BAND SURGERY ON KNOTS AND LINKS

TAIZO KANENOBU

Citation	OCAMI Preprint Series
Issue Date	2009
Type	Preprint
Textversion	Author
Rights	For personal use only. No other uses without permission.
Relation	Preprint of an article submitted for consideration in Journal of Knot Theory and Its Ramifications © 2010 World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr .
Is version of	https://doi.org/10.1142/S0218216510008522

From: Osaka City University Advanced Mathematical Institute

<http://www.sci.osaka-cu.ac.jp/OCAMI/publication/preprint/preprint.html>

BAND SURGERY ON KNOTS AND LINKS

TAIZO KANENOBU

*Department of Mathematics, Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
kanenobu@sci.osaka-cu.ac.jp*

ABSTRACT

We give some relationships of the Jones and Q polynomials between two links which are related by a band surgery. Then we consider two applications: The first one is to an evaluation of the ribbon-fusion number, the least fusion number of a ribbon knot. The second one is to DNA knot theory, helping us to understand the action of the Xer site-specific recombination at *psi* site.

Keywords: Knot; Link; Band surgery; Jones polynomial; Q polynomial; ribbon-fusion number; DNA knot; Xer recombination.

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

Let L be an oriented link, and $b : I \times I \rightarrow S^3$ an embedding such that $b(I \times I) \cap L = b(I \times \partial I)$, where I is a closed interval. Let $L' = (L - b(I \times \partial I)) \cup b(\partial I \times I)$, which is another link. If L' has the orientation compatible with the orientation of $L - b(I \times I) \cap L$ and $b(\partial I \times I)$, then L' is called the link obtained from L by the *band surgery* along the band b . Then there is a relation between the signatures of L and L' due to Murasugi; see Eq. (2.2). In this paper, we give further relationships in terms of the Jones polynomial (Theorem 2.2) and the Q polynomial (Theorem 3.1). Then we apply these relations in two ways: The first application is to estimate the ribbon-fusion number of a ribbon knot. A knot is a *ribbon knot* if it is a knot obtained from a trivial $(m + 1)$ -component link by doing band surgery along m bands for some m . We call the least number of such m the *ribbon-fusion number*. There is an estimation for this number due to Sakuma, which is given in terms of the Nakanishi index (Proposition 4.2). Using the above-mentioned relationships we deduce Theorems 4.3 and 4.4, which can give a sharper estimation (Examples 4.6, 4.7).

The second application is to consider a problem whether a given knot with $(2n + 1)$ crossings is related to a $(2, 2n)$ torus link or not by a band surgery, which was brought from the study of a DNA site-specific recombination. More precisely, Bath, Sherratt, and Colloms [1] have shown that the action of the Xer site-specific

recombination at *psi* site is the change from a $(2, 2n)$ torus link to a $(2n+1)$ -crossing knot by a band surgery. So characterizing such change is an important problem. Applying Theorems 2.2 and 3.1, we will show the 7 crossing knots 7_3 , 7_6 cannot be obtained from a $(2, 6)$ torus link (Proposition 5.4), and the 9 crossing knots 9_{15} , 9_{17} , 9_{31} cannot be obtained from a $(2, 8)$ torus link (Propositions 5.6 and 5.7).

Notation. For knots with up to 10 crossings we use Rolfsen notation [23, Appendix C].

2. The Jones Polynomial

In this section, we give a relationship of the Jones polynomials of two links that are related by a band surgery. Before that we review a classical result for the signature of these links due to Murasugi. Let L_+ , L_- , L_0 be three links that are identical except near one point where they are as in Fig. 1; we call (L_+, L_-, L_0) a *skein triple*. Then Murasugi [19, Lemma 7.1] has shown:

$$|\sigma(L_+) - \sigma(L_-)| \leq 1. \quad (2.1)$$

Since we may consider the link L_+ or L_- is obtained from L_0 by a band surgery, and vice versa, two links L and L' which are related by a band surgery satisfy:

$$|\sigma(L) - \sigma(L')| \leq 1. \quad (2.2)$$

Fig. 1. A skein triple.

The *Jones polynomial* $V(L; t) \in \mathbb{Z}[t^{\pm 1/2}]$ [8], is an invariant of the isotopy type of an oriented link L , which is defined by the following formulas:

$$V(U; t) = 1; \quad (2.3)$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = (t^{1/2} - t^{-1/2})V(L_0; t), \quad (2.4)$$

where U is the unknot and (L_+, L_-, L_0) is a skein triple.

We put $\omega = e^{i\pi/3}$. For a knot K , Lickorish and Millett [14, Theorem 3] have shown:

$$V(L; \omega) = \pm i^{c(L)-1} (i\sqrt{3})^d, \quad (2.5)$$

where $c(L)$ is the number of the components of L , $d = \dim H_1(\Sigma(L); \mathbb{Z}_3)$ with $\Sigma(L)$ the double cover of S^3 branched over L ; cf. [15]. Note that $V(L; \omega)$ means the value of $V(L; t)$ at $t^{1/2} = e^{i\pi/6}$, whence $t^{1/2} - t^{-1/2} = i$.

The following lemma is due to Miyazawa [17].

Lemma 2.1.

$$\frac{V(L_+; \omega)}{V(L_-; \omega)} \in \left\{ \pm 1, i\sqrt{3}^{\pm 1} \right\} \quad (2.6)$$

Proof. For the skein triple (L_+, L_-, L_0) , we consider another oriented link L_∞ which is one of the diagram of Fig. 2, the choice being (i) if $c(L_+) < c(L_0)$ and (ii) if $c(L_+) > c(L_0)$.

Fig. 2. Two choices of the oriented link L_∞ .

Then by [2, Theorem 2] for the case (i) we have

$$V(L_+; t) - tV(L_-; t) + t^{3\lambda}(t-1)V(L_\infty; t) = 0, \quad (2.7)$$

where λ is the linking number of the right-hand component of L_0 in Fig. 1 with the remainder of L_0 , and for the case (ii) we have

$$V(L_+; t) - tV(L_-; t) + t^{3(\mu-\frac{1}{2})}(t-1)V(L_\infty; t) = 0, \quad (2.8)$$

where μ is the linking number of the bottom-right and top-left component L_+ in Fig. 1 with the remainder of L_+ .

We consider the case (i). Putting $t = \omega$ in (2.7), we have

$$x - \omega + (-1)^\lambda(\omega - 1)y = 0, \quad (2.9)$$

where $x = V(L_+; \omega)/V(L_-; \omega)$ and $y = V(L_\infty; \omega)/V(L_-; \omega)$. Then by Eq. (2.5) there are four cases:

- (a) $(x, y) = (\alpha, \beta)$;
- (b) $(x, y) = (\alpha, \beta i)$;
- (c) $(x, y) = (\alpha i, \beta)$;
- (d) $(x, y) = (\alpha i, \beta i)$,

where α, β are real numbers. For the case (a), we have $\alpha = 1, \beta = (-1)^\lambda$; for the case (b), we have $\alpha = -1, \beta = (-1)^{\lambda+1}\sqrt{3}$; for the case (c), we have $\alpha = \sqrt{3}, \beta = (-1)^{\lambda+1}$; for the case (d), we have $\alpha = \sqrt{3}^{-1}, \beta = (-1)^{\lambda+1}\sqrt{3}^{-1}$, obtaining the result.

For the case (ii) we can prove similarly. $\square$

Theorem 2.2. *Let L and L' be two links related with a band surgery such that $c(L) < c(L')$. Then*

$$\frac{V(L; \omega)}{V(L'; \omega)} \in \left\{ \pm i, -\sqrt{3}^{\pm 1} \right\} \quad (2.10)$$

Proof. From the condition there is a skein triple (L_+, L_-, L_0) such that L_+ and L_0 are isotopic to L and L' , respectively. Put $x = V(L_+; \omega)/V(L_-; \omega)$ and $z = V(L_0; \omega)/V(L_-; \omega)$. Then by Eq. (2.4), we have

$$\omega^{-1}x - \omega = iz, \quad (2.11)$$

and so

$$\frac{V(L; \omega)}{V(L'; \omega)} = \frac{x}{z} = \frac{ix}{\omega^{-1}x - \omega}. \quad (2.12)$$

By Lemma 2.1 we obtain (2.2). $\square$

By using Eq. (2.5), Theorem 2.2 immediately implies the following.

Corollary 2.3. *Suppose that a knot K is obtained from a 2-component link L by a band surgery. Then*

$$V(K; \omega) \in \begin{cases} \{\pm 1, -i\sqrt{3}\epsilon\} & \text{if } V(L; \omega) = i\epsilon; \\ \{-\epsilon, \pm i\sqrt{3}, -3\epsilon\} & \text{if } V(L; \omega) = \sqrt{3}\epsilon, \end{cases} \quad (2.13)$$

where $\epsilon = \pm 1$.

3. The Q Polynomial

In this section, we give a relationship of the Q polynomials of two links that are related by a band surgery. The Q polynomial $Q(L; z) \in \mathbb{Z}[z^{\pm 1}]$ [4, 6] is an invariant of the isotopy type of an unoriented link L , which is defined by the following formulas:

$$Q(U; z) = 1; \quad (3.1)$$

$$Q(L_+; z) + Q(L_-; z) = z(Q(L_0; z) + Q(L_\infty; z)), \quad (3.2)$$

where U is the unknot and L_+, L_-, L_0, L_∞ are four unoriented links that are identical except near one point where they are as in Fig. 3. We call $(L_+, L_-, L_0, L_\infty)$ an *unoriented skein quadruple*.

We put $\rho(K) = Q(K; (\sqrt{5} - 1)/2)$. For a knot K , Jones [9] has shown:

$$\rho(K) = \pm \sqrt{5}^r, \quad (3.3)$$

where $r = \dim H_1(\Sigma(K); \mathbb{Z}_5)$ with $\Sigma(K)$ the double cover of S^3 branched over K . Furthermore, Rong [24] deduced some information on the values $\rho(L_-)/\rho(L_\infty)$,

Fig. 3. An unoriented skein quadruple.

$\rho(L_0)/\rho(L_\infty)$, $\rho(L_+)/\rho(L_\infty)$, where $(L_+, L_-, L_0, L_\infty)$ is an unoriented skein quadruple. Using these values, we have the following, which is analogous to a criterion on the unknotting number of a knot due to Stoimenow [25, Theorem 4.1]; cf. [10].

Theorem 3.1. *If two links L and L' are related by a band surgery, then*

$$\rho(L)/\rho(L') \in \left\{ \pm 1, \sqrt{5}^{\pm 1} \right\}. \quad (3.4)$$

Proof. From the condition there is an unoriented skein quadruple $(L_+, L_-, L_0, L_\infty)$ such that L_0 and L_∞ are isotopic to L and L' , respectively. Then from the proof of Theorem 2 in [24], we have $\rho(L_0)/\rho(L_\infty) \in \{\pm 1, \sqrt{5}^{\pm 1}\}$, obtaining the result. $\square$

By using (3.3), Theorem 3.1 immediately implies the following.

Corollary 3.2. *Suppose that a knot K is obtained from a link L by a band surgery. Then*

$$\rho(K) \in \begin{cases} \{\pm 1, \sqrt{5}\epsilon\} & \text{if } \rho(L) = \epsilon; \\ \{1, \pm\sqrt{5}, 5\} & \text{if } \rho(L) = \sqrt{5}, \end{cases} \quad (3.5)$$

where $\epsilon = \pm 1$.

4. The Ribbon-Fusion Number of a Ribbon Knot

In this section, we apply the theorems given in the previous sections to an evaluation of the ribbon-fusion number of a ribbon knot. A knot is said to be a *ribbon knot of m -fusions* if it is a knot obtained from a trivial $(m+1)$ -component link by doing band surgery along m bands. More precisely, it has the form

$$S_0^1 \cup S_1^1 \cup \cdots \cup S_m^1 \cup \bigcup_{i=1}^m f_i(\partial I \times I) - \text{int} \left(\bigcup_{i=1}^m f_i(I \times \partial I) \right), \quad (4.1)$$

where $S_0^1 \cup S_1^1 \cup \cdots \cup S_m^1$ is a trivial link of m components and $f_i : I \times I \rightarrow S^3$ ($i = 1, 2, \dots, m$) are disjoint embeddings such that

$$f_i(I \times \partial I) \cup S_j = \begin{cases} f_i(I, 0) & \text{if } j = 0; \\ f_i(I, 1) & \text{if } j = i; \\ \emptyset & \text{if otherwise.} \end{cases} \quad (4.2)$$

By a *ribbon knot* we mean a ribbon knot of m -fusions for some m ; see [16, 27]. The least number of such m is the *ribbon-fusion number* of K , which we denote by $\text{rf}(K)$.

Remark 4.1. In [3, 22, 26] the ribbon-fusion number is called the ribbon number.

If K and K' are ribbon knots, then it is easy to see

$$\text{rf}(K \# K') \leq \text{rf}(K) + \text{rf}(K'). \quad (4.3)$$

Also, for any n -bridge knot K , the connected sum of K and its mirror image $K^!$, $K \# K^!$ is a ribbon knot (cf. [23, 8E30]), which satisfies

$$\text{rf}(K \# K^!) \leq n - 1. \quad (4.4)$$

Bleiler and Eudave-Muñoz [3] have shown a composite knot with ribbon-fusion number one has a summand that is two-bridge. Then Tanaka [26] proved that there exist composite ribbon-fusion number one knots with arbitrarily large bridge numbers.

The *Nakanishi index* of a knot K , denoted by $m(K)$, is the minimum size among all square Alexander matrix of K , provided that $m(K) = 0$ if and only if an Alexander matrix of K is equivalent to the 1×1 matrix with entry 1 as presentation matrices; see [11, p. 72]. Then Makoto Sakuma has given a lower bound of the ribbon-fusion number using the Nakanishi index of a knot [22, Proposition 2].

Proposition 4.2. *For a ribbon knot K ,*

$$\text{rf}(K) \geq m(K)/2. \quad (4.5)$$

As applications of Theorems 2.2 and 3.1 we give other lower bounds for the ribbon-fusion number.

Theorem 4.3. *If $\text{rf}(K) = n$, then*

$$V(K; \omega) \in \left\{ 1, \pm(i\sqrt{3})^k, 3^n \mid k = 1, 2, \dots, 2n - 1 \right\}. \quad (4.6)$$

In particular, if K is a ribbon knot with $V(K; \omega) = -3^n$, then $\text{rf}(K) > n$.

Proof. We use induction on n . If $\text{rf}(K) = 1$, then K is obtained from the trivial 2-component link U^2 by a band surgery. Thus since $V(U^2; \omega) = -\sqrt{3}$, by Corollary 2.3 we obtain Eq. (4.6) with $n = 1$.

Suppose that Eq. (4.6) holds for $n = j$. If $\text{rf}(K) = j + 1$, then K is obtained from the split union of a knot K' with $\text{rf}(K') = j$ and the trivial knot, $K' \sqcup U$, by a band surgery. Then since $V(K' \sqcup U; \omega) = -\sqrt{3}V(K'; \omega)$, by Theorem 2.2 we have $V(K; \omega)/V(K'; \omega) \in \{1, \pm i\sqrt{3}, 3\}$. Hence we obtain Eq. (4.6) with $n = j + 1$. $\square$

Theorem 4.4. *If $\text{rf}(K) = n$, then*

$$\rho(K) \in \left\{ 1, \pm\sqrt{5}^k, 5^n \mid k = 1, 2, \dots, 2n-1 \right\}. \quad (4.7)$$

In particular, if K is a ribbon knot with $\rho(K) = -5^n$, then $\text{rf}(K) > n$.

Proof. We use induction on n . If $\text{rf}(K) = 1$, then K is obtained from the trivial 2-component link U^2 by a band surgery. Thus since $\rho(U^2) = \sqrt{5}$, by Corollary 3.2 we obtain Eq. (4.7) with $n = 1$.

Suppose that Eq. (4.7) holds for $n = j$. If $\text{rf}(K) = j + 1$, then K is obtained from the split union of a knot K' with $\text{rf}(K') = j$ and the trivial knot, $K' \sqcup U$, by a band surgery. Then by Theorem 3.1 we have $\rho(K)/\rho(K' \sqcup U) \in \{\pm 1, \sqrt{5}^{\pm 1}\}$. Since $Q(U^2; z) = 2z^{-1} - 1$ and $\rho(U^2) = \sqrt{5}$, $\rho(K' \sqcup U) = \rho(K')\rho(U^2) = \sqrt{5}\rho(K')$, and so we have $\rho(K)/\rho(K') \in \{1, \pm\sqrt{5}, 5\}$. Hence we obtain Eq. (4.7) with $n = j + 1$. $\square$

Theorems 4.3 and 4.4 immediately imply:

Corollary 4.5. *If a knot K satisfies either $V(K; \omega) = -1$ or $\rho(K) = -1$, then K is not a ribbon knot.*

We denote the connected sum of n copies of a knot K by $\#^n K$.

Example 4.6. Let $J_{r,s}$ be the connected sum of r copies of the knot 6_1 and s copies of its mirror image $6_1!$. Suppose that $r \geq s$. Then putting $J_{r,s} = \left(\#^{r-s} 6_1 \right) \# \left(\#^s (6_1 \# 6_1!) \right)$, we have $\text{rf}(J_{r,s}) \leq r$. In fact, the knot 6_1 is a ribbon knot of 1-fusion (see [11, Appendix F.5]), and also the connected sum $6_1 \# 6_1!$ is a ribbon knot of 1-fusion since 6_1 is a 2-bridge knot. On the other hand, by Proposition 4.2, $\text{rf}(J_{r,s}) \geq (r + s)/2$. Let us consider the case $s = r - 2$. Since $V(6_1; \omega) = i\sqrt{3}$, $V(6_1!; \omega) = -i\sqrt{3}$ (cf. [12, Table 3.1]), we have $V(J_{r,r-2}; \omega) = -3^{r-1}$. Thus by Theorem 4.3 $\text{rf}(J_{r,r-2}) \geq r$, and so $\text{rf}(J_{r,r-2}) = r$, which cannot be deduced from Proposition 4.2.

Example 4.7. Let K_n be the connected sum of n copies of the knot 8_8 , $(n-1)$ copies of the knot $8_8!$, and the knot 8_9 ;

$$K_n = 8_9 \# 8_8 \# \left(\#^{n-1} (8_8 \# 8_8!) \right) \quad (4.8)$$

Then we have $\text{rf}(K_n) = n + 1$. In fact, the knots 8_8 and 8_9 are ribbon knots of 1-fusion (see [11, Appendix F.5]), and the connected sum $8_8 \# 8_8!$ is also a ribbon knot of 1-fusion since 8_8 is a 2-bridge knot. Thus $\text{rf}(K_n) \leq n + 1$. On the other hand,

$\rho(8_8) = \sqrt{5}$, $\rho(8_9) = -\sqrt{5}$ (cf. [4, Table]) and so $\rho(K_n) = -5^n$. Thus by Theorem 4.4 $\text{rf}(K_n) > n$. Note that using Proposition 4.2, we only have $\text{rf}(K_n) \geq n$.

5. Band Surgery from a $(2, 2n)$ Torus Link to a $(2n + 1)$ -Crossing Knot

The motivation of this section is the study of Bath, Sherratt, and Colloms [1] of a DNA site-specific recombination; they showed that the action of the Xer site-specific recombination at *psi* site is the change from a $(2, 2n)$ torus link to a $(2n + 1)$ -crossing knot by a band surgery. So characterizing such change is an important problem. In this section, we consider a problem whether a given knot with $(2n + 1)$ crossings is related to a $(2, 2n)$ torus link or not by a band surgery. Also, DNA knots or links are mainly of 2 bridge, so we consider this problem for 2-bridge knots with 7 or 9 crossings. Applying Corollary 2.3 or Corollary 3.2 we can conclude that some knot cannot be related with a $(2, 2n)$ torus link by a band surgery.

5.1. Torus links

First, we calculate some values for torus links needed to apply Eq. (2.2) and Corollaries 2.3 and 3.2. For a positive integer m , we denote by T_m the oriented torus knot or link of type $(2, m)$ with m crossings as shown in Fig. 4.

Fig. 4. The oriented torus knot or link of type $(2, m)$, T_m .

If m is even, then T_m is a 2-component link. We denote by T'_{2n} the oriented torus link obtained from T_{2n} by reversing the orientation of one component, and $T_{2n}!$, $T'_{2n}!$ the mirror images of T_{2n} , T'_{2n} , respectively. Fig. 5 shows torus links T_6 , T'_6 , $T_6!$, $T'_6!$.

Fig. 5. Oriented torus links of type $(2, 6)$.

Lemma 5.1. *The torus links T_{2n} , T'_{2n} , $T_{2n}!$, $T'_{2n}!$ have the linking numbers, the signatures, and the values of the Jones polynomials at $t = \omega$ as in Table 1.*

Table 1. The linking number, the signature, and the Jones polynomial at $t = \omega$ of the torus links of type $(2, 2n)$.

L	$\text{lk}(L)$	$\sigma(L)$	$V(L; \omega) \pmod{6}$					
			$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$	$n \equiv 4$	$n \equiv 5$
T_{2n}	$-n$	$2n - 1$	$-\sqrt{3}$	i	i	$\sqrt{3}$	$-i$	$-i$
T'_{2n}	n	-1	$-\sqrt{3}$	$-i$	i	$-\sqrt{3}$	$-i$	i
$T_{2n}!$	n	$-2n + 1$	$-\sqrt{3}$	$-i$	$-i$	$\sqrt{3}$	i	i
$T'_{2n}!$	$-n$	1	$-\sqrt{3}$	i	$-i$	$-\sqrt{3}$	i	$-i$

Proof. The signatures of T_{2n} and $T_{2n}!$ are given in [5, Theorem 5.2]; cf. [21, Theorem 7.5.1]. We obtain the signatures of T'_{2n} and $T'_{2n}!$ by the following formula due to Murasugi [20, Theorem 1]:

$$\sigma(L') = \sigma(L) + 2\text{lk}(L), \quad (5.1)$$

where L is an oriented 2-component link with linking number $\text{lk}(L)$ and L' a link obtained from L by reversing the orientation of one component

Now we consider the Jones polynomial of T'_n . Since (T'_{2n}, T'_{2n-2}, U) is a skein triple, from Eq. (2.4) we have

$$t^{-1}V(T'_{2n}; t) - tV(T'_{2n-2}; t) = t^{1/2} - t^{-1/2}. \quad (5.2)$$

Then

$$\begin{aligned} V(T'_{2n}; t) - \mu^{-1} &= t^2 (V(T'_{2n-2}; t) - \mu^{-1}) \\ &= t^{2n} (V(T'_0; t) - \mu^{-1}) \\ &= t^{2n} (\mu - \mu^{-1}), \end{aligned} \quad (5.3)$$

where $\mu = V(U^2; t) = -t^{1/2} - t^{-1/2}$. Then

$$\begin{aligned} V(T'_{2n}; \omega) &= -\sqrt{3}^{-1} + \omega^{2n} (-\sqrt{3} + \sqrt{3}^{-1}) \\ &= \begin{cases} -\sqrt{3} & \text{if } n \equiv 0 \pmod{3}; \\ -i & \text{if } n \equiv 1 \pmod{3}; \\ i & \text{if } n \equiv 2 \pmod{3}. \end{cases} \end{aligned} \quad (5.4)$$

Since $V(T'_{2n}!; t) = V(T'_{2n}; t^{-1})$ [7, Theorem 3] and $\omega^{-1} = \bar{\omega}$, the complex conjugate of ω , we have $V(T'_{2n}!; \omega) = \bar{V}(T'_{2n}; \omega)$. Since $V(T_{2n}; t) = t^{-3n}V(T'_{2n}; t)$ [13, 18], we have $V(T_{2n}; \omega) = (-1)^n V(T'_{2n}; \omega)$. Similarly, we have $V(T_{2n}!; \omega) = (-1)^n V(T'_{2n}!; \omega)$. Then we obtain Table 1. $\square$

Let $\rho_m = \rho(T_m)$. Then we have the following.

Lemma 5.2.

$$\rho_m = \begin{cases} \sqrt{5} & \text{if } m \equiv 0 \pmod{5}; \\ 1 & \text{if } m \equiv 1, 4 \pmod{5}; \\ -1 & \text{if } m \equiv 2, 3 \pmod{5}. \end{cases} \quad (5.5)$$

Proof. From an unoriented skein quadruple $(T_{m+1}, T_{m-1}, T_m, U^2)$, where U^2 is the trivial 2-component link, by Eq. (3.2) we have

$$Q(T_{m+1}; z) + Q(T_{m-1}; z) = z (Q(T_m; z) + Q(U^2; z)). \quad (5.6)$$

Since $\rho(U^2) = \sqrt{5}$, we have

$$\rho_{m+1} + \rho_{m-1} = \frac{\sqrt{5} - 1}{2} (\rho_m + \sqrt{5}). \quad (5.7)$$

Using $\rho_0 = \sqrt{5}$, $\rho_1 = 1$, we obtain Eq. (5.5). $\square$

Combining Theorem 3.1 and Lemma 5.2, we obtain immediately the following.

Corollary 5.3. *Suppose that a knot K is obtained from a torus link of type $(2, 2n)$ by a band surgery. Then*

$$\rho(K) \in \begin{cases} \{1, \pm\sqrt{5}, 5\} & \text{if } n \equiv 0 \pmod{5}; \\ \{\pm 1, -\sqrt{5}\} & \text{if } n \equiv 1, 4 \pmod{5}; \\ \{\pm 1, \sqrt{5}\} & \text{if } n \equiv 2, 3 \pmod{5}. \end{cases} \quad (5.8)$$

5.2. 7-crossing 2-bridge knots

We consider the problem whether a 7-crossing 2-bridge knot is related to a $(2, 6)$ torus link or not by a band surgery. According to Shimokawa, the knots $7_1, 7_2, 7_4$ are obtained from a $(2, 6)$ torus link.

First, we consider applying Corollary 2.3. The γ -values of $(2, 6)$ torus links are $\pm i\sqrt{3}$ from Table 1. Then we can apply Corollary 2.3 for a knot K with $V(K; \omega) = \pm 1$. Note that the determinant of such a knot is $\not\equiv 0 \pmod{3}$; see Eq. (2.5). Since the determinants of the knots $7_3, 7_5, 7_6, 7_7$ are 13, 17, 19, 21, respectively, we should test this method except for the knot 7_7 . Then for the knot 7_3 and 7_6 we can obtain the result.

Proposition 5.4. *The knots 7_3 and 7_6 cannot be obtained from a $(2, 6)$ torus link by a band surgery.*

Proof. Suppose that the knot 7_6 is related with a $(2, 6)$ torus link by a band surgery. Since $\sigma(7_6) = 2$ (cf. [12, Table 8.1]), by Eq. (2.2) $T_6!$ should be such a torus link. From Table 1 $V(T_6!; \omega) = -\sqrt{3}$, and so by Corollary 2.3 $V(7_6; \omega) \in \{1, \pm i\sqrt{3}, 3\}$, which is a contradiction since $V(7_6; \omega) = -1$ (cf. [12, Table 3.1]).

For the knot 7_3 , the proof is similar. Since $\sigma(7_3) = -4$, we have to consider the link $T_6!$. Suppose that 7_3 is related with $T_6!$ by a band surgery. Since $V(T_6!; \omega) = \sqrt{3}$, $V(7_3; \omega) \in \{-1, \pm i\sqrt{3}, -3\}$, which is a contradiction since $V(7_3; \omega) = 1$. $\square$

Remark 5.5. Kawauchi has proved that 7_3 and 7_7 cannot be obtained from a $(2, 6)$ torus link by a band surgery using the Alexander invariants. Also, Darcy, Ishihara, Shimokawa have given a characterization of band surgery for the knots 7_2 and 7_4 . So the question whether the knot 7_5 , whose signature is -4 , is related by a band surgery to a $(2, 6)$ torus link $T_6!$ or not remains open.

For a 7 crossing knot, we cannot apply Corollary 5.3. In fact, in order to apply Corollary 5.3 the knot should satisfy $\rho(K) = -\sqrt{5}$. Then the determinant of such a knot is $\equiv 0 \pmod{5}$; see Eq. (3.3).

5.3. 9-crossing 2-bridge knots

We consider the problem whether a 9-crossing 2-bridge knot is related to a $(2, 8)$ torus link or not by a band surgery. Since $(2, 8)$ torus links have signatures ± 1 or ± 7 (Table 1), a knot with signature ± 4 is never related to $(2, 8)$ torus links by a band surgery by Eq. (2.2). The following knots have signature ± 4 : $9_4, 9_7, 9_{10}, 9_{11}, 9_{13}, 9_{18}, 9_{20}, 9_{23}$; see [11, Appendix F.3]. Also, it is easy to see that the knots $9_1, 9_2$ are related to a $(2, 8)$ torus link by a band surgery.

First, we consider applying Corollary 2.3. The γ -values of $(2, 8)$ torus links are ± 1 (Table 1), and so we can apply Corollary 2.3 for a knot K with $V(K; \omega) = \pm i\sqrt{3}$. Note that the determinant of such a knot is $\equiv 0 \pmod{3}$; see Eq. (2.5). Thus we apply this method for the knots $9_6, 9_{15}, 9_{17}$, whose determinants are 27, 39, 39, respectively.

Proposition 5.6. *The knots 9_{15} and 9_{17} cannot be obtained from a $(2, 8)$ torus link by a band surgery.*

Proof. The proof is similar to that of Proposition 5.4. We list the necessary data:

$$\sigma(9_{15}) = -2, \quad V(9_{15}; \omega) = -i\sqrt{3}; \quad (5.9)$$

$$\sigma(9_{17}) = 2, \quad V(9_{17}; \omega) = i\sqrt{3}. \quad (5.10)$$

$\square$

Next, we consider applying Corollary 5.3. We can apply Corollary 2.3 for a knot K with $\rho(K) = -\sqrt{5}$. Note that the determinant of such a knot is $\equiv 0 \pmod{5}$. Thus we apply this method for the knots $9_6, 9_{15}, 9_{17}$.

Proposition 5.7. *The knot 9_{31} cannot be obtained from a $(2, 8)$ torus link by a band surgery.*

Proof. Suppose that the knot 9_{31} is related with a $(2, 8)$ torus link by a band surgery. By Lemma 5.2 $\rho(T_8) = -1$, and so by Corollary 3.2 $\rho(9_{31}) \in \{\pm 1, -\sqrt{5}\}$, which is a contradiction since $\rho(9_{31}) = \sqrt{5}$. (Note that $Q(9_{31}) = -7 + 12z + 36z^2 - 22z^3 - 58z^4 - 4z^5 + 28z^7 + 14z^8 + 2z^9$, which is obtained from the Kauffman F polynomial listed in [11, Appendix F.6].) $\square$

For the following 9 crossing 2-bridge knots we cannot decide whether they are related to a $(2, 8)$ torus link or not by a band surgery using our method:

$$9_k, \quad k = 3, 5, 6, 8, 9, 12, 14, 19, 21, 26, 27. \quad (5.11)$$

Acknowledgments

The author would like to thank Professor Koya Shimokawa for introducing the problem in Sect. 5. This work was partially supported by Grant-in-Aid for Scientific Research (C) (No. 21540092), Japan Society for the Promotion of Science.

References

- [1] J. Bath, D. J. Sherratt and S. D. Colloms, Topology of Xer recombination on catenanes produced by lambda integrase, *J. Mol. Biol.* **289** (1999) 873–883.
- [2] J. S. Birman and T. Kanenobu, Jones’ braid-plat formula and a new surgery triple, *Proc. Amer. Math. Soc.* **102** (1988) 687–695.
- [3] S. A. Bleiler and M. Eudave-Muñoz, Composite ribbon number one knots have two-bridge summands, *Trans. Amer. Math. Soc.* **321** (1990) 231–243.
- [4] R. D. Brandt, W. B. R. Lickorish and K. Millett, A polynomial invariant for unoriented knots and links, *Inv. Math.* **84** (1986) 563–573.
- [5] C. McA. Gordon, R. A. Litherland and K. Murasugi, Signatures of covering links, *Canad. J. Math.* **33** (1981) 381–415.
- [6] C. F. Ho, A polynomial invariant for knots and links—preliminary report, *Abstracts Amer. Math. Soc.* **6** (1985) 300.
- [7] V. F. R. Jones, A polynomial invariant for knots via von Neumann algebra, *Bull. Amer. Math. Soc.* **12** (1985) 103–111.
- [8] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* **126** (1987) 335–388.
- [9] V. F. R. Jones, On a certain value of the Kauffman polynomial, *Comm. Math. Phys.* (1989) 103–111.
- [10] T. Kanenobu and Y. Miyazawa, $H(2)$ -unknotting number of a knot, *Communications in Mathematical Research* (to appear).
- [11] A. Kawachi, *A Survey of Knot Theory*, (Birkhäuser Verlag, Berlin, 1996).
- [12] W. B. R. Lickorish, *An introduction to Knot Theory*, Graduate Texts in Math., 175 (Springer-Verlag, New York, 1997).
- [13] W. B. R. Lickorish and K. C. Millett, The reversing formula for the Jones polynomial, *Pacific J. Math.* **124** (1986) 173–176.

- [14] W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helv. **61** (1986), 349–359.
- [15] A. S. Lipson, An evaluation of a link polynomial, *Math. Proc. Camb. Phil. Soc.* **100** (1986) 361–364.
- [16] Y. Marumoto, On ribbon 2-knots of 1-fusion, *Math. Sem. Notes, Kobe Univ.* **5** (1977) 59–68.
- [17] Y. Miyazawa, Talk at the conference “The 5th East Asian School of Knots and Related Topic” (Gyeongju, Korea, 2009).
- [18] H. R. Morton, The Jones polynomial for unoriented links, *Quart. J. Math. Oxford* **37** (1986) 55–60.
- [19] K. Murasugi, On the certain numerical invariant of links, *Trans. Amer. Math. Soc.* **117** (1965) 387–422.
- [20] K. Murasugi, On the signature of links, *Topology* **9** (1970) 283–298.
- [21] K. Murasugi, *Knot Theory and Its Applications* (Birkhäuser, 1996).
- [22] Y. Nakanishi and Y. Nakagawa, On ribbon knots, *Math. Sem. Notes, Kobe Univ.* **10** (1982) 423–430.
- [23] D. Rolfsen, *Knots and Links*, (AMS Chelsea Pub. 2003).
- [24] Y. Rong, The Kauffman polynomial and the two-fold cover of a link, *Indiana Univ. Math. J.* **40** (1991) 321–331.
- [25] A. Stoimenow, Polynomial values, the linking form and unknotting numbers, *Math. Res. Lett.* **11** (2004) 755–769.
- [26] T. Tanaka, On bridge numbers of composite ribbon knots, *J. Knot Theory Ramifications* **9** (2000) 423–430.
- [27] T. Yanagawa, On ribbon 2-knots. I. The 3-manifold bounded by the 2-knots. *Osaka J. Math.* **6** (1969) 447–464.