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Nondegeneracy of least energy solutions for an elliptic problem with nearly critical nonlinearity

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Abstract

Consider the problem $-\Delta u = c_0 K(x) u^{p_\varepsilon}$, $u > 0$ in Ω , $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 3$), $c_0 = N(N-2)$, $p_\varepsilon = (N+2)/(N-2)-\varepsilon$ and K is a smooth positive function on $\bar{\Omega}$.

We prove that least energy solutions of the above problem are non-degenerate for $\varepsilon > 0$ small, under some assumptions on the coefficient function K . This is a generalization of the recent result by Grossi [6] for $K \equiv 1$, and needs precise estimates and a new argument.

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Key words: nondegeneracy, critical exponent, least energy solution

1 Introduction

We consider the problem

$$(P_{\varepsilon,K}) \begin{cases} -\Delta u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a smooth bounded domain, $c_0 = N(N-2)$, $p_\varepsilon = p - \varepsilon$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small parameter. Here, K is a positive function in $C^2(\overline{\Omega})$.

Since $(P_{\varepsilon,K})$ is a subcritical problem, there exists a least energy solution u_ε such that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega K(x) |u_\varepsilon|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega K(x) |u|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} =: S_{\varepsilon,K}$$

for any small $\varepsilon > 0$.

In the following, we put an assumption on the coefficient function K .

(K): $K \in C^2(\overline{\Omega})$, $0 < K(x) \leq 1$, $K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and x_0 is a nondegenerate critical point of K .

It is easy to see that

$$S_{\varepsilon,K} \rightarrow \left(\max_{\overline{\Omega}} K\right)^{-(N-2)/N} S_N = S_N$$

as $\varepsilon \rightarrow 0$, where S_N is the best Sobolev constant with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. Since S_N is never achieved on bounded domains, we have $\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. It is also known that the maximum point x_ε of u_ε converges to a maximum point of K in $\overline{\Omega}$. Thus under the assumption (K), we have $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$ where x_0 is the unique interior maximum point of K .

In this paper, we prove a nondegeneracy property of least energy solutions to $(P_{\varepsilon,K})$ when $\varepsilon > 0$ is sufficiently small, under the assumption (K).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a smooth bounded domain. Assume (K). Then the least energy solution u_ε to $(P_{\varepsilon,K})$ is nondegenerate for $\varepsilon > 0$ small, in the sense that the linearized problem around u_ε :*

$$(L_{\varepsilon,K}) \begin{cases} -\Delta v = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

admits only the trivial solution $v \equiv 0$.

When $K \equiv 1$, Grossi [6] obtained a nondegeneracy result for solutions $\{u_\varepsilon\}$ to $(P_{\varepsilon,K})$ satisfying

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} \rightarrow S_N \quad \text{as } \varepsilon \rightarrow 0. \quad (1.1)$$

It is known ([8], [10]) that a solution sequence with the property (1.1) blows up at one point in the domain, and the blow up point is a critical point of the Robin function associated to the Dirichlet Green function. Under the assumption that the blow up point is a nondegenerate critical point of the Robin function, Grossi obtained the nondegeneracy of solutions satisfying (1.1) for $\varepsilon > 0$ small. This result was former obtained in [1] when $N \geq 4$ by another method. More recently, Grossi and Pacella [7] studied the linearized eigenvalue problem associated with the blow up solutions satisfying (1.1), and obtained the same nondegeneracy result, again when $K \equiv 1$.

When $K \not\equiv 1$, corresponding results to [1] or [7] are still not known. Main purpose of this paper is to generalize the result in [6] to the inhomogeneous case $K \not\equiv 1$. In this case, Hebey [9] obtained the precise asymptotic behavior of least energy solutions as $\varepsilon \rightarrow 0$ under a stronger assumption than (K). We remark here that the same asymptotic result as [9] can be obtained under the assumption (K) by using the local blow up analysis of YanYan Li [11]; see Remark after Theorem 2.3.

Based on Hebey's result, we prove Theorem 1.1 with a new argument. Note that, even in the case $K \equiv 1$, our argument simplifies the proof in [6].

2 Preliminaries

In this section, we prepare some facts which are needed in the sequel. Let $G = G(x, z)$ denote the Green function of $-\Delta$ under the Dirichlet boundary

condition:

$$\begin{cases} -\Delta G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

Define the (positive) Robin function R associated to the Green function

$$R(z) = \lim_{x \rightarrow z} \left[\frac{1}{(N-2)\sigma_N} |x-z|^{2-N} - G(x, z) \right],$$

here σ_N denotes the volume of the unit sphere in $\mathbb{R}^N$.

Lemma 2.1 *The identities*

$$\int_{\partial\Omega} ((x-z) \cdot \nu_x) \left(\frac{\partial G(x, z)}{\partial \nu_x} \right)^2 ds_x = (N-2)R(z) \quad (2.1)$$

and

$$\int_{\partial\Omega} \left(\frac{\partial G}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left(\frac{\partial G}{\partial z_j} \right) (x, z) ds_x = \frac{1}{2} \frac{\partial^2 R}{\partial z_i \partial z_j} (z) \quad (2.2)$$

hold true for any $z \in \Omega$. Here, ν_x is the outer unit normal at $x \in \partial\Omega$.

Proof: See [2]:Theorem 4.3 for (2.1) and [6]:Lemma 3.2 for (2.2). $\square$

Lemma 2.2 *Let u_ε be a solution to $(P_{\varepsilon, K})$ and v_ε be a solution to $(L_{\varepsilon, K})$. Then the following identities hold true:*

$$\int_{\partial\Omega} ((x-z) \cdot \nu_x) \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) ds_x = c_0 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon ((x-z) \cdot \nabla K(x)) dx \quad (2.3)$$

for any $z \in \mathbb{R}^N$ and

$$\int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) ds_x = c_0 \int_{\Omega} \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx, \quad i = 1, 2, \dots, N. \quad (2.4)$$

Proof: Set $w_\varepsilon(x) = (x-z) \cdot \nabla u_\varepsilon(x) + \frac{2}{p_\varepsilon-1} u_\varepsilon(x)$. Direct computation yields that

$$-\Delta w_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} w_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} (x-z) \cdot \nabla K(x).$$

Since v_ε satisfies $-\Delta v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon$, we have

$$(\Delta v_\varepsilon) w_\varepsilon - (\Delta w_\varepsilon) v_\varepsilon = c_0 u_\varepsilon^{p_\varepsilon} v_\varepsilon (x - z) \cdot \nabla K(x).$$

Integrating this identity on Ω , using integration by parts and noting $w_\varepsilon(x) = (x - z) \cdot \nu_x \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right)$ for $x \in \partial\Omega$, we have (2.3).

On the other hand, differentiating the equation in $(P_{\varepsilon,K})$ with respect to x_i , we have

$$-\Delta \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) + c_0 \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon}.$$

From this equation and the equation in $(L_{\varepsilon,K})$, we obtain

$$(\Delta v_\varepsilon) \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) - \left(\Delta \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right) v_\varepsilon = c_0 \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon.$$

Finally, integration by parts yields (2.4). $\square$

Next is a part of the main theorem of [9], with a result of [11]. In what follows, we abbreviate $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)}$.

Theorem 2.3 (*Hebey [9]*) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a smooth bounded domain. Assume (K). Let u_ε be a least energy solution to $(P_{\varepsilon,K})$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$.*

Then there exists a constant $C > 0$ independent of ε such that for any $R_\varepsilon \rightarrow \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-(\frac{p_\varepsilon-1}{2})} \rightarrow 0$, the following estimates hold true:

$$u_\varepsilon(x) \leq \begin{cases} \frac{\|u_\varepsilon\|}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-2}} |x - x_\varepsilon|^2\right)^{\frac{N-2}{2}}}, & \text{for } |x - x_\varepsilon| \leq r_\varepsilon, \\ \frac{C}{\|u_\varepsilon\|} \frac{1}{|x - x_\varepsilon|^{N-2}}, & \text{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \end{cases} \quad (2.5)$$

Furthermore, after passing to a subsequence, we have

$$\begin{cases} |x_\varepsilon - x_0| = O(\|u_\varepsilon\|^{-2}) & N = 3, \\ |x_\varepsilon - x_0| = o(\|u_\varepsilon\|^{-2/(N-2)}) & N \geq 4, \end{cases} \quad (2.6)$$

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|^\varepsilon = 1 \quad (2.7)$$

and

$$\|u_\varepsilon\| u_\varepsilon \rightarrow (N-2) \sigma_N G(\cdot, x_0) \quad \text{in } C^1(\omega) \quad (2.8)$$

for any neighborhood ω of $\partial\Omega$ not containing x_0 .

Remark: Hebey assumed in [9] that Ω is star-shaped with respect to a point x_0 in Ω , $K \in C^2(\overline{\Omega})$, $0 < K(x) \leq 1$, $K(x_0) = 1$ and $(x - x_0) \cdot \nabla K(x) \leq 0$ for any $x \in \Omega$. However, crucial pointwise estimate (2.5) can be obtained by the theory of *isolated simple blow up point* due to YanYan Li [11], since in our case, the blow up point of least energy solutions has to be a unique interior maximum point of K , and thus to be an isolated simple blow up point in the sense of [11]. Once the crucial pointwise estimate (2.5) is obtained, the rest of the proof in Hebey [9] is still valid under the assumption (K).

Now, let us consider the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}} (\Omega - x_\varepsilon). \quad (2.9)$$

Then $0 < \tilde{u}_\varepsilon \leq 1$, $\tilde{u}_\varepsilon(0) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{u}_\varepsilon = c_0 K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where $K_\varepsilon(y) = K \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right)$. Since $\|u_\varepsilon\| \rightarrow \infty$ and $x_\varepsilon \rightarrow x_0 \in \Omega$, we see $\Omega_\varepsilon \rightarrow \mathbb{R}^N$ and $K_\varepsilon \rightarrow K(x_0) = 1$ compact uniformly on $\mathbb{R}^N$ as $\varepsilon \rightarrow 0$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$\tilde{u}_\varepsilon \rightarrow U \quad \text{compact uniformly in } \mathbb{R}^N \quad (2.10)$$

for some function U . Passing to the limit, we obtain that U is a solution of

$$\begin{cases} -\Delta U = c_0 U^p & \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \quad U(0) = 1, \\ \lim_{|y| \rightarrow \infty} U(y) = 0. \end{cases}$$

According to the uniqueness theorem by Caffarelli, Gidas and Spruck [4], we obtain

$$U(y) = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-2}{2}}.$$

Note that by (2.7), the estimate (2.5) is written as

$$\tilde{u}_\varepsilon(y) \leq \begin{cases} CU(y) & \text{for } \{|y| \leq R_\varepsilon\} \cap \Omega_\varepsilon, \\ C|y|^{2-N} & \text{for } \{|y| > R_\varepsilon\} \cap \Omega_\varepsilon. \end{cases} \quad (2.11)$$

We recall here the classification theorem by Bianchi and Egnell [3].

Lemma 2.4 *Let v_0 be a solution to*

$$\begin{cases} -\Delta v_0 = c_0 p U^{p-1} v_0 & \text{in } \mathbb{R}^N, \\ v_0 \in D^{1,2}(\mathbb{R}^N) \end{cases}$$

where $D^{1,2}(\mathbb{R}^N) = \{v \in L^{2N/(N-2)}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\nabla v|^2 dy < \infty\}$.

Then there exist constants a_j ($j = 1, 2, \dots, N$) and b in $\mathbb{R}$ such that v_0 has the form

$$v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1 + |y|^2)^{N/2}} + b \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}}. \quad (2.12)$$

Next lemma concerns a well-known unique solvability for the linear first order PDE's with the initial condition. The proof will be done by the standard method of characteristics, so we omit it.

Lemma 2.5 *Let $a = (a_1, a_2, \dots, a_N) \neq 0$ is a constant vector and $f, g \in C^1(\mathbb{R}^N)$. Let $\Gamma_a = \{x \in \mathbb{R}^N \mid a \cdot x = 0\}$ be the hyperplane perpendicular to a through the origin. Then there exists a unique solution of the following initial value problem of the linear first order PDE*

$$\begin{aligned} a \cdot \nabla u &= f, \\ u|_{\Gamma_a} &= g. \end{aligned}$$

More precisely, this solution is obtained as

$$u(x) = \int_0^{\phi(x)} f(\tau a + \alpha(\psi(x))) d\tau + g(\alpha(\psi(x))), \quad x \in \mathbb{R}^N$$

where

$$\begin{aligned} \phi(x) &= \frac{a \cdot x}{|a|^2}, \quad \psi(x) = (\psi_1(x), \dots, \psi_{N-1}(x)), \\ \psi_j(x) &= \frac{|a|^2 x_j - (a \cdot x) a_j}{|a|^2}, \quad (j = 1, \dots, N-1) \\ \alpha(s) &= (s, -\frac{1}{a_N} \sum_{j=1}^{N-1} a_j s_j) \in \mathbb{R}^N, \quad s = (s_1, \dots, s_{N-1}) \in \mathbb{R}^{N-1}, \end{aligned}$$

if we assume (w.l.o.g) $a_N \neq 0$. Furthermore, if $f(x) = O(|x|^\beta), g(x) = O(|x|^\beta)$ as $|x| \rightarrow \infty$, then $u(x) = O(|x|^{\beta+1})$ as $|x| \rightarrow \infty$.

3 The nondegeneracy result

In this section, we will prove Theorem 1.1. In the course of proof, we need precise estimates and a new argument which are not in [6].

We argue by contradiction and assume that there exists a non-trivial solution v_ε to $(L_{\varepsilon,K})$ satisfying $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$, without loss of generality.

Let us consider the scaled function

$$\tilde{v}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} v_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}} (\Omega - x_\varepsilon). \quad (3.1)$$

Then $0 < \tilde{v}_\varepsilon \leq 1$ and $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{v}_\varepsilon = c_\varepsilon(y) \tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1 \end{cases} \quad (3.2)$$

where $c_\varepsilon(y) := c_0 p_\varepsilon K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1}(y)$. By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$ and the elliptic estimate, we see there exists v_0 such that

$$\tilde{v}_\varepsilon \rightarrow v_0 \quad \text{uniformly on compact subsets of } \mathbb{R}^N \quad (3.3)$$

and v_0 satisfies

$$-\Delta v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Now, we claim that

$$\int_{\mathbb{R}^N} |\nabla v_0|^2 dy \leq C \quad (3.4)$$

for some $C > 0$.

Indeed, let $0 < \delta < \min(2, 4/(N-2) - 2\varepsilon)$. By (3.2) and the Sobolev inequality, we have

$$S_N \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{2/(p+1)} \leq \int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy = \int_{\Omega_\varepsilon} c_\varepsilon(y) \tilde{v}_\varepsilon^2 dy \leq \int_{\Omega_\varepsilon} |c_\varepsilon(y)| |\tilde{v}_\varepsilon|^{2-\delta} dy,$$

here, the last inequality comes from the fact that $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq 1$. Now, by the Hölder inequality and (2.11), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |c_\varepsilon(y)| |\tilde{v}_\varepsilon|^{2-\delta} dy &\leq \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{(2-\delta)/(p+1)} \left(\int_{\Omega_\varepsilon} |c_\varepsilon(y)|^{(p+1)/(p-1+\delta)} dy \right)^{(p-1+\delta)/(p+1)} \\ &\leq C \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{(2-\delta)/(p+1)} \left(\int_{\Omega_\varepsilon} U(y)^{(p_\varepsilon-1)(p+1)/(p-1+\delta)} dy \right)^{(p-1+\delta)/(p+1)}, \end{aligned}$$

thus we obtain

$$\left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{\delta/(p+1)} \leq C \left(\int_{\mathbb{R}^N} U(y)^{(p_\varepsilon-1)(p+1)/(p-1+\delta)} dy \right)^{(p-1+\delta)/(p+1)}.$$

Note that $(N-2)(p_\varepsilon-1)(p+1)/(p-1+\delta) > N$ if $\delta < 4/(N-2) - 2\varepsilon$, so the last integral is bounded by a constant. Therefore, we have

$$\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \leq C. \quad (3.5)$$

Finally, again by the Hölder inequality, (3.5) and (2.11), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy &\leq \int_{\Omega_\varepsilon} c_\varepsilon(y) |\tilde{v}_\varepsilon|^{2-\delta} dy \\ &\leq \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{(2-\delta)/(p+1)} \left(\int_{\Omega_\varepsilon} |c_\varepsilon(y)|^{(p+1)/(p-1+\delta)} dy \right)^{(p-1+\delta)/(p+1)} \\ &\leq C. \end{aligned}$$

Thus we have confirmed

$$\int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy \leq C \quad (3.6)$$

for some $C > 0$ independent of $\varepsilon > 0$. (3.6) and Fatou's lemma implies (3.4).

Now, by (3.4) and Lemma 2.4, we have (2.12), i.e.

$$v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1+|y|^2)^{N/2}} + b \frac{1-|y|^2}{(1+|y|^2)^{N/2}}. \quad (3.7)$$

In the following, we divide the proof into several steps.

Step 1. $b = 0$ in (3.7).

Step 2. $a_j = 0, j = 1, \dots, N$ in (3.7).

Step 3. $v_0 = 0$ leads to a contradiction.

In the sequel, we need the following pointwise estimate for $\tilde{v}_\varepsilon$.

Lemma 3.1 *Let $\tilde{v}_\varepsilon$ be a solution of (3.2). Then we have the estimate*

$$|\tilde{v}_\varepsilon(y)| \leq C \left(\frac{1}{1+|y|^2} \right)^{(N-2)/2}, \quad \forall y \in \Omega_\varepsilon \quad (3.8)$$

for some $C > 0$.

Proof: Consider the Kelvin transformation of $\tilde{v}_\varepsilon$:

$$\tilde{v}_\varepsilon^*(z) = |z|^{2-N} \tilde{v}_\varepsilon\left(\frac{z}{|z|^2}\right), \quad z \in \Omega_\varepsilon^* := \left\{ \frac{y}{|y|^2} : y \in \Omega_\varepsilon \right\}. \quad (3.9)$$

To prove (3.8), it will be enough to show that $|\tilde{v}_\varepsilon^*|$ is bounded in $B(0, R) \cap \Omega_\varepsilon^*$ for some $R > 0$, since $|\tilde{v}_\varepsilon(y)| \leq 1$ for $y \in \Omega_\varepsilon$, $|y|$ small. Direct calculation shows that

$$\begin{aligned} \Delta_z \tilde{v}_\varepsilon^*(z) &= |z|^{-2-N} \Delta_y \tilde{v}_\varepsilon(y), \quad z \in \Omega_\varepsilon^*, \\ \int_{\Omega_\varepsilon^*} |\tilde{v}_\varepsilon^*|^{p+1} dz &= \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy. \end{aligned}$$

Thus by (3.2), $\tilde{v}_\varepsilon^*$ satisfies the equation

$$\begin{cases} -\Delta \tilde{v}_\varepsilon^* &= |z|^{-4} c_\varepsilon\left(\frac{z}{|z|^2}\right) \tilde{v}_\varepsilon^* & \text{in } \Omega_\varepsilon^*, \\ \tilde{v}_\varepsilon^* &= 0 & \text{on } \partial\Omega_\varepsilon^*. \end{cases} \quad (3.10)$$

We claim that

$$a_\varepsilon(z) := |z|^{-4} c_\varepsilon\left(\frac{z}{|z|^2}\right) \in L^\infty(\Omega_\varepsilon^*). \quad (3.11)$$

Indeed, since $\Omega_\varepsilon \subset B(0, \gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/2})$ for some $\gamma > 0$, the domain Ω_ε^* satisfies $\Omega_\varepsilon^* \subset \mathbb{R}^N \setminus B(0, \frac{1}{\gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/2}})$. By (2.11), we have

$$|c_\varepsilon(y)| \leq C U^{p_\varepsilon-1}(y) \quad \text{for } y \in \Omega_\varepsilon.$$

Therefore, we have

$$\begin{aligned} |z|^{-4} c_\varepsilon\left(\frac{z}{|z|^2}\right) &\leq C |z|^{-4} \left(\frac{|z|^2}{1+|z|^2} \right)^{(\frac{N-2}{2})(p_\varepsilon-1)} \\ &= C |z|^{-4+(N-2)(p_\varepsilon-1)} \frac{1}{(1+|z|^2)^{2-\varepsilon(\frac{N-2}{2})}} \\ &\leq C |z|^{-4+(N-2)(p_\varepsilon-1)} = C |z|^{-\varepsilon(N-2)} \end{aligned}$$

Since $|z| \geq \frac{1}{\gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/2}}$ for $z \in \Omega_\varepsilon^*$, we have

$$|z|^{-\varepsilon(N-2)} \leq \gamma^{\varepsilon(N-2)} \|u_\varepsilon\|^{\varepsilon(N-2)(p_\varepsilon-1)/2} \rightarrow 1$$

as $\varepsilon \rightarrow 0$ by (2.7). From these, we confirm that the claim (3.11).

Now, for any $R > 0$, we have

$$\begin{aligned} \int_{\Omega_\varepsilon^* \cap B(0, 2R)} |\tilde{v}_\varepsilon^*|^{p+1} dz &\leq \int_{\Omega_\varepsilon^*} |\tilde{v}_\varepsilon^*|^{p+1} dz = \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \\ &\leq \left(S_N^{-1} \int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy \right)^{(p+1)/2} \leq C, \end{aligned}$$

here we have used the Sobolev inequality and (3.6). Then by a result of classical elliptic regularity ([5] Theorem 8.17), we obtain

$$\sup_{B(0, R) \cap \Omega_\varepsilon^*} |\tilde{v}_\varepsilon^*| \leq C \left[\frac{1}{R^N} \int_{B(0, 2R) \cap \Omega} |\tilde{v}_\varepsilon^*|^{p+1} dz \right]^{1/(p+1)} \leq C$$

for some $R > 0$. □

By Lemma 3.1, we have the following convergence result.

Lemma 3.2 *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing x_0 . Then we have*

$$\|u_\varepsilon\|v_\varepsilon \rightarrow -(N-2)\sigma_N bG(\cdot, x_0) \quad \text{in } C^1(\omega). \quad (3.12)$$

Proof: We see

$$-\Delta(\|u_\varepsilon\|v_\varepsilon) = \|u_\varepsilon\|c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon =: f_\varepsilon(x) \quad (3.13)$$

for $x \in \Omega$ with the boundary condition $\|u_\varepsilon\|v_\varepsilon = 0$ on $\partial\Omega$. By using (2.11), (3.8), (2.7) and the dominated convergence theorem, we obtain

$$\begin{aligned} \int_{\Omega} f_\varepsilon(x) dx &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1}(y) \tilde{v}_\varepsilon(y) dy \\ &\rightarrow c_0 p \int_{\mathbb{R}^N} U^{p-1} v_0 dy = c_0 p b \int_{\mathbb{R}^N} \frac{1-|y|^2}{(1+|y|^2)^{N/2+2}} dy \\ &= c_0 p b \sigma_N \left(\int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N/2+2}} dr - \int_0^\infty \frac{r^{N+1}}{(1+r^2)^{N/2+2}} dr \right) \\ &= -(N-2)b\sigma_N. \end{aligned}$$

Note that the integral involving the a_j terms of v_0 must vanish by the oddness of the integrand. Last integral can be computed by the formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{\Gamma((\alpha+1)/2)\Gamma(\beta-(\alpha+1)/2)}{2\Gamma(\beta)}$$

where $\alpha > 0, \beta > 0$ with $\beta - (\alpha+1)/2 > 0$. Furthermore, for any $x \neq x_0$, we have by (2.5) and (3.8),

$$\begin{aligned} f_\varepsilon(x) &\leq C \|u_\varepsilon\| \frac{\|u_\varepsilon\|^{p_\varepsilon}}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-2}} |x - x_\varepsilon|^2\right)^{\frac{N-2}{2} p_\varepsilon}} \\ &\leq C \frac{\|u_\varepsilon\|^{-(p_\varepsilon-1)}}{|x - x_\varepsilon|^{(N-2)p_\varepsilon}} \rightarrow 0 \end{aligned}$$

since $-(p_\varepsilon - 1) = -4/(N-2) + \varepsilon < 0$ for $\varepsilon > 0$ small. In conclusion, we confirm that

$$f_\varepsilon \rightarrow -(N-2)\sigma_N b \delta_{x_0} \quad (3.14)$$

in the sense of distributions. On the other hand, from the equation (3.13) with the boundary condition, we have the uniform boundary $C^{1,\alpha}$ -estimate ([8] Lemma 2)

$$\| \|u_\varepsilon\| v_\varepsilon \|_{C^{1,\alpha}(\omega)} \leq C(\omega) (\|f_\varepsilon\|_{L^1(\Omega)} + \|f_\varepsilon\|_{L^\infty(\omega')}) ,$$

here $\omega \subset\subset \omega'$ is a neighborhood of $\partial\Omega$ not containing 0. Since the RHS of the above estimate is bounded by a constant independent of ε , Ascoli-Arzelà theorem implies that the function $\|u_\varepsilon\| v_\varepsilon$ converges to some function in $C^{1,\alpha}$ -topology. Finally, (3.14) implies that this limit function is $-(N-2)\sigma_N b G(x, x_0)$. $\square$

Assume for the moment that the proof of Step 1 and Step 2 is finished. Then the proof of Step 3 is as follows. By Step 1 and Step 2, we deduce that $\lim_{\varepsilon \rightarrow 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$. Since $\tilde{v}_\varepsilon \rightarrow v_0 \equiv 0$ uniformly on compact sets of $\mathbb{R}^N$, we must have $|y_\varepsilon| \rightarrow \infty$. But this is not possible because of Lemma 3.1. $\square$

Proof of Step 1.

First, we treat the case $N = 3$ or $N = 4$. Putting $z = x_0$ in (2.3) and multiplying $\|u_\varepsilon\|^2$, we have

$$\begin{aligned} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| v_\varepsilon}{\partial \nu_x} \right) ds_x \\ = c_0 \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) (x - x_0) \cdot \nabla K(x) dx. \end{aligned} \quad (3.15)$$

By Theorem 2.3 (2.8) and (3.12), we see

$$\begin{aligned} (\text{LHS}) \text{ of (3.15)} &\rightarrow -(N-2)^2 \sigma_N^2 b \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial G(x, x_0)}{\partial \nu_x} \right)^2 ds_x \\ &= -(N-2)^3 \sigma_N^2 b R(x_0) \end{aligned} \quad (3.16)$$

as $\varepsilon \rightarrow 0$. Here we have used (2.1) in Lemma 2.1.

Also by Taylor's theorem, we have

$$K(x) = 1 + \frac{1}{2} \sum_{i,j=1}^N b_{ij} (x_i - x_{0i})(x_j - x_{0j}) + O(|x - x_0|^3)$$

and

$$\frac{\partial K}{\partial x_j}(x) = \sum_{i=1}^N b_{ij} (x_i - x_{0i}) + O(|x - x_0|^2)$$

where $b_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)$. Thus

$$(x - x_0) \cdot \nabla K(x) = \sum_{i,j=1}^N b_{ij} (x_i - x_{0i})(x_j - x_{0j}) + O(|x - x_0|^3).$$

When $N = 3$, we write

$$\begin{aligned} (\text{RHS}) \text{ of (3.15)} \\ = \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon O(|x - x_\varepsilon|^2) dx + \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon O(|x_\varepsilon - x_0|^2) dx \\ =: A_1 + A_2. \end{aligned}$$

By a change of variables, we see

$$\begin{aligned} |A_1| &\leq \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} |v_\varepsilon| O(|x - x_\varepsilon|^2) dx \\ &= \|u_\varepsilon\|^{3+p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (p_\varepsilon-1)} \int_{\Omega_\varepsilon} O(|y|^2) \tilde{u}_\varepsilon^{p_\varepsilon}(y) |\tilde{v}_\varepsilon(y)| dy. \end{aligned}$$

By (2.11), (3.8), (2.7) and the dominated convergence theorem, we have

$$\int_{\Omega_\varepsilon} O(|y|^2) \tilde{u}_\varepsilon^{p_\varepsilon}(y) |\tilde{v}_\varepsilon(y)| dy \rightarrow \int_{\mathbb{R}^N} O(|y|^2) U^p(y) |v_0(y)| dy$$

which is finite if $N \geq 3$. On the other hand, the exponent of $\|u_\varepsilon\|$ is $3 + p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (p_\varepsilon - 1) = (2N - 8)/(N - 2) + \varepsilon N/2 < 0$ when $N = 3$ and $\varepsilon > 0$ small, thus $A_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Similarly by Theorem 2.3 (2.6) and the dominated convergence theorem, we see

$$\begin{aligned} A_2 &= \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon O(|x_\varepsilon - x_0|^2) dx \\ &= \|u_\varepsilon\|^2 \times \left(\int_{\mathbb{R}^N} U^p v_0(y) dy + o(1) \right) \times O(\|u_\varepsilon\|^{-4}) \\ &= o(1) \end{aligned}$$

as $\varepsilon \rightarrow 0$. Thus together with (3.16), we have

$$-\sigma_3^2 b R(x_0) = 0$$

when $N = 3$, which leads to $b = 0$.

When $N = 4$, we write

$$\begin{aligned} (\text{RHS}) \text{ of (3.15)} &= c_0 \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) \sum_{i,j=1}^N b_{ij}(x_i - x_{0i})(x_j - x_{0j}) dx \\ &\quad + \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x - x_0|^3) dx. \end{aligned}$$

As before, by Theorem 2.3 (2.6), we see

$$\begin{aligned} &\|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x - x_0|^3) dx \\ &= \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x - x_\varepsilon|^3) dx + \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x_\varepsilon - x_0|^3) dx \\ &= \|u_\varepsilon\|^{3+p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - 3(\frac{p_\varepsilon-1}{2})} \left(\int_{\mathbb{R}^N} O(|y|^3) U^p(y) |v_0(y)| dy + o(1) \right) \\ &\quad + \|u_\varepsilon\|^2 \times \left(\int_{\mathbb{R}^N} U^p(y) |v_0(y)| dy + o(1) \right) \times o(\|u_\varepsilon\|^{-3}) \\ &= o(1) \end{aligned}$$

as $\varepsilon \rightarrow 0$, since $3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - 3(\frac{p_\varepsilon - 1}{2}) = \frac{2(N-5)}{N-2} + \varepsilon \frac{N+1}{2} < 0$ for $N = 4$ and $\varepsilon > 0$ small.

On the other hand, by a change of variables, we have

$$\begin{aligned}
& c_0 \|u_\varepsilon\|^2 \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) \sum_{i,j=1}^N b_{ij}(x_i - x_{0i})(x_j - x_{0j}) dx \\
&= c_0 \|u_\varepsilon\|^{3+p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \times \\
& \quad \sum_{i,j=1}^N b_{ij} \left(\frac{y_i}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_{\varepsilon i} - x_{0i} \right) \left(\frac{y_j}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_{\varepsilon j} - x_{0j} \right) dy \\
&=: B_1 + B_2 + B_3
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= c_0 \|u_\varepsilon\|^{3+p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (p_\varepsilon - 1)} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy, \\
B_2 &= 2c_0 \|u_\varepsilon\|^{3+p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (\frac{p_\varepsilon - 1}{2})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i (x_{\varepsilon j} - x_{0j}) dy, \\
B_3 &= c_0 \|u_\varepsilon\|^{3+p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} (x_{\varepsilon i} - x_{0i})(x_{\varepsilon j} - x_{0j}) dy.
\end{aligned}$$

The exponents are

$$\begin{aligned}
3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (p_\varepsilon - 1) &= 2\varepsilon, \\
3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (\frac{p_\varepsilon - 1}{2}) &= 1 + (3/2)\varepsilon, \\
3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N &= 2 + \varepsilon
\end{aligned}$$

when $N = 4$. Thus by Theorem 2.3 (2.6), (2.7) and the dominated convergence theorem as before, we see

$$\begin{aligned}
|B_2| &\leq 2c_0 \|u_\varepsilon\|^{1+\frac{3}{2}\varepsilon} \left| \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i dy \right| o(\|u_\varepsilon\|^{-1}) \\
&= o(\|u_\varepsilon\|^{(3/2)\varepsilon}) = o(1) \quad \text{as } \varepsilon \rightarrow 0
\end{aligned}$$

and

$$\begin{aligned} |B_3| &\leq c_0 \|u_\varepsilon\|^{2+\varepsilon} \left| \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) dy \right| o(\|u_\varepsilon\|^{-2}) \\ &= o(\|u_\varepsilon\|^\varepsilon) = o(1) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Now, we treat the term B_1 . By (2.11), (3.8), (2.7) and the dominated convergence theorem as before, we have

$$\begin{aligned} B_1 &= c_0 \|u_\varepsilon\|^{2\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy \\ &\rightarrow c_0 \int_{\mathbb{R}^N} U^p(y) v_0(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy \\ &= c_0 \int_{\mathbb{R}^N} U^p(y) b \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}} \sum_{i,j=1}^N b_{ij} y_i y_j dy \\ &= c_0 \int_{\mathbb{R}^N} U^p(y) b \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}} \sum_{i=1}^N b_{ii} y_i^2 dy \\ &= \frac{c_0}{N} \int_{\mathbb{R}^N} U^p(y) b \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}} \Delta K(x_0) |y|^2 dy \\ &= (N - 2) b \Delta K(x_0) \left(\int_{\mathbb{R}^N} \frac{|y|^2}{(1 + |y|^2)^{N+1}} dy - \int_{\mathbb{R}^N} \frac{|y|^4}{(1 + |y|^2)^{N+1}} dy \right) \\ &= (N - 2) b \Delta K(x_0) \times -\sigma_N \frac{\Gamma(\frac{N}{2} + 1) \Gamma(\frac{N}{2} - 1)}{\Gamma(N + 1)} \\ &= -\frac{b \Delta K(x_0) \sigma_4}{6} \quad \text{when } N = 4. \end{aligned}$$

Note that by the oddness of the integrand,

$$\int_{\mathbb{R}^N} U^p(y) \frac{1}{(1 + |y|^2)^{N/2}} y_i y_j y_k dy = 0$$

for any $i, j, k \in \{1, \dots, N\}$, so the integral involving a_j terms in v_0 must vanish.

Returning to (3.16), we obtain

$$-8\sigma_4^2 b R(x_0) = -\frac{b \Delta K(x_0) \sigma_4}{6}.$$

Since $\Delta K(x_0) \leq 0$ and $R(x_0) > 0$, we conclude that $b = 0$ when $N = 4$.

Next, we treat the case $N \geq 5$. In this case, we multiply (2.3) by $\|u_\varepsilon\|^{\frac{4}{N-2}}$ to get

$$\begin{aligned} & \|u_\varepsilon\|^{\frac{4}{N-2}-2} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| v_\varepsilon}{\partial \nu_x} \right) dS_x \\ &= c_0 \|u_\varepsilon\|^{\frac{4}{N-2}} \int_{\Omega} u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) (x - x_0) \cdot \nabla K(x) dx. \end{aligned} \quad (3.17)$$

Since $\frac{4}{N-2} < 2$ if $N \geq 5$, the LHS of (3.17) converges to 0 as $\varepsilon \rightarrow 0$. On the other hand, by Taylor's formula and the change of variables, we write

$$(\text{RHS}) \text{ of (3.17)} =: C_1 + C_2 + C_3 + C_4$$

where

$$\begin{aligned} C_1 &= c_0 \|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N-(p_\varepsilon-1)} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy, \\ C_2 &= 2c_0 \|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N-(\frac{p_\varepsilon-1}{2})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i (x_{\varepsilon j} - x_{0j}) dy, \\ C_3 &= c_0 \|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} (x_{\varepsilon i} - x_{0i}) (x_{\varepsilon j} - x_{0j}) dy, \\ C_4 &= c_0 \|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left(O \left(\left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon - x_0 \right|^3 \right) \right) dy. \end{aligned}$$

Again by (2.11), (3.8), (2.7), (2.6) and the dominated convergence theorem, we see

$$\begin{aligned} C_2 &= O(\|u_\varepsilon\|^{\frac{2}{N-2}+\frac{N-1}{2}\varepsilon}) \times O \left(\int_{\mathbb{R}^N} U^p v_0(y) |y| dy + o(1) \right) \times o(\|u_\varepsilon\|^{-\frac{2}{N-2}}) = o(1), \\ C_3 &= O(\|u_\varepsilon\|^{\frac{4}{N-2}+\frac{N-2}{2}\varepsilon}) \times O \left(\int_{\mathbb{R}^N} U^p v_0(y) dy + o(1) \right) \times o(\|u_\varepsilon\|^{-\frac{4}{N-2}}) = o(1), \\ C_4 &= O(\|u_\varepsilon\|^{\frac{4}{N-2}+\frac{N-2}{2}\varepsilon}) \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left(O \left(\left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} \right|^3 \right) + O(|x_\varepsilon - x_0|^3) \right) \\ &= O(\|u_\varepsilon\|^{\frac{4}{N-2}}) \times O(\|u_\varepsilon\|^{-\frac{6}{N-2}}) \times O \left(\int_{\mathbb{R}^N} U^p v_0(y) (|y|^3 + 1) dy + o(1) \right) \\ &= O(\|u_\varepsilon\|^{-\frac{2}{N-2}}) \end{aligned}$$

as $\varepsilon \rightarrow 0$. As for C_1 , we see just as in the estimate of B_1 ,

$$\begin{aligned} C_1 &= c_0 \|u_\varepsilon\|^{(N/2)\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy \\ &\rightarrow = -(N-2)b \Delta K(x_0) \sigma_N \frac{\Gamma(\frac{N}{2}+1)\Gamma(\frac{N}{2}-1)}{\Gamma(N+1)}. \end{aligned}$$

Thus letting $\varepsilon \rightarrow 0$ in (3.17), we have

$$0 = \Delta K(x_0) \times b.$$

Hence we obtain $b = 0$, because our nondegeneracy assumption of x_0 assures that $\Delta K(x_0) < 0$ strictly. This proves step 1 for all cases.

Proof of Step 2.

In this step, we prove $a_j = 0, j = 1, 2, \dots, N$ in (3.7). For this purpose, we need a lemma, which is not in [6].

Lemma 3.3 *Assume $b = 0$ and $a = (a_1, \dots, a_N) \neq 0$ in (3.7). Then we have*

$$\|u_\varepsilon\|^{N/(N-2)} v_\varepsilon \rightarrow \sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x, z) \right) \Big|_{z=x_0}$$

in $C_{loc}^1(\bar{\Omega} \setminus \{x_0\})$.

Proof. For any $x \in \bar{\Omega} \setminus \{x_0\}$, the Green representation formula to $(L_{\varepsilon,K})$ and a change of variables imply that

$$\begin{aligned} v_\varepsilon(x) &= c_0 p_\varepsilon \int_{\Omega} G(x, z) K(z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz \\ &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} G_\varepsilon(x, y) K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) dy \end{aligned}$$

where $G_\varepsilon(x, y) = G(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon)$ and $K_\varepsilon(y) = K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon)$ for $y \in \Omega_\varepsilon$.

By (2.10) and (3.3) with $b = 0$, we see

$$\begin{aligned} \tilde{u}_\varepsilon^{p_\varepsilon-1}(y) &\rightarrow U^{p-1}(y), \\ \tilde{v}_\varepsilon(y) &\rightarrow v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1+|y|^2)^{N/2}} = \frac{-1}{(N-2)} \sum_{j=1}^N a_j \frac{\partial U}{\partial y_j}(y) \end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^N$, thus

$$\tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) \rightarrow \sum_{j=1}^N a_j \left(\frac{\partial}{\partial y_j} \frac{-1}{(N+2)} U^p(y) \right)$$

uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N \quad (3.18)$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{(N+2)} U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. The RHS of (3.18) should be understood as 0 outside of Ω_ε . By Lemma 2.5, the solution w_ε satisfies the estimate $w_\varepsilon(y) = O(|y|^{-(N+1)})$ as $|y| \rightarrow \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon}(y)) = O(|y|^{-(N+2)})$ by (2.11) and (3.8). Also we have

$$w_\varepsilon \rightarrow \frac{-1}{(N+2)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \rightarrow \frac{-1}{(N+2)} \int_{\mathbb{R}^N} U^p dy = \frac{-1}{N(N+2)} \sigma_N$$

by the dominated convergence theorem.

Using integration by parts, we have

$$\begin{aligned} v_\varepsilon(x) &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} G_\varepsilon(x, y) K_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\ &= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \{G_\varepsilon(x, y) K_\varepsilon(y)\} w_\varepsilon(y) dy \\ &= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (\frac{p_\varepsilon-1}{2})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial z_j} \{G(x, z) K(z)\} \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon} w_\varepsilon(y) dy. \end{aligned}$$

Now, we see

$$\begin{aligned}
& \left. \frac{\partial}{\partial z_j} \{G(x, z)K(z)\} \right|_{z=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon} \\
& \rightarrow \left(\frac{\partial G}{\partial z_j}(x, x_0) \right) K(x_0) + G(x, x_0) \left(\frac{\partial K}{\partial z_j}(x_0) \right) \\
& = \frac{\partial G}{\partial z_j}(x, x_0)
\end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \rightarrow 0$, since x_0 is a critical point of K with $K(x_0) = 1$. Also we note that $\frac{N}{N-2} + p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (\frac{p_\varepsilon-1}{2}) = (\frac{N-1}{2})\varepsilon$. Therefore, we have the convergence

$$\begin{aligned}
& \|u_\varepsilon\|^{N/(N-2)} v_\varepsilon(x) \\
& = -c_0 p_\varepsilon \|u_\varepsilon\|^{\frac{N}{N-2} + p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (\frac{p_\varepsilon-1}{2})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial z_j} \{G(x, z)K(z)\} \Big|_{z=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon} w_\varepsilon(y) dy \\
& \rightarrow -c_0 p \sum_{j=1}^N a_j \frac{\partial G}{\partial z_j}(x, x_0) \times \left(\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} w_\varepsilon(y) dy \right) \\
& = \sigma_N \sum_{j=1}^N a_j \frac{\partial G}{\partial z_j}(x, x_0)
\end{aligned}$$

for any $x \in \bar{\Omega} \setminus \{x_0\}$. Elliptic estimates implies this convergence holds true in $C_{loc}^1(\bar{\Omega} \setminus \{x_0\})$. This proves Lemma. $\square$

Now, multiply both sides of (2.4) by $\|u_\varepsilon\|^{N/(N-2)} \times \|u_\varepsilon\|^{-1}$. Letting $\varepsilon \rightarrow 0$, we see the LHS is

$$\begin{aligned}
& \|u_\varepsilon\|^{-2} \int_{\partial\Omega} \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{N/(N-2)} v_\varepsilon}{\partial \nu_x} \right) ds_x \\
& = \|u_\varepsilon\|^{-2} \left((N-2) \sigma_N^2 \int_{\partial\Omega} \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial x_i} \right) (x, x_0) \frac{\partial}{\partial \nu_x} \left(\frac{\partial G}{\partial z_j} \right) (x, z) \Big|_{z=x_0} ds_x + o(1) \right) \\
& = \|u_\varepsilon\|^{-2} \left(\frac{N-2}{2} \sigma_N^2 \sum_{j=1}^N a_j \frac{\partial^2 R}{\partial z_i \partial z_j}(x_0) + o(1) \right) \rightarrow 0,
\end{aligned}$$

here we have used (2.8), Lemma 3.3 and (2.2).

On the other hand, using Lemma 2.5 again, we solve the linear PDE

$$\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N \quad (3.19)$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{2N} U^{p+1}(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here as before, the RHS of (3.19) is understood as 0 outside of Ω_ε .

Lemma 2.5 implies that the solution w_ε satisfies the estimate $w_\varepsilon(y) = O(|y|^{-2N+1})$ as $|y| \rightarrow \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon+1}(y)) = O(|y|^{-2N})$ by (2.11) and (3.8). As before, we have

$$w_\varepsilon \rightarrow \frac{-1}{2N} U^{p+1} \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \rightarrow \frac{-1}{2N} \int_{\mathbb{R}^N} U^{p+1} dy = \frac{-1}{2N} \sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} dr = \frac{\Gamma(N/2)^2}{2\Gamma(N)}$.

Thus, (RHS of (2.4)) $\times \|u_\varepsilon\|^{\frac{N}{N-2}-1}$ is

$$\begin{aligned} & c_0 \|u_\varepsilon\|^{-1+\frac{N}{N-2}} \int_{\Omega} \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \\ &= c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i} \right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right) \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon dy \\ &= c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i} \right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\ &= -c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \left(\frac{\partial K}{\partial x_i} \right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right) \right\} w_\varepsilon(y) dy \\ &= -c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N-(\frac{p_\varepsilon-1}{2})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \left(\frac{\partial^2 K}{\partial x_i \partial x_j} \right) (x) \Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon} w_\varepsilon(y) dy \\ &\rightarrow -c_0 \left(\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|^{(\frac{N-1}{2})\varepsilon} \right) \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j} (x_0) \left(\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} w_\varepsilon(y) dy \right) \\ &= \frac{N-2}{2} \sigma_N C_N \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j} (x_0). \end{aligned}$$

Thus we have

$$\sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) = 0.$$

By our assumption of the nondegeneracy of x_0 , the matrix $\left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right)(x_0)$ is invertible. Therefore we obtain that $a_j = 0$ for all $j = 1, \dots, N$. Thus we have proved Step 2, and consequently, Theorem 1.1. $\square$

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