Cords and 1-handles attached to surface-knots

Seiichi Kamada

CORDS AND 1-HANDLES ATTACHED TO SURFACE-KNOTS

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ABSTRACT. J. Boyle classified 1-handles attached to surface-knots, that are closed and connected surfaces embedded in the Euclidean 4-space, in the case that the surfaces are oriented and 1-handles are orientable with respect to the orientations of the surfaces. In that case, the equivalence classes of 1-handles correspond to the equivalence classes of cords attached to the surface-knot, and correspond to the double cosets of the peripheral subgroup of the knot group. In this paper, we classify cords and cords with local orientations attached to (possibly non-orientable) surface-knots. And we classify 1-handles attached to surface-knots in the case that the surface-knots are oriented and 1-handles are non-orientable, and in the case that the surface-knots are non-orientable.

Dedicated to Professor Francisco González-Acuña on his seventieth birthday

1. INTRODUCTION

By a *surface-knot* we mean a closed (possibly non-orientable) and connected surface embedded in $\mathbb{R}^4$. F. Hosakawa and A. Kawauchi [3] studied unknotted surface-knots in $\mathbb{R}^4$ and surgery along 1-handles attached to surface-knots. They proved that an oriented surface-knot F in $\mathbb{R}^4$ satisfies that the knot group $\pi_1(\mathbb{R}^4 - F)$ is infinite cyclic if and only if an unknotted surface-knot can be obtained from F by surgery along trivial 1-handles. A similar result holds for a non-orientable surface-knot in $\mathbb{R}^4$ (cf. [5]). Surgery along a 1-handle is a method of constructing a surface-knot from another with lower genus. The knot type of the surface-knot obtained from a surface-knot in $\mathbb{R}^4$ by surgery along a 1-handle depends on the equivalence class of the 1-handle. Classifying 1-handles attached to a surface-knot F is important in order to consider the knot types obtained from F by surgery along 1-handles. J. Boyle [2] classified such 1-handles in the case that F is oriented, and 1-handles are orientable with respect to the orientation of F. This case, say (Case 1), is sufficient when we work on oriented surface-knots in R^4. When we work on non-orientable surface-knots, we should also consider the following two cases: (Case 2) F is oriented and 1-handles are non-orientable with respect to the orientation of F, and (Case 3) F is non-orientable. In this paper we give a classification theorem to each of these two cases (Case 2) and (Case 3), which is analogous to Boyle's classification in (Case 1).

In order to classify 1-handles attached to a surface-knot, we first classify cords and cords with local orientations attached to a surface-knot. Roughly speaking, the equivalence classes of 1-handles attached to a surface-knot F correspond to the

equivalence classes of cords attached to F in Cases 1 and 2, or correspond to the equivalence classes of cords with local orientations at the endpoints attached to F in Case 3.

We work in the PL category and all embedded surfaces in 4-manifolds are assumed to be locally flat. The results in this paper are also valid in the smooth category.

Throughout this paper, B^n denotes the unit n-ball in $\mathbb{R}^n$ and $0 \in B^n$ is the center.

An *ambient isotopy* of a space X is an isotopy $(f_s \,|\, s \in [0,1])$ such that for each $s \in [0,1]$, $f_s : X \to X$ is a homeomorphism and f_0 is the identity map of X. Two subsets A and A' of X are *ambient isotopic* if there is an ambient isotopy $(f_s \,|\, s \in [0,1])$ of X with $f_1(A) = A'$. Two maps $g : Y \to X$ and $g' : Y \to X$ are *ambient isotopic* if there is an ambient isotopy $(f_s \,|\, s \in [0,1])$ of X with $f_1 \circ g = g'$.

Some results of this paper are given partially in Section 5.2 of [7], written in Japanese. This paper completes it.

2. Definitions on 1-handles

There are two notions of 1-handles, one is defined by embeddings (cf. [2]) and the other is defined by 3-cells in $\mathbb{R}^4$ (cf. [3]). To distinguish these two, we call a 1-handle as an embedding a 1-*handle map* in this paper.

Let F be a surface-knot.

Definition 2.1. A 1-*handle map* attached to F is an embedding $h : [0,1] \times B^2 \to \mathbb{R}^4$ with $F \cap h([0,1] \times B^2) = h(\{0,1\} \times B^2)$. The restriction of h to $[0,1] \times \{0\}$ $(= [0,1])$ is denoted by $h^c : [0,1] \to R^4$ and called the *core map*. The image of h^c is called the *core* of h.

For a 1-handle map $h : [0,1] \times B^2 \to \mathbb{R}^4$, the *reverse* of h is a 1-handle map $\mathrm{rev}(h) : [0,1] \times B^2 \to \mathbb{R}^4$ with $\mathrm{rev}(h)(t,x) = h(1-t,x)$.

Definition 2.2. Let $h : [0,1] \times B^2 \to \mathbb{R}^4$ and $h' : [0,1] \times B^2 \to \mathbb{R}^4$ be 1-handle maps attached to F.

 (1) h and h' are *equivalent* if they are ambient isotopic in $\mathbb{R}^4$ by an ambient isotopy of $\mathbb{R}^4$ keeping F setwise fixed.
 (2) h and h' are *equivalent up to reversion* if h is equivalent to h' or $\mathrm{rev}(h')$.

For a 1-handle map h attached to F, we denote by $\mathrm{h}^1(F; h)$ the surface-knot

$$(F - h(\{0,1\} \times \mathrm{int} B^2)) \cup h([0,1] \times \partial B^2),$$

which we call the surface-knot obtained from F by *surgery* along h. The surgery is also called a 1-handle surgery or a hyperboloidal transformation ([3]). The symbol h^1 stands for a 1-handle surgery. In [2] it is denoted by $F + h$.

If h and h' are equivalent or equivalent up to reversion attached to a surface-knot F, then $\mathrm{h}^1(F; h)$ and $\mathrm{h}^1(F; h')$ are ambient isotopic in $\mathbb{R}^4$.

Definition 2.3. Assume that F is an orientable surface-knot. A 1-handle map h attached to F is *orientable* (or *non-orientable*, resp.) if $\mathrm{h}^1(F; h)$ is orientable (or non-orientable, resp.).

When F is oriented and h is orientable, the surface-knot $\mathrm{h}^1(F; h)$ is assumed to have an orientation that coincides, over $F - h(\{0, 1\} \times \mathrm{int} B^2)$, with the orientation of F.

Now we recall the notion of a 1-handle as a 3-cell in $\mathbb{R}^4$ from [3].

Definition 2.4. A 1-*handle* attached to F is a 3-cell B in $\mathbb{R}^4$ such that $B \cap F = \partial B \cap F$ and this intersection is the union of disjoint two 2-cells. A properly embedded arc C in B is called a *core* of B if it is a strong deformation retract of B and it connects an interior point of one 2-cell of $B \cap F$ with another interior point of the other 2-cell.

For a 1-handle map h attached to F, the image of h is a 1-handle attached to F, say B, and the core of h is a core of B. Conversely, for a 1-handle B and a core C of B, there is a 1-handle map h whose image is B and its core is C.

Definition 2.5. Two 1-handles B and B' attached to F are *equivalent* if they are ambient isotopic in $\mathbb{R}^4$ by an ambient isotopy of $\mathbb{R}^4$ keeping F setwise fixed.

Lemma 2.6. *For* 1-*handles B and B' attached to F, let h and h' be 1-handle maps attached to F whose images are B and B', respectively. B and B' are equivalent if and only if h and h' are equivalent up to reversion.*

This lemma follows from Lemma 2.8 stated below.

Definition 2.7. A 1-*handle with an oriented core* attached to F is a pair (B, C) of a 1-handle B attached to F and an oriented core C of B. Two 1-handles with oriented cores (B, C) and (B', C') attached to F are *equivalent* if they are ambient isotopic in $\mathbb{R}^4$ by an ambient isotopy of $\mathbb{R}^4$ keeping F setwise fixed. (Here we assume that C is mapped to C' with respect to the orientations.)

For a 1-handle map h attached to F, let B be the image of h, which is a 1-handle attached to F, and let C be the core of h. Using the core map $h^c : [0, 1] \to \mathbb{R}^4$, we give an orientation to the core C. Then we say that the 1-handle with an oriented cord (B, C) is *determined by* h.

Lemma 2.8. *For* 1-*handles with oriented cores (B, C) and (B', C') attached to F, let h and h' be 1-handle maps attached to F determining (B, C) and (B', C'), respectively. (B, C) and (B', C') are equivalent if and only if h and h' are equivalent.*

Proof. The if part is obvious. We prove the only if part. It is sufficient to prove this in the case that $(B, C) = (B', C')$. Let ∂h and $\partial h'$ be the restrictions of h and h' to $\partial([0, 1] \times B^2)$, respectively. Then $\partial h(\partial([0, 1] \times B^2)) = \partial h'(\partial([0, 1] \times B^2)) = \partial B$, the initial point of C is $\partial h((0, 0)) = \partial h'((0, 0))$ and the terminal point of C is $\partial h((1, 0)) = \partial h'((1, 0))$. By a standard argument, so-called Alexander's trick, we see that ∂h is ambient isotopic to $\partial h'$ in ∂B keeping $h((0, 0))$ and $h((1, 0))$ fixed and keeping $F \cap B$ setwise fixed. This ambient isotopy is extended to an ambient isotopy of $\mathbb{R}^4$ keeping F setwise fixed. So we may assume that $\partial h = \partial h'$. By Alexander's trick, we may change h so that $h = h'$, by an ambient isotopy of B keeping $F \cap B$ setwise fixed, which is extended by an ambient isotopy of $\mathbb{R}^4$ keeping F setwise fixed. Thus h is equivalent to h'. $\square$

Proof of Lemma 2.6. Let C and C' be the oriented cores of B and B' such that (B, C) and (B', C') are determined by h and h', respectively. The 1-handle

B is equivalent to B' if and only if (B, C) is equivalent to (B', C') or $(B', -C')$. By Lemma 2.8, (B, C) is equivalent to (B', C') if and only if h is equivalent to h', and (B, C) is equivalent to $(B', -C')$ if and only if h is equivalent to the reverse of h'. $\qquad\square$

For a 1-handle B attached to F, we denote by $\mathrm{h}^1(F; B)$ the surface-knot

$$(F - \mathrm{int}(F \cap \partial B)) \cup (\partial B - \mathrm{int}(F \cap \partial B)),$$

which we call the surface-knot obtained from F by *surgery* along B. The surgery is also called a 1-handle surgery or a hyperboloidal transformation ([3]).

If B and B' are equivalent 1-handles attached to F, then $\mathrm{h}^1(F; B)$ and $\mathrm{h}^1(F; B')$ are ambient isotopic in $\mathbb{R}^4$.

Definition 2.9. Assume that F is an orientable surface-knot. A 1-handle B attached to F is *orientable* (or *non-orientable*, resp.) if $\mathrm{h}^1(F; B)$ is orientable (or non-orientable, resp.).

3. Cords and 1-handles attached to a surface-knot

Let F be a surface-knot.

Definition 3.1. A simple arc C in $\mathbb{R}^4$ is a *cord* attached to F if $C \cap F = \partial C \cap F$ and if this intersection consists of two distinct points of F. An *oriented cord* is a cord with an orientation as a 1-manifold. Two cords C and C' attached to F are *equivalent* if they are ambient isotopic in $\mathbb{R}^4$ by an ambient isotopy of $\mathbb{R}^4$ keeping F setwise fixed.

Let F be a surface-knot and let C be a cord attached to F. Let $N(C)$ be a regular neighborhood of C in $\mathbb{R}^4$, and put $F \cap N(C) =: M = M_- \cup M_+$, where M_- and M_+ are disjoint 2-cells on F. (When C is oriented, we assume that the orientation of C is from M_- toward M_+.)

Let B be a 1-handle attached to F with core C. We assume that B is contained in $\mathrm{int}N(C)$. We denote by $\mathrm{h}^1(M; B)$ the surface

$$(M - \mathrm{int}(M \cap \partial B)) \cup (\partial B - \mathrm{int}(M \cap \partial B)),$$

which we call the surface obtained from M by *surgery* along B. Then $\mathrm{h}^1(M; B) = \mathrm{h}^1(F; B) \cap N(C)$.

Definition 3.2. In the above situation, let o be an orientation of $M = F \cap N(C)$. We say that B is *compatible with* o if we can give an orientation to the surface $\mathrm{h}^1(M; B)$ such that the restriction to $M - \mathrm{int}(M \cap \partial B)$ of the orientation coincides with that of o. Otherwise, we say that B is *incompatible with* o.

Lemma 3.3. *For any cord C attached to F and for any orientation o of $M = F \cap N(C)$, there exists a 1-handle B attached to F with core C contained in $\mathrm{int}N(C) \subset \mathbb{R}^4$ which is compatible with the orientation o. Moreover, such a 1-handle is unique up to ambient isotopy $(f_s \mid s \in [0, 1])$ of $N(C)$ keeping $\partial N(C) \cup C$ pointwise fixed and M setwise fixed.*

Proof. By an ambient isotopy of $\mathbb{R}^4$, we move F, C, $N(C)$, $M = M_- \cup M_+$ so that $C = \{(0, 0, 0, t) \in \mathbb{R}^4 \mid t \in [-1, 1]\}$, $N(C) = \{(x, y, z, t) \mid x^2 + y^2 + z^2 \leq 2,\ t \in [-2, 2]\}$, $M_- = \{(x, y, 0, -1) \mid x^2 + y^2 \leq 2\}$, $M_+ = \{(x, y, 0, 1) \mid x^2 + y^2 \leq 2\}$, and

that the orientation of o restricted to M_- is opposite to that restricted to M_+. It is sufficient to prove the lemma in the case where F, C, $N(C)$, M and o are in this situation.

Let $N'(C) = \{(x, y, z, t) \,|\, x^2 + y^2 + z^2 \leq 1,\, t \in [-1 - \epsilon, 1 + \epsilon]\}$ for a small positive number ϵ, which is a smaller tubular neighborhood of C. Let $B = \{(x, y, 0, t) \,|\, x^2 + y^2 \leq 1,\, t \in [-1, 1]\}$. It is a 1-handle with core C which is compatible with o. Let B' be another 1-handle in int$N(C)$ with core C which is compatible with o. By an ambient isotopy $(f_s \,|\, s \in [0, 1])$ of $N(C)$ keeping $\partial N(C) \cup C$ pointwise fixed and M setwise fixed, we may assume that $B' = \cup\{X_t \times \{t\} \,|\, t \in [0, 1]\}$ where X_{-1} and X_1 are the standard 2-ball $B^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3$, and for each $t \in (-1, 1)$, X_t is a unit 2-disk in $\mathbb{R}^3$ with center 0. We give X_{-1} an orientation such that the orientation is the same with o restricted to $M_- \cap B' = X_{-1} \times \{-1\}$. For each $t \in (-1, 1]$, we give X_t an orientation induced from the orientation of X_{-1} continuously. (Note that the orientation of X_1 is opposite to the orientation o restricted to $M_- \cap B' = X_{-1} \times \{1\}$, since B' is compatible with o.) The one-parameter family $(X_t \,|\, t \in [-1, 1])$ of oriented disks determines a family $(H_t \,|\, t \in [-1, 1])$ of oriented 2-planes H_t in $\mathbb{R}^3$ with $H_{-1} = H_1 = \mathbb{R}^2$. It induces a map $\theta : [-1, 1] \to G_{3,2}; t \mapsto H_t$ to the Grassmann manifold $G_{3,2}$ with $\theta(-1) = \theta(1) = \mathbb{R}^2$. Since $G_{3,2}$ is homeomorphic to S^2, the loop θ is homotopic to the trivial map. Hence by rotating the 3-balls $B^3 \times \{t\}$ for $t \in [-1, 1]$ relative to $B^3 \times \{-1\} \cup B^3 \times \{1\}$, we can move B' to B. Using a collar neighborhood of $\partial N'(C)$ in $N(C)$, we may extend the rotations to an ambient isotopy $(g_s \,|\, s \in [0, 1])$ of $N(C)$ keeping $\partial N(C) \cup C$ pointwise fixed and M setwise fixed. $\square$

Lemma 3.4. *For any cord C attached to F and for any orientation o of $M = F \cap N(C)$, there exists a 1-handle B attached to F with core C contained in* int$N(C) \subset \mathbb{R}^4$ *which is "incompatible" with the orientation o. Moreover, such a 1-handle is unique up to ambient isotopy $(f_s \,|\, s \in [0, 1])$ of $N(C)$ keeping $\partial N(C) \cup C$ pointwise fixed and M setwise fixed.*

Proof. By reversing the orientation of M_+ in the proof of Lemma 3.3, we see the result. $\square$

We say that two cords C and C' attached to F are *homotopic* if there is a homotopy $(C_s \,|\, s \in [0, 1])$ consisting of arcs in $\mathbb{R}^4$ (possibly with self-intersection) with $C_0 = C$ and $C_1 = C'$ such that for each $s \in [0, 1]$, $C_s \cap F = \partial C_s \cap F$ and the intersection consists of two distinct points of F.

Lemma 3.5. *Two cords C and C', with the same endpoints, $\partial C = \partial C'$, attached to F are equivalent rel ∂C if and only if they are homotopic rel ∂C. Two cords C and C' attached to F are equivalent if and only if they are homotopic.*

Proof. The former assertion follows from Theorem 4 of [4]. The latter assertion is easily seen from the former. $\square$

Let C and C' be cords attached to F which are homotopic as cords attached to F by a homotopy $(C_s \,|\, s \in [0, 1])$ with $C_0 = C$ and $C_1 = C'$. Let o and o' be orientations of $M = F \cap N(C)$ and $M' = F \cap N(C')$, respectively. For $s \in [0, 1]$, let o_s be the orientation of $M_s = F \cap N(C_s)$ induced from $o = o_0$ by the 1-parameter family $(M_t \,|\, t \in [0, s])$. If $o' = o_1$, then we say that the homotopy $(C_s \,|\, s \in [0, 1])$ *respects the orientations o and o'*, and that C and C' are *homotopic with respect to the orientations o and o'*.

Definition 3.6. Let C and C' be cords attached to F which are equivalent as cords attached to F, and let $(f_s \,|\, s \in [0,1])$ be an ambient isotopy of $\mathbb{R}^4$ carrying C to C'. Consider the induced homotopy $(C_s \,|\, s \in [0,1])$ with $C_s = f_s(C)$. We say that C and C' are *equivalent with respect to the orientations o and o'* if the homotopy $(C_s \,|\, s \in [0,1])$ respects o and o'.

Definition 3.7. Let C be an (oriented) cord attached to F and o be an orientation of $M = F \cap N(C)$. We call o a *local orientation of F at the endpoints of C*, and the pair (C, o) an *oriented cord attached to F with a local orientation at the endpoints*. Let (C', o') be an oriented cord attached to F with local orientation at the endpoints. We say that (C, o) is *equivalent to (C', o')* if C is equivalent to C' with respect to o and o'.

By the same reason of Lemma 3.5, we have the following lemma.

Lemma 3.8. *Two cords C and C' attached to F are equivalent with respect to o and o' (in other words, (C, o) is equivalent to (C', o')) if and only if they are homotopic with respect to o and o'.*

The following is a key lemma for classification of 1-handles.

Lemma 3.9. *Let C and C' be cords attached to F, and let o and o' be orientations of $M = F \cap N(C)$ and $M' = F \cap N(C')$. Let B and B' be 1-handles attached to F with core C and C' such that they are compatible with o and o', respectively. If C and C' are homotopic with respect to o and o', then B and B' are equivalent.*

Proof. If C and C' are homotopic with respect to o and o', then by Lemma 3.8 they are equivalent with respect to o and o'. Let $(f_s \,|\, s \in [0,1])$ be an ambient isotopy of $\mathbb{R}^4$ carrying C to C' and keeping F setwise fixed such that the induced homotopy $(C_s \,|\, s \in [0,1])$ respects o and o', where $C_s := f_s(C)$. Let $f_1(B) = B''$. By definition, B is equivalent to B''. Since B'' has core C' and it is compatible with o', by Lemma 3.3, we see that B'' is equivalent to B'. Hence B and B' are equivalent. $\square$

As a corollary, we have the following.

Corollary 3.10. *Let F be an oriented surface-knot. Let B and B' be orientable 1-handles attached to F, and let C and C' be their cores.*

(1) *B and B' are equivalent if and only if C and C' are equivalent.*
(2) *Suppose that C and C' are oriented. (B, C) and (B', C') are equivalent as 1-handles with oriented cores if and only if C and C' are equivalent as oriented cords.*

Corollary 3.11. *Let F be an oriented surface-knot. Let B and B' be non-orientable 1-handles attached to F, and let C and C' be their cores.*

(1) *B and B' are equivalent if and only if C and C' are equivalent.*
(2) *Suppose that C and C' are oriented. (B, C) and (B', C') are equivalent as 1-handles with oriented cores if and only if C and C' are equivalent as oriented cords.*

4. Classification of oriented cords

Let F be a surface-knot, $N(F)$ be a tubular neighborhood of F in $\mathbb{R}^4$, and $E(F)$ be the exterior $\mathbb{R}^4 - \mathrm{int} N(F)$. The tubular neighborhood $N(F)$ is a B^2-bundle over F. Let $p : N(F) \to F$ be the projection map. A fiber $p^{-1}(y)$ $(y \in F)$ is called a *meridian disk* over y.

Take a point x in $\partial N(F) = \partial E(F)$ and put $G(F) := \pi_1(E(F), x)$, which is the *knot group* of F.

Let $\pi_1^+(\partial N(F), x)$ be the subgroup of $\pi_1(\partial N(F), x)$ consisting of all elements represented by loops in $\partial N(F)$ with base point x such that their images under p are orientation-preserving loops in F. If F is orientable, then $\pi_1^+(\partial N(F), x) = \pi_1(\partial N(F), x)$. If F is non-orientable, then $\pi_1^+(\partial N(F), x)$ is a subgroup of $\pi_1(\partial N(F), x)$ of index 2.

Let P and P^+ denote subgroups of $G(F)$ that are the images of $\pi_1(\partial N(F), x)$ and $\pi_1^+(\partial N(F), x)$, respectively, under the inclusion-induced homomorphism $i_* : \pi_1(\partial N(F), x) \to G(F) = \pi_1(E(F), x)$. The subgroup P is called the *peripheral subgroup* of $G(F)$, and we call P^+ the *positive peripheral subgroup*.

Let C be an oriented cord attached to F. We assume that $C \cap N(F)$ consists of two arcs each of which is contained in a meridian disk. The restriction of C to $E(F)$ is an oriented simple arc in $E(F)$, which we denote by $\overline{C}$. Take a path $\alpha : [0, 1] \to \partial N(F)$ such that $\alpha(0) = x$ and $\alpha(1)$ is the initial point of $\overline{C}$, and take a path $\beta : [0, 1] \to \partial N(F)$ such that $\beta(0) = x$ and $\beta(1)$ is the terminal point of $\overline{C}$. The composition $\alpha \overline{C} \beta^{-1}$ is a path in $E(F)$ with base point x, where we regard $\overline{C}$ as a path. We have an element $[\alpha \overline{C} \beta^{-1}]$ of $G(F)$.

Definition 4.1. In the above situation, we call the element $[\alpha \overline{C} \beta^{-1}]$ of $G(F)$ the *element determined from C with (α, β).*

Lemma 4.2 (cf. [2]). *The double coset $P[\alpha \overline{C} \beta^{-1}]P \in P \backslash G(F)/P$ does not depend on a choice of (α, β).*

Proof. Let (α', β') be another choice of paths. Then
$$P[\alpha \overline{C} \beta^{-1}]P = P[\alpha \alpha'^{-1} \alpha' \overline{C} \beta'^{-1} \beta' \beta^{-1}]P = P[\alpha' \overline{C} \beta'^{-1}]P. \qquad \square$$

For an oriented cord C attached to F, we denote the double coset $P[\alpha \overline{C} \beta^{-1}]P \in P \backslash G(F)/P$ by $P(C)P$.

The idea of the following theorem is essentially due to Boyle [2].

Theorem 4.3 (cf. [2]). *Let C and C' be oriented cords attached to F. The cord C is equivalent to C' if and only if $P(C)P = P(C')P$.*

Proof. First we show that if C is equivalent to C', then $P(C)P = P(C')P$. Without loss of generality, we may assume that C is ambient isotopic to C' by an ambient isotopy of $\mathbb{R}^4$ keeping F and $N(F)$ setwise fixed and keeping the base point x fixed. Let (α, β) be a pair of paths for C as in Definition 4.1. By the ambient isotopy of $\mathbb{R}^4$, let (α, β) be mapped to (α', β') and $\overline{C}$ be mapped to $\overline{C'}$. Then $[\alpha \overline{C} \beta^{-1}] = [\alpha' \overline{C'} \beta'^{-1}]$ in $G(F)$. Thus $P(C)P = P(C')P$.

Suppose that $P(C)P = P(C')P$. Let U be a regular neighborhood of x in $\partial N(F)$, and let α_0 and β_0 be short paths in U with $\alpha_0(0) = \beta_0(0) = x$ and $\alpha_0(1) \neq \beta_0(1)$. Let C and C' be oriented cords attached to F with $P(C)P = P(C')P$. By moving

C and C' up to equivalence, without loss of generality, we may assume that the starting points of $\overline{C}$ and $\overline{C'}$ are $\alpha_0(1)$ and the terminal points of $\overline{C}$ and $\overline{C'}$ are $\beta_0(1)$. Then $P[\alpha_0\overline{C}\beta_0^{-1}]P = P[\alpha_0\overline{C'}\beta_0^{-1}]P$, and hence $[\alpha_0\overline{C}\beta_0^{-1}] = g[\alpha_0\overline{C'}\beta_0^{-1}]g'$ in $G(F)$ for some elements $g, g' \in G(F)$. This implies that $\overline{C}$ is homotopic to $\overline{C'}$ in $E(F)$ after sliding the endpoints suitably, and we see that C is homotopic to C'. By Lemma 3.5, C is equivalent to C'. $\qquad\square$

Theorem 4.4. *Let φ be the map from the set of equivalence classes of oriented cords attached to F to the double cosets $P \setminus G(F)/P$ that sends the equivalence class of C to $P(C)P$. The map φ is a bijection.*

Proof. By Theorem 4.3, φ is well defined and injective. We show that φ is surjective. Let U be a regular neighborhood of x in $\partial N(F)$, and let α_0 and β_0 be short paths in U with $\alpha_0(0) = \beta_0(0) = x$ and $\alpha_0(1) \neq \beta_0(1)$. Let g be an element of $G(F)$. There is a simple path $\gamma : [0,1] \to E(F)$ such that $g = [\alpha_0\gamma\beta_0^{-1}]$ in $G(F)$. Let C be an oriented core attached to F such that $\overline{C}$ is the image of γ. Then $P(C)P = PgP$. Thus the map φ is surjective. $\qquad\square$

Now we consider oriented cords attached to F with local orientations of F at the endpoints.

Let C be an oriented cord attached to F, and let y_- and y_+ be the initial point and the terminal point of C, respectively. Let $M = M_- \cup M_+ = F \cap N(C)$, where M_- is a 2-cell in F containing y_- and M_+ is a 2-cell containing y_+. We denote by y the image $p(x)$ of x, and let U_y be a regular neighborhood of y in F. Let o_y be an orientation of U_y, and let o be an orientation of M.

Let (α, β) be a pair of paths for C as in Definition 4.1. Then $(p\alpha, p\beta)$ is a pair of paths in F with $(p\alpha)(0) = (p\beta)(0) = y$, $(p\alpha)(1) = y_-$ and $(p\beta)(1) = y_+$.

Definition 4.5. We say that (α, β) is *compatible with o_y and o* if the local orientation of F at y_- determined from o coincides with the local orientation at y_- obtained from the local orientation o_y at y by translating along $p\alpha$ and if the local orientation of F at y_+ determined from o coincides with the local orientation at y_+ obtained from the local orientation o_y at y by translating along $p\beta$. Otherwise, we say that it is *incompatible with o_y and o*.

Lemma 4.6. *Let o_y be an orientation of U_y. Let C be an oriented cord attached to F, and o be an orientation of $M = F \cap N(C)$. Let (α, β) be a pair of paths as in Definition 4.1. Suppose that (α, β) is compatible with o_y and o. The double coset $P^+[\alpha\overline{C}\beta^{-1}]P^+ \in P^+ \setminus G(F)/P^+$ does not depend on a choice of (α, β).*

Proof. Let (α', β') be another choice of paths that is compatible with o_y and o. Since $(p\alpha)(p\alpha'^{-1})$ is an orientation-preserving loop in F, we have that $[\alpha\alpha'^{-1}] \in G(F)$ belongs to P^+. Similarly, $[\beta'\beta^{-1}]$ belongs to P^+. Then

$$P^+[\alpha\overline{C}\beta^{-1}]P^+ = P^+[\alpha\alpha'^{-1}\alpha'\overline{C}\beta'^{-1}\beta'\beta^{-1}]P^+ = P^+[\alpha'\overline{C}\beta'^{-1}]P^+. \qquad\square$$

In the situation of Lemma 4.6, we denote the double coset $P^+[\alpha\overline{C}\beta^{-1}]P^+ \in P^+G(F)P^+$ by $P^+(C, o_y, o)P^+$.

Theorem 4.7. *Let o_y be an orientation of U_y. Let (C, o) and (C', o') oriented cords attached to F with local orientations at the endpoints. (C, o) is equivalent to (C', o') if and only if $P^+(C, o_y, o)P^+ = P^+(C', o_y, o')P^+$.*

The proof is analogous to the proof of Theorem 4.3, which is left to the reader.

Theorem 4.8. *Let o_y be an orientation of U_y. Let ψ be the map from the set of equavalence classes of oriented cords attached to F with local orientations at the endpoints to the double cosets $P^+ \setminus G(F)/P^+$ that sends the equivalence class of (C, o) to $P^+(C, o_y, o)P^+$. The map ψ is a bijection.*

Proof. (The proof is analogous to the proof of Theorem 4.4.) By Theorem 4.7, ψ is well defined and injective. We show that ψ is surjective. Let U be a regular neighborhood of x in $\partial N(F)$, and let α_0 and β_0 be short paths in U with $\alpha_0(0) = \beta_0(0) = x$ and $\alpha_0(1) \neq \beta_0(1)$. Let g be an element of $G(F)$. There is a simple path $\gamma : [0, 1] \to E(F)$ such that $g = [\alpha_0 \gamma \beta_0^{-1}]$ in $G(F)$. Let C be an oriented core attached to F such that $\overline{C}$ is the image of γ. Let o be a local orientation of F at the endpoints of C such that it is obtained from o_y by translating o_y along $p\alpha_0$ and $p\beta_0$. Then $\psi(C, o) = P^+(C, o_y, o)P^+ = P^+gP^+$. Thus the map ψ is surjective. $\square$

5. Classification of 1-handles in Case 1

In this section we consider Case 1: F is oriented and 1-handles are orientable.

First we classify 1-handles with oriented cores. This case is due to Boyle [2].

Let F be an oriented surface-knot, $G(F)$ be the knot group $\pi_1(E(F), x)$, and P be the peripheral subgroup.

Let (B, C) be an orientable 1-handle with an oriented core attached to F. Let $P(C)P$ be the double coset $[\alpha \overline{C} \beta^{-1}] \in P \setminus G(F)/P$ where $\overline{C} = C \cap E(F)$ and (α, β) is a pair of paths for C as in Definition 4.1.

Theorem 5.1 (Boyle [2]). *Two orientable 1-handles with oriented cores (B, C) and (B', C') attached to F are equivalent if and only if $P(C)P = P(C')P$. Moreover, a map sending the equivalence class of (B, C) to $P(C)P$ is a bijection from the set of equivalence classes of orientable 1-handles with oriented cores attached to F to the double cosets $P \setminus G(F)/P$.*

Proof. By Corollary 3.10, the equivalence class of (B, C) corresponds to the equivalence class of the oriented core C. By Theorem 4.4, we have the result. $\square$

Definition 5.2. Let B be an orientable 1-handle attached to F. Define $P(B)P$ by an unordered pair $\{P(C)P, P(-C)P\}$, where C is an oriented core C of B.

Note that $P(B)P$ does not depend on a choice of an oriented core of B.

Theorem 5.3. *Two orientable 1-handles B and B' attached to F are equivalent if and only if $P(B)P = P(B')P$.*

Proof. Let C and C' be oriented cores of B and B', respectively. Then B and B' are equivalent if and only if (B, C) is equivalent to $(B', \epsilon C')$ for some $\epsilon \in \{+1, -1\}$. The latter statement holds if and only if (B, C) is equivalent to $(B', \epsilon C')$ and $(B, -C)$ is equivalent to $(B', -\epsilon C')$, and hence by Theorem 5.1, if and only if $P(C)P = P(\epsilon C')P$ and $P(-C)P = P(-\epsilon C')P$. The last statement holds if and only if $P(B)P = P(B')P$. $\square$

For a set X and a positive integer n, the *unordered n-fold product of X* means the set of unordered n-tuples of elements $\{x_1, \ldots, x_n\}$ of X.

Note that $P(B)P$ is an element of the unordered 2-fold product of $P \setminus G(F)/P$.

By Theorem 5.3, a map sending the equivalence class of B to $P(B)P$ from the set of equivalence classes of orientable 1-handles attached to F to the unordered 2-fold product of $P \setminus G(F)/P$ is well defined and injective. However this map is not surjective in general. The image of this map is characterized as follows.

Proposition 5.4. *The image of the map sending the equivalence class of B to $P(B)P$ from the set of equivalence classes of orientable 1-handles attached to F to the unordered 2-fold product of $P \setminus G(F)/P$ is the subset consisting of the elements $\{PgP, Pg^{-1}P\}$ for all $g \in G(F)$.*

Proof. Let B be an orientable 1-handle attached to F, and C an oriented core. By definition, $P(C)P = P[\alpha \overline{C} \beta^{-1}]P$ as before. Put $g = [\alpha \overline{C} \beta^{-1}] \in G(F)$. Then $P(-C)P = P[\beta \overline{C}^{-1} \alpha^{-1}]P = Pg^{-1}P$. Hence $P(B)P = \{PgP, Pg^{-1}P\}$ for some $g \in G(F)$. Conversely for any $g \in G(F)$, there is an orientable 1-handle with an oriented core C such that $P(C)P = PgP$. Then $P(B)P = \{PgP, Pg^{-1}P\}$.   □

Corollary 5.5. *Let F be an oriented surface-knot with $P \neq G(F)$. Then the map sending the equivalence class of B to $P(B)P$ from the set of equivalence classes of orientable 1-handles attached to F to the unordered 2-fold product of $P \setminus G(F)/P$ is not surjective.*

Proof. Let g be an element of $G(F) - P$. We show that $\{PgP, P1P\}$ is not obtained from any 1-handle. Assume that $\{PgP, P1P\} = \{Pg'P, Pg'^{-1}P\}$ for some $g' \in G(F)$. Replacing g' with g'^{-1} if it is necessary, we may assume that $PgP = Pg'P$. Since $g \notin P$, we have $g' \notin P$. On the other hand, $P1P = Pg'^{-1}P$ implies that $g' \in P$. This is a contradiction.   □

For example, every non-trivial 2-knot satisfies that $P \neq G(F)$.

6. Classification of 1-handles in Case 2

In this section we consider Case 2: F is oriented and 1-handles are non-orientable.

This case is completely analogus to Case 1. Let F be an oriented surface-knot, $G(F)$ be the knot group $\pi_1(E(F), x)$, and P be the peripheral subgroup.

Let (B, C) be a non-orientable 1-handle with an oriented core attached to F. Let $P(C)P$ be the double coset $[\alpha \overline{C} \beta^{-1}] \in P \setminus G(F)/P$ where $\overline{C} = C \cap E(F)$ and (α, β) is a pair of paths for C as in Definition 4.1.

Theorem 6.1. *Two non-orientable 1-handles with oriented cores (B, C) and (B', C') attached to F are equivalent if and only if $P(C)P = P(C')P$. Moreover, a map sending the equivalence class of (B, C) to $P(C)P$ is a bijection from the set of equivalence classes of non-orientable 1-handles with oriented cores attached to F to the double cosets $P \setminus G(F)/P$.*

Proof. By Corollary 3.11, the equivalence class of (B, C) corresponds to the equivalence class of the oriented core C. By Theorem 4.4, we have the result.   □

Definition 6.2. Let B be a non-orientable 1-handle attached to F. Define $P(B)P$ by an unordered pair $\{P(C)P, P(-C)P\}$, where C is an oriented core C of B.

Note that $P(B)P$ does not depend on a choice of an oriented core of B.

Theorem 6.3. *Two non-orientable* 1-*handles B and B' attached to F are equivalent if and only if $P(B)P = P(B')P$.*

Proof. The proof is the same with the proof of Theorem 5.3, where we use Theorem 6.1 instead of Theorem 5.1. $\qquad\square$

By Theorem 6.3, a map sending the equivalence class of B to $P(B)P$ from the set of equivalence classes of non-orientable 1-handles attached to F to the unordered 2-fold product of $P \backslash G(F)/P$ is well defined and injective. This map is not surjective in general. The image of this map is exactly the same with the subset given in Proposition 5.4. An analogous statement to Corollary 5.5 is also valid for non-orientable 1-handles.

7. Classification of 1-handles in Case 3

In this section we consider Case 3: F is non-orientable.

Let F be a non-oriented surface-knot, $G(F)$ be the knot group $\pi_1(E(F), x)$, and P^+ be the positive peripheral subgroup. Let o_y be an orientation of a regular neighborhood U_y of $y = p(x)$.

First we classify 1-handles with oriented cores.

Let (B, C) be a 1-handle with an oriented core attached to F. Let o be an orientation of $M = F \cap N(C)$ such that the 1-handle B is compatible with o (Definition 3.2). Let (α, β) be a pair of paths for C as in Definition 4.1 such that it is compatible with o_y and o (Definition 4.5). Let $P^+(C, o_y, o)P^+$ be the double coset $[\alpha\overline{C}\beta^{-1}] \in P^+ \backslash G(F)/P^+$ where $\overline{C} = C \cap E(F)$.

By $-o_y$ and $-o$, we denote the reversed orientations of o_y and o, respectively.

Lemma 7.1. *In the situation above, we have the following.*

(1) *The* 1-*handle B is compatible with an orientation o' of M if and only if $o' = o$ or $o' = -o$.*

(2) *$P^+(C, -o_y, o)P^+ = P^+(C, o_y, -o)P^+$, and*
$P^+(C, -o_y, -o)P^+ = P^+(C, o_y, o)P^+$.

Proof. (1) Since $M = F \cap N(C) = M_- \cup M_+$, there are four orientations of M. Let $o|M_-$ and $o|M_+$ be the orientations of M_- and M_+ that are restrictions of o. Then the four orientations are $o = (o|M_-, o|M_+), -o = (-o|M_-, -o|M_+), (o|M_-, -o|M_+)$ and $(-o|M_-, o|M_+)$. If B is compatible with o then it is compatible with $-o$ and it is incompatible with the other two.

(2) It is obvious by the definition of $P^+(C, o_y, o)P^+$. $\qquad\square$

Definition 7.2. Let (B, C) be a 1-handle with an oriented core attached to a non-orientable surface-knot F. In the situation above, we define $P^+(B, C)P^+$ by an unordered pair $\{P^+(C, o_y, o)P^+, P^+(C, o_y, -o)P^+\}$, which is an element of the unordered 2-fold product of $P^+ \backslash G(F)/P^+$.

By Lemma 7.1, $P^+(B, C)P^+$ does not depend on a choice of o_y and o.

Theorem 7.3. *Let F be a non-orientable surface-knot. Two* 1-*handle with oriented cores (B, C) and (B', C') attached to F are equivalent if and only $P^+(B, C)P^+ = P^+(B', C')P^+$.*

Proof. Let o and o' be orientations of $M = F \cap N(C)$ and $M' = F \cap N(C')$ such that B and B' are compatible with o and o', respectively.

Note that (B, C) and (B', C') are equivalent if and only if (C, o) is equivalent to $(C', \epsilon o')$ for some $\epsilon \in \{+1, -1\}$, and $(C, -o)$ is equivalent to $(C', -\epsilon o')$. By Theorem 4.7, this condition is equivalent to that $P^+(C, o_y, o)P^+ = P^+(C', o_y, \epsilon o')P^+$ and $P^+(C, o_y, -o)P^+ = P^+(C', o_y, -\epsilon o')P^+$. It is equivalent to that $P^+(B, C)P^+ = P^+(B', C')P^+$. $\square$

Definition 7.4. Let θ be a map sending the equivalence class of (B, C) to $P^+(B, C)P^+$ from the set of equivalence classes of 1-handles with oriented cores attached to F to the unordered 2-fold product of $P^+ \setminus G(F)/P^+$.

By Theorem 7.3, the map θ is well defined and injective. In general, it is not surjective.

Lemma 7.5. *Let m be an element of $\pi_1(\partial N(F), x) - \pi_1^+(\partial N(F), x)$. The image of the map θ is the subset consisting of the elements $\{P^+ g P^+, P^+ i_*(m) g \, i_*(m) P^+\}$ for all $g \in G(F)$.*

Proof. We may fix an orientation o_y. Let ν be a loop in $\partial N(F)$ with base point x representing m. Let (B, C) be a 1-handle with an oriented core attached to F. Let o be an orientation of $M = F \cap N(C)$ such that B is compatible with o. (Then B is also compatible with $-o$.) Let (α, β) be a pair of paths in $N(C)$ for C as in Definition 4.1 such that it is compatible with o_y and o. Then by definition, $P^+(C, o_y, o)P^+ = P^+[\alpha \overline{C} \beta^{-1}]P^+$, where $\overline{C} = C \cap E(F)$ and we regard $\overline{C}$ as a simple path. Note that $(\nu \alpha, \nu^{-1} \beta)$ is a pair of paths in $N(F)$ for C such that it is compatible with o_y and $-o$. Thus $P^+(C, o_y, -o)P^+ = P^+[\nu \alpha \overline{C} \beta^{-1} \nu]P^+$. When we put $g = [\alpha \overline{C} \beta^{-1}] \in G(F)$, we have $P^+(C, o_y, o)P^+ = P^+ g P^+$ and $P^+(C, o_y, -o)P^+ = P^+ i_*(m) g \, i_*(m) P^+$. Hence $P^+(B, C)P^+ = \{P^+ g P^+, P^+ i_*(m) g \, i_*(m) P^+\}$.

Conversely for any $g \in G(F)$, there is a 1-handle with an oriented core (B, C) attached to F such that $P^+(C, o_y, o)P^+ = P^+ g P^+$. This is verified by the same argument in the proof of Theorem 4.8. Then, as shown above, we see that $P^+(B, C)P^+ = \{P^+ g P^+, P^+ i_*(m) g \, i_*(m) P^+\}$. $\square$

Proposition 7.6. *The image of the map θ is characterized as follows:*

(1) *If $P^+ \neq P$, then the image consists of the elements*

$$\{P^+ g P^+, P^+ n g n P^+\}$$

   for all $g \in G(F)$, where n is an element of $P - P^+$.
(2) *If $P^+ = P$, then the image consists of the elements*

$$\{P^+ g P^+, P^+ g P^+\}$$

   for all $g \in G(F)$.

Proof. (1) Suppose that $P^+ \neq P$ and let $n \in P - P^+$. Take an element $m \in \pi_1(\partial N(F), x) - \pi_1^+(\partial N(F), x)$ with $i_*(m) = n$. By Lemma 7.5, we have the result.

(2) Suppose that $P^+ = P$. There is an element $m \in \pi_1(\partial N(F), x) - \pi_1^+(\partial N(F), x)$ with $i_*(m) = 1$. By Lemma 7.5, we have the result. $\square$

Corollary 7.7. *Let F be a non-orientable surface-knot with $P^+ \neq P$. The map θ is not surjective.*

Proof. Let n be an element of $P - P^+$. We show that $\{P^+nP^+, P^+1P^+\}$ is not in the image of θ. Assume that $\{P^+nP^+, P^+1P^+\}$ is in the image of θ. Then by Proposition 7.6 (1), $\{P^+nP^+, P^+1P^+\} = \{P^+gP^+, P^+ngnP^+\}$ for some $g \in G(F)$.

(i) Suppose that $P^+nP^+ = P^+gP^+$ and $P^+1P^+ = P^+ngnP^+$. Since $n \in P - P^+$, $P^+nP^+ = P^+gP^+$ implies that $g \in P - P^+$. On the other hand, $P^+1P^+ = P^+ngnP^+$ implies that $g \in P^+$. This is a contradiciton.

(ii) Suppose that $P^+nP^+ = P^+ngnP^+$ and $P^+1P^+ = P^+gP^+$. Since $n \in P - P^+$, $P^+nP^+ = P^+ngnP^+$ implies that $g \in P - P^+$. On the other hand, $P^+1P^+ = P^+gP^+$ implies that $g \in P^+$. This is a contradiciton. $\square$

A surface-knot F is said to be *incompressible* if the inclusion-induced homomorphism $i_* : \pi_1(\partial N(F), x) \to G(F)$ is injective. A method of constructing incompressible Klein bottles in R^4 is given in [6]. (The method in [6] relied on the existence of incompressible tori in $\mathbb{R}^4$, which is shown in [1, 8].)

Incompressible non-orientable surface-knots satisfy that $P^+ \neq P$.

Corollary 7.8. *Let F be a non-orientable surface-knot with $P^+ = P \neq G(F)$. The map θ is not surjective.*

Proof. Let g be an element of $G(F) - P^+$. We show that $\{P^+gP^+, P^+1P^+\}$ is not in the image of θ. Assume that $\{P^+gP^+, P^+1P^+\}$ is in the image of θ. Then by Proposition 7.6 (2), $\{P^+gP^+, P^+1P^+\} = \{P^+g'P^+, P^+g'P^+\}$ for some $g' \in G(F)$. This implies that $g \in P^+$. This contradicts to $g \in G(F) - P^+$. $\square$

If F is a non-orientable surface-knot obtained from a connected sum of a standard projective plane in $\mathbb{R}^4$ (cf. [3]) and a surface-knot, then F satisfies that $P^+ = P$. Using this, one can obtain a lot of examples of non-orientable surface-knots with $P^+ = P \neq G(F)$.

Now we consider 1-handles attached to F.

Definition 7.9. Let B be a 1-handle attached to F. Define $P^+(B)P^+$ by an unordered pair of unordered pairs $\{P^+(B, C)P^+, P^+(B, -C)P^+\}$, where C is an oriented core C of B.

Note that $P^+(B)P^+$ does not depend on a choice of an oriented core of B.

Theorem 7.10. *Two 1-handles B and B' attached to F are equivalent if and only if $P^+(B)P^+ = P^+(B')P^+$.*

Proof. Let C and C' be oriented cores of B and B', respectively. Then B and B' are equivalent if and only if (B, C) is equivalent to $(B', \epsilon C')$ for some $\epsilon \in \{+1, -1\}$. The latter statement holds if and only if (B, C) is equivalent to $(B', \epsilon C')$ and $(B, -C)$ is equivalent to $(B', -\epsilon C')$, and hence by Theorem 7.3, if and only if $P^+(B, C)P^+ = P^+(B', \epsilon C')P^+$ and $P^+(B, -C)P^+ = P^+(B', -\epsilon C')P^+$. The last statement holds if and only if $P^+(B)P^+ = P^+(B')P^+$. $\square$

Let J be the image of the map θ defined in Definition 7.4, which is characterized in Lemma 7.5 and Proposition 7.6.

Definition 7.11. Let Θ be a map sending the equivalence class of B to $P^+(B)P^+$ from the set of equivalence classes of 1-handles attached to F to the unordered 2-fold product of J.

By Theorem 7.10, the map Θ is well defined and injective.

By Lemma 7.5 and Proposition 7.6, we have the following.

Lemma 7.12. *Let m be an element of $\pi_1(\partial N(F), x) - \pi_1^+(\partial N(F), x)$. The image of the map Θ is the subset consisting of the elements*

$$\{\{P^+gP^+, P^+i_*(m)\, g\, i_*(m)P^+\}, \{P^+g^{-1}P^+, P^+i_*(m)\, g^{-1}\, i_*(m)P^+\}\}$$

for all $g \in G(F)$.

Proof. This follows from Lemma 7.5 by a similar argument as in the proof of Proposition 5.4. $\square$

Lemma 7.13. *The image of the map Θ is characterized as follows:*

(1) *If $P^+ \neq P$, then the image consists of the elements*

$$\{\{P^+gP^+, P^+ngnP^+\}, \{P^+g^{-1}P^+, P^+ng^{-1}nP^+\}\}$$

for all $g \in G(F)$, where n is an element of $P - P^+$.

(2) *If $P^+ = P$, then the image consists of the elements*

$$\{\{P^+gP^+, P^+gP^+\}, \{P^+g^{-1}P^+, P^+g^{-1}P^+\}\}$$

for all $g \in G(F)$.

Proof. This follows from Proposition 7.6 by a similar argument as in the proof of Proposition 5.4. $\square$

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