Improved Rellich Type Inequalities in R^N

Megumi Sano, Futoshi Takahashi

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1 Introduction

Let $N \geq 2, 1 \leq p < N$, and let Ω be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, or $\Omega = \mathbb{R}^N$. The classical Hardy inequality

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \tag{1}$$

holds for all $u \in W_0^{1,p}(\Omega)$, or $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ when $\Omega = \mathbb{R}^N$. Here $W_0^{1,p}(\Omega)$ (resp. $\mathcal{D}^{1,p}(\mathbb{R}^N)$) is the completion of $C_0^\infty(\Omega)$ (resp. $C_0^\infty(\mathbb{R}^N)$) with respect to the norm $\|\nabla \cdot\|_{L^p(\Omega)}$ (resp. $\|\nabla \cdot\|_{L^p(\mathbb{R}^N)}$). It is known that for $1 < p < N$, the best constant $(\frac{N-p}{p})^p$ is never attained in $W_0^{1,p}(\Omega)$, or in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Therefore, one can expect the existence of remainder terms on the right-hand side of the inequality (1). Indeed, there are many papers that deal with remainder terms for (1) when Ω is a smooth bounded domain (see [1], [8], [9], [12], [13], [21], to name a few). For example, Brezis and Vázquez [8] show that the inequality

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx + z_0^2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega |u|^2 dx \tag{2}$$

holds true for all $u \in W_0^{1,2}(\Omega)$ where $z_0 = 2.4048\cdots$ is the first zero of the Bessel function of the first kind.

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On the other hand, when $\Omega = \mathbb{R}^N$, the remainder term in (2) becomes trivial and does not provide better inequality than the classical one. More generally, Ghoussoub and Moradifam [14] show that there is no strictly positive $V \in C^1((0,+\infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx$$

holds for all $u \in W^{1,2}(\mathbb{R}^N)$. One of the reasons of it is the lack of the Poincaré inequality: $\|\nabla u\|_{L^2(\Omega)} \ge C\|u\|_{L^2(\Omega)}$ when $\Omega = \mathbb{R}^N$. Although there is a result of refining the Hardy type inequality on the whole space (see Maz'ya's book [18], pp. 139, Corollary 3.), we cannot expect the same type of remainder terms as in (2) on the whole space.

In spite of this fact, the authors of the present paper recently showed the following result [23] : Let $2 \le p < N$ and $q > 2$. Set $\alpha = \alpha(p,q,N) = \frac{N}{2}(q-2) - \frac{pq}{2} + 2$. Then there exists $D = D(p,q,N) > 0$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \ge \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx + D\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^N} |u^\#|^p |x|^{2-p}} dx\right)^{\frac{2}{q-2}} \tag{3}$$

holds for all $u \in W^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$. Here $u^\#$ denotes the Schwartz symmetrization of a function u on $\mathbb{R}^N$:

$$u^\#(x) = u^\#(|x|) = \inf\left\{\lambda > 0 \,\middle|\, \left|\{x \in \mathbb{R}^N \mid |u(x)| > \lambda\}\right| \le |B_{|x|}(0)|\right\},$$

where $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$ (see e.g., [17]). Note that the integral $\int_{\mathbb{R}^N} |u^\#|^p |x|^{2-p} dx$ is finite for any $u \in W^{1,p}(\mathbb{R}^N)$.

In this paper, we focus on the higher-order case. A higher-order generalization of (1) was first proved by Rellich [22]: it holds

$$\int_\Omega |\Delta u|^2 dx \ge \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{|u|^2}{|x|^4} dx$$

for all $u \in W_0^{2,2}(\Omega)$, where Ω is a domain in $\mathbb{R}^N$, $N \ge 5$. More generally, let $k, m \in \mathbb{N}$ and $k < kp < N$. Define

$$|u|_{k,p}^p = \begin{cases} \int_\Omega |\Delta^m u|^p \, dx & \text{if } k = 2m, \\ \int_\Omega |\nabla(\Delta^m u)|^p \, dx & \text{if } k = 2m+1, \end{cases} \quad \text{and}$$

$$C_{k,p} = \begin{cases} p^{-2m} \prod_{j=1}^m (N-2jp)\{N(p-1)+2(j-1)p\} & \text{if } k = 2m, \\ (N-p)p^{-2(m+1)} \prod_{j=1}^m (N-(2j+1)p)\{N(p-1)+(2j-1)p\} & \text{if } k = 2m+1. \end{cases}$$

We put $|u|_{0,p} = \|u\|_{L^p(\mathbb{R}^N)}$ and $C_{0,p} = 1$, $C_{1,p} = \frac{N-p}{p}$ for the convenience of description. Then the inequality

$$|u|_{k,p}^p \geq C_{k,p}^p \int_\Omega \frac{|u|^p}{|x|^{kp}} dx \tag{4}$$

holds for all $u \in W_0^{k,p}(\Omega)$. It is also known that $C_{k,p}^p$ is optimal (see [10], [19], or Proposition 1 in Appendix) and never attained in $W_0^{k,p}(\Omega)$. Furthermore, Gazzola-Grunau-Mitidieri [13] prove the following inequality on a smooth bounded domain: there exist positive constants $A, B > 0$ such that the inequality

$$|u|_{2,2}^2 \geq C_{2,2}^2 \int_\Omega \frac{|u|^2}{|x|^4} dx + A \int_\Omega \frac{|u|^2}{|x|^2} dx + B \int_\Omega |u|^2 dx$$

holds for all $u \in W_0^{2,2}(\Omega)$, where $N \geq 5$. In addition to this, there are many papers that deal with various types of Rellich inequalities with remainder terms on bounded domains (see [2], [3], [4], [5], [6], [7], [11], [15], [20], [25], [26] etc.).

A main aim of this paper is to obtain remainder terms for the inequality (4) when $\Omega = \mathbb{R}^N$. Note that the inequalities (1) and (4) have the scale invariance under the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u\left(\frac{x}{\lambda}\right) \tag{5}$$

for $\lambda > 0$ when $\Omega = \mathbb{R}^N$. Therefore the possible remainder term to (4) should be invariant under the scaling (5) when $\Omega = \mathbb{R}^N$. In the following, ω_N will denote the area of the unit sphere in $\mathbb{R}^N$, $\|\cdot\|_r = \|\cdot\|_{L^r(\mathbb{R}^N)}$ and $\mathcal{D}^{k,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $|\cdot|_{k,p}$.

Our main results are as follows:

Theorem 1. *(Radial case) Let $k \geq 2$ be an integer, $k < kp < N$ and $q > 2$. Set $\alpha_k = \frac{N}{2}(q-2) - \frac{kpq}{2} + 2$. Then there exists a constant $C > 0$ such that the inequality*

$$|u|_{k,p}^p \geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}} dx + C \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha_k} dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-kp} dx} \right)^{\frac{2}{q-2}} \tag{6}$$

holds for all radial function $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $u \neq 0$.

In the non-radial case, we obtain only partial results for $k = 2, 3$.

Theorem 2. *(Non-radial case) For $k = 2$ or $k = 3$, let $k < kp < N$ and $q > 2$. Set $\alpha_k = \frac{N}{2}(q-2) - \frac{kpq}{2} + 2$ and $r = \frac{Np}{N+2p}$ (i.e. $\frac{1}{p} = \frac{1}{r} - \frac{2}{N}$). Then there exists a constant $C > 0$ such that the inequality*

$$|u|_{k,p}^p \geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_k} dx}{|u|_{k,p}^{\frac{kp-2}{k}} \|\Delta u\|_r^{\frac{2}{k}}} \right)^{\frac{2}{q-2}} \tag{7}$$

holds for all $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap \mathcal{D}^{2,p}(\mathbb{R}^N) \cap \mathcal{D}^{2,r}(\mathbb{R}^N)$, $u \neq 0$.

Remark 1. The remainder term of the inequalities (6) and (7) are scale invariant under the scaling (5) on $\mathbb{R}^N$: $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u(y)$, $y = \frac{x}{\lambda}, x \in \mathbb{R}^N$. Indeed, it holds $|u_\lambda|_{k,p} = |u|_{k.p}$ and for $a, b \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} |u_\lambda(x)|^a |x|^b dx = \lambda^{-\left(\frac{N-kp}{p}\right)a+b+N} \int_{\mathbb{R}^N} |u(y)|^a |y|^b dy. \tag{8}$$

Thus by taking $a = \frac{pq}{2}$ and $b = \alpha_k$, or $a = p$ and $b = 2 - kp$ in (8), we have

$$\int_{\mathbb{R}^N} |u_\lambda(x)|^{\frac{pq}{2}} |x|^{\alpha_k} dx = \lambda^2 \int_{\mathbb{R}^N} |u(y)|^{\frac{pq}{2}} |y|^{\alpha_k} dy,$$
$$\int_{\mathbb{R}^N} |u_\lambda(x)|^p |x|^{2-kp} dx = \lambda^2 \int_{\mathbb{R}^N} |u(y)|^p |y|^{2-kp} dy.$$

Therefore the remainder term in the inequality (6) has the scale invariance.

Furthermore from Proposition 2 in Appendix, we obtain

$$\int_{\mathbb{R}^N} |(u_\lambda)^\#|^{\frac{pq}{2}} |x|^{\alpha_k}\, dx = \int_{\mathbb{R}^N} |(u^\#)_\lambda|^{\frac{pq}{2}} |x|^{\alpha_k}\, dx = \lambda^2 \int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_k}\, dx,$$
$$\|\Delta u_\lambda\|_{L^r(\mathbb{R}^N)}^{\frac{2}{k}} = \lambda^2 \|\Delta u\|_{L^r(\mathbb{R}^N)}^{\frac{2}{k}}.$$

Thus the remainder term in the inequality (7) also has the scale invariance.

Remark 2. If $\alpha_k \le 0$ in Theorem 2, then $u^\#$ in the RHS of (7) can be replaced by u thanks to the Hardy-Littlewood inequality: $\int_{\mathbb{R}^N} g^\# h^\# \ge \int_{\mathbb{R}^N} gh$ (see e.g., [17]), and the fact $(|x|^{\alpha_k})^\# = |x|^{\alpha_k}$.

2 Proofs of Main results

In this section, we prove Theorem 1 and Theorem 2. The next simple lemma is used in the proof.

Lemma 1. *Let $p \ge 1$ and $a, b \in \mathbb{R}$. Then it holds*

$$|a - b|^p - |a|^p \ge -p|a|^{p-2} ab.$$

Proof. . First, we assume $a \ge 0$. We use the mean value theorem for the function $f(t) = (a - t)^p$, which is defined for $t \le a$. When $b \le a$, we have

$$f(b) - f(0) = (a - b)^p - a^p = pc^{p-1}(-b) \ge -pa^{p-1} b,$$

where $c \in \mathbb{R}$ satisfies $0 \le a - b \le c \le a$ if $b \ge 0$, or $0 \le a \le c \le a - b$ if $b \le 0$. When $b \ge a$, then $2a - b \le a$ and we have

$$f(2a - b) - f(0) = (b - a)^p - a^p = pc^{p-1}(b - 2a) \ge -pa^{p-1} b,$$

where $c \in \mathbb{R}$ satisfies $0 \le a \le c \le b-a$ if $b-2a \ge 0$, or $0 \le b-a \le c \le a$ if $b-2a \le 0$. This implies the result when $a \ge 0$.

The case $a \le 0$ follows by considering $a = -\tilde{a}$, $\tilde{a} \ge 0$ and $b = -\tilde{b}$, $\tilde{b} \in \mathbb{R}$.

□

Proof. of Theorem 1.

We show the inequality (6) for all radial function $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. By density argument, we may assume $u \in C_0^\infty(\mathbb{R}^N)$ without loss of generality.

First, note that the inequality

$$|u|_{k,p}^p = |\Delta u|_{k-2,p}^p \ge C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx \tag{9}$$

holds from Rellich's inequality (4). Actually when $k = 2$, this is the equality. Thus, in order to prove Theorem, it is enough to show the RHS of (9) is bounded from below by the RHS of (6).

Since u is radial, u can be written as $u(x) = \tilde{u}(|x|)$ where $\tilde{u} \in C_0^\infty([0,+\infty))$. We define the new function v as follows:

$$\tilde{v}(r) = r^{\frac{N-kp}{p}}\tilde{u}(r), \quad r \in [0,\infty), \quad \text{and} \quad v(y) = \tilde{v}(|y|), \quad y \in \mathbb{R}^2. \tag{10}$$

Note that $\tilde{v}(0) = 0$ and also $\tilde{v}(+\infty) = 0$ since the support of u is compact. We claim that if $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, then $v \in L^p(\mathbb{R}^2)$. Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |v(y)|^p dy &= \omega_2 \int_0^\infty |\tilde{v}(r)|^p r dr \\
&= \omega_2 \int_0^\infty |\tilde{u}(r)|^p r^{N-kp+1}\, dr = \frac{\omega_2}{\omega_N} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp-2}}\, dx \\
&\le \frac{\omega_2}{\omega_N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx \right)^{\frac{kp-2}{kp}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{\frac{2}{kp}} \\
&\le \frac{\omega_2}{\omega_N} C_{k,p}^{\frac{2-kp}{k}} |u|_{k,p}^{\frac{kp-2}{k}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{\frac{2}{kp}} < \infty,
\end{aligned} \tag{11}$$

here we have used Hölder's inequality, Rellich's inequality (4), and the assumption $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Therefore we have checked $v \in L^p(\mathbb{R}^2)$.

For $k \ge 2, k \in \mathbb{N}$ and $k < kp < N$, put

$$\theta_k = \theta(k,N,p) = 2k + \frac{N(p-2)}{p}, \quad \text{and}$$

$$\Delta_{\theta_k} f = f''(r) + \frac{\theta_k - 1}{r} f'(r)$$

for a smooth function $f = f(r)$. Define

$$A_{k,p} = \frac{(N-kp)[(k-2)p + (p-1)N]}{p^2}.$$

Then we see $C_{k-2,p}A_{k,p} = C_{k,p}$ and a direct calculation shows that

$$-\Delta \tilde{u} = r^{k-2-\frac{N}{p}} \left(A_{k,p}\tilde{v}(r) - r^2 \Delta_{\theta_k} \tilde{v}(r) \right).$$

Define

$$J = \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx - A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx. \tag{12}$$

Now applying Lemma 1 with the choice

$$a = A_{k,p}\tilde{v}(r) \quad \text{and} \quad b = r^2 \Delta_{\theta_k}\tilde{v}(r),$$

and using the fact $\int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'\, dr = 0$ since $\tilde{v}(0) = \tilde{v}(+\infty) = 0$, we have

$$\begin{aligned}
J &= \omega_N \int_0^\infty |-\Delta\tilde{u}(r)|^p\, r^{N-1-(k-2)p}\, dr - A_{k,p}^p \omega_N \int_0^\infty |\tilde{u}(r)|^p r^{N-kp-1}\, dr \\
&= \omega_N \int_0^\infty \left(\left| A_{k,p}\tilde{v}(r) - r^2\Delta_{\theta_k}\tilde{v}(r)\right|^p - |A_{k,p}\tilde{v}(r)|^p \right) r^{-1}\, dr \\
&\geq -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\Delta_{\theta_k}\tilde{v}\, r\, dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\left(\tilde{v}'' + \frac{\theta_k - 1}{r}\tilde{v}' \right) r\, dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}''\, r\, dr.
\end{aligned}$$

Moreover by integration by parts, we observe that

$$\begin{aligned}
-\int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}''\, r\, dr &= (p-1)\int_0^\infty |\tilde{v}|^{p-2}(\tilde{v}')^2\, r\, dr + \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'\, dr \\
&= \frac{4(p-1)}{p^2}\int_0^\infty |(|\tilde{v}|^{\frac{p-2}{2}}\tilde{v})'|^2\, r\, dr \\
&= \frac{4(p-1)}{p^2\omega_2}\int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}}v)|^2\, dy.
\end{aligned}$$

Combining these, we have

$$J \geq \frac{4(p-1)\omega_N}{p\omega_2} A_{k,p}^{p-1} \int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}}v)|^2\, dy. \tag{13}$$

Now, we apply the Gagliardo-Nirenberg inequality to $|v|^{\frac{p-2}{2}}v \in L^2(\mathbb{R}^2)$: for $q > 2$, there exists a constant $C(q) > 0$ such that it holds

$$\||v|^{\frac{p}{2}}\|_{L^q(\mathbb{R}^2)} \leq C(q)\, \||v|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \|\nabla(|v|^{\frac{p-2}{2}}v)\|_{L^2(\mathbb{R}^2)}^{\frac{q-2}{q}}. \tag{14}$$

Combining (13) and (14), we obtain

$$
\begin{aligned}
J &\geq \frac{4(p-1)\omega_N A_{k,p}^{p-1}}{p\omega_2} C(q)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^2} |v(y)|^{\frac{pq}{2}} dy}{\int_{\mathbb{R}^2} |v(y)|^p dy} \right)^{\frac{2}{q-2}} \\
&= \frac{4(p-1)\omega_N A_{k,p}^{p-1}}{p\omega_2} C(q)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha_k} dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-kp} dx} \right)^{\frac{2}{q-2}}. \qquad (15)
\end{aligned}
$$

Consequently, from (9), (12), (15) and $C_{k-2,p}A_{k,p} = C_{k,p}$, we obtain

$$
\begin{aligned}
|u|_{k,p}^p &\geq C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}} dx \\
&= C_{k-2,p}^p \left(A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}} dx + J \right) \\
&\geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}} dx + C \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha_k} dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-kp} dx} \right)^{\frac{2}{q-2}}
\end{aligned}
$$

where $C = \frac{4(p-1)\omega_N}{p\omega_2} C_{k-2,p} C_{k,p}^{p-1} C(q)^{-\frac{2q}{q-2}}$. This proves Theorem 1.

□

Proof. of Theorem 2.

First, we treat the case $k = 2$. We show the inequality

$$
\int_{\mathbb{R}^N} |\Delta u|^p dx \geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_2} dx}{||\Delta u||_p^{p-1} ||\Delta u||_r} \right)^{\frac{2}{q-2}} \qquad (16)
$$

for all $u \in \mathcal{D}^{2,p}(\mathbb{R}^N) \cap \mathcal{D}^{2,r}(\mathbb{R}^N)$. Set $f = -\Delta u \in L^p(\mathbb{R}^N)$ and $w(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-2}} dy$. Since $w(Ox) = w(x)$ for any $O \in O(N)$, the group of orthogonal matrices in $\mathbb{R}^N$, we see w is a radial function. Also since $f^\# \in L^p(\mathbb{R}^N)$, the Calderon-Zygmund inequality (see [16] Theorem 9.9.) implies that $w \in \mathcal{D}^{2,p}(\mathbb{R}^N)$ and satisfies $-\Delta w = f^\#$ a.e. in $\mathbb{R}^N$. Therefore we have

$$
||\Delta w||_p = ||\Delta u||_p. \qquad (17)
$$

By Talenti's comparison principle [24], we know $w \geq u^\# \geq 0$. Hence we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |w|^\beta |x|^\gamma dx &\geq \int_{\mathbb{R}^N} |u^\#|^\beta |x|^\gamma dx \quad \text{if } \beta \geq 0, \qquad (18) \\
&\geq \int_{\mathbb{R}^N} |u|^\beta |x|^\gamma dx \quad \text{if } \beta \geq 0 \text{ and } \gamma \leq 0.
\end{aligned}
$$

where the second inequality comes from the Hardy-Littlewood inequality. Furthermore there exists a constant $H > 0$ such that the inequality

$$
||w||_p \leq H||f^\#||_r = H||(-\Delta u)^\#||_r = H||(-\Delta u)||_r \qquad (19)
$$

holds from the Hardy-Littlewood-Sobolev inequality, where $\frac{1}{p} = \frac{1}{r} - \frac{2}{N}$. From (17), Theorem 1, (18) and (19), we obtain

$$
\begin{aligned}
|u|_{2,p}^p &= |w|_{2,p}^p \\
&\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{2p}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |w|^{\frac{pq}{2}} |x|^{\alpha_2}\,dx}{\int_{\mathbb{R}^N} |w|^p |x|^{2-2p}\,dx} \right)^{\frac{2}{q-2}} \\
&\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_2}\,dx}{C_{2,p}^{1-p} \|\varDelta w\|_p^{p-1} \|w\|_p} \right)^{\frac{2}{q-2}} \\
&\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_2}\,dx}{\|\varDelta u\|_p^{p-1} \|\varDelta u\|_r} \right)^{\frac{2}{q-2}},
\end{aligned}
$$

which concludes (16).

Next, we treat the case $k = 3$. As before, set $f = -\varDelta u \in L^p(\mathbb{R}^N) \cap \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $w(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-2}}\,dy$. Again we obtain $w \in \mathcal{D}^{2,p}(\mathbb{R}^N)$, w radial, $w \geq u^\# > 0$ and $-\varDelta w = f^\#$ a.e. in $\mathbb{R}^N$. By Pólya-Szegö inequality (see e.g., [17]), we have

$$
|u|_{3,p}^p = \int_{\mathbb{R}^N} |\nabla \varDelta u|^p dx = \int_{\mathbb{R}^N} |\nabla f|^p dx \geq \int_{\mathbb{R}^N} |\nabla f^\#|^p dx = |w|_{3,p}^p.
$$

In the same way as $k = 2$ case, we use Theorem 1. Then we obtain

$$
\begin{aligned}
|u|_{3,p}^p &\geq |w|_{3,p}^p \\
&\geq C_{3,p}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{3p}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |w|^{\frac{pq}{2}} |x|^{\alpha_3}\,dx}{\int_{\mathbb{R}^N} |w|^p |x|^{2-3p}\,dx} \right)^{\frac{2}{q-2}} \\
&\geq C_{3,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{3p}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_3}\,dx}{C_{3,p}^{\frac{2-3p}{3}} |w|_{3,p}^{\frac{3p-2}{3}} \|w\|_p^{\frac{2}{3}}} \right)^{\frac{2}{q-2}} \\
&\geq C_{3,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{3p}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_3}\,dx}{|u|_{3,p}^{\frac{3p-2}{3}} \|\varDelta u\|_r^{\frac{2}{3}}} \right)^{\frac{2}{q-2}},
\end{aligned}
$$

which concludes (7).

□

Remark 3. Up to now, we do not obtain the result for $k \geq 4$ in Theorem 2. For example, put $f = -\varDelta u \in \mathcal{D}^{2,p}(\mathbb{R}^N)$ for $u \in \mathcal{D}^{4,p}(\mathbb{R}^N)$. Since we do not know the validity of the inequality

$$
\int_{\mathbb{R}^N} |\varDelta f|^p\,dx \geq \int_{\mathbb{R}^N} |\varDelta f^\#|^p\,dx,
$$

the argument of the proof of Theorem 2 does not work for $k = 4$ case. Instead, if we define $f = (-\Delta)^2 u \in L^p(\mathbb{R}^N)$ and $w(x) = C_N \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-4}}\,dy$, then we obtain $(-\Delta)^2 w = f^\#$ in $\mathbb{R}^N$ and $|u|^p_{4,p} = |w|^p_{4,p}$. However in this case, we do not know whether the comparison $u^\# \le w$ hold or not, which invalidates the proof of Theorem 2.

3 Another improved Rellich inequality

In this section, we prove another improved Rellich inequality on the whole space. In Theorem 1, we have used the Gagliardo-Nirenberg inequality as a substitute for the Poincaré inequality, which is usually used to improve the Rellich inequality on bounded domains. In the next theorem, we will employ *the logarithmic Sobolev inequality* on the whole space.

Theorem 3. *Let $k \ge 2$ be a integer and $k \le kp < N$. Then the inequality*

$$|u|^p_{k,p} - C^p_{k,p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{kp}}\,dx$$
$$\ge BE(u)\exp\left(1 + E(u)^{-1}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{kp-2}}\log\left(\frac{\omega_N|x|^{N-kp}|u|^p}{\omega_2 E(u)}\right)dx\right) \tag{20}$$

holds for all radial function $u \in W^{k,p}(\mathbb{R}^N)$, where $B = \frac{4\pi(p-1)}{p}C^p_{k-2,p}A^{p-1}_{k,p}$ and $E(u) = \int_{\mathbb{R}^N}|u|^p|x|^{2-kp}dx$.

Proof. of Theorem 3. We proceed as in the proof of Theorem 1. From the proof of Theorem 1, we observe that

$$|u|^p_{k,p} - C^p_{k,p}\int_{\mathbb{R}^N}\frac{|\Delta u|^p}{|x|^{kp}}\,dx \ge B(k,p,N)\int_{\mathbb{R}^2}\left|\nabla|v|^{\frac{p-2}{2}}v\right|^2 dy, \tag{21}$$

where $B(k,p,N) = \frac{4(p-1)\omega_N}{p\omega_2}C_{k-2,p}C^{p-1}_{k,p}$. Differently from the proof of Theorem 1, here, instead of the Gagliardo-Nirenberg inequality, we apply the logarithmic Sobolev inequality (see [27]) on $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} f^2(y)\log f^2(y)\,dy \le \log\left(\frac{1}{\pi e}\int_{\mathbb{R}^2}|\nabla f(y)|^2\,dy\right) \tag{22}$$

for the function $f = \|v\|^{-\frac{p}{2}}_{L^p(\mathbb{R}^2)}|v|^{\frac{p-2}{2}}v$, $\|f\|_{L^2(\mathbb{R}^2)} = 1$, where v is defined in (10). By (21) and (22), we obtain

$$
\begin{aligned}
&|u|^p_{k,p} - C^p_{k,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx \geq B(k,p,N) \int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}} v)|^2\,dy \\
&\geq \pi B(k,p,N)\|v\|^p_{L^p(\mathbb{R}^2)} \exp\left(1 + \int_{\mathbb{R}^2} \frac{|v(y)|^p}{\|v\|^p_{L^p(\mathbb{R}^2)}} \log\left(\frac{|v(y)|^p}{\|v\|^p_{L^p(\mathbb{R}^2)}}\right) dy\right) \\
&= \pi B(k,p,N)\frac{\omega_2}{\omega_N} E(u) \exp\left(1 + \frac{\omega_N}{E(u)} \int_0^\infty r^{N-kp}|u(r)|^p \log\left(\frac{\omega_N r^{N-kp}|u(r)|^p}{\omega_2 E(u)}\right) r\,dr\right) \\
&= BE(u)\exp\left(1 + E(u)^{-1} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp-2}} \log\left(\frac{\omega_N |x|^{N-kp}|u|^p}{\omega_2 E(u)}\right) dx\right)
\end{aligned}
$$

where $E(u) = \int_{\mathbb{R}^N} |u|^p|x|^{2-kp}dx = \frac{\omega_N}{\omega_2}\|v\|^p_{L^p(\mathbb{R}^2)}$. Hence the inequality (20) holds. □

Remark 4. The inequality (20) has an invariance under the scaling $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u(y)$ where $y = \frac{x}{\lambda}$, $(\lambda > 0, x \in \mathbb{R}^N)$. Indeed, we have $E(u_\lambda) = \lambda^2 E(u)$ and

$$
\begin{aligned}
&E(u_\lambda)\exp\left(1 + E(u_\lambda)^{-1} \int_{\mathbb{R}^N} \frac{|u_\lambda(x)|^p}{|x|^{kp-2}} \log\left(\frac{\omega_N |x|^{N-kp}|u_\lambda(x)|^p}{\omega_2 E(u_\lambda)}\right) dx\right) \\
&= \lambda^2 E(u)\exp\left(1 + E(u)^{-1} \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|y|^{kp-2}} \left(\log \lambda^{-2} + \log\left(\frac{\omega_N |y|^{N-kp}|u(y)|^p}{\omega_2 E(u)}\right)\right) dy\right) \\
&= E(u)\exp\left(1 + E(u)^{-1} \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|y|^{kp-2}} \log\left(\frac{\omega_N |y|^{N-kp}|u(y)|^p}{\omega_2 E(u)}\right) dy\right),
\end{aligned}
$$

so the inequality (20) also enjoys a scale invariance.

4 Appendix

Davies-Hinz [10] showed that the constant $C^p_{k,p}$ in the inequality (4) is optimal when $\Omega = \mathbb{R}^N$. In this Appendix, we will show the fact when Ω is a general bounded domain.

Proposition 1. *Let $k \in \mathbb{N}$, $k < kp < N$ and let Ω be a bounded domain with $0 \in \Omega$ in $\mathbb{R}^N$. Then the constant $C^p_{k,p}$ in the inequality (4) is optimal. That is*

$$
\inf_{0 \neq u \in W^{k,p}_0(\Omega)} \frac{|u|^p_{k,p}}{\int_\Omega \frac{|u(x)|^p}{|x|^{kp}}\,dx} = C^p_{k,p}.
$$

Proof. of Proposition 1.

By the scaling (5) and zero extension, we may assume $B_1(0) \subset\subset \Omega$ without loss of generality. First, we show the optimality of $C^p_{k,p}$ in the even case $k = 2m$, $m \in \mathbb{N}$. For $0 < \varepsilon \ll 1$, we define the function $u_\varepsilon \in W^{2m,p}_0(\Omega)$ as follows:

$$u_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{N-2mp}{p}} \log \frac{1}{\varepsilon}, & \text{if } 0 \le |x| \le \varepsilon \\ |x|^{-\frac{N-2mp}{p}} \log \frac{1}{|x|}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

Let $\alpha = \frac{N-2mp}{p}$. By using the formula

$$\Delta r^{-\alpha} = \alpha(\alpha - N + 2) r^{-\alpha-2},$$
$$\Delta\left(r^{-\alpha} \log \frac{1}{r}\right) = \alpha(\alpha - N + 2) r^{-\alpha-2} \log \frac{1}{r} + (2\alpha - N + 2) r^{-\alpha-2},$$

we compute that

$$\Delta^m u_\varepsilon = \begin{cases} 0, & \text{if } 0 \le |x| \le \varepsilon, \\ A_m |x|^{-(\alpha+2m)} \log \frac{1}{|x|} + B_m |x|^{-(\alpha+2m)}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0), \end{cases}$$

where $A_m(\alpha)$ and $B_m(\alpha)$ are determined by the iterative formula:

$$A_1(\alpha) = \alpha(\alpha - N + 2),$$
$$A_{j+1}(\alpha) = (\alpha + 2j)(\alpha + 2(j+1) - N) A_j, \quad j = 1, 2, \ldots,$$
$$B_1(\alpha) = 2\alpha - N + 2,$$
$$B_{j+1}(\alpha) = (\alpha + 2j)(\alpha + 2(j+1) - N) B_j + 2\alpha + 2(2j+1) - N \quad j = 1, 2, \ldots.$$

Thus we have

$$A_m = A_m(\alpha) = \prod_{j=0}^{m-1} (\alpha + 2j)(\alpha + 2(j+1) - N), \quad |A_m(\alpha)| = C_{2m,p}.$$

We compute

$$\int_\Omega |\Delta^m u_\varepsilon(x)|^p \, dx = \omega_N \int_\varepsilon^1 \left| A_m \log \frac{1}{r} + B_m \right|^p r^{-(\alpha+2m)p+N-1} dr$$
$$= \omega_N \left(\frac{1}{A_m}\right) \int_{B_m}^{B_m + A_m \log \frac{1}{\varepsilon}} |t|^p \, dt$$
$$= \omega_N \left(\frac{1}{A_m(p+1)}\right) \left(\left| B_m + A_m \log \frac{1}{\varepsilon} \right|^p (B_m + A_m \log \frac{1}{\varepsilon}) - |B_m|^p B_m \right). \tag{23}$$

On the other hand, we have

$$
\begin{aligned}
&\int_\Omega \frac{|u_\varepsilon(x)|^p}{|x|^{2mp}}\,dx \\
&= \omega_N\,\varepsilon^{-\alpha p}\left(\log\frac{1}{\varepsilon}\right)^p \int_0^\varepsilon r^{N-2mp-1}\,dr + \omega_N \int_\varepsilon^1 r^{-1}\left(\log\frac{1}{r}\right)^p dr \\
&= \omega_N\,\frac{\varepsilon^{N-2mp}}{N-2mp}\left(\log\frac{1}{\varepsilon}\right)^p + \omega_N \int_0^{\log\frac{1}{\varepsilon}} t^p\,dt \\
&= \omega_N\,\frac{\varepsilon^{N-2mp}}{N-2mp}\left(\log\frac{1}{\varepsilon}\right)^p + \omega_N\,\frac{1}{p+1}\left(\log\frac{1}{\varepsilon}\right)^{p+1}. \qquad (24)
\end{aligned}
$$

By (23), (24) and the fact $|A_m| = C_{2m,p}$, we obtain

$$
\frac{\int_{B_1(0)} |\Delta^m u_\varepsilon(x)|^p\,dx}{\int_{B_1(0)} \frac{|u_\varepsilon(x)|^p}{|x|^{2mp}}\,dx} \to |A_m(\alpha)|^p = C_{2m,p}^p \text{ as } \varepsilon \to 0,
$$

which implies the optimality of $C_{2m,p}^p$.

Next, in the odd case $k = 2m+1$, $m \in \mathbb{N}$, we consider the function $u_\varepsilon \in W_0^{2m+1,p}(B_1(0))$ as follows:

$$
u_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{N-(2m+1)p}{p}} \log\frac{1}{\varepsilon}, & \text{if } 0 \le |x| \le \varepsilon, \\ |x|^{-\frac{N-(2m+1)p}{p}} \log\frac{1}{|x|}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0). \end{cases}
$$

Let $\beta = \frac{N-(2m+1)p}{p}$. Note that

$$
\nabla(\Delta^m u_\varepsilon) = \begin{cases} 0, \quad \text{if } 0 \le |x| \le \varepsilon, \\ |x|^{-(\beta+2m+2)} x\left\{-A_m(\beta)(\beta+2m)\log\frac{1}{|x|} - (A_m(\beta) + (\beta+2m)B_m(\beta))\right\}, \\ \qquad \text{if } \varepsilon \le |x| \le 1, \\ 0, \quad \text{if } x \in \Omega \setminus B_1(0). \end{cases}
$$

If we make a calculation similar to the even case, we obtain

$$
\frac{\int_\Omega |\nabla(\Delta^m u_\varepsilon)(x)|^p\,dx}{\int_\Omega \frac{|u_\varepsilon(x)|^p}{|x|^{(2m+1)p}}\,dx} \to |A_m(\beta)|^p(\beta+2m)^p \text{ as } \varepsilon \to 0,
$$

which implies the optimality of $C_{2m+1,p}^p$ by $\beta + 2m = \frac{N-p}{p}$ and $C_{2m+1,p}^p = |A_m(\beta)|^p(\beta + 2m)^p$.

□

Proposition 2. *Put $r = |x|, x \in \mathbb{R}^N$ and let*

$$
u^\#(r) = \inf\{\tau > 0 \mid \mu_u(\tau) \le |B_r(0)|\}
$$

be the symmetric decreasing rearrangement of a function u, where μ_u is a distribution function of u: $\mu_u(\tau) = \left|\{x \in \mathbb{R}^N \mid |u(x)| > \tau\}\right|$, $\tau \geq 0$. Define $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u\left(\frac{x}{\lambda}\right)$ for $\lambda > 0$. Then the equality

$$(u_\lambda)^{\#}(r) = (u^{\#})_\lambda(r) \tag{25}$$

holds for any $r, \lambda > 0$.

Proof. of Proposition 2. The distribution function of u_λ can be written as

$$\begin{aligned}
\mu_{u_\lambda}(\tau) &= \left|\{x \in \mathbb{R}^N \mid |u_\lambda(x)| > \tau\}\right| \\
&= \left|\{x \in \mathbb{R}^N \;\middle|\; \lambda^{-\frac{N-kp}{p}} \left|u\left(\frac{x}{\lambda}\right)\right| > \tau\}\right| \\
&= |\{\lambda y \in \mathbb{R}^N \mid |u(y)| > \lambda^{\frac{N-kp}{p}} \tau\}| \\
&= \lambda^N |\{y \in \mathbb{R}^N \mid |u(y)| > \lambda^{\frac{N-kp}{p}} \tau\}| \\
&= \lambda^N \mu_u(\lambda^{\frac{N-kp}{p}} \tau).
\end{aligned} \tag{26}$$

Hence by the definition of $(u_\lambda)^{\#}$ and (26), we obtain

$$\begin{aligned}
(u_\lambda)^{\#}(r) &= \inf\{\tau > 0 \mid \mu_{u_\lambda}(\tau) \leq |B_r|\} \\
&= \inf\{\tau > 0 \mid \lambda^N \mu_u(\lambda^{\frac{N-kp}{p}} \tau) \leq |B_r|\} \\
&= \inf\{\lambda^{-\frac{N-kp}{p}} \tilde{\tau} > 0 \mid \mu_u(\tilde{\tau}) \leq \lambda^{-N} |B_r|\} \\
&= \lambda^{-\frac{N-kp}{p}} \inf\{\tilde{\tau} > 0 \mid \mu_u(\tilde{\tau}) \leq |B_{\frac{r}{\lambda}}|\} \\
&= \lambda^{-\frac{N-kp}{p}} u^{\#}\left(\frac{r}{\lambda}\right) = (u^{\#})_\lambda(r).
\end{aligned}$$

The proof of Proposition 2 is now complete.

□

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