^2)P_1(\lambda,\alpha) = \varepsilon^2+\alpha+\lambda^2, \\ Q_{01}(\lambda,\alpha) &= (\varepsilon^2+\lambda^2)Q_1(\lambda,\alpha) \end{aligned} \tag{37}$$

are polynomials in λ. Note that $\widehat{h_n}(\lambda) = \widehat{h(t/n)}(\lambda) = n\widehat{h}(n\lambda)$. Thus if $\varepsilon^2+\alpha_{\nu_0}\neq 0$, then we have

$$\frac{\iint_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n|^2|x|^{2\gamma}dx}{\iint_{\mathbb{R}^N} |\boldsymbol{u}_n|^2|x|^{2\gamma-2}dx} = \frac{\int_{\mathbb{R}} Q_{01}(\lambda,\alpha_{\nu_0})|\widehat{h}(n\lambda)|^2d\lambda}{\int_{\mathbb{R}} P_{01}(\lambda,\alpha_{\nu_0})|\widehat{h}(n\lambda)|^2d\lambda}$$
$$\to \frac{Q_{01}(0,\alpha_{\nu_0})}{P_{01}(0,\alpha_{\nu_0})} = \lim_{|\lambda|\to+0}\frac{Q_1(\lambda,\alpha_{\nu_0})}{P_1(\lambda,\alpha_{\nu_0})}$$

as $n\to\infty$. In the case $\varepsilon = 0 = \alpha_{\nu_0}$, by using

$$P_{01}(\lambda,0)=\lambda^2, \quad Q_{01}(\lambda,0) = N\lambda^2+\lambda^4,$$

we can check that

$$\frac{\iint_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n|^2|x|^{2\gamma}dx}{\iint_{\mathbb{R}^N} |\boldsymbol{u}_n|^2|x|^{2\gamma-2}dx} = \frac{\int_{\mathbb{R}} Q_{01}(\lambda,0)|\widehat{h}(n\lambda)|^2d\lambda}{\int_{\mathbb{R}} P_{01}(\lambda,0)|\widehat{h}(n\lambda)|^2d\lambda} = \frac{\int_{\mathbb{R}}(N\lambda^2+\lambda^4)|\widehat{h}(n\lambda)|^2d\lambda}{\int_{\mathbb{R}}\lambda^2|\widehat{h}(n\lambda)|^2d\lambda}$$
$$\to N = \lim_{|\lambda|\to+0}\frac{Q_1(\lambda,0)}{P_1(\lambda,0)}$$

as $n\to\infty$. Thus we have proved (34) which shows the optimality of $H_{N,\gamma}$ in the class of curl-free vector fields in $C_c^\infty(\mathbb{R}^N)^N$. □

4. Proof of Theorem 3

Let $\gamma\neq 2-N/2$ be a real number and put $\varepsilon = 3-N/2-\gamma\neq 1$. Under the transformation $\boldsymbol{v} = \rho^{1-\varepsilon}\boldsymbol{u}$ in (21), the gradient vector field is transformed as

$$\nabla\boldsymbol{v} = \nabla(\rho^{1-\varepsilon}\boldsymbol{u}) = (1-\varepsilon)\rho^{-\varepsilon}\boldsymbol{e}_\rho\otimes\boldsymbol{u}+\rho^{1-\varepsilon}\nabla\boldsymbol{u},$$

which leads to

$$|\rho\nabla\boldsymbol{v}|^2 = (1-\varepsilon)^2|\rho^{1-\varepsilon}\boldsymbol{u}|^2+2(1-\varepsilon)\rho^{2-2\varepsilon}\boldsymbol{u}\cdot\rho\partial_\rho\boldsymbol{u}+\rho^{2-2\varepsilon}|\rho\nabla\boldsymbol{u}|^2. \tag{38}$$

On the other hand, the assumption $\int_{\mathbb{R}^N}|\boldsymbol{x}|^{2-2\varepsilon-N}|\boldsymbol{u}|^2dx<\infty$ and the smoothness of $\boldsymbol{u}$ imply that

$$\boldsymbol{u}(\boldsymbol{x}) = O\left(|\boldsymbol{x}|^m\right), \quad \nabla\boldsymbol{u}(\boldsymbol{x}) = O\left(|\boldsymbol{x}|^{m-1}\right) \quad \text{as} \quad |\boldsymbol{x}|\searrow 0$$

for some integer $m>\varepsilon-1$ if $\varepsilon>1$. Therefore, we see that $\boldsymbol{v}$ must satisfy

$$|\boldsymbol{v}(0)| = \lim_{\rho\searrow 0}|\rho\nabla\boldsymbol{v}| = 0 \tag{39}$$

by (38) when $\varepsilon > 1$.

Next, we see the $\Delta \boldsymbol{u}$ is written in terms of $\boldsymbol{v}$ as follows:

$$\Delta \boldsymbol{u} = \Delta(\rho^{\varepsilon-1}\boldsymbol{v}) = \rho^{\varepsilon-3}\left(\alpha_{\varepsilon-1}\boldsymbol{v} + (2\varepsilon+N-4)\partial_t \boldsymbol{v} + \partial_t^2 \boldsymbol{v} + \Delta_{\boldsymbol{\sigma}}\boldsymbol{v}\right), \tag{40}$$

here we have used (15) and $\Delta\rho^{\varepsilon-1} = \alpha_{\varepsilon-1}\rho^{\varepsilon-3}$. Note that $\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \partial_t \boldsymbol{v}\cdot\boldsymbol{v} d\sigma dt = \iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \partial_t^2 \boldsymbol{v}\cdot\partial_t \boldsymbol{v} d\sigma dt = 0$ and $\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \boldsymbol{v}\cdot\partial_t^2\boldsymbol{v} d\sigma dt = -\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} |\partial_t \boldsymbol{v}|^2 d\sigma dt$ by (39). Thus by using (40), Lemma 9, and noting $(2\varepsilon+N-4)^2 - 2\alpha_{\varepsilon-1} = (N-2)^2 + 2\alpha_{\varepsilon-1}$, we find that the both integrals of the Rellich-Leray inequality (4) are written as

$$\begin{aligned}
\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\Delta \boldsymbol{u}|^2 dx &= \int_{\mathbb{R}^N} |\boldsymbol{x}|^{6-2\varepsilon-N}|\Delta \boldsymbol{u}|^2 dx \\
&= \int_{\mathbb{S}^{N-1}} d\sigma \int_0^\infty \left|\alpha_{\varepsilon-1}\boldsymbol{v} + (2\varepsilon+N-4)\partial_t \boldsymbol{v} + \partial_t^2\boldsymbol{v} + \Delta_{\boldsymbol{\sigma}}\boldsymbol{v}\right|^2 \frac{d\rho}{\rho} \\
&= \int_{\mathbb{S}^{N-1}} d\sigma \int_{\mathbb{R}} \Big(\alpha_{\varepsilon-1}^2 |\boldsymbol{v}|^2 + \left((N-2)^2 + 2\alpha_{\varepsilon-1}\right)|\partial_t \boldsymbol{v}|^2 + |\partial_t^2 \boldsymbol{v}|^2 \\
&\qquad\qquad - 2\alpha_{\varepsilon-1}|\nabla_{\boldsymbol{\sigma}}\boldsymbol{v}|^2 + 2|\partial_t\nabla_{\boldsymbol{\sigma}}\boldsymbol{v}|^2 + |\Delta_{\boldsymbol{\sigma}}\boldsymbol{v}|^2\Big) dt \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(\Big(\alpha_{\varepsilon-1}^2 + \left((N-2)^2 + 2\alpha_{\varepsilon-1}\right)\lambda^2 + \lambda^4\Big)|\widehat{\boldsymbol{v}}|^2 \\
&\qquad\qquad + 2\left(\lambda^2 - \alpha_{\varepsilon-1}\right)|\nabla_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2 + |\Delta_{\boldsymbol{\sigma}}\widehat{\boldsymbol{v}}|^2\Big) d\lambda d\sigma \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \overline{f}\Big(\left(\alpha_{\varepsilon-1}^2 + \left((N-2)^2 + 2\alpha_{\varepsilon-1}\right)\lambda^2 + \lambda^4\right) P_1(\lambda, -\Delta_{\boldsymbol{\sigma}}) \\
&\qquad\qquad + 2\left(\lambda^2 - \alpha_{\varepsilon-1}\right) P_2(\lambda, -\Delta_{\boldsymbol{\sigma}}) + P_3(\lambda, -\Delta_{\boldsymbol{\sigma}})\Big) f d\lambda d\sigma,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}|^2 dx &= \int_{\mathbb{R}^N} |\boldsymbol{x}|^{2-2\varepsilon-N}|\boldsymbol{u}|^2 dx \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \overline{f} P_1(\lambda, -\Delta_{\boldsymbol{\sigma}}) f d\lambda d\sigma.
\end{aligned} \tag{42}$$

Therefore, by (41) and (42), the optimal constant in (4) can be expressed as

$$R_{N,\gamma} = \inf_{\boldsymbol{u}\neq 0, \operatorname{curl}\boldsymbol{u}=\boldsymbol{0}} \frac{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\Delta \boldsymbol{u}|^2 dx}{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}|^2 dx} = \inf_{f\neq 0} \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \overline{f} Q_2(\lambda, -\Delta_{\boldsymbol{\sigma}}) f d\lambda d\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \overline{f} P_1(\lambda, -\Delta_{\boldsymbol{\sigma}}) f d\lambda d\sigma} \tag{43}$$

with the polynomial $Q_2(\lambda, \alpha)$ given by

$$\begin{aligned}
Q_2(\lambda, \alpha) &= \left(\alpha_{\varepsilon-1}^2 + \left((N-2)^2 + 2\alpha_{\varepsilon-1}\right)\lambda^2 + \lambda^4\right) P_1(\lambda, \alpha) \\
&\quad + 2\left(\lambda^2 - \alpha_{\varepsilon-1}\right) P_2(\lambda, \alpha) + P_3(\lambda, \alpha).
\end{aligned} \tag{44}$$

Calculation of a lower bound. As in (29), we expand f in terms of eigenfunctions of $-\Delta_{\boldsymbol{\sigma}}$. Then by (43), (44), and Lemma 10, we find

$$R_{N,\gamma} \geq \inf_{\nu\in\mathbb{N}\cup\{0\}} \inf_{\lambda\in\mathbb{R}\setminus\{0\}} \frac{Q_2(\lambda, \alpha_\nu)}{P_1(\lambda, \alpha_\nu)} = \inf_{\nu\in\mathbb{N}\cup\{0\}} \inf_{\kappa>0} F(\kappa, \alpha_\nu),$$

where for $\kappa > 0$ and $\alpha \geq 0$, $F(\kappa, \alpha)$ is defined as

$$F(\kappa,\alpha) = \frac{Q_2(\sqrt{\kappa},\alpha)}{P_1(\sqrt{\kappa},\alpha)}$$
$$= \alpha_{\varepsilon-1}^2 + \left((N-2)^2 + 2\alpha_{\varepsilon-1}\right)\kappa + \kappa^2 + \frac{2\,(\kappa-\alpha_{\varepsilon-1})\,P_2(\sqrt{\kappa},\alpha) + P_3(\sqrt{\kappa},\alpha)}{P_1(\sqrt{\kappa},\alpha)} .$$

By directly calculating further, we can check that

$$F(\kappa,\alpha) = \kappa^2 + \frac{4\alpha(1-\varepsilon)(N+2\varepsilon-2)^2\kappa}{(\varepsilon^2+\alpha)(\kappa+\varepsilon^2+\alpha)} + \left(\frac{N^2}{2} + 2\left(\varepsilon + \frac{N-4}{2}\right)^2 + 2\alpha\right)\kappa + \frac{(\varepsilon-2)^2+\alpha}{\varepsilon^2+\alpha}(\alpha_\varepsilon - \alpha)^2$$

for $\varepsilon = 3 - N/2 - \gamma \neq 0$, and

$$F(\kappa,\alpha) = \kappa^2 + \frac{4(N-2)^2\kappa}{\kappa+\alpha} + \left((N-2)^2 + 4 + 2\alpha\right)\kappa + (4+\alpha)\alpha$$

for $\varepsilon = 0$. We also define $F(0,\alpha)$ as

$$F(0,\alpha) = \lim_{|\lambda|\searrow+0} \frac{Q_2(\lambda,\alpha)}{P_1(\lambda,\alpha)} = \lim_{\kappa\searrow+0} F(\kappa,\alpha)$$

$$= \begin{cases} \frac{(\varepsilon-2)^2+\alpha}{\varepsilon^2+\alpha}(\alpha_\varepsilon-\alpha)^2, & \text{for } \varepsilon \neq 0, \alpha \geq 0, \\ (4+\alpha)\alpha, & \text{for } \varepsilon = 0, \alpha > 0, \\ 4(N-2)^2, & \text{for } \varepsilon = 0, \alpha = 0. \end{cases} \tag{45}$$

In these settings, from now on we evaluate the expression

$$\inf_{\nu\in\mathbb{N}\cup\{0\}} \inf_{\kappa>0} F(\kappa,\alpha_\nu).$$

If $\varepsilon < 1$, it is clear that the map $0 < \kappa \mapsto F(\kappa,\alpha)$ is increasing for any fixed $\alpha \geq 0$. Also, if $\varepsilon > 1$, estimating $\partial_\kappa F(\kappa,\alpha)$ from below by

$$\begin{aligned} \frac{\partial F(\kappa,\alpha)}{\partial\kappa} &= 2\kappa - \frac{4\alpha(\varepsilon-1)(N+2\varepsilon-2)^2}{(\kappa+\varepsilon^2+\alpha)^2} + \frac{N^2}{2} + 2\left(\varepsilon+\frac{N-4}{2}\right)^2 + 2\alpha \\ &\geq -\frac{4\alpha(\varepsilon-1)(N+2\varepsilon-2)^2}{(\varepsilon^2+\alpha)^2} + \frac{N^2}{2} + 2\left(\varepsilon+\frac{N-4}{2}\right)^2 + 2\alpha \\ &\geq -\frac{\varepsilon-1}{\varepsilon^2}(N+2\varepsilon-2)^2 + \frac{N^2}{2} + 2\left(\varepsilon+\frac{N-4}{2}\right)^2 + 2\alpha \\ &\geq -\frac{1}{4}(N+2\varepsilon-2)^2 + \frac{N^2}{2} + 2\left(\varepsilon+\frac{N-4}{2}\right)^2 + 2\alpha \\ &= \left(\varepsilon+\frac{N}{2}-3\right)^2 + \frac{N^2-4}{2} + 2\alpha \geq 0, \end{aligned}$$

we see again that $F(\kappa,\alpha)$ is increasing with respect to $\kappa > 0$ for any $\alpha \geq 0$. Therefore we have

$$\inf_{\kappa>0} F(\kappa,\alpha) = F(0,\alpha)$$

for all $\varepsilon \neq 1$, which implies

$$\inf_{\nu\in\mathbb{N}\cup\{0\}} \inf_{\kappa>0} F(\kappa,\alpha_\nu) = \inf_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu) .$$

Moreover, we can check that

$$\frac{\partial F(0,\alpha)}{\partial \alpha} \geq 0, \quad \alpha \geq \max\{\alpha_1, \alpha_\varepsilon\},$$

see Lemma 11. This implies that $\inf_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu)$ is attained. Therefore, we have the desired estimate:

$$R_{N,\gamma} \geq \min_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu) \qquad \text{with} \quad F(0,\alpha_\nu) = \lim_{|\lambda|\searrow +0} \frac{Q_2(\lambda,\alpha_\nu)}{P_1(\lambda,\alpha_\nu)}\ . \tag{46}$$

Furthermore, we see that $\min_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu)$ is given by

$$\min_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu) = \min_{\nu\in\mathbb{N}\cup\{0\}} \frac{(\varepsilon-2)^2+\alpha_\nu}{\varepsilon^2+\alpha_\nu}(\alpha_\varepsilon-\alpha_\nu)^2$$

for $\varepsilon = 3 - N/2 - \gamma \neq 0$, and

$$\begin{aligned}\min_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu) &= \min\left\{4(N-2)^2, (4+\alpha_1)\alpha_1\right\} \\ &= \begin{cases} 4(N-2)^2 = F(0,\alpha_0) & \text{for} \quad N=2,3,4, \\ (N+3)(N-1) = F(0,\alpha_1) & \text{for} \quad N\geq 5 \end{cases}\end{aligned}$$

for $\varepsilon = 3-N/2-\gamma = 0$. This gives the lower bound of $R_{N,\gamma}$. In the next subsection we will show that the above inequality is indeed the equality.

Optimality for $R_{N,\gamma}$. To show that the inequality (46) is indeed the equality, let $\nu_0 \in \mathbb{N}\cup\{0\}$ be such that $F(0,\alpha_{\nu_0}) = \min_{\nu\in\mathbb{N}\cup\{0\}} F(0,\alpha_\nu)$ is satisfied. We use the sequence of curl-free vector fields $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ in (36) again with (35), however for $\varepsilon = 3 - N/2 - \gamma$. Then, as in the proof of Theorem 1, we obtain the following expression:

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\Delta\boldsymbol{u}_n|^2 dx}{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}_n|^2 dx} &= \frac{\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \overline{f_n} Q_2(\lambda, -\Delta_{\boldsymbol{\sigma}}) f_n d\sigma d\lambda}{\iint_{\mathbb{S}^{N-1}\times\mathbb{R}} \overline{f_n} P_1(\lambda, -\Delta_{\boldsymbol{\sigma}}) f_n d\sigma d\lambda} \\ &= \frac{\int_{\mathbb{R}}(\varepsilon^2+\lambda^2)Q_2(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2 d\lambda}{\int_{\mathbb{R}}(\varepsilon^2+\lambda^2)P_1(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2 d\lambda} \\ &= \frac{\int_{\mathbb{R}} Q_{02}(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2 d\lambda}{\int_{\mathbb{R}} P_{01}(\lambda,\alpha_{\nu_0})|\widehat{h_n}(\lambda)|^2 d\lambda},\end{aligned}$$

where $P_{01}(\lambda,\alpha)$ is the same as in (37) and

$$Q_{02}(\lambda,\alpha) = (\varepsilon^2+\lambda^2)Q_2(\lambda,\alpha)$$

is a polynomial in λ. When $\varepsilon = 0$ and $\alpha_{\nu_0} = 0$, by using the facts

$$Q_{02}(\lambda,0) = 4(N-2)^2\lambda^2 + (N^2-4N+8)\lambda^4 + \lambda^6$$

and $P_{01}(\lambda,0) = \lambda^2$, we prove that

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\Delta\boldsymbol{u}_n|^2 dx}{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}_n|^2 dx} = 4(N-2)^2 = F(0,0).$$

Thus as in the proof of Theorem 1, we can show that

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma}|\Delta \boldsymbol{u}_n|^2 dx}{\int_{\mathbb{R}^N} |\boldsymbol{x}|^{2\gamma-4}|\boldsymbol{u}_n|^2 dx} = F(0, \alpha_{\nu_0})$$

for all cases $\varepsilon^2 + \alpha_{\nu_0} \neq 0$ and $\varepsilon^2 + \alpha_{\nu_0} = 0$. This leads to the optimality of $R_{N,\gamma}$. □

5. Appendix.

In this appendix, we prove technical lemmas.

Lemma 10. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\xi, \eta : X \to \mathbb{R}$ be a μ-measurable function such that $\xi \neq 0$ μ-a.e. Suppose $g : X \to \mathbb{R}$ is a μ-measurable function satisfying, $\xi g \geq 0$ μ-a.e., $0 < \int_X \xi g d\mu < \infty$, and $\int_X |\eta g| d\mu < \infty$. Then we have*

$$\frac{\int_X \eta g d\mu}{\int_X \xi g d\mu} \geq \operatorname{ess} \inf_{x\in X} \frac{\eta(x)}{\xi(x)}.$$

Proof. Let $I = \operatorname{ess\,inf}_{x\in X} \frac{\eta(x)}{\xi(x)}$. Then $\frac{\eta}{\xi} \geq I$ μ-a.e. Multiply the both sides by $\xi g \geq 0$, we have $\eta g = \frac{\eta}{\xi}\xi g \geq I\xi g$ μ-a.e.. By integrating over X, we obtain

$$\int_X \eta g d\mu \geq I \int_X \xi g d\mu$$

which leads the result. □

Lemma 11. *Let $F(0, \alpha)$ be given by (45). Then we have*

$$\frac{\partial F(0, \alpha)}{\partial \alpha} \geq 0 \quad \text{for} \quad \alpha \geq \max\{\alpha_1, \alpha_\varepsilon\}.$$

Proof. Recall $\alpha_1 = N - 1$ and $\alpha_\varepsilon = \varepsilon(\varepsilon + N - 2)$. It is enough to show the lemma when $\varepsilon \neq 0$ and $F(0, \alpha) = \frac{(\varepsilon-2)^2+\alpha}{\varepsilon^2+\alpha}(\alpha_\varepsilon - \alpha)^2$. A direct computation shows that

$$\frac{\partial F(0, \alpha)}{\partial \alpha} = \frac{2(\alpha - \alpha_\varepsilon)}{(\alpha + \varepsilon^2)^2} f_\varepsilon(\alpha), \quad \text{where}$$
$$f_\varepsilon(\alpha) = \alpha^2 + 2(\varepsilon^2 - \varepsilon + 1)\alpha + \varepsilon^2(\varepsilon - 1)^2 + 2\alpha_\varepsilon(1 - \varepsilon).$$

Since $\varepsilon^2 - \varepsilon + 1 > 0$ for any $\varepsilon \in \mathbb{R}$, we see that f_ε is strictly increasing for $\alpha \geq 0$. Thus if we show (i) $f_\varepsilon(\alpha_\varepsilon) \geq 0$ if $\alpha_\varepsilon \geq \alpha_1$, and (ii) $f_\varepsilon(\alpha_1) \geq 0$ if $\alpha_1 \geq \alpha_\varepsilon$, then $f_\varepsilon(\alpha) \geq 0$ for any $\alpha \geq \max\{\alpha_1, \alpha_\varepsilon\}$, which concludes the lemma.

To prove (i), we observe that $f_\varepsilon(\alpha_\varepsilon) = (\alpha_\varepsilon + \varepsilon^2)(\alpha_\varepsilon + (\varepsilon - 2)^2)$. Thus if $\alpha_\varepsilon \geq \alpha_1 = N - 1 > 0$, clearly we have $f_\varepsilon(\alpha_\varepsilon) > 0$.

To prove (ii), we observe that $f_\varepsilon(\alpha_1) = f_\varepsilon(N-1) = \varepsilon^4 - 6\varepsilon^3 + 8\varepsilon^2 - 2\varepsilon + N^2 - 1$. We need to prove this quartic function is nonnegative for $\varepsilon \in \mathbb{R}$ such that $\alpha_1 \geq \alpha_\varepsilon$, i.e., $-(N - 1) \leq \varepsilon \leq 1$. However, this is an elementary fact. □

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