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Continuous wavelet transforms for vector-valued functions*

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Abstract. We consider continuous wavelet transforms associated to unitary representations of the semi-direct product of a vector group with a linear Lie group realized on the Hilbert spaces of square-integrable vector-valued functions. In particular, we give a concrete example of an admissible vector-valued function (vector field) for the 3-dimensional similitude group.

Keywords: continuous wavelet transforms · unitary representations · similitude group · divergence-free vector fields

1 Introduction

Let ϕ be a square-integrable function on $\mathbb{R}$ such that $2\pi \int_{\mathbb{R}} |\mathcal{F}\phi(\xi)|^2 \frac{d\xi}{|\xi|} = 1$ (see below for the definition of the Fourier transform $\mathcal{F}\phi$). Then for any $f \in L^2(\mathbb{R})$, we have the reproducing formula $f = \int_{\mathbb{R}\times(\mathbb{R}\setminus\{0\})} (f|\phi_{b,a})_{L^2} \phi_{b,a} \frac{db da}{|a|^2}$, where $\phi_{b,a}(x) := |a|^{-1/2}\phi(\frac{x-b}{a})$ $(a \neq 0,\, b \in \mathbb{R})$, and the integral is interpreted in the weak sense. Let $\mathrm{Aff}(\mathbb{R})$ be the group of invertible affine transforms $g_{b,a}(x) := b + ax$ $(a \neq 0,\, b \in \mathbb{R})$. We have a natural unitary representation π of $\mathrm{Aff}(\mathbb{R})$ on $L^2(\mathbb{R})$ given by $\pi(g_{b,a})f := f_{b,a}$. Then the reproducing formula above means that the continuous wavelet transform $W_\phi : L^2(\mathbb{R}) \ni f \mapsto (f|\pi(\cdot)\phi) \in L^2(G)$ is an isometry, and as was pointed out in [6], the isometry property follows from the orthogonal relation for square-integrable irreducible unitary representations. After [6], theory of continuous wavelet transforms is generalized to a wide class of groups and representations (cf. [1], [3]). In particular, the continuous wavelet transform associated to the quasi-regular representation of the semi-direct product $G = \mathbb{R}^n \rtimes H$, where H is a Lie subgroup of $GL(n,\mathbb{R})$, defined on the Hilbert space $L^2(\mathbb{R}^n)$ is studied thoroughly, whereas the assumption of the irreducibility of the representation is relaxed by [5].

In the present paper, we consider continuous wavelet transforms for vector-valued functions in a reasonably general setting. We show in Theorem 3 that,

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under certain assumptions (A1), (A2) and (A3), one can construct a continuous wavelet transform with an appropriate admissible vector. The conditions (A1) and (A2) seem common in the wavelet theory, whereas the condition (A3) is specific in our setting. The results can be applied to the natural action of the similitude group $\mathbb{R}^n \rtimes (\mathbb{R}_{>0} \times SO(n))$ on vector fields and tensor fields on $\mathbb{R}^n$ (see Section 3). In particular, we present a concrete example of admissible vector for the 3-dimensional vector field (see (11)). Our example can be used for the decomposition of vector fields into a sum of divergence-free and curl-free vector fields. This result will shed some insight to the wavelet theory for such vector fields ([7], [9]). A part of the contents of this paper is contained in the second author's doctoral thesis [8], where one can find other concrete examples of admissible vectors.

Let us fix some notations used in this paper. The transposition of a matrix A is denoted by $A^\top$. For $w, w' \in \mathbb{C}^m$, the standard inner product is defined by $(w|w') := w^\top \bar{w}' = \sum_{i=1}^m w_i \bar{w}'_i$, where w and w' are regarded as column vectors. Write $\|w\| := \sqrt{(w|w)}$. A $\mathbb{C}^m$-valued function f on $\mathbb{R}^n$ is said to be square-integrable if $\|f\|_{L^2}^2 := \int_{\mathbb{R}^n} \|f(x)\|^2\, dx < \infty$, and the condition is equivalent to that each of its components is square-integrable. Similarly, we say that f is rapidly decreasing if so is each of its components. The space $L^2(\mathbb{R}^n, \mathbb{C}^m)$ of square-integrable $\mathbb{C}^m$-valued functions forms a Hilbert space, where the inner product is given by $(f_1|f_2)_{L^2} = \int_{\mathbb{R}^n} (f_1(x)|f_2(x))\, dx \quad (f_1, f_2 \in L^2(\mathbb{R}^n, \mathbb{C}^m))$. The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n, \mathbb{C}^m) \to L^2(\mathbb{R}^n, \mathbb{C}^m)$ is defined as a unitary isomorphism such that $\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\xi|x)} f(x)\, dx \quad (\xi \in \mathbb{R}^n)$ if f is rapidly decreasing. For two unitary representations π_1 and π_2 of a group, we write $\pi_1 \simeq \pi_2$ if π_1 and π_2 are equivalent.

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2 General results

Let $H \subset GL(n, \mathbb{R})$ be a linear Lie group. Then the semi-direct product $G := \mathbb{R}^n \rtimes H$ acts on $\mathbb{R}^n$ as affine transforms:

$$g \cdot x := v + hx \qquad (g = (v, h) \in G,\ x, v \in \mathbb{R}^n,\ h \in H).$$

Let $\sigma : H \to U(m)$ be a unitary representation of H on $\mathbb{C}^m$. We define a unitary representation π of G on the Hilbert space $L^2(\mathbb{R}^n, \mathbb{C}^m)$ by

$$\pi(g)f(x) := |\det h|^{-1/2} \sigma(h) f(h^{-1}(x - v)) \\ (g = (v, h) \in G,\ f \in L^2(\mathbb{R}^n, \mathbb{C}^m),\ x \in \mathbb{R}^n).$$

We introduce another unitary representation $\hat{\pi}$ of G on $L^2(\mathbb{R}^n, \mathbb{C}^m)$ defined by $\hat{\pi}(g) := \mathcal{F} \circ \pi(g) \circ \mathcal{F}^{-1}$ for $g \in G$. By a straightforward calculation, we have

$$\hat{\pi}(g)\varphi(\xi) = e^{i(\xi|v)} |\det h|^{1/2} \sigma(h) \varphi(h^\top \xi) \\ (g = (v, h) \in G,\ \varphi \in L^2(\mathbb{R}^n, \mathbb{C}^m),\ \xi \in \mathbb{R}^n). \tag{1}$$

This formula implies that the contragredient action ρ of H on $\mathbb{R}^n$ given by $\rho(h)\xi := (h^\top)^{-1}\xi$ ($h \in H$, $\xi \in \mathbb{R}^n$) plays crucial roles for the study of the representation $\pi \simeq \hat{\pi}$ (cf. [1], [5]). For $\xi \in \mathbb{R}^n$, we write $\mathcal{O}_\xi$ for the orbit $\rho(H)\xi = \{\, \rho(h)\xi\,;\, h \in H \,\} \subset \mathbb{R}^n$. Now we pose the first assumption:
(A1): There exists an element $\xi \in \mathbb{R}^n$ for which $\mathcal{O}_\xi$ is an open set in $\mathbb{R}^n$.
Let $\mathfrak{h} \subset \mathrm{Mat}(n, \mathbb{R})$ be the Lie algebra of H, and $\{X_1, \dots, X_N\}$ ($N := \dim \mathfrak{h}$) be a basis of $\mathfrak{h}$. For $\xi \in \mathbb{R}^n$, we consider the matrix $R(\xi) \in \mathrm{Mat}(n, N, \mathbb{R})$ whose i-th column is $X_i^\top \xi \in \mathbb{R}^n$ for $i = 1, \dots, N$. Then the tangent space of $\mathcal{O}_\xi$ at ξ equals $\mathrm{Image}\, R(\xi) \subset \mathbb{R}^n$. In particular, $\mathcal{O}_\xi$ is open if and only if the matrix $R(\xi)$ is of rank n, and the condition is equivalent to $\det(R(\xi)R(\xi)^\top) \neq 0$. Since $p(\xi) := \det(R(\xi)R(\xi)^\top)$ is a polynomial function of ξ, the set $\{\, \xi \in \mathbb{R}^n\,;\, p(\xi) \neq 0 \,\}$ has a finite number of connected components with respect to the classical topology by [10]. On the other hand, we see from a connectedness argument that each of the connected components is an open $\rho(H)$-orbit. Summarizing these observations, we have the following.

Lemma 1. *There exist elements* $\xi^{[1]}, \dots \xi^{[K]} \in \mathbb{R}^n$ *such that*
(a) The orbit $\mathcal{O}_k := \rho(H)\xi^{[k]}$ *is open for* $k = 1, \dots, K$*.*
(b) If $k \neq l$*, then* $\mathcal{O}_k \cap \mathcal{O}_l = \emptyset$*.*
(c) The disjoint union $\bigsqcup_{k=1}^K \mathcal{O}_k$ *is dense in* $\mathbb{R}^n$*.*

For $k = 1, \dots, K$, put $L_k := \left\{\, f \in L^2(\mathbb{R}^n, \mathbb{C}^m)\,;\, \mathcal{F}f(\xi) = 0 \text{ (a.a. } \xi \in \mathbb{R}^n \setminus \mathcal{O}_k) \,\right\}$. By (1) and Lemma 1, we have an orthogonal decomposition $L^2(\mathbb{R}^n, \mathbb{C}^m) = \sum^\oplus_{1 \le k \le K} L_k$, which gives also a decomposition of the unitary representation π. We shall decompose each (π, L_k) further into a direct sum of irreducible subrepresentations. Let H_k be the isotropy subgroup $\left\{\, h \in H\,;\, \rho(h)\xi^{[k]} = \xi^{[k]} \,\right\}$ of H at $\xi^{[k]}$, and decompose the restriction $\sigma|_{H_k}$ of the representation σ orthogonally as

$$\mathbb{C}^m = \sum_{\alpha \in A_k}^{\oplus} W_{k,\alpha}, \tag{2}$$

where A_k is a finite index set, and $W_{k,\alpha}$ is an irreducible subspace of $\mathbb{C}^m$. We write $\sigma_{k,\alpha}$ for the irreducible representation $H_k \ni h \mapsto \sigma(h)|_{W_{k,\alpha}} \in U(W_{k,\alpha})$. For $\xi \in \mathcal{O}_k$, take $h \in H$ for which $\xi = \rho(h)\xi^{[k]}$ and put $W_{\xi,\alpha} := \sigma(h)W_{k,\alpha}$. Note that the space $W_{\xi,\alpha}$ is independent of the choice of $h \in H$. For $\alpha \in A_k$, define

$$L_{k,\alpha} = \{\, f \in L_k\,;\, \mathcal{F}f(\xi) \in W_{\xi,\alpha} \text{ (a.a. } \xi \in \mathcal{O}_k) \,\}\,. \tag{3}$$

Then we have an orthogonal decomposition $L_k = \sum^\oplus_{\alpha \in A_k} L_{k,\alpha}$, and each $L_{k,\alpha}$ is an invariant subspace thanks to (1). Moreover, we can show that the subrepresentation $(\pi, L_{k,\alpha})$ is equivalent to the induced representation $\mathrm{Ind}^G_{\mathbb{R}^n \rtimes H_k} \tilde{\sigma}_{k,\alpha}$, where $(\tilde{\sigma}_{k,\alpha}, W_{k,\alpha})$ is a unitary representation of $\mathbb{R}^n \rtimes H_k$ defined by $\tilde{\sigma}_{k,\alpha}(v, h) := e^{i(\xi^{[k]}|v)}\sigma_{k,\alpha}(h)$ ($v \in \mathbb{R}^n, h \in H_k$). It follows from the Mackey theory (see [4, Theorem 6.42]) that $(\pi, L_{k,\alpha})$ is irreducible. Therefore we obtain:

Theorem 1. *Under the assumption* (A1)*, an irreducible decomposition of the representation* $(\pi, L^2(\mathbb{R}^n, \mathbb{C}^m))$ *is given by*

$$L^2(\mathbb{R}^n, \mathbb{C}^m) = \sum_{1 \le k \le K}^{\oplus} \sum_{\alpha \in A_k}^{\oplus} L_{k,\alpha}. \tag{4}$$

Let d_H be a left Haar measure on the Lie group H. Then a measure d_G on G defined by $d_G(g) := |\det h|^{-1} dv\, d_H(h)$ $(g = (v, h) \in G)$ is a left Haar measure.

Theorem 2. *If the group H_k is compact, then the representation $(\pi, L_{k,\alpha})$ of G is square-integrable for all $\alpha \in A_k$. In this case, one has*

$$\int_G |(f|\pi(g)\phi)_{L^2}|^2\, d_G(g) = (2\pi)^n (\dim W_{k,\alpha})^{-1} \|f\|_{L^2}^2 \int_H \|\mathcal{F}\phi(h^\top \xi^{[k]})\|^2\, d_H(h) \tag{5}$$

for $f, \phi \in L_{k,\alpha}$.

Proof. In order to show (5), it is sufficient to consider the case where f and ϕ are rapidly decreasing. For $g = (v, h) \in G$, the isometry property of $\mathcal{F}$ together with (1) implies that

$$\begin{aligned}(f|\pi(v,h)\phi)_{L^2} &= \int_{\mathbb{R}^n} \big(\mathcal{F}f(\xi) \,|\, \mathcal{F}[\pi(v,h)\phi](\xi)\big)\, d\xi \\ &= \int_{\mathbb{R}^n} e^{-i(\xi|v)} |\det h|^{1/2} \big(\mathcal{F}f(\xi) \,|\, \sigma(h)\mathcal{F}\phi(h^\top \xi)\big)\, d\xi.\end{aligned}$$

Note that the right-hand is the inverse Fourier transform of $(2\pi)^{n/2}|\det h|^{1/2}$ $(\mathcal{F}f(\xi)|\sigma(h)\mathcal{F}\phi(h^\top\xi))$, which is a rapidly decreasing function of $\xi \in \mathbb{R}^n$, so that the Plancherel formula tells us that

$$\int_{\mathbb{R}^n} |(f|\pi(v,h)\phi)_{L^2}|^2 dv = (2\pi)^n |\det h| \int_{\mathbb{R}^n} |\big(\mathcal{F}f(\xi) \,|\, \sigma(h)\mathcal{F}\phi(h^\top \xi)\big)|^2 d\xi. \tag{6}$$

Thus, to obtain the formula (5), it is enough to show that

$$\begin{aligned}&\int_H |\big(\mathcal{F}f(\xi) \,|\, \sigma(h)\mathcal{F}\phi(h^\top \xi)\big)|^2 d_H(h) \\ &\quad = (\dim W_{k,\alpha})^{-1} \|\mathcal{F}f(\xi)\|^2 \int_H \|\mathcal{F}\phi(h^\top \xi^{[k]})\|^2 d_H(h)\end{aligned} \tag{7}$$

for each $\xi \in \mathcal{O}_k$. Let us take $h_1 \in H$ for which $\xi = \rho(h_1)\xi^{[k]}$. Then $h^\top \xi = \rho(h^{-1}h_1)\xi^{[k]} = \rho(h_1^{-1}h)^{-1}\xi^{[k]}$. By the left-invariance of the Haar measure d_H, the left-hand side of (7) equals

$$\begin{aligned}&\int_H |\big(\mathcal{F}f(\xi) \,|\, \sigma(h_1 h)\mathcal{F}\phi(\rho(h)^{-1}\xi^{[k]})\big)|^2\, d_H(h) \\ &= \int_H |\big(\sigma(h_1)^{-1}\mathcal{F}f(\xi) \,|\, \sigma(h)\mathcal{F}\phi(\rho(h)^{-1}\xi^{[k]})\big)|^2\, d_H(h).\end{aligned}$$

Writing d_{H_k} for the normalized Haar measure on the compact group H_k, we rewrite the right-hand side as

$$\begin{aligned}&\int_H \int_{H_k} |\big(\sigma(h_1)^{-1}\mathcal{F}f(\xi) \,|\, \sigma(h'h)\mathcal{F}\phi(\rho(h'h)^{-1}\xi^{[k]})\big)|^2\, d_{H_k}(h') d_H(h) \\ &= \int_H \int_{H_k} |\big(\sigma(h_1)^{-1}\mathcal{F}f(\rho(h_1)\xi^{[k]}) \,|\, \sigma(h')\sigma(h)\mathcal{F}\phi(\rho(h)^{-1}\xi^{[k]})\big)|^2\, d_{H_k}(h') d_H(h).\end{aligned}$$

Now put $w_1 := \sigma(h_1)^{-1}\mathcal{F}f(\rho(h_1)\xi^{[k]})$ and $w_2 := \sigma(h)\mathcal{F}\phi(\rho(h)^{-1}\xi^{[k]})$. Since $f, \phi \in L_{k,\alpha}$, we see that w_1 and w_2 belong to $W_{k,\alpha}$ for almost all $h, h_1 \in H$, so that the Schur orthogonality relation for the representation $\sigma_{k,\alpha}$ yields

$$\begin{aligned}
&\int_{H_k} |\big(\sigma(h_1)^{-1}\mathcal{F}f(\rho(h_1)\xi^{[k]}) \,|\, \sigma(h')\sigma(h)\mathcal{F}\phi(\rho(h)^{-1}\xi^{[k]})\big)|^2 \, d_{H_k}(h') \\
&= \int_{H_k} |(w_1|\sigma_{k,\alpha}(h')w_2)|^2 \, d_{H_k}(h') = (\dim W_{k,\alpha})^{-1}\|w_1\|^2\|w_2\|^2 \\
&= (\dim W_{k,\alpha})^{-1}\|\sigma(h_1)^{-1}\mathcal{F}f(\rho(h_1)\xi^{[k]})\|^2\|\sigma(h)\mathcal{F}\phi(\rho(h)^{-1}\xi^{[k]})\|^2 \\
&= (\dim W_{k,\alpha})^{-1}\|\mathcal{F}f(\xi)\|^2\|\mathcal{F}\phi(h^\top\xi^{[k]})\|^2,
\end{aligned}$$

which leads us to (7).

Note that the formula (5) is valid whether both sides converge or not. On the other hand, we can define a function $\phi \in L_{k,\alpha}$ for which (5) converges for all $f \in L_{k,\alpha}$ as follows. In view of the homeomorphism $H/H_k \ni hH_k \mapsto \rho(h)\xi^{[k]} \in \mathcal{O}_k$, we take a Borel map $h_k : \mathcal{O}_k \to H$ such that $\xi = \rho(h_k(\xi))\xi^{[k]}$ for all $\xi \in \mathcal{O}_k$. Let ψ be a non-negative continuous function on H with compact support such that $\int_H \psi(h^{-1})^2 \, d_H(h) = 1$. Take a unit vector $e_{k,\alpha} \in W_{k,\alpha}$ and define $\phi_{k,\alpha} \in L^2(\mathbb{R}^n, \mathbb{C}^m)$ by

$$\phi_{k,\alpha}(x) := (\dim W_{k,\alpha})^{1/2}(2\pi)^{-n}\int_{\mathcal{O}_k} e^{-i(\xi|x)}\psi(h_k(\xi))\sigma(h_k(\xi))e_{k,\alpha}\, d\xi \quad (x \in \mathbb{R}^n). \tag{8}$$

Note that the integral above converges because the support of the integrand is compact. We can check that $\phi_{k,\alpha} \in L_{k,\alpha}$ and

$$(2\pi)^n(\dim W_{k,\alpha})^{-1}\int_H \|\mathcal{F}\phi_{k,\alpha}(h^\top\xi^{[k]})\|^2 \, d_H(h) = 1. \tag{9}$$

The formula (5) together with (9) tells us that

$$\int_G |(f|\pi(g)\phi_{k,\alpha})_{L^2}|^2 \, d_G(g) = \|f\|^2_{L^2} \qquad (f \in L_{k,\alpha}). \tag{10}$$

In particular, the left-hand side converges for all $f \in L_{k,\alpha}$, which means that the unitary representation $(\pi, L_{k,\alpha})$ is square-integrable. □

Let us recall the decomposition (2). The multiplicity of the representation $\sigma_{k,\alpha}$ of H_k in $\sigma|_{H_k}$, denoted by $[\sigma|_{H_k} : \sigma_{k,\alpha}]$, is defined as the number of $\beta \in A_k$ for which $\sigma_{k,\beta}$ is equivalent to $\sigma_{k,\alpha}$. If $\sigma_{k,\beta}$ is equivalent to $\sigma_{k,\alpha}$, there exists a unitary intertwining operator $U_{k,\alpha,\beta} : W_{k,\beta} \to W_{k,\alpha}$ between $\sigma_{k,\beta}$ and $\sigma_{k,\alpha}$, and $U_{k,\alpha,\beta}$ is unique up to scalar multiples by the Schur lemma. Moreover, by the Schur orthogonality relation, we have

$$\int_{H_k} (w_\alpha|\sigma(h)w'_\alpha)\,(\sigma(h)w'_\beta|w_\beta) d_{H_k}(h) = (\dim W_{k,\alpha})^{-1}(w_\alpha|U_{k,\alpha,\beta}w_\beta)\,(U_{k,\alpha,\beta}w'_\beta|w'_\alpha).$$

Now we assume:
(A2) The group H_k is compact for all $k = 1, \dots, K$.

(A3) $[\sigma|_{H_k} : \sigma_{k,\alpha}] \le \dim W_{k,\alpha}$ for all $k = 1, \dots, K$ and $\alpha \in A_k$.
Because of (A3), we can take a unit vector $e_{k,\alpha} \in W_{k,\alpha}$ for each $k = 1, \dots, K$ and $\alpha \in A_k$ in such a way that if $\sigma_{k,\alpha}$ and $\sigma_{k,\beta}$ are equivalent with $\alpha \ne \beta$, then $(e_{k,\alpha}|U_{k,\alpha,\beta}e_{k,\beta}) = 0$. As in (8), we construct $\phi_{k,\alpha} \in L_{k,\alpha}$ from the vector $e_{k,\alpha}$. Put $\tilde{\phi} := \sum_{k=1}^{K} \sum_{\alpha \in A_k} \phi_{k,\alpha}$.

Theorem 3. *The transform* $W_{\tilde{\phi}} : L^2(\mathbb{R}^n, \mathbb{C}^m) \ni f \mapsto (f|\pi(\cdot)\tilde{\phi})_{L^2} \in L^2(G)$ *is an isometry. In other words, one has*

$$f = \int_G (f|\pi(g)\tilde{\phi})_{L^2}\, \pi(g)\tilde{\phi}\, d_G(g) \qquad (f \in L^2(\mathbb{R}^n, \mathbb{C}^m)),$$

where the integral of the right-hand side is defined in the weak sense.

Proof. Along (4), we write $f = \sum_{k=1}^{K} \sum_{\alpha \in A_k} f_{k,\alpha}$ with $f_{k,\alpha} \in L_{k,\alpha}$. Then

$$\begin{aligned}
\int_G |(f|\pi(g)\tilde{\phi})_{L^2}|^2\, d_G(g) &= \int_G \Big| \sum_{k=1}^{K} \sum_{\alpha \in A_k} (f_{k,\alpha}|\pi(g)\phi_{k,\alpha})_{L^2} \Big|^2 d_G(g) \\
= \sum_{(k,\alpha)} \int_G |(f_{k,\alpha}|\pi(g)\phi_{k,\alpha})_{L^2}|^2\, d_G(g) \\
+ \sum_{(k,\alpha) \ne (l,\beta)} \int_G (f_{k,\alpha}|\pi(g)\phi_{k,\alpha})_{L^2} \overline{(f_{l,\beta}|\pi(g)\phi_{l,\beta})_{L^2}}\, d_G(g).
\end{aligned}$$

By (10), we have $\int_G |(f_{k,\alpha}|\pi(g)\phi_{k,\alpha})_{L^2}|^2\, d_G(g) = \|f_{k,\alpha}\|^2$. On the other hand, if $(k,\alpha) \ne (l,\beta)$, we have

$$\int_G (f_{k,\alpha}|\pi(g)\phi_{k,\alpha})_{L^2} \overline{(f_{l,\beta}|\pi(g)\phi_{l,\beta})_{L^2}}\, d\mu_G(g) = 0.$$

In fact, similarly to (6) we have for all $h \in H$

$$\begin{aligned}
&\int_{\mathbb{R}^n} (f_{k,\alpha}|\pi(v,h)\phi_{k,\alpha})_{L^2} \overline{(f_{l,\beta}|\pi(v,h)\phi_{l,\beta})_{L^2}}\, dv \\
&= (2\pi)^n |\det h| \int_{\mathbb{R}^n} (\mathcal{F}f_{k,\alpha}(\xi)|\sigma(h)\mathcal{F}\phi_{k,\alpha}(h^\top\xi)) \overline{(\mathcal{F}f_{l,\beta}(\xi)|\sigma(h)\mathcal{F}\phi_{l,\beta}(h^\top\xi))}\, d\xi.
\end{aligned}$$

If $k \ne l$, this quantity vanishes because the integrand in the right-hand side equals 0 by $\mathcal{O}_k \cap \mathcal{O}_l = \emptyset$. If $k = l$ and $\alpha \ne \beta$, by the same argument as the proof of Theorem 2, we have for all $\xi \in \mathcal{O}_k$

$$\int_H (\mathcal{F}f_{k,\alpha}(\xi)|\sigma(h)\mathcal{F}\phi_{k,\alpha}(h^\top\xi)) \overline{(\mathcal{F}f_{k,\beta}(\xi)|\sigma(h)\mathcal{F}\phi_{k,\beta}(h^\top\xi))}\, d_H(h) = 0$$

whether $\sigma_{k,\alpha} \simeq \sigma_{k,\beta}$ or not, by the Schur orthogonality relation. Thus we get

$$\int_G |(f|\pi(g)\tilde{\phi})_{L^2}|^2\, d_G(g) = \sum_{(k,\alpha)} \|f_{k,\alpha}\|^2 = \|f\|^2,$$

which completes the proof. $\square$

3 The 3-dimensional Similitude group case

We consider the case where H is the group $\{\, cA\,;\, c>0, A\in SO(n)\,\}\simeq \mathbb{R}_{>0}\times SO(n)$ on $\mathbb{R}^n$. Let σ be an irreducible unitary representation of $SO(n)$. We extend σ to a representation of H by $\sigma(cA):=\sigma(A)$. The contragredient action ρ of H on $\mathbb{R}^n$ has one open orbit $\mathcal{O}:=\mathbb{R}^n\setminus\{0\}$, and the isotropy subgroup H_1 at any $\xi^{[1]}\in\mathcal{O}$ is isomorphic to $SO(n-1)$. Moreover, since $SO(n-1)$ is a multiplicity free subgroup of $SO(n)$ by [2], we have $[\sigma|_{H_1}:\sigma_{1,\alpha}]=1$ for all $\alpha\in A_1$. Thus the assumptions (A1), (A2) and (A3) are satisfied, and we can find an admissible vector $\tilde{\phi}$ in Theorem 3. A natural unitary representation of G on the space of square-integrable (r,s)-tensor fields on $\mathbb{R}^n$ corresponds to the case where σ is the tensor product $\sigma_0\otimes\cdots\otimes\sigma_0$ ($r+s$ times) of the natural representation σ_0 of $SO(n)$. Then the assumption (A3) is not satisfied in general. In this case, we decompose the value of $f\in L^2(\mathbb{R}^n,(\mathbb{C}^n)^{\otimes(r+s)})$ into the sum of irreducible summands of $(\sigma,(\mathbb{C}^n)^{\otimes(r+s)})$ as a pretreatment, and we can consider a continuous wavelet transform for each component.

Now let us present a concrete example of $\tilde{\phi}$ for the case where $n=3$ and σ is the natural representation of $SO(3)$. For $1\le i<j\le 3$, put $X_{ij}:=-E_{ij}+E_{ji}\in \mathrm{Mat}(3,\mathbb{R})$. The Lie algebra of $SO(3)$ is spanned by X_{12},X_{23}, and X_{13}. Put $\xi^{[1]}:=(0,0,1)^\top$. Then $H_1=\exp\mathbb{R}X_{12}\simeq SO(2)$. Define $e_{1,\pm1}:=\frac{1}{\sqrt{2}}(1,\mp i,0)^\top$ and $e_{1,0}:=(0,0,1)^\top$, so that we have $\sigma(\exp tX_{12})e_{1,\alpha}=e^{i\alpha t}e_{1,\alpha}$ for $\alpha=0,\pm1$ and $t\in\mathbb{R}$, which means that we have the decomposition (2) with $A_1=\{0,\pm1\}$ and $W_{1,\alpha}:=\mathbb{C}e_{1,\alpha}$. Note that $\dim W_{1,\alpha}=1$ in this case.

Let us consider the spherical coordinate $\xi=(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ with $r\ge0$, $\theta\in[0,\pi]$, $\varphi\in[0,2\pi)$. If $\xi\in\mathbb{R}^3\setminus\mathbb{R}\xi^{[1]}$, then θ and φ are uniquely determined. In this case, we put $h_1(\xi):=r^{-1}\exp(\varphi X_{12})\exp(-\theta X_{13})\in H$ so that $\rho(h_1(\xi))\xi^{[1]}=\xi$. Then $e_{\xi,\alpha}:=\sigma(h_1(\xi))e_{1,\alpha}$ for $\alpha\in A_1$ is computed as

$$e_{\xi,0}=\frac{1}{\|\xi\|}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix},\quad e_{\xi,\pm1}=\frac{1}{\|\xi\|\sqrt{2(\xi_1^2+\xi_2^2)}}\begin{pmatrix}\xi_1\xi_3\pm i\xi_2\|\xi\|\\ \xi_2\xi_3\mp i\xi_1\|\xi\|\\ -(\xi_1^2+\xi_2^2)\end{pmatrix}.$$

As in (3), the irreducible summand $L_{1,\alpha}$ is the space of $f\in L^2(\mathbb{R}^3,\mathbb{C}^3)$ such that $\mathcal{F}f(\xi)\in\mathbb{C}e_{\xi,\alpha}$ for almost all $\xi\in\mathbb{R}^3\setminus\mathbb{R}\xi^{[1]}$. We see easily that a vector field belonging to $L_{1,0}$ is curl-free, while one belonging to $L_{1,1}\oplus L_{1,-1}$ is divergence-free. Putting $\psi_0(\xi):=C_0e^{-\|\xi\|^2/2}\|\xi\|^3$, $\psi_{\pm1}(\xi):=\pm iC_1e^{-\|\xi\|^2/2}\|\xi\|^2\sqrt{\xi_1^2+\xi_2^2}$ with $C_0:=(4\pi)^{-1}$ and $C_1:=(32\pi^2/3)^{-1/2}$, we define $\phi_{1,\alpha}\in L_{1,\alpha}$ for $\alpha\in A_1$ by

$$\phi_{1,\alpha}(x):=(2\pi)^{-3}\int_{\mathbb{R}^3\setminus\mathbb{R}\xi^{[1]}}e^{-i(x|\xi)}\psi_\alpha(\xi)e_{\xi,\alpha}\,d\xi\quad(x\in\mathbb{R}^3).$$

We write $d_{SO(3)}$ for the normalized Haar measure on $SO(3)$, and define $d_H(h):=\frac{dc}{c}d_{SO(3)}(A)$ for $h=cA\in H$ with $c>0$ and $A\in SO(3)$. Then we have $\int_H\varphi(h^\top\xi^{[1]})\,d_H(h)=4\pi\int_{\mathbb{R}^3}\varphi(\xi)\frac{d\xi}{\|\xi\|^3}$ for a non-negative measurable function φ on $\mathbb{R}^3$. Thus the left-hand side of (9) equals $4\pi\int_{\mathbb{R}^3}|\psi_\alpha(\xi)|^2\frac{d\xi}{\|\xi\|^3}$. If $\alpha=0$, this becomes $4\pi C_0^2\int_{\mathbb{R}^3}e^{-\|\xi\|^2}\|\xi\|^3\,d\xi=1$, and if $\alpha=\pm1$, it is $4\pi C_1^2\int_{\mathbb{R}^3}e^{-\|\xi\|^2}\|\xi\|(\xi_1^2+$

$\xi_2^2)\,d\xi = 1$. Namely (9) holds for $\alpha \in A_1$. We compute

$$\phi_{1,0}(x) = (2\pi)^{-3} C_0 \int_{\mathbb{R}^3} e^{-i(\xi|x)} e^{-\|\xi\|^2/2} \|\xi\|^2 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} d\xi = C(\|x\|^2 - 5) e^{-\|x\|^2/2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with $C := 2^{-7/2}\pi^{-5/2}$. On the other hand, although we don't have explicit expression of $\phi_{1,\pm1}$, we see that $\phi_{1,\sharp}(x) := \phi_{1,1}(x) + \phi_{1,-1}(x)$ equals

$$(2\pi)^{-3} \frac{C_1}{\sqrt{2}} \int_{\mathbb{R}^3} e^{-i(\xi|x)} e^{-\|\xi\|^2/2} \|\xi\|^2 \begin{pmatrix} -2\xi_2 \\ 2\xi_1 \\ 0 \end{pmatrix} d\xi = C(\|x\|^2 - 5) e^{-\|x\|^2/2} \begin{pmatrix} -\sqrt{3}x_2 \\ \sqrt{3}x_1 \\ 0 \end{pmatrix}.$$

Therefore we obtain a concrete admissible vector field $\tilde{\phi} := \phi_{1,0} + \phi_{1,\sharp}$ given by

$$\tilde{\phi}(x) = 2^{-7/2}\pi^{-5/2}(\|x\|^2 - 5) e^{-\|x\|^2} \begin{pmatrix} x_1 - \sqrt{3}x_2 \\ x_2 + \sqrt{3}x_1 \\ x_3 \end{pmatrix} \quad (x \in \mathbb{R}^3). \tag{11}$$

If a vector field $f \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ is decomposed as $\sum_{\alpha \in A_1} f_{1,\alpha}$, then $f_{1,1} + f_{1,-1}$ is the divergence-free part of f, which equals $\int_G (f|\pi(g)\phi_{1,\sharp})_{L_2} \pi(g)\phi_{1,\sharp}\, d_G(g)$. A similar formula holds for the curl-free part $f_{1,0}$ and $\phi_{1,0}$.

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