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Morse indices and the number of maximum points of some solutions to a two-dimensional elliptic problem

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Abstract. In this note, we consider the problem

$$-\Delta u = u^p \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

on a smooth bounded domain Ω in \mathbb{R}^2 for $p > 1$. Let u_p be a positive solution of the above problem with Morse index less than or equal to $m \in \mathbb{N}$. We prove that if u_p further satisfies the assumption $p \int_{\Omega} |\nabla u_p|^2 dx = O(1)$ as $p \rightarrow \infty$, then the number of maximum points of u_p is less than or equal to m for p sufficiently large. If Ω is convex, we also show that a solution of Morse index one satisfying the above assumption has a unique critical point and the level sets are star-shaped for p sufficiently large.

Keywords: Morse index, maximum point, semilinear elliptic equation.

2000 Mathematics Subject Classifications: 35B40, 35J20, 35J25.

1. Introduction.

In this note we consider the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^2 and $p > 1$. Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for any $p > 1$, the existence of at least one solution is easy to obtain. In fact, if we define

$$S_p = \inf_{u \in H_0^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 dx \mid \int_{\Omega} |u|^{p+1} dx = 1 \right\},$$

then a standard variational method implies that S_p is achieved by a positive function $\bar{u}_p \in H_0^1(\Omega)$ and $u_p = S_p^{1/(p-1)}\bar{u}_p$ is a solution of (1.1).

For the least energy solution u_p obtained in this way, several studies on the asymptotic behavior have been done in [6], [7], [3] and [1]. In particular, in [6] and [7], along a suitable subsequence $p \rightarrow \infty$, least energy solution u_p is shown to develop a single spiky pattern at an interior point of the domain. More precisely, u_p satisfies a uniform L^∞ -norm estimate

$$C_1 \leq \|u_p\|_{L^\infty(\Omega)} \leq C_2$$

for some $0 < C_1 \leq C_2 < \infty$ independent of p , and to “concentrate” at an interior point of the domain, i.e.,

$$\frac{u_p^p}{\int_\Omega u_p^p dx} \rightarrow \delta(x_0) \quad \text{as } p \rightarrow \infty$$

for some $x_0 \in \Omega$ in the sense of Radon measures. Moreover, the estimate

$$p \int_\Omega |\nabla u_p|^2 dx \rightarrow 8\pi e \quad \text{as } p \rightarrow \infty$$

is proved for least energy solution u_p . Recently, for any $m \in \mathbb{N}$, a solution sequence $\{u_p\}$ which exhibits the asymptotic behavior

$$p \int_\Omega |\nabla u_p|^2 dx \rightarrow 8\pi m e \quad \text{as } p \rightarrow \infty$$

has been constructed in [5] under some topological assumption of the domain.

In the following, we restrict our attention to the solution u_p of (1.1) which satisfies the assumption

$$p \int_\Omega |\nabla u_p|^2 dx = O(1) \quad \text{as } p \rightarrow \infty. \tag{1.2}$$

Before stating the results in this paper, we recall that the *Morse index* of a solution u of (1.1) is the number of negative eigenvalues of the linearized operator $L_u = -\Delta - pu^{p-1}$ acting on $H_0^1(\Omega)$. In this paper, we prove the following theorems.

Theorem 1. *Let $\{u_p\}$ be a solution sequence of (1.1) satisfying the assumption (1.2) with the Morse index less than or equal to $m, m \in \mathbb{N}$. Then the*

number of maximum points of u_p is less than or equal to m for p sufficiently large.

Theorem 2. *Let $\{u_p\}$ be a solution sequence of (1.1) satisfying the assumption (1.2) with the Morse index one. If Ω is convex, then u_p has only one critical point x_p which is the global maximum point of u_p , and*

$$(x - x_p) \cdot \nabla u_p(x) < 0, \quad \forall x \in \Omega \setminus \{x_p\}$$

holds for p sufficiently large. In particular, the level sets of u_p are strict star-shaped with respect to x_p .

In [4], El Mehdi and Pacella treated the problem

$$\begin{cases} -\Delta u = N(N-2)u^{p-\varepsilon} - \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded, star-shaped domain in \mathbb{R}^N , $N \geq 3$, $p = (N+2)/(N-2)$, $\varepsilon > 0$ and $\lambda \geq 0$. They proved similar results on the relation between the Morse index and the number of maximum points of blowing-up solutions $\{u_\varepsilon\}$ to this problem via a blow-up analysis. Note that in this case, it holds $\|u_\varepsilon\|_{L^\infty(\Omega)} \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Theorems in this paper are two-dimensional counterparts to the results in [4]. However, in our situation, solutions may not blow up in the L^∞ -norm sense, so the usual blow-up analysis as in the higher-dimensional case does not work. To overcome this difficulty, we combine the arguments in [4] and the two-dimensional blow-up technique by Adimurthi and Grossi [1].

2. Proof of Theorem 1.

Let $x_p^1 \in \Omega$ be a maximum point of u_p for p large, that is, $\|u_p\|_\infty = u_p(x_p^1)$. First, we recall a result in [6] that for any solution u_p of (1.1), there holds an estimate

$$\|u_p\|_\infty \geq \lambda_1^{1/(p-1)}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition. Proof of this fact in [6] is simple, so we recall it here for the readers' convenience. Let $\phi_1 > 0$ be the first eigenfunction associated to λ_1 .

Then we see

$$0 = \int_{\Omega} (u_p \Delta \phi_1 - \phi_1 \Delta u_p) dx = \int_{\Omega} \phi_1 u_p (u_p^{p-1} - \lambda_1) dx,$$

thus we have $\|u_p\|_{\infty}^{p-1} \geq \lambda_1$.

From this fact, we see

$$\varepsilon_p := \frac{1}{\sqrt{p} \|u_p\|_{\infty}^{(p-1)/2}} \rightarrow 0 \quad (p \rightarrow \infty). \quad (2.1)$$

Next, we claim that x_p^1 is away from $\partial\Omega$ uniformly in p sufficiently large. Indeed, (1.1) and (1.2) imply that $p \int_{\Omega} u_p^{p+1} dx = O(1)$ as $p \rightarrow \infty$. Then we have

$$p \int_{\Omega} u_p^p dx \leq \left(p \int_{\Omega} u_p^{p+1} dx \right)^{p/(p+1)} p^{1/(p+1)} |\Omega|^{1/(p+1)} \leq C$$

uniformly in p , thus $\int_{\Omega} u_p^p dx = O(1/p)$ as $p \rightarrow \infty$. Therefore, at any maximum point x_p of u_p , $v_p(x) := u_p(x) / \int_{\Omega} u_p^p dx$ satisfies

$$v_p(x_p) = \frac{u_p(x_p)}{\int_{\Omega} u_p^p dx} \geq \frac{\lambda_1^{1/(p-1)}}{\int_{\Omega} u_p^p dx} \rightarrow +\infty$$

as $p \rightarrow \infty$. On the other hand, applying Lemma 4.1 in [6] and the elliptic L^1 estimate to v_p , we have, as in Lemma 4.2 in [6], that v_p is bounded in L^∞ near $\partial\Omega$ uniformly in p . Thus any maximum point x_p of u_p cannot approach to $\partial\Omega$ and for some neighborhood ω of $\partial\Omega$, we have $\{x_p \in \Omega | u_p(x_p) = \|u_p\|_{\infty}\} \subset \Omega \setminus \omega$ for large p .

Now, we define the scaled function

$$\tilde{u}_p(y) = \frac{p}{\|u_p\|_{\infty}} \{u_p(\varepsilon_p y + x_p^1) - u_p(x_p^1)\}, \quad y \in \Omega_p^1 = \frac{\Omega - x_p^1}{\varepsilon_p} \quad (2.2)$$

as in [1], which satisfies

$$\begin{cases} -\Delta \tilde{u}_p(y) = \left(1 + \frac{\tilde{u}_p(y)}{p}\right)^p & \text{in } \Omega_p^1, \\ 0 < 1 + \frac{\tilde{u}_p(y)}{p} \leq 1 & \text{in } \Omega_p^1, \\ \tilde{u}_p(y) = -p & y \in \partial\Omega_p^1. \end{cases} \quad (2.3)$$

By the above claim, the limit domain of Ω_p^1 as $p \rightarrow \infty$ is \mathbb{R}^2 . As in [1] p.1015, we can pass to the limit in (2.3) to obtain some function $U \in C^2(\mathbb{R}^2)$ such that $\tilde{u}_p \rightarrow U$ as $p \rightarrow \infty$ in $C_{loc}^2(\mathbb{R}^2)$. Passing to the limit in (2.3), we see that U satisfies

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2, \quad \max_{y \in \mathbb{R}^2} U(y) = U(0) = 0.$$

Moreover, by the assumption (1.2) and Fatou's lemma, we can check that

$$\int_{\mathbb{R}^2} e^U dy < +\infty. \quad (2.4)$$

In fact, since $\tilde{u}_p(y) \rightarrow U(y)$ a.e. $y \in \mathbb{R}^2$, we see

$$p \log \left(1 + \frac{\tilde{u}_p(y)}{p} \right) \rightarrow U(y) \quad \text{a.e. } y \in \mathbb{R}^2$$

and

$$e^{p \log \left(1 + \frac{\tilde{u}_p(y)}{p} \right)} \rightarrow e^{U(y)} \quad \text{a.e. } y \in \mathbb{R}^2.$$

Thus Fatou's lemma and a simple change of variables using (2.2) imply that

$$\begin{aligned} \int_{\mathbb{R}^2} e^U dy &\leq \liminf_{p \rightarrow \infty} \int_{\Omega_p^1} \left(1 + \frac{\tilde{u}_p(y)}{p} \right)^p dy \\ &= \liminf_{p \rightarrow \infty} \int_{\Omega} \left(\frac{u_p(x)}{\|u_p\|_{\infty}} \right)^p p \|u_p\|_{\infty}^{p-1} dx \leq (1/C) \liminf_{p \rightarrow \infty} p \int_{\Omega} u_p^p dx, \end{aligned}$$

where we have used the fact $\|u_p\|_{\infty} \geq C > 0$ uniformly for p large. The last term is bounded as before thanks to (1.2), so we get (2.4).

At this point, we see by a result of Chen and Li [2] that

$$U(y) = -2 \log \left(1 + \frac{|y|^2}{8} \right). \quad (2.5)$$

Next, we define two elliptic operators

$$L_p := -\Delta_x - p u_p^{p-1}(x) \cdot : H_0^1(\Omega) \rightarrow H^{-1}(\Omega), \quad (2.6)$$

$$\tilde{L}_p := -\Delta_y - \left(1 + \frac{\tilde{u}_p(y)}{p} \right)^{p-1} \cdot : H_0^1(\Omega_p^1) \rightarrow H^{-1}(\Omega_p^1). \quad (2.7)$$

Note that the operators (2.6) and (2.7) are related to each other by a simple scaling

$$\varepsilon_p^2 L_p \Big|_{u_p(x)=\|u_p\|_\infty \left(1+\frac{\tilde{u}_p}{p}(y)\right)} = \tilde{L}_p,$$

here, $x = \varepsilon_p y + x_p^1$ for $x \in \Omega$ and $y \in \Omega_p^1$. Further if we write the j -th eigenvalue of an elliptic operator L acting on $H_0^1(D)$ for a bounded domain D as $\lambda_j(L, D)$, $j \in \mathbb{N}$, then

$$\varepsilon_p^2 \lambda_j(L_p, D) = \lambda_j(\tilde{L}_p, D_p), \quad D_p = \frac{D - x_p^1}{\varepsilon_p}. \quad (2.8)$$

Now, we claim that there exists $R > 0$ such that $\lambda_1(L_p, B(x_p^1, \varepsilon_p R)) < 0$ for p sufficiently large.

Indeed, define

$$w_p(y) = y \cdot \nabla \tilde{u}_p(y) + \frac{2p}{p-1} \left(1 + \frac{\tilde{u}_p}{p}(y)\right), \quad y \in \Omega_p^1. \quad (2.9)$$

Then we have $-\Delta w_p(y) = \left(1 + \frac{\tilde{u}_p(y)}{p}\right)^{p-1} w_p(y)$ for $y \in \Omega_p^1$. Note that $w_p(0) = 2p/(p-1) \rightarrow 2$. Also, since $\tilde{u}_p \rightarrow U$ in $C_{loc}^2(\mathbb{R}^2)$, we have $w_p(y) \rightarrow 2(8 - |y|^2)/(8 + |y|^2) < 0$ if $|y| = R > 2\sqrt{2}$ as $p \rightarrow \infty$. Now, set $A_p = \{y \in B(0, R) : w_p(y) > 0\}$, $A_p \neq \emptyset$, and

$$\bar{w}_p(y) = \begin{cases} w_p(y) & y \in A_p, \\ 0 & y \in B(0, R) \setminus \overline{A_p}. \end{cases}$$

Testing

$$\lambda_1(\tilde{L}_p, B(0, R)) = \inf_{v \in H_0^1(B(0, R))} \frac{\int_{B(0, R)} |\nabla v|^2 dy - \int_{B(0, R)} \left(1 + \frac{\tilde{u}_p}{p}(y)\right)^{p-1} v^2 dy}{\int_{B(0, R)} v^2 dy}$$

by $\bar{w}_p \in H_0^1(B(0, R))$, we see that $\lambda_1(\tilde{L}_p, B(0, R)) \leq 0$. Strict inequality $\lambda_1(\tilde{L}_p, B(0, R)) < 0$ actually follows since if equality holds, \bar{w}_p would be the first eigenfunction of \tilde{L}_p on $B(0, R)$, so it must be strictly positive on $B(0, R)$, which contradicts to the fact that \bar{w}_p is 0 near $\partial B(0, R)$. By scaling (2.8), we prove the claim.

If there is another maximum point $x_p^2 \neq x_p^1$, we repeat the same procedure as before to obtain the ball $B(x_p^2, \varepsilon_p R)$ such that $\lambda_1(L_p, B(x_p^2, \varepsilon_p R)) < 0$.

We claim that $B(x_p^1, \varepsilon_p R)$ and $B(x_p^2, \varepsilon_p R)$ are disjoint for p large. Indeed, since $\tilde{u}_p \rightarrow U$ in $C_{loc}^2(\mathbb{R}^2)$, $U(y) = -2 \log(1 + |y|^2/8)$ is strictly concave and $y \cdot \nabla \tilde{u}_p(y) \rightarrow y \cdot \nabla U(y) < 0$ on $B(0, R) \setminus \{0\}$, we see that u_p is also strictly concave and $(x - x_p^1) \cdot \nabla u_p(x) < 0$ on $B(x_p^1, \varepsilon_p R) \setminus \{x_p^1\}$ for p large. The same property holds for u_p on $B(x_p^2, \varepsilon_p R) \setminus \{x_p^2\}$, and this concavity property means the claim.

Now, if there are N maximum points x_p^1, \dots, x_p^N of u_p , we have N open balls B^1, \dots, B^N , $B^j = B(x_p^j, \varepsilon_p R)$, which are disjoint, and

$$\lambda_1(L_p, B^j) < 0 \quad \text{for } j = 1, \dots, N. \quad (2.10)$$

By a variational characterization of N -th eigenvalue of L_p and the well-known argument as in a proof of Courant's Nodal Domain Theorem, we see that

$$\lambda_N(L_p, \Omega) \leq \sum_{j=1}^N \lambda_1(L_p, B^j). \quad (2.11)$$

From (2.10) and (2.11), we have $\lambda_N(L_p, \Omega) < 0$. On the other hand, the Morse index of u_p is less than or equal to m by assumption, we have $\lambda_{m+1}(L_p, \Omega) \geq 0$. Therefore we must have $N \leq m$, and we have proved Theorem 1. \square

3. Proof of Theorem 2.

Assume u_p is a solution of (1.1) of Morse index one with the property (1.2) as $p \rightarrow \infty$. By Theorem 1, we know there exists only one maximum point $x_p = x_p^1$ of u_p and on the ball $B = B(x_p, \varepsilon_p R) \subset \Omega$,

$$\lambda_1(L_p, B) < 0 \quad \text{and } (x - x_p) \cdot \nabla u_p(x) < 0 \quad (\forall x \in B \setminus \{x_p\}).$$

Now, we claim that $(x - x_p) \cdot \nabla u_p(x) < 0$ for all $x \in \Omega \setminus \overline{B}$. Indeed, assume there exists some $\bar{x} \in \Omega \setminus B$ such that $(\bar{x} - x_p) \cdot \nabla u_p(\bar{x}) \geq 0$. By the variational characterization of the second eigenvalue, we have

$$\lambda_2(L_p, \Omega) \leq \lambda_1(L_p, B) + \lambda_1(L_p, \Omega \setminus \overline{B}).$$

Since u_p is a solution of Morse index one, we have $\lambda_2(L_p, \Omega) \geq 0$, and thus $\lambda_1(L_p, \Omega \setminus \overline{B}) > 0$ for p large. On the other hand, by a scaling $\bar{x} = \varepsilon_p \bar{y} + x_p$ and $\nabla_{\bar{y}} = \varepsilon_p \nabla_x$, we have $\bar{y} \cdot \nabla \tilde{u}_p(\bar{y}) \geq 0$. By the convexity of Ω , the scaled domain

$\Omega_p = (\Omega - x_p)/\varepsilon_p$ is star-shaped with respect to 0. Hence the Hopf lemma implies that w_p in (2.9) satisfies $w_p(y) < 0$ for $y \in \partial\Omega_p$. Since $w_p(\bar{y}) > 0$, there would exist a connected component $C_p \subset \Omega_p \setminus \overline{B}(0, R)$, $C_p \cap \partial\Omega_p = \emptyset$ such that $w_p > 0$ on C_p . Then we would have that $\lambda_1(\tilde{L}_p, C_p) \leq 0$, and by (2.8), $\lambda_1(L_p, \varepsilon_p(C_p + \{x_p\})) \leq 0$. Note that $\varepsilon_p(C_p + \{x_p\}) \subset \Omega \setminus \overline{B}$. This contradicts to the monotonicity $0 < \lambda_1(L_p, \Omega \setminus \overline{B}) \leq \lambda_1(L_p, \varepsilon_p(C_p + \{x_p\}))$. Thus we have proved $(x - x_p) \cdot \nabla u_p(x) < 0$ for all $x \in \Omega \setminus \{x_p\}$. The rest of the statement in Theorem 2 is a simple consequence of this inequality. \square

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