CLASSIFICATION OF REAL BOTT MANIFOLDS

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Abstract. A real Bott manifold is the total space of a sequence of $\mathbb{R}P^1$ bundles starting with a point, where each $\mathbb{R}P^1$ bundle is the projectivization of a Whitney sum of two real line bundles. A real Bott manifold is a real toric manifold which admits a flat riemannian metric. An upper triangular $(0, 1)$ matrix with zero diagonal entries uniquely determines such a sequence of $\mathbb{R}P^1$ bundles but different matrices may produce diffeomorphic real Bott manifolds. In this paper we determine when two such matrices produce diffeomorphic real Bott manifolds. The argument also proves that any graded ring isomorphism between the cohomology rings of real Bott manifolds with $\mathbb{Z}/2$ coefficients is induced by an affine diffeomorphism between the real Bott manifolds. In particular, this implies the main theorem of [3] which asserts that two real Bott manifolds are diffeomorphic if and only if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings. We also prove that the decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors.

1. Introduction

A *real Bott tower* of height n, which is a real analogue of a Bott tower introduced in [2], is a sequence of $\mathbb{R}P^1$ bundles

$$M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} M_0 = \{\text{a point}\}$$

such that $M_j \to M_{j-1}$ for $j = 1, \dots, n$ is the projective bundle of the Whitney sum of a real line bundle L_{j-1} and the trivial real line bundle over M_{j-1}, and we call M_n a *real Bott manifold*. A real Bott manifold naturally supports an action of an elementary abelian 2-group and provides an example of a real toric manifold which admits a flat

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riemannian metric invariant under the action. Conversely, it is shown in [3] that a real toric manifold which admits a flat riemannian metric invariant under an action of an elementary abelian 2-group is a real Bott manifold.

Real line bundles are classified by their first Stiefel-Whitney classes as is well-known and $H^1(M_{j-1};\mathbb{Z}/2)$, where $\mathbb{Z}/2=\{0,1\}$, is isomorphic to $(\mathbb{Z}/2)^{j-1}$ through a canonical basis, so the line bundle L_{j-1} is determined by a vector A_j in $(\mathbb{Z}/2)^{j-1}$. We regard A_j as a column vector in $(\mathbb{Z}/2)^n$ by adding zero's and form an $n\times n$ matrix A by putting A_j as the j-th column. This gives a bijective correspondence between the set of real Bott towers of height n and the set $\mathfrak{B}(n)$ of $n\times n$ upper triangular $(0,1)$ matrices with zero diagonal entries. Because of this reason, we may denote the real Bott manifold M_n by $M(A)$.

Although $M(A)$ is determined by the matrix A, it happens that two different matrices in $\mathfrak{B}(n)$ produce (affinely) diffeomorphic real Bott manifolds. In this paper we introduce three operations on $\mathfrak{B}(n)$ and say that two elements in $\mathfrak{B}(n)$ are *Bott equivalent* if one is transformed to the other through a sequence of the three operations. Our first main result is the following.

Theorem 1.1. *The following are equivalent for A, B in $\mathfrak{B}(n)$:*

1. (1) *A and B are Bott equivalent.*
2. (2) *$M(A)$ and $M(B)$ are affinely diffeomorphic.*
3. (3) *$H^*(M(A);\mathbb{Z}/2)$ and $H^*(M(B);\mathbb{Z}/2)$ are isomorphic as graded rings.*

Moreover, any graded ring isomorphism from $H^*(M(A);\mathbb{Z}/2)$ to $H^*(M(B);\mathbb{Z}/2))$ is induced by an affine diffeomorphism from $M(B)$ to $M(A)$.

In particular, we obtain the following main theorem of [3].

Corollary 1.2 ([3])**.** *Two real Bott manifolds are diffeomorphic if and only if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings.*

It is asked in [3] whether Corollary 1.2 holds for any real toric manifolds but a counterexample is given in [5].

We say that a real Bott manifold is *indecomposable* if it is not diffeomorphic to a product of more than one real Bott manifolds. Using Corollary 1.2 together with an idea used to prove Theorem 1.1, we are able to prove our second main result.

Theorem 1.3. *The decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors.*

In particular, we have

Corollary 1.4 (Cancellation Property)**.** *Let M and M' be real Bott manifolds. If $S^1 \times M$ and $S^1 \times M'$ are diffeomorphic, then M and M' are diffeomorphic.*

It would be interesting to ask whether Theorem 1.3 and Corollary 1.4 hold for any real toric manifolds.

The author learned from Y. Kamishima that Corollary 1.4 can also be obtained from the method developed in [4] and [7] and that the cancellation property above fails to hold for general compact flat riemannian manifolds, see [1].

This paper is organized as follows. In Section 2 we describe $M(A)$ and its cohomology rings explicitly in terms of the matrix A. In Section 3 we introduce the three operations on $\mathfrak{B}(n)$. To each operation we associate an affine diffeomorphism between real Bott manifolds in Section 4, which implies the implication (1) $\Rightarrow$ (2) in Theorem 1.1. The implication (2) $\Rightarrow$ (3) is trivial. In Section 5 we prove the latter statement in Theorem 1.1. The argument also establishes the implication (3) $\Rightarrow$ (1). In the proof we introduce a notion of eigen-element and eigen-space in the first cohomology group of a real Bott manifold using the multiplicative structure of the cohomology ring and they play an important role on the analysis of isomorphisms between cohomology rings. Using this notion, we prove Theorem 1.3 in Section 6.

2. Real Bott manifolds and their cohomology rings

As mentioned in the Introduction, a real Bott manifold $M(A)$ of dimension n is associated to a matrix $A \in \mathfrak{B}(n)$. In this section we give an explicit description of $M(A)$ and its cohomology ring.

We set up some notation. Let S^1 denote the unit circle consisting of complex numbers with unit length. For elements $z \in S^1$ and $a \in \mathbb{Z}/2$ we use the following notation

$$z(a) := \begin{cases} z & \text{if } a = 0 \\ \bar{z} & \text{if } a = 1. \end{cases}$$

For a matrix A we denote by A^i_j the (i, j) entry of A and by A^i (resp. A_j) the i-th row (resp. j-th column) of A.

Now we take A from $\mathfrak{B}(n)$ and define involutions a_i's on $T^n := (S^1)^n$ by

$$a_i(z_1, \ldots, z_n) := (z_1, \ldots, z_{i-1}, -z_i, z_{i+1}(A^i_{i+1}), \ldots, z_n(A^i_n)) \tag{2.1}$$

for $i = 1, \ldots, n$. These involutions a_i's commute with each other and generate an elementary abelian 2-group of rank n, denoted by $G(A)$.

The action of $G(A)$ on T^n is free and the orbit space is the desired real Bott manifold $M(A)$.

$M(A)$ is a flat riemannian manifold. In fact, Euclidean motions s_i's ($i=1,\dots,n$) on $\mathbb{R}^n$ defined by

$$s_i(u_1,\dots,u_n) := (u_1,\dots,u_{i-1},u_i+\frac{1}{2},(-1)^{A^i_{i+1}}u_{i+1},\dots,(-1)^{A^i_n}u_n)$$

generate a crystallographic group $\Gamma(A)$, where the subgroup generated by $s_1^2,\dots,s_n^2$ consists of all translations by $\mathbb{Z}^n$, and the action of $\Gamma(A)$ on $\mathbb{R}^n$ is free and the orbit space $\mathbb{R}^n/\Gamma(A)$ agrees with $M(A)$ through an identification $\mathbb{R}/\mathbb{Z}$ with S^1 via an exponential map $u\to\exp(2\pi\sqrt{-1}u)$. $M(A)$ admits an action of an elementary abelian 2-group defined by $(u_1,\dots,u_n)\to(\pm u_1,\dots,\pm u_n)$ and this action preserves the flat riemannian metric on $M(A)$.

Let G_k ($k=1,\dots,n$) be a subgroup of $G(A)$ generated by $a_1,\dots,a_k$. Needless to say $G_n=G(A)$. Let $T^k:=(S^1)^k$ be a product of first k-factors in $T^n=(S^1)^n$. Then G_k acts on T^k by restricting the action of G_k on T^n to T^k and the orbit space T^k/G_k is a real Bott manifold of dimension k. Natural projections $T^k\to T^{k-1}$ for $k=1,\dots,n$ produce a real Bott tower

$$M(A)=T^n/G_n\to T^{n-1}/G_{n-1}\to\cdots\to T^1/G_1\to\{\text{a point}\}.$$

The graded ring structure of $H^*(M(A);\mathbb{Z}/2)$ can be described explicitly in terms of the matrix A. We shall recall it. For a homomorphism $\lambda\colon G(A)\to\mathbb{Z}_2=\{\pm1\}$ we denote by $\mathbb{R}(\lambda)$ the real one-dimensional $G(A)$-module associated with λ. Then the orbit space of $T^n\times\mathbb{R}(\lambda)$ by the diagonal action of $G(A)$, denoted by $L(\lambda)$, defines a real line bundle over $M(A)$ with the first projection. Let $\lambda_j\colon G(A)\to\mathbb{Z}_2$ ($j=1,\dots,n$) be a homomorphism sending a_i to -1 for $i=j$ and 1 for $i\neq j$, and we set

$$x_j=w_1(L(\lambda_j))$$

where w_1 denotes the first Stiefel-Whitney class.

Lemma 2.1 (see [3, Lemma 2.1] for example)**.** *As a graded ring*

$$H^*(M(A);\mathbb{Z}/2)=\mathbb{Z}/2[x_1,\dots,x_n]/(x_j^2=x_j\sum_{i=1}^n A^i_jx_i\mid j=1,\dots,n).$$

Let B be another element of $\mathfrak{B}(n)$. Since $M(A)=T^n/G(A)$ and $M(B)=T^n/G(B)$, an affine automorphism $\tilde f$ of T^n together with a group isomorphism $\phi\colon G(B)\to G(A)$ induces an affine diffeomorphism $f\colon M(B)\to M(A)$ if $\tilde f$ is ϕ-equivariant, i.e., $\tilde f(gz)=\phi(g)\tilde f(z)$ for $g\in G(B)$ and $z\in T^n$. Since the actions of $G(A)$ and $G(B)$ on T^n are free, the isomorphism ϕ will be uniquely determined by $\tilde{f}$ if it exists. We shall use b_i and y_j for $M(B)$ in place of a_i and x_j for $M(A)$.

Lemma 2.2. *If* $\phi(b_i) = \prod_{j=1}^n a_j^{F_j^i}$ *with* $F_j^i \in \mathbb{Z}/2$*, then* $f^*(x_j) = \sum_{i=1}^n F_j^i y_i$.

Proof. A map $T^n \times \mathbb{R}(\lambda \circ \phi) \to T^n \times \mathbb{R}(\lambda)$ sending (z, u) to $(\tilde{f}(z), u)$ induces a bundle map $L(\lambda \circ \phi) \to L(\lambda)$ covering $f\colon M(B) \to M(A)$. Since $(\lambda_j \circ \phi)(b_i) = F_j^i$, this implies the lemma. □

3. Three matrix operations

In this section we introduce three operations on matrices used in later sections to analyze when $M(A)$ and $M(B)$ (resp. $H^*(M(A); \mathbb{Z}/2)$ and $H^*(M(B); \mathbb{Z}/2)$) are diffeomorphic (resp. isomorphic) for $A, B \in \mathfrak{B}(n)$. In the following A will denote an element of $\mathfrak{B}(n)$.

1st operation (Op1). For a permutation matrix S of size n we define

$$\Phi_S(A) := SAS^{-1}.$$

To be more precise, there is a permutation σ on a set $\{1, \dots, n\}$ such that $S_j^i = 1$ if $i = \sigma(j)$ and $S_j^i = 0$ otherwise. We note that if we set $B = \Phi_S(A)$, then $SA = BS$ and

$$A_j^i = (SA)_j^{\sigma(i)} = (BS)_j^{\sigma(i)} = B_{\sigma(j)}^{\sigma(i)}. \tag{3.1}$$

$\Phi_S(A)$ may not be in $\mathfrak{B}(n)$ but we will perform the operation Φ_S on A only when $\Phi_S(A)$ stays in $\mathfrak{B}(n)$.

2nd operation (Op2). For $k \in \{1, \dots, n\}$ we define a square matrix $\Phi^k(A)$ of size n by

$$\Phi^k(A)_j := A_j + A_j^k A_k \quad \text{for } j = 1, \dots, n. \tag{3.2}$$

$\Phi^k(A)$ stays in $\mathfrak{B}(n)$ and since the diagonal entries of A are all zero and we are working over $\mathbb{Z}/2$, the composition $\Phi^k \circ \Phi^k$ is the identity; so Φ^k is bijective on $\mathfrak{B}(n)$.

3rd operation (Op3). Let I be a subset of $\{1, \dots, n\}$ such that $A_i = A_j$ for $i, j \in I$ and $A_i \neq A_j$ for $i \in I$ and $j \notin I$. Since the diagonal entries of A are all zero, the condition $A_i = A_j$ for $i, j \in I$ implies that $A_j^i = 0$ for $i, j \in I$. Let $C = (C_k^i)_{i,k \in I}$ with $C_k^i \in \mathbb{Z}/2$ be an invertible matrix of size $|I|$. Then we define a square matrix $\Phi_C^I(A)$ of size n by

$$\Phi_C^I(A)_j^i := \begin{cases} \sum_{k \in I} C_k^i A_j^k & (i \in I) \\ A_j^i & (i \notin I). \end{cases} \tag{3.3}$$

$\Phi^I_C(A)$ stays in $\mathfrak{B}(n)$ and since C is invertible, Φ^I_C is bijective on $\mathfrak{B}(n)$.

Definition. We say that two elements in $\mathfrak{B}(n)$ are *Bott equivalent* if one is transformed to the other through a sequence of the three operations (Op1), (Op2) and (Op3).

Example 3.1. $\mathfrak{B}(2)$ has two elements and they are not Bott equivalent. $\mathfrak{B}(3)$ has $2^3 = 8$ elements and they are classified into four Bott equivalence classes as follows:

(1) The zero matrix of size 3

(2) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(4) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\mathfrak{B}(4)$ has $2^6 = 64$ elements and one can check that it has twelve Bott equivalence classes, see [3] and [7]. Furthermore, $\mathfrak{B}(5)$ has $2^{10} = 1024$ elements and one can check that it has 54 Bott equivalence classes. The author learned from Admi Nazra that he classified real Bott manifolds of dimension 5 up to diffeomorhism from a different viewpoint (see [4], [7]) and found the 54 Bott equivalence classes in $\mathfrak{B}(5)$. The author does not know the number of Bott equivalence classes in $\mathfrak{B}(n)$ for $n \geq 6$ although it is in between $2^{(n-2)(n-3)/2}$ and $2^{n(n-1)/2}$ (see Example 3.3 below).

Example 3.2. Let $\mathfrak{B}_k(n)$ $(1 \leq k \leq n-1)$ be a subset of $\mathfrak{B}(n)$ such that $A \in \mathfrak{B}(n)$ is in $\mathfrak{B}_k(n)$ if and only if A has exactly k non-zero columns. There is only one Bott equivalence class in $\mathfrak{B}_1(n)$ and the corresponding real Bott manifold is the product of a Klein bottle and $(\mathbb{R}P^1)^{n-2}$. $\mathfrak{B}_2(3)$ has two Bott equivalence classes represented by

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

But $\mathfrak{B}_2(n)$ for $n \geq 4$ has four Bott equivalence classes; two of them are represented by $n \times n$ matrices with the above 3×3 matrices at the right-low corner and 0 in others, and the other two are represented by $n \times n$ matrices with the following 4×4 matrices at the right-low corner and 0 in others

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3.3. Let $\Delta(n)$ be a subset of $\mathfrak{B}(n)$ such that $A \in \mathfrak{B}(n)$ is in $\Delta(n)$ if and only if $A^i_{i+1} = 1$ for $i = 1, \dots, n-1$. Only the operation (Op2) is available on $\Delta(n)$ and one can change $(i, i+2)$ entry into 0 for $i = 1, \dots, n-2$ using the operation, so that A is Bott equivalent to a matrix $\bar{A}$ of this form

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \bar{A}^1_4 & \bar{A}^1_5 & \dots & \bar{A}^1_{n-1} & \bar{A}^1_n \\ 0 & 0 & 1 & 0 & \bar{A}^2_5 & \dots & \bar{A}^2_{n-1} & \bar{A}^2_n \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & \bar{A}^{n-4}_{n-1} & \bar{A}^{n-4}_n \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \bar{A}^{n-3}_n \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\bar{A}$ is uniquely determined by A and two elements $A, B \in \Delta(n)$ are Bott equivalent if and only if $\bar{A} = \bar{B}$. Therefore there are exactly $2^{(n-2)(n-3)/2}$ Bott equivalent classes in $\Delta(n)$ for $n \geq 2$.

Remark. As remarked above $\Phi_S(A)$ may not stay in $\mathfrak{B}(n)$. This awkwardness can be resolved if we consider the union of $\Phi_S(\mathfrak{B}(n))$ over all permutation matrices S. The three operations above preserve the union and are bijective on it. This union is a natural object. In fact, it is shown in [6, Lemma 3.3] that a square matrix A of size n with entries in $\mathbb{Z}/2$ lies in the union if and only if all principal minors of $A + E$ (even the determinant of $A + E$ itself) are one in $\mathbb{Z}/2$ where E denotes the identity matrix of size n.

4. Affine diffeomorphisms

In this section we associate an affine diffeomorphism between real Bott manifolds to each operation introduced in the previous section, and prove the implication (1) ⇒ (2) in Theorem 1.1, that is

Proposition 4.1. *If $A, B \in \mathfrak{B}(n)$ are Bott equivalent, then the associated real Bott manifolds $M(A)$ and $M(B)$ are affinely diffeomorphic.*

We set $B = \Phi_S(A), \Phi^k(A), \Phi^I_C(A)$ respectively for the three operations introduced in the previous section. In order to prove the proposition above, it suffices to find a group isomorphism $\phi\colon G(B) \to G(A)$ and a ϕ-equivariant affine automorphism $\tilde{f}$ of T^n which induces an affine diffeomorphism from $M(B)$ to $M(A)$.

The case of the operation (Op1). Let S and σ be as before. We define a group isomorphism $\phi_S\colon G(B) \to G(A)$ by

$$\phi_S(b_{\sigma(i)}) := a_i \tag{4.1}$$

and an affine automorphism $\tilde f_S$ of T^n by

$$\tilde f_S(z_1,\dots,z_n) := (z_{\sigma(1)},\dots,z_{\sigma(n)}).$$

Then it follows from (2.1) (applied to $b_{\sigma(i)}$) that the j-th component of $\tilde f_S(b_{\sigma(i)}(z))$ is $z_{\sigma(j)}(B^{\sigma(i)}_{\sigma(j)})$ for $j\neq i$ and $-z_{\sigma(i)}$ for $j=i$ while that of $a_i(\tilde f_S(z))$ is $z_{\sigma(j)}(A^i_j)$ for $j\neq i$ and $-z_{\sigma(i)}$ for $j=i$. Since $A^i_j = B^{\sigma(i)}_{\sigma(j)}$ by (3.1), this shows that $\tilde f_S$ is ϕ_S-equivariant.

It follows from Lemma 2.2 and (4.1) that the affine diffeomorphism $f_S\colon M(B)\to M(A)$ induced from $\tilde f_S$ satisfies

$$f_S^*(x_j) = y_{\sigma(j)} \quad \text{for } j=1,\dots,n. \tag{4.2}$$

The case of the operation (Op2). We define a group isomorphism $\phi^k\colon G(B)\to G(A)$ by

$$\phi^k(b_i) := a_i a_k^{A^i_k} \tag{4.3}$$

and an affine automorphism $\tilde f^k$ of T^n by

$$\tilde f^k(z_1,\dots,z_n) := (z_1,\dots,z_{k-1},\sqrt{-1}z_k,z_{k+1},\dots,z_n).$$

We shall check that $\tilde f^k$ is ϕ^k-equivariant, i.e.,

$$\tilde f^k(b_i(z)) = a_i a_k^{A^i_k}(\tilde f^k(z)). \tag{4.4}$$

The identity is obvious when $i=k$ because $A^k_k=0$ and $B^k_j=A^k_j$ for any j by (3.2). Suppose $i\neq k$. Then the j-th component of the left hand side of (4.4) is given by

$$\begin{cases} z_j(B^i_j) & \text{for } j\neq i,k,\\ -z_i & \text{for } j=i,\\ \sqrt{-1}(z_k(B^i_k)) & \text{for } j=k,\end{cases}$$

while that of the right hand side of (4.4) is given by

$$\begin{cases} z_j(A^i_j + A^k_j A^i_k) & \text{for } j\neq i,k,\\ -z_i(A^k_i A^i_k) & \text{for } j=i,\\ (-1)^{A^i_k}(\sqrt{-1}z_k)(A^i_k) & \text{for } j=k.\end{cases}$$

Since $B^i_j = A^i_j + A^k_j A^i_k$ by (3.2), the j-th components above agree for $j\neq i,k$. They also agree for $j=i$ because either A^k_i or A^i_k is zero. We note that $B^i_k = A^i_k$ by (3.2), and the k-th components above are both $\sqrt{-1}z_k$ when $B^i_k=A^i_k=0$ and $\sqrt{-1}\bar z_k$ when $B^i_k=A^i_k=1$. Thus the j-th components above agree for any j.

Since $A^i_k = B^i_k$ for any i, it follows from Lemma 2.2 and (4.3) that the affine diffeomorphism $f^k\colon M(B) \to M(A)$ induced from $\tilde f^k$ satisfies

$$(f^k)^*(x_j) = y_j \quad \text{for } j \neq k, \qquad (f^k)^*(x_k) = y_k + \sum_{i=1}^n B^i_k y_i. \tag{4.5}$$

The case of the operation (Op3). The homomorphism $\mathrm{GL}(m;\mathbb{Z}) \to \mathrm{GL}(m;\mathbb{Z}/2)$ induced from the surjective homomorphism $\mathbb{Z} \to \mathbb{Z}/2$ is known (and easily proved) to be surjective. We take a lift of the matrix $C = (C^i_k)_{i,k\in I}$ to $\mathrm{GL}(|I|,\mathbb{Z})$ and denote the lift by $\tilde C$. Then we define a group isomorphism $\phi^I_C\colon G(B) \to G(A)$ by

$$\phi^I_C(b_i) := \begin{cases} \prod_{k\in I} a_k^{C^i_k} & \text{for } i \in I, \\ a_i & \text{for } i \notin I, \end{cases} \tag{4.6}$$

and the j-th component of an affine automorphism $\tilde f^I_{\tilde C}$ of T^n by

$$\tilde f^I_{\tilde C}(z)_j := \begin{cases} \prod_{\ell\in I} z_\ell^{\tilde C^\ell_j} & \text{for } j \in I, \\ z_j & \text{for } j \notin I. \end{cases} \tag{4.7}$$

We shall check that $\tilde f^I_{\tilde C}$ is ϕ^I_C-equivariant. To simplify notation we abbreviate $\tilde f^I_{\tilde C}$ and ϕ^I_C as $\tilde f$ and ϕ respectively. What we prove is the identity

$$\tilde f(b_i(z))_j = \phi(b_i)\tilde f(z)_j. \tag{4.8}$$

We distinguish four cases.

Case 1. The case where $i, j \in I$. As remarked in the definition of (Op3), $A^k_\ell = 0$ whenever $k, \ell \in I$, so $B^i_\ell = 0$ for any $\ell \in I$ by (3.3). It follows from (4.6) and (4.7) that

$$\tilde f(b_i(z))_j = (-z_i)^{\tilde C^i_j} \prod_{\ell\in I,\ell\neq i} z_\ell (B^i_\ell)^{\tilde C^\ell_j} = (-1)^{C^i_j} \prod_{\ell\in I} z_\ell^{\tilde C^\ell_j}$$

while

$$\phi(b_i)\tilde f(z)_j = (\prod_{k\in I} a_k^{C^i_k})\tilde f(z)_j = (-1)^{C^i_j} \prod_{\ell\in I} z_\ell^{\tilde C^\ell_j}.$$

Case 2. The case where $i \in I$ but $j \notin I$. In this case we have

$$\tilde f(b_i(z))_j = z_j(B^i_j)$$

while

$$\phi(b_i)\tilde f(z)_j = z_j(\sum_{k\in I} C^i_k A^k_j) = z_j(B^i_j)$$

where the last identity follows from (3.3).

Case 3. The case where $i \notin I$ but $j \in I$. In this case we have

$$\tilde{f}(b_i(z))_j = \prod_{\ell \in I} z_\ell (B^i_\ell)^{\tilde{C}^\ell_j}$$

while

$$\phi(b_i)\tilde{f}(z)_j = (\prod_{\ell \in I} z_\ell^{\tilde{C}^\ell_j})(A^i_j) = \prod_{\ell \in I} z_\ell (A^i_j)^{\tilde{C}^\ell_j}.$$

Since $B^i_\ell = A^i_\ell$ for $i \notin I$ by (3.3), the above verifies (4.8).

Case 4. The case where $i, j \notin I$. In this case

$$\tilde{f}(b_i(z))_j = z_j(B^i_j)$$

while

$$\phi(b_i)\tilde{f}(z)_j = z_j(A^i_j).$$

Since $B^i_j = A^i_j$ for $i \notin I$ by (3.3), the above verifies (4.8).

It follows from Lemma 2.2 and (4.6) that the affine diffeomorphism $f^I_C \colon M(B) \to M(A)$ induced from $\tilde{f}^I_C$ satisfies

$$(f^I_C)^*(x_j) = \begin{cases} \sum_{i \in I} C^i_j y_i & \text{for } j \in I, \\ y_j & \text{for } j \notin I. \end{cases} \tag{4.9}$$

5. Cohomology isomorphisms

In this section we prove the latter statement in Theorem 1.1 and the implication (3) $\Rightarrow$ (1) at the same time, i.e. the purpose of this section is to prove the following.

Proposition 5.1. *Any isomorphism $H^*(M(A);\mathbb{Z}/2) \to H^*(M(B);\mathbb{Z}/2)$ is induced from a composition of affine diffeomorphisms corresponding to the three operations (Op1), (Op2) and (Op3), and if $H^*(M(A);\mathbb{Z}/2)$ and $H^*(M(B);\mathbb{Z}/2)$ are isomorphic as graded rings, then A and B are Bott equivalent.*

We introduce a notion and prepare a lemma. Remember that
(5.1)

$$H^*(M(A);\mathbb{Z}/2) = \mathbb{Z}/2[x_1,\dots,x_n]/(x_j^2 = x_j \sum_{i=1}^n A^i_j x_i \mid j = 1,\dots,n).$$

One easily sees that products $x_{i_1}\dots x_{i_q}$ $(1 \le i_1 < \dots < i_q \le n)$ form a basis of $H^q(M(A);\mathbb{Z}/2)$ as a vector space over $\mathbb{Z}/2$ so that the dimension of $H^q(M(A);\mathbb{Z}/2)$ is $\binom{n}{q}$ (see [6, Lemma 5.3]).

We set

$$\alpha_j = \sum_{i=1}^n A^i_j x_i \quad \text{for } j = 1,\dots,n$$

where $\alpha_1 = 0$ since A is an upper triangular matrix with zero diagonal entries. Then the relations in (5.1) are written as

$$x_j^2 = \alpha_j x_j \quad \text{for } j = 1, \dots, n. \tag{5.2}$$

Motivated by this identity we introduce the following notion.

Definition. We call an element $\alpha \in H^1(M(A); \mathbb{Z}/2)$ an *eigen-element* of $H^*(M(A); \mathbb{Z}/2)$ if there exists $x \in H^1(M(A); \mathbb{Z}/2)$ such that $x^2 = \alpha x$, $x \neq 0$ and $x \neq \alpha$. The set of all elements $x \in H^1(M(A); \mathbb{Z}/2)$ satisfying the equation $x^2 = \alpha x$ is a vector subspace of $H^1(M(A); \mathbb{Z}/2)$ which we call the *eigen-space* of α and denote by $\mathcal{E}_A(\alpha)$. We also introduce a notation $\bar{\mathcal{E}}_A(\alpha)$ which is the quotient of $\mathcal{E}_A(\alpha)$ by the subspace spanned by α, and call it the *reduced eigen-space* of α.

Eigen-elements and (reduced) eigen-spaces are invariants preserved under graded ring isomorphisms. By (5.2) α_j's are eigen-elements of $H^*(M(A); \mathbb{Z}/2)$ and the following lemma shows that these are the only eigen-elements.

Lemma 5.2. *If α is an eigen-element of $H^*(M(A); \mathbb{Z}/2)$, then $\alpha = \alpha_j$ for some j and the eigen-space $\mathcal{E}_A(\alpha)$ of α is generated by α and x_i's with $\alpha_i = \alpha$.*

Proof. By the definition of eigen-element there exists a non-zero element $x \in H^1(M(A); \mathbb{Z}/2)$ different from α such that $x^2 = \alpha x$. Since both x and $x + \alpha$ are non-zero, there exist i and j such that $x = x_i + p_i$ and $x + \alpha = x_j + q_j$ where p_i is a linear combination of $x_1, \dots, x_{i-1}$ and q_j is a linear combination of $x_1, \dots, x_{j-1}$. Then

$$x_i x_j + x_i q_j + x_j p_i + p_i q_j = 0$$

because $x(x + \alpha) = 0$. As remarked above, products $x_{i_1} x_{i_2}$ $(1 \leq i_1 < i_2 \leq n)$ form a basis of $H^2(M(A); \mathbb{Z}/2)$, so i must be equal to j for the identity above to hold. Then as $x_j^2 = x_j \alpha_j$, it follows from the identity above that $\alpha_j = q_j + p_i$ (and $p_i q_j = 0$). This implies that $\alpha = \alpha_j$, proving the former statement of the lemma.

We express a non-zero element $x \in \mathcal{E}_A(\alpha)$ as $\sum_{i=1}^n c_i x_i$ $(c_i \in \mathbb{Z}/2)$ and let m be the maximum number among i's with $c_i \neq 0$.

Case 1. The case where x_m appears when we express α as a linear combination of $x_1, \dots, x_n$. We express $x(x+\alpha)$ as a linear combination of the basis elements $x_{i_1} x_{i_2}$ $(1 \leq i_1 < i_2 \leq n)$. Since x_m appears in both x and α, it does not appear in $x + \alpha$. Therefore the term in $x(x + \alpha)$ which contains x_m is $x_m(x + \alpha)$ and it must vanish because $x(x + \alpha) = 0$. Therefore $x = \alpha$.

Case 2. The case where x_m does not appear in the linear expression of α. In this case, the term in $x(x + \alpha)$ which contains x_m is $x_m(x_m +$

$\alpha) = x_m(\alpha_m + \alpha)$ since $x_m^2 = \alpha_m x_m$, and it must vanish because $x(x + \alpha) = 0$. Therefore $\alpha_m = \alpha$. The sum $x + x_m$ is again an element of $\mathcal{E}_A(\alpha)$. If $x \neq x_m$ (equivalently $x + x_m$ is non-zero), then the same argument applied to $x + x_m$ shows that there exists $m_1(\neq m)$ such that $\alpha_{m_1} = \alpha$ and $x + x_m + x_{m_1}$ is again an element of $\mathcal{E}_A(\alpha)$. Repeating this argument, x ends up with a linear combination of x_i's with $\alpha_i = \alpha$. □

With this preparation we shall prove Proposition 5.1.

Proof of Proposition 5.1. Let B be another element of $\mathfrak{B}(n)$. We denote the canonical basis of $H^*(M(B); \mathbb{Z}/2)$ by $y_1, \dots, y_n$ and the elements in $H^1(M(B); \mathbb{Z}/2)$ corresponding to α_j's by β_j's, i.e., $\beta_j = \sum_{i=1} B^i_j y_i$ for $j = 1, \dots, n$.

Let $\varphi\colon H^*(M(A); \mathbb{Z}/2) \to H^*(M(B); \mathbb{Z}/2)$ be a graded ring isomorphism. It preserves the eigen-elements and (reduced) eigen-spaces. In the following we shall show that we can change φ into the identity map by composing isomorphisms induced from affine diffeomorphisms corresponding to the three operations (Op1), (Op2) and (Op3).

Through the operation (Op1) we may assume that $\varphi(\alpha_j) = \beta_j$ for any j because of (4.2). Then φ restricts to an isomorphism $\mathcal{E}_A(\alpha_j) \to \mathcal{E}_B(\beta_j)$ between eigen-spaces and induces an isomorphism $\bar{\mathcal{E}}_A(\alpha_j) \to \bar{\mathcal{E}}_B(\beta_j)$ between reduced eigen-spaces.

Let α (resp. β) stand for α_j (resp. β_j) and suppose that $\varphi(\alpha) = \beta$. Let I be a subset of $\{1, \dots, n\}$ such that $\alpha_i = \alpha$ if and only if $i \in I$. We denote the image of x_i (resp. y_i) in $\bar{\mathcal{E}}_A(\alpha)$ (resp. $\bar{\mathcal{E}}_B(\beta)$) by $\bar{x}_i$ (resp. $\bar{y}_i$). The $\bar{x}_i$'s (resp. $\bar{y}_i$'s) for $i \in I$ form a basis of $\bar{\mathcal{E}}_A(\alpha)$ (resp. $\bar{\mathcal{E}}_B(\beta)$) by Lemma 5.2, so if we express $\varphi(\bar{x}_j) = \sum_{i \in I} C^i_j \bar{y}_i$ with $C^i_j \in \mathbb{Z}/2$, then the matrix $C = (C^i_j)_{i,j \in I}$ is invertible. Therefore, through the operation (Op3), we may assume that C is the identity matrix because of (4.9). This means that we may assume that $\varphi(x_j) = y_j$ or $y_j + \beta_j$ for each $j = 1, \dots, n$. Finally through the operation (Op2), we may assume that $\varphi(x_j) = y_j$ for any j because of (4.5) and hence $A = B$ (and φ is the identity) because $\varphi(\alpha_j) = \beta_j$, $\alpha_j = \sum_{i=1}^n A^i_j x_i$ and $\beta_j = \sum_{i=1}^n B^i_j y_i$ for any j, proving the proposition. □

6. Unique decomposition of real Bott manifolds

We say that a real Bott manifold is *indecomposable* if it is not diffeomorphic to a product of more than one real Bott manifolds. The purpose of this section is to prove Theorem 1.3 in the Introduction, that is

Theorem 6.1. *The decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations* *of the indecomposable factors. Namely, if* $\prod_{i=1}^{k} M_i$ *is diffeomorphic to* $\prod_{j=1}^{\ell} N_j$ *where* M_i *and* N_j *are indecomposable real Bott manifolds, then* $k = \ell$ *and there is a permutation* σ *on* $\{1, \dots, k = \ell\}$ *such that* M_i *is diffeomorphic to* $N_{\sigma(i)}$ *for* $i = 1, \dots, k$.

$H^*(\prod_{i=1}^{k} M_i; \mathbb{Z}/2) = \bigotimes_{i=1}^{k} H^*(M_i; \mathbb{Z}/2)$ by Künneth formula and the diffeomorphism types of real Bott manifolds are detected by cohomology rings with $\mathbb{Z}/2$ coefficient by Corollary 1.2, so the theorem above reduces to a problem on the decomposition of a cohomology ring into tensor products over $\mathbb{Z}/2$.

We call a graded ring over $\mathbb{Z}/2$ a *Bott ring* of rank n if it is isomorphic to the cohomology ring with $\mathbb{Z}/2$ coefficient of a real Bott manifold of dimension n. Let $\mathcal{H}$ be a Bott ring of rank n, so it has an expression

$$\mathcal{H} = \mathbb{Z}/2[x_1, \dots, x_n]/(x_j^2 = x_j \sum_{i=1}^{n} A_j^i x_i \mid j = 1, \dots, n) \tag{6.1}$$

with $A \in \mathfrak{B}(n)$. The eigen-elements of $\mathcal{H}$ are

$$\alpha_j = \sum_{i=1}^{n} A_j^i x_i \quad (j = 1, \dots, n). \tag{6.2}$$

We denote by $\mathcal{H}^q$ the degree q part of $\mathcal{H}$ and define

$N(\mathcal{H}) := \{x \in \mathcal{H}^1 \mid x^2 = 0\}$, and

$S(\mathcal{H}) := \{x \in \mathcal{H}^1 \backslash \{0\} \mid \exists \overline{x} \in \mathcal{H}^1 \backslash \{0\} \text{ with } x\overline{x} = 0 \text{ and } \overline{x} \neq x\}$.

In terms of eigen-elements and eigen-spaces, $N(\mathcal{H})$ is the eigen-space of the zero eigen-element. Also, if we write $\overline{x} = x + \alpha$ with $\alpha \in \mathcal{H}^1$, then $x\overline{x} = 0$ means that $x^2 = \alpha x$; so $S(\mathcal{H})$ with the zero element added is the union of eigen-spaces of all non-zero eigen-elements in $\mathcal{H}$. The latter statement in Lemma 5.2 shows that the eigen-element α is uniquely determined by x, hence so is $\overline{x}$.

$N(\mathcal{H}) = \mathcal{H}^1$ if and only if A in (6.1) is the zero matrix. Unless $N(\mathcal{H}) = \mathcal{H}^1$, $S(\mathcal{H}) \neq \emptyset$.

Lemma 6.2. *The graded subring* $\mathcal{H}_S$ *of a Bott ring* $\mathcal{H}$ *generated by* $S(\mathcal{H})$ *is a Bott ring.*

Proof. The isomorphism class of $\mathcal{H}$ does not change through the three operations (Op1), (Op2) and (Op3). Through (Op1) we may assume that the first ℓ columns of the matrix A in (6.1) are all zero but none of the remaining columns is zero. If the maximum number of linearly independent vectors in the first ℓ rows of A is m, then we may assume that the first $\ell - m$ rows are zero by applying the operation (Op3) to the first ℓ columns. Then $\mathcal{H}_S$ is the Bott ring associated with the $(n-\ell+m)\times(n-\ell+m)$ submatrix of A at the right-low corner of A. $\square$

Let $\mathcal{H}_S$ be as in Lemma 6.2 and let V be a subspace of $N(\mathcal{H})$ complementary to $N(\mathcal{H})\cap\mathcal{H}_S^1$. The dimension of V is $\ell-m$ in the proof of Lemma 6.2. The graded subalgebra of $\mathcal{H}$ generated by V is an exterior algebra $\Lambda(V)$, so

$$\mathcal{H}=\Lambda(V)\otimes\mathcal{H}_S. \tag{6.3}$$

We say that a Bott ring $\mathcal{H}$ is *semisimple* if $\mathcal{H}$ is generated by $S(\mathcal{H})$. Clearly $\mathcal{H}_S$ is semisimple and $\mathcal{H}$ is semisimple if and only if $\mathcal{H}=\mathcal{H}_S$.

Lemma 6.3. *Let $\mathcal{H}$ be a Bott ring. If $\mathcal{H}=\bigotimes_{i=1}^r\mathcal{H}_i$ with Bott subrings $\mathcal{H}_i$'s of $\mathcal{H}$, then $S(\mathcal{H})=\coprod_{i=1}^r S(\mathcal{H}_i)$. Therefore $\mathcal{H}$ is semisimple if and only if all $\mathcal{H}_i$'s are semisimple.*

Proof. Let $x\in S(\mathcal{H})$ and write $x=\sum_{i=1}^r y_i$ and $\overline{x}=\sum_{i=1}^r z_i$ with $y_i,z_i\in\mathcal{H}_i$. Since $x\overline{x}=0$, we have

$$y_iz_j+y_jz_i=0 \text{ for all } i\neq j.$$

Suppose that $y_i\neq 0$ and $z_j\neq 0$ for some $i\neq j$. Then $y_i=z_i$ and $y_j=z_j$ to satisfy the equations above. This shows that $x=\overline{x}$, which contradicts the fact that $x\in S(\mathcal{H})$. Therefore $x=y_i$ and $\overline{x}=z_i$ for some i, proving the lemma. $\square$

Recall that a Bott ring $\mathcal{H}$ has a decomposition $\Lambda(V)\otimes\mathcal{H}_S$ in (6.3).

Corollary 6.4. *If $\mathcal{H}$ has another decomposition $\Lambda(U)\otimes\mathcal{S}$ where U is a subspace of $N(\mathcal{H})$ and $\mathcal{S}$ is a semisimple subring of $\mathcal{H}$, then $\dim U=\dim V$ and $\mathcal{S}=\mathcal{H}_S$.*

Proof. Since both $S(\Lambda(U))$ abd $S(\Lambda(V))$ are empty, $S(\mathcal{H})=S(\mathcal{S})$ by Lemma 6.3 and this implies the corollary. $\square$

Lemma 6.5. *Let $\mathcal{H}=\bigotimes_{i=1}^r\mathcal{H}_i$ be as in Lemma 6.3 and $\pi_i\colon\mathcal{H}\to\mathcal{H}_i$ be the projection. Let $\mathcal{L}$ be a semisimple Bott ring and let $\psi\colon\mathcal{L}\to\mathcal{H}$ be a graded ring monomorphism. If the composition $\pi_i\circ\psi\colon\mathcal{L}\to\mathcal{H}_i$ is an isomorphism for some i, then $\psi(\mathcal{L})=\mathcal{H}_i$.*

Proof. Let $y\in S(\mathcal{L})$. Then $\psi(y)\in S(\mathcal{H})$ because ψ is a graded ring monomorphism, and it is actually in $S(\mathcal{H}_i)$ by Lemma 6.3 since $(\pi_i\circ\psi)(y)\neq 0$. This shows that $\psi(S(\mathcal{L}))\subset S(\mathcal{H}_i)$ but since $\pi_i\circ\psi$ is an isomorphism, the inclusion should be the equality. Therefore $\psi(\mathcal{L})=\mathcal{H}_i$ because $\mathcal{L}$ and $\mathcal{H}_i$ are both semisimple. $\square$

We say that a semisimple Bott ring is *simple* if it is not isomorphic to the tensor product (over $\mathbb{Z}/2$) of more than one semisimple Bott rings, in other words, a simple Bott ring is a Bott ring isomorphic to the cohomology ring (with $\mathbb{Z}/2$ coefficient) of an indecomposable real Bott manifold different from S^1. A Bott ring isomorphic to the cohomology ring of the Klein bottle with $\mathbb{Z}/2$ coefficient is simple and we call it especially a *Klein ring.* If an element $x \in S(\mathcal{H})$ satisfies $(x+\overline{x})^2 = 0$, then the subring generated by x and $\overline{x}$ is a Klein ring and we call such a pair $\{x, \overline{x}\}$ a *Klein pair.* We note that x and $\overline{x}$ have the same eigen-element and $\{x, \overline{x}\}$ is a Klein pair if and only if the eigen-element of x and $\overline{x}$, that is $x + \overline{x}$, lies in $N(\mathcal{H})$.

Lemma 6.6. *If $S(\mathcal{H}) \neq \emptyset$, then a Klein pair exists in $\mathcal{H}$ and the quotient of $\mathcal{H}$ by the ideal generated by a Klein pair is again a Bott ring.*

Proof. Let $\mathcal{H}$ be of the form (6.1). The assumption $S(\mathcal{H}) \neq \emptyset$ is equivalent to A being non-zero as remarked before. As in the proof of Lemm 6.2, we may assume through the operation (Op1) that the first ℓ columns of A are zero and none of the remaining columns is zero. Then $x_1, \dots, x_\ell$ are elements of $N(\mathcal{H})$ and the eigen-element $\alpha_{\ell+1}$ of $x_{\ell+1}$ is a linear combination of $x_1, \dots, x_\ell$, so $\alpha_{\ell+1}$ lies in $N(\mathcal{H})$ which means that $\{x_{\ell+1}, \overline{x}_{\ell+1}\}$ is a Klein pair.

If $\{x, \overline{x}\}$ is a Klein pair, then the eigen-element of x is non-zero and belongs to $N(\mathcal{H})$, so through the operation (Op1) we may assume that it is $\alpha_{\ell+1}$. Then, applying the operation (Op3) to the eigen-space of $\alpha_{\ell+1}$, we may assume $x = x_{\ell+1}$. We further may assume $\alpha_{\ell+1} = x_\ell$ by applying the operation (Op3) to $N(\mathcal{H})$. The quotient ring of $\mathcal{H}$ by the ideal generated by the Klein pair $\{x, \overline{x}\}$ is then nothing but to take $x_\ell = x_{\ell+1} = 0$ in $\mathcal{H}$, so it is a Bott ring associated with a $(n-2) \times (n-2)$ matrix obtained from A by deleting ℓ-th and $\ell+1$-st columns and rows. ☐

Now we are in a position to prove the unique decomposition of a semisimple Bott ring into a tensor product of simple Bott rings.

Proposition 6.7. *Let $\mathcal{A}_i$ $(i = 1, \dots, p)$ and $\mathcal{B}_j$ $(j = 1, \dots, q)$ be simple Bott rings. If there exists a graded ring isomorphism*

$$\varphi\colon \bigotimes_{i=1}^{p} \mathcal{A}_i \to \bigotimes_{j=1}^{q} \mathcal{B}_j, \tag{6.4}$$

then $p = q$ and φ preserves the factors, i.e. there is a permutation ρ on $\{1, \dots, p = q\}$ such that $\varphi(\mathcal{A}_i) = \mathcal{B}_{\rho(i)}$ for $i = 1, \dots, p$.

Proof. We set $\mathcal{A} = \bigotimes_{i=1}^{p} \mathcal{A}_i$ and $\mathcal{B} = \bigotimes_{j=1}^{q} \mathcal{B}_j$. If either $\mathcal{A}$ or $\mathcal{B}$ is simple (i.e. $p = 1$ or $q = 1$), then both of them must be simple and the proposition is trivial. In the sequel we will assume that both $\mathcal{A}$ and $\mathcal{B}$ are not simple (so that $p \geq 2$ and $q \geq 2$), and prove the proposition by induction on the rank of $\mathcal{A}$, that is, $\dim \mathcal{A}^1$.

If $\varphi(\mathcal{A}_i) = \mathcal{B}_j$ for some i and j, say $\varphi(\mathcal{A}_p) = \mathcal{B}_q$, then we factorize them so that φ induces an isomorphism $\bar{\varphi}\colon \bigotimes_{i=1}^{p-1} \mathcal{A}_i \to \bigotimes_{j=1}^{q-1} \mathcal{B}_j$. By the induction assumption, we conclude $p = q$ and may assume that $\bar{\varphi}(\mathcal{A}_i) = \mathcal{B}_i$ for $i = 1, \dots, p-1$ if necessary by permuting the suffixes of $\mathcal{B}_j$'s. Then it follows from Lemma 6.5 that $\varphi(\mathcal{A}_i) = \mathcal{B}_i$ for $i = 1, \dots, p-1$. This together with $\varphi(\mathcal{A}_p) = \mathcal{B}_q$ where $p = q$ proves the statement in the lemma. In the sequel, it suffices to show that $\varphi(\mathcal{A}_i) = \mathcal{B}_j$ for some i and j when we have an isomorphism φ in the proposition.

Case 1. The case where some $\mathcal{A}_i$ or $\mathcal{B}_j$ is a Klein ring. We may assume that $\mathcal{A}_p$ is a Klein ring without loss of generality. Let $\{x, \overline{x}\}$ be a Klein pair in $\mathcal{A}_p$. Its image by φ sits in some $\mathcal{B}_j$ by Lemma 6.3 and we may assume that it sits in $\mathcal{B}_q$. If $\mathcal{B}_q$ is also a Klein ring, then $\varphi(\mathcal{A}_p) = \mathcal{B}_q$. Therefore we may assume that $\mathcal{B}_q$ is not a Klein ring in the following.

Our isomorphism φ induces an isomorphism

$$\bar{\varphi}\colon \mathcal{A}/(x, \overline{x}) = \bigotimes_{i=1}^{p-1} \mathcal{A}_i \cong \bigotimes_{j=1}^{q-1} \mathcal{B}_j \otimes (\mathcal{B}_q/(\varphi(x), \varphi(\overline{x})))$$

where (u, v) denotes the ideal generated by the elements u and v and $\mathcal{B}_q/(\varphi(x), \varphi(\overline{x}))$ is a Bott ring by Lemma 6.6. Since $\operatorname{rank}(\mathcal{A}/(x, \overline{x})) = \operatorname{rank} \mathcal{A} - 2$, it follows from the induction assumption that $p-1 \geq q$ and we may assume that $\bar{\varphi}(\mathcal{A}_i) = \mathcal{B}_i$ for $i = 1, \dots, q-1$ and $\bar{\varphi}(\otimes_{i=q}^{p-1} \mathcal{A}_i) = \mathcal{B}_q/(\varphi(x), \varphi(\overline{x}))$, in particular, $\bar{\varphi}(\mathcal{A}_1) = \mathcal{B}_1$ as $q \geq 2$. Then, it follows from Lemma 6.5 that $\varphi(\mathcal{A}_1) = \mathcal{B}_1$.

Case 2. The case where none of $\mathcal{A}_i$'s and $\mathcal{B}_j$'s is a Klein ring. Let $\{x, \overline{x}\}$ be a Klein pair of $\mathcal{A}_p$ and we may assume that its image by φ sits in $\mathcal{B}_q$ as before. Then φ induces an isomorphism

$$\bar{\varphi}\colon \mathcal{A}/(x, \overline{x}) = \bigotimes_{i=1}^{p-1} \mathcal{A}_i \otimes (\mathcal{A}_p/(x, \overline{x})) \to \bigotimes_{j=1}^{q-1} \mathcal{B}_j \otimes (\mathcal{B}_q/(\varphi(x), \varphi(\overline{x}))),$$

where the quotients $\mathcal{A}_p/(x, \overline{x})$ and $\mathcal{B}_q/(\varphi(x), \varphi(\overline{x}))$ are both Bott rings by Lemma 6.6. The induction assumption can be applied to this situation as before. If $\bar{\varphi}(\mathcal{A}_i) = \mathcal{B}_j$ for some $1 \leq i \leq p-1$ and $1 \leq j \leq q-1$, then $\varphi(\mathcal{A}_i) = \mathcal{B}_j$ by Lemma 6.5. If $\bar{\varphi}(\bigotimes_{i=1}^{p-1} \mathcal{A}_i) = \mathcal{B}_q/(\varphi(x), \varphi(\overline{x}))$ and

$\bar{\varphi}(\mathcal{A}_p/(x,\overline{x})) = \bigotimes_{j=1}^{q-1}\mathcal{B}_j$, then φ restricts to an isomorphism

$$(\bigotimes_{i=1}^{p-1}\mathcal{A}_i)\otimes\langle x,\overline{x}\rangle\to\mathcal{B}_q$$

where $\langle x,\overline{x}\rangle$ denotes the Klein ring generated by x and $\overline{x}$, and this contradicts the fact that $\mathcal{B}_q$ is simple as $p\geq 2$. □

Now Theorem 6.1 follows from Corollaries 1.2, 6.4 and Proposition 6.7.

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