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Abstract. We completely characterize real Bott manifolds up to diffeomorphism in terms of three simple matrix operations on square binary matrices obtained from strictly upper triangular matrices by permuting rows and columns simultaneously. We also prove that any graded ring isomorphism between the cohomology rings of real Bott manifolds with $\mathbb{Z}/2$ coefficients is induced by an affine diffeomorphism between the real Bott manifolds.

This characterization can also be described combinatorially in terms of graph operations on directed acyclic graphs. Using this combinatorial interpretation, we prove that the decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors. Finally, we produce some numerical invariants of real Bott manifolds from the viewpoint of graph theory and discuss their topological meaning. As a by-product, we prove that the toral rank conjecture holds for real Bott manifolds.

1. Introduction

A manifold M is called a *real Bott manifold* if there is a sequence of $\mathbb{R}P^1$ bundles

$$M = M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} M_0 = \{\text{a point}\}, \tag{1.1}$$

such that for each $j \in \{1, 2, \dots, n\}$, $M_j \to M_{j-1}$ is the projective bundle of the Whitney sum of a real line bundle L_{j-1} and the trivial real line bundle over M_{j-1}. The sequence (1.1) is called a *real Bott tower* of height n and it is a real analogue of a Bott tower introduced by Grossberg and Karshon [12]. A real Bott manifold naturally supports an action of an elementary abelian 2-group. In fact, Kamishima and Masuda [15] proved that a manifold is a real Bott manifold if and only if it is a real toric manifold admitting a flat riemannian metric invariant under the action.

It is well known that real line bundles are classified by their first Stiefel-Whitney classes. With the binary field $\mathbb{Z}/2 = \{0, 1\}$, $H^1(M_{j-1}; \mathbb{Z}/2)$ is isomorphic to $(\mathbb{Z}/2)^{j-1}$ through a canonical basis. Therefore the line bundle L_{j-1} is determined by a vector

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A_j in $(\mathbb{Z}/2)^{j-1}$. We regard A_j as a column vector in $(\mathbb{Z}/2)^n$ by adding zero's and form a strictly upper triangular binary matrix A of size n by putting A_j as the j-th column. Since the real Bott manifold M_n is determined by the matrix A, we may denote it by $M(A)$. We note that two different matrices may produce (affinely) diffeomorphic real Bott manifolds.

In fact, one can describe $M(A)$ as the quotient of the n-dimensional torus T^n by a smooth free action of an elementary abelian 2-group of rank n, where the action is defined by A and A is not necessarily a strictly upper triangular binary matrix and may be a binary matrix conjugate by a permutation matrix to a strictly upper triangular binary matrix. We call such a matrix a *Bott matrix* and denote by $\mathfrak{B}(n)$ the set of Bott matrices of size n.

The cohomology ring of $M(A)$ can be described explicitly in terms of A, and three operations on $\mathfrak{B}(n)$, denoted by (Op1), (Op2) and (Op3), naturally arise when we analyze isomorphism classes of cohomology rings of real Bott manifolds. The operation (Op1) is a conjugation by a permutation matrix, (Op2) is a variant of simultaneous addition of a column vector to other column vectors and (Op3) is addition of a row vector to another row vector under some condition. We say that two matrices in $\mathfrak{B}(n)$ are *Bott equivalent* if one is transformed to the other through a sequence of these three operations. Our first main result is the following.

Theorem 1.1. *The following are equivalent for Bott matrices A, B in $\mathfrak{B}(n)$.*
(1) A and B are Bott equivalent.
(2) $M(A)$ and $M(B)$ are affinely diffeomorphic.
(3) $H^*(M(A);\mathbb{Z}/2)$ and $H^*(M(B);\mathbb{Z}/2)$ are isomorphic as graded rings.
Moreover, every graded ring isomorphism from $H^*(M(A);\mathbb{Z}/2)$ to $H^*(M(B);\mathbb{Z}/2)$ is induced by an affine diffeomorphism from $M(B)$ to $M(A)$.

In particular, we obtain the following main theorem of Kamishima and Masuda [15].

Corollary 1.2. *Two real Bott manifolds are diffeomorphic if and only if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings.*

To a matrix A of $\mathfrak{B}(n)$, one can associate an *acyclic digraph* (a directed graph with no directed cycles) whose adjacency matrix is A. This correspondence is a bijection from $\mathfrak{B}(n)$ to the set of acyclic digraphs on vertices $\{1, 2, \ldots, n\}$. Through the bijection, the three operations (Op1), (Op2) and (Op3) on $\mathfrak{B}(n)$ can be described as operations on acyclic digraphs. (Op1) corresponds to permuting labels of vertices. To our surprise, (Op2) corresponds to a known operation in graph theory called a *local complementation* while the operation corresponding to (Op3) seems not studied and we call it a *slide*. As far as we know, a local complementation on digraphs was first introduced by Bouchet [4]. Fon-Der-Flaass [11] surveyed this operation. This operation also appears in the coding theory [10] and quantum information theory [23]. Our result adds another application of this operation in topology.

We prove that real Bott manifolds of dimension n up to diffeomorphism can be identified with non-isomorphic acyclic digraphs on n vertices up to local complementation and slide. This combinatorial interpretation enables us to efficiently count

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{D}_n$	1	2	4	12	54	472	8,512	328,416	?	?
$\mathcal{O}_n$	1	1	2	3	8	29	222	3,607	131,373	?
$\mathcal{S}_n$		1		2		6		31		416

Table 1. The numbers $\mathcal{D}_n$, $\mathcal{O}_n$, $\mathcal{S}_n$ of n-dimensional real Bott manifolds, orientable real Bott manifolds and symplectic real Bott manifolds up to diffeomorphism, respectively.

the number $\mathcal{D}_n$ of real Bott manifolds of dimension n up to diffeomorphism. We list $\mathcal{D}_n$ in Table 1 for $n \leq 8$. Previously, $\mathcal{D}_n$ was known for $n \leq 5$ and it was a hard task to find $\mathcal{D}_5$ using a geometrical method ([20]). The computation of $\mathcal{D}_8$ takes less than 10 minutes by a regular desktop computer if we use the list of non-isomorphic acyclic digraphs provided by B. D. McKay.* In addition to $\mathcal{D}_n$, we also list the number $\mathcal{O}_n$ and $\mathcal{S}_n$ of n-dimensional orientable and symplectic, respectively, real Bott manifolds in Table 1 for small values of n.

Our classification of real Bott manifolds helps us to prove the topologically unique decomposition property for real Bott manifolds as follows. We say that a real Bott manifold is *indecomposable* if it is not diffeomorphic to a product of more than one real Bott manifolds.

Theorem 1.3 (Unique Decomposition Property). *The decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors.*

In particular, since S^1 is a real Bott manifold $\mathbb{R}P^1$, we have the following corollary.

Corollary 1.4 (Cancellation Property). *Let M and M' be real Bott manifolds. If $S^1 \times M$ and $S^1 \times M'$ are diffeomorphic, then M and M' are diffeomorphic.*

As remarked before, a real Bott manifold admits a flat riemannian metric. We note that the cancellation property above fails to hold for general compact flat riemannian manifolds [6]. It would be interesting to ask whether Theorem 1.3 and Corollary 1.4 hold for any (real) toric manifolds.

Our combinatorial interpretation allows us to discover several numerical invariants of real Bott manifolds up to diffeomorphism. Interestingly, those invariants can be thought of as a refinement of the topological and geometrical properties of real Bott manifolds. We will discuss them in Section 8. In particular, we prove the following theorem, which confirms that the toral rank conjecture [1, p.280] holds for real Bott manifolds.

Theorem 1.5. *Let A be a Bott matrix in $\mathfrak{B}(n)$. If $M(A)$ admits an effective topological action of a torus T^k of dimension k, then*

$$\sum_{i=0}^{n} \dim_{\mathbb{Q}} H^i(M(A);\mathbb{Q}) \geq 2^k.$$

* `http://cs.anu.edu.au/~bdm/data/digraphs.html`

This paper is organized as follows. In Section 2 we describe $M(A)$ and its cohomology ring explicitly in terms of a Bott matrix A. We introduce three operations on $\mathfrak{B}(n)$ in Section 3 and translate them into operations on acyclic digraphs in Section 4. To each operation we associate an affine diffeomorphism between real Bott manifolds in Section 5, which implies the implication (1) $\Rightarrow$ (2) in Theorem 1.1. The implication (2) $\Rightarrow$ (3) is trivial. In Section 6 we prove the latter statement in Theorem 1.1. The argument also establishes the implication (3) $\Rightarrow$ (1). We prove Theorem 1.3 in Section 7. In Section 8 we produce numerical invariants of real Bott manifolds from the viewpoint of graph theory. In particular, in Section 8.2, we prove Theorem 1.5.

Note. This paper is a combination of preprints [8] and [17]. After the second author wrote the paper [17], the first and third authors wrote the paper [8] which relates results in [17] to acyclic digraphs based on the observation in [7], simplifies the operation (Op3) in [17] and produces many numerical invariants of real Bott manifolds. In this paper, we also re-prove Theorem 1.3 from a graph theoretical viewpoint. Readers may find an algebraic proof of Theorem 1.3 in [17]. These two proofs are completely different. Some parts of this paper including Theorem 1.5 are new. We hope that this combination will make the subject and results more appealing to the readers.

2. Real Bott manifolds and their cohomology rings

The real Bott manifold $M(A)$ associated with a strictly upper triangular $n \times n$ binary matrix A can be described as the quotient of the n-dimensional torus by a free action of an elementary abelian 2-group of rank n. The free action is uniquely determined by the matrix A. In addition, this quotient construction also works if A is conjugate by a permutation matrix to a strictly upper triangular binary matrix. Motivated by this, we make the following definition.

Definition. A binary square matrix A is a *Bott matrix* if

$$A = PBP^{-1}$$

for a permutation matrix P and a strictly upper triangular binary matrix B. In other words, a Bott matrix is conjugate by a permutation matrix to a strictly upper triangular binary matrix. We denote by $\mathfrak{B}(n)$ the set of all Bott matrices of size n.[†]

Masuda and Panov [18, Lemma 3.3] showed that a binary square matrix A is a Bott matrix if and only if every principal minor of $A + I$ is 1 over $\mathbb{Z}/2 = \{0, 1\}$, where I is the identity matrix.

Let us recall the quotient construction and the structure of the cohomology ring of $M(A)$ for a Bott matrix A in $\mathfrak{B}(n)$. Let S^1 denote the unit circle consisting of complex numbers with absolute value 1. For $z \in S^1$ and $a \in \mathbb{Z}/2$, we use the following notation

$$z(a) := \begin{cases} z & \text{if } a = 0, \\ \bar{z} & \text{if } a = 1. \end{cases}$$

[†]In [17] $\mathfrak{B}(n)$ is defined to be the set of strictly upper triangular binary matrices of size n.

For a matrix A, let A^i_j be the (i,j) entry of A and let A^i, A_j be the i-th row vector, the j-th column vector, respectively, of A. We define the involutions $a_1, a_2, \ldots, a_n$ on $T^n := (S^1)^n$ by

$$a_i(z_1, \ldots, z_n) := (z_1(A^i_1), \ldots, z_{i-1}(A^i_{i-1}), -z_i, z_{i+1}(A^i_{i+1}), \ldots, z_n(A^i_n)). \tag{2.1}$$

These involutions $a_1, a_2, \ldots, a_n$ commute with each other and generate an elementary abelian 2-group of rank n, denoted by $G(A)$.

Lemma 2.1. *The action of $G(A)$ on T^n is free.*

Proof. Let A be a Bott matrix in $\mathfrak{B}(n)$. If A is strictly upper triangular, then $z_1(A^i_1) = z_1, \ldots, z_{i-1}(A^i_{i-1}) = z_{i-1}$ in (2.1) because $A^i_1 = \cdots = A^i_{i-1} = 0$. Therefore, for an element $t = a_{i_1} \cdots a_{i_\ell}$ of $G(A)$ with $i_1 < \cdots < i_\ell$, the i_1-th component of $t(z_1, \ldots, z_n)$ is $-z_{i_1}$. Hence, the action of $G(A)$ on T^n is clearly free when A is strictly upper triangular.

Now let us assume that A is not strictly upper triangular. There is a permutation σ on $\{1, 2, \ldots, n\}$ with its permutation matrix P such that $B = PAP^{-1}$ is strictly upper triangular, where

$$P^i_j = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

Since $PA = BP$, we have

$$A^i_j = (PA)^{\sigma(i)}_j = (BP)^{\sigma(i)}_j = B^{\sigma(i)}_{\sigma(j)}. \tag{2.2}$$

This together with (2.1) means that if we change the suffix of the coordinate $(z_1, \ldots, z_n)$ by σ, then the involution a_i in (2.1) is the same as the involution $b_{\sigma(i)}$ associated with B for each i. Since the action of $G(B)$ on T^n is free as B is strictly upper triangular, so is the action of $G(A)$ on T^n. $\square$

We define $M(A)$ to be the orbit space $T^n/G(A)$. By Lemma 2.1, $M(A)$ is a closed smooth manifold. Moreover it is a flat riemannian manifold. In fact, the Euclidean motions $s_1, s_2, \ldots, s_n$ on $\mathbb{R}^n$ defined by

$$s_i(u_1, \ldots, u_n) := ((-1)^{A^i_1} u_1, \ldots, (-1)^{A^i_{i-1}} u_{i-1}, u_i + \frac{1}{2}, (-1)^{A^i_{i+1}} u_{i+1}, \ldots, (-1)^{A^i_n} u_n)$$

generate a crystallographic group $\Gamma(A)$, where the subgroup generated by $s_1^2, \ldots, s_n^2$ consists of all translations by $\mathbb{Z}^n$, and $\Gamma(A)/\mathbb{Z}^n = G(A)$. The action of $\Gamma(A)$ on $\mathbb{R}^n$ is free. Through the identification $\mathbb{R}/\mathbb{Z}$ with S^1 via an exponential map

$$u \mapsto \exp(2\pi\sqrt{-1}u),$$

the orbit space $\mathbb{R}^n/\mathbb{Z}^n$ agrees with T^n and the orbit space $\mathbb{R}^n/\Gamma(A)$ agrees with $M(A) = T^n/G(A)$.

For $k = 1, 2, \ldots, n$, let G_k be the subgroup of $G(A)$ generated by $a_1, \ldots, a_k$. Obviously $G_n = G(A)$. Let $T^k := (S^1)^k$ be the product of first k-factors in $T^n = (S^1)^n$. Then G_k acts on T^k by restricting the action of G_k on T^n to T^k and the orbit space T^k/G_k is a manifold of dimension k. If A is strictly upper triangular, then the natural projections $T^k \to T^{k-1}$ for $k = 1, 2, \ldots, n$ produce a real Bott tower

$$M(A) = T^n/G_n \to T^{n-1}/G_{n-1} \to \cdots \to T^1/G_1 \to \{\text{a point}\},$$

which agrees with (1.1) in Section 1 (see [15]).

The graded ring structure of $H^*(M(A);\mathbb{Z}/2)$ can be described explicitly in terms of the matrix A. We shall recall it. For a homomorphism $\lambda\colon G(A)\to\{\pm1\}$ we denote by $\mathbb{R}(\lambda)$ the real one-dimensional $G(A)$-module associated with λ. Then the orbit space of $T^n\times\mathbb{R}(\lambda)$ by the diagonal action of $G(A)$, denoted by $L(\lambda)$, defines a real line bundle over $M(A)$ with the first projection. For $j\in\{1,2,\dots,n\}$, let $\lambda_j\colon G(A)\to\{\pm1\}$ be a homomorphism such that

$$\lambda_j(a_i)=\begin{cases}-1 & \text{if } i=j,\\ 1 & \text{otherwise.}\end{cases}$$

We set

$$x_j=w_1(L(\lambda_j))$$

where w_1 denotes the first Stiefel-Whitney class.

Lemma 2.2. *Let A be a Bott matrix in $\mathfrak{B}(n)$. Then*

$$H^*(M(A);\mathbb{Z}/2)=\mathbb{Z}/2[x_1,\dots,x_n]/(x_j^2=x_j\sum_{i=1}^n A^i_j x_i\mid j=1,\dots,n)$$

as graded rings. Moreover,

(i) $M(A)$ is orientable if and only if $\sum_{j=1}^n A^i_j=0$ in $\mathbb{Z}/2$ for every $i\in\{1,2,\dots,n\}$,

(ii) $M(A)$ admits a symplectic form if and only if $|\{k\mid A_k=A_j\}|$ is even for every $j\in\{1,2,\dots,n\}$.

Proof. Since $A\in\mathfrak{B}(n)$ is conjugate by a permutation matrix to a strictly upper triangular matrix, we may assume that A is strictly upper triangular (see the proof of Lemma 2.1). Then the first two statements are proved in [15, Lemmas 2.1 and 2.2] and the last statement is proved in [14]. □

Let A, B be Bott matrices in $\mathfrak{B}(n)$. Since $M(A)=T^n/G(A)$ and $M(B)=T^n/G(B)$, an affine automorphism $\tilde f$ of T^n together with a group isomorphism $\phi\colon G(B)\to G(A)$ induces an affine diffeomorphism $f\colon M(B)\to M(A)$ if $\tilde f$ is ϕ-*equivariant*, that is

$$\tilde f(gz)=\phi(g)\tilde f(z) \text{ for } g\in G(B) \text{ and } z\in T^n.$$

Since the actions of $G(A)$ and $G(B)$ on T^n are free, the isomorphism ϕ will be uniquely determined by $\tilde f$ if it exists. We shall use b_i and y_j for $M(B)$ in place of a_i and x_j for $M(A)$.

Lemma 2.3. *For Bott matrices A, B in $\mathfrak{B}(n)$, let $f\colon M(B)\to M(A)$ be the affine diffeomorphism induced by a ϕ-equivariant affine automorphism $\tilde f$ of T^n, where $\phi\colon G(B)\to G(A)$ is a group isomorphism. If $\phi(b_i)=\prod_{j=1}^n a_j^{F^i_j}$ with $F^i_j\in\mathbb{Z}/2$, then $f^*(x_j)=\sum_{i=1}^n F^i_j y_i$.*

Proof. A map $T^n\times\mathbb{R}(\lambda\circ\phi)\to T^n\times\mathbb{R}(\lambda)$ sending (z,u) to $(\tilde f(z),u)$ induces a bundle map $L(\lambda\circ\phi)\to L(\lambda)$ covering $f\colon M(B)\to M(A)$. Since $(\lambda_j\circ\phi)(b_i)=F^i_j$, this implies the lemma. □

3. Three matrix operations

In this section we introduce three operations on Bott matrices used in later sections to analyze when $M(A)$ and $M(B)$ are diffeomorphic and when $H^*(M(A);\mathbb{Z}/2)$ and $H^*(M(B);\mathbb{Z}/2)$ are isomorphic for two Bott matrices A, B in $\mathfrak{B}(n)$.

Operation (Op1). For a permutation matrix P of a permutation σ on $\{1, 2, \dots, n\}$, we define a map Φ_P on $n \times n$ matrices such that

$$\Phi_P(A) := PAP^{-1}.$$

Thus if we set $B = \Phi_P(A)$, then

$$A^i_j = B^{\sigma(i)}_{\sigma(j)} \tag{3.1}$$

as observed in (2.2).

Operation (Op2). For $k \in \{1, 2, \dots, n\}$, we define Φ_k to be an operation to add the k-th column of an $n \times n$ matrix to every column having 1 in the k-th row. In other words, for an $n \times n$ binary matrix A, the $n \times n$ matrix $\Phi_k(A)$ is given by

$$\Phi_k(A)_j := A_j + A^k_j A_k \quad \text{for } j \in \{1, 2, \dots, n\}. \tag{3.2}$$

We note that if $A \in \mathfrak{B}(n)$, then $\Phi_k(A) \in \mathfrak{B}(n)$. In fact, if A is strictly upper triangular, then so is $\Phi_k(A)$, and the general case reduces to the strictly upper triangular case by (3.1). Since every diagonal entry of a Bott matrix A is zero, $(\Phi_k \circ \Phi_k)(A) = A$ and therefore Φ_k is a bijection on $\mathfrak{B}(n)$.

Operation (Op3). For distinct ℓ, m in $\{1, 2, \dots, n\}$, we define $\Phi^{\ell,m}$ on $n \times n$ matrices A with $A_\ell = A_m$ to be an operation to add the ℓ-th row to the m-th row. In other words, when $A_\ell = A_m$, then $\Phi^{\ell,m}(A)$ is defined to be an $n \times n$ matrix by

$$\Phi^{\ell,m}(A)^i := \begin{cases} A^\ell + A^m & \text{if } i = m, \\ A^i & \text{otherwise.} \end{cases} \tag{3.3}$$

Since the diagonal entries of a Bott matrix A are all zero, the condition $A_\ell = A_m$ implies that

$$0 = A^\ell_\ell = A^\ell_m \quad \text{and} \quad A^m_\ell = A^m_m = 0 \tag{3.4}$$

and one can check that $\Phi^{\ell,m}(A)$ stays in $\mathfrak{B}(n)$. Note that if $A_\ell = A_m$, then $\Phi^{\ell,m}(A)_\ell = \Phi^{\ell,m}(A)_m$ and $(\Phi^{\ell,m} \circ \Phi^{\ell,m})(A) = A$.

Definition. Two Bott matrices in $\mathfrak{B}(n)$ are *Bott equivalent* if one can be transformed into the other through a sequence of the three operations (Op1), (Op2) and (Op3).

We note that every Bott equivalence class in $\mathfrak{B}(n)$ has a representative of a strictly upper triangular matrix (not necessarily unique) because of the operation (Op1).

Example 3.1. There are two 2×2 strictly upper triangular binary matrices and they are not Bott equivalent. There are $2^3 = 8$ strictly upper triangular binary matrices of size 3 and they are classified into the following four Bott equivalence classes:

(1) The zero matrix of size 3

(2) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(4) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

There are $2^6 = 64$ strictly upper triangular binary matrices of size 4 and they are classified into 12 Bott equivalence classes, see [15] and [19]. Furthermore, there are $2^{10} = 1024$ strictly upper triangular binary matrices of size 5 and they are classified into 54 Bott equivalence classes; see Table 1.

Example 3.2. Let $\Delta(n)$ be the set of all $n \times n$ strictly upper triangular binary matrices A such that $A^1_2 = A^2_3 = \cdots = A^{n-1}_n = 1$. One can change $(i, i+2)$ entry into 0 for $i = 1, \ldots, n-2$ using the operation (Op2), so that A is Bott equivalent to the matrix $\bar{A}$ of the following form

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \bar{A}^1_4 & \bar{A}^1_5 & \ldots & \bar{A}^1_{n-1} & \bar{A}^1_n \\ 0 & 0 & 1 & 0 & \bar{A}^2_5 & \ldots & \bar{A}^2_{n-1} & \bar{A}^2_n \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 & \bar{A}^{n-4}_{n-1} & \bar{A}^{n-4}_n \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \bar{A}^{n-3}_n \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\bar{A}$ is uniquely determined by A and two matrices $A, B \in \Delta(n)$ are Bott equivalent if and only if $\bar{A} = \bar{B}$. Therefore $\Delta(n)$ has exactly $2^{(n-2)(n-3)/2}$ Bott equivalent classes for $n \geq 2$.

4. Acyclic digraphs

A *directed graph* (simply *digraph*) D consists of a finite set $V(D)$ of elements called *vertices* and a set $A(D)$ of ordered pairs of distinct vertices called *arcs*. Two digraphs D and H are *isomorphic* if there is a bijection $\psi\colon V(D) \to V(H)$ such that $(u, v) \in A(D)$ if and only if $(\psi(u), \psi(v)) \in A(H)$. If $(u, v) \in A(D)$, then v is called an *out-neighbor* of u and u is called an *in-neighbor* of v. For a vertex v of D, we denote by $N^+_D(v)$ and $N^-_D(v)$ the set of all out-neighbors and in-neighbors, respectively, of v. The *out-degree* $\deg^+_D(v)$ and the *in-degree* $\deg^-_D(v)$ of $v \in V(D)$ are the number of out-neighbors and in-neighbors, respectively, of v. An ordering $v_1, v_2, \ldots, v_n$ of the vertices is called *acyclic* if $i < j$ whenever $(v_i, v_j) \in A(D)$. A digraph is called *acyclic* if it admits an acyclic ordering. Equivalently, a digraph is acyclic if and only if it has no directed cycle; see [2, Proposition 1.4.3].

For a digraph D with an ordering of the vertices $v_1, v_2, \ldots, v_n$, the *adjacency matrix* A_D of D is an $n \times n$ binary matrix such that $(A_D)^i_j = 1$ if and only if $(v_i, v_j) \in A(D)$. To a Bott matrix A in $\mathfrak{B}(n)$, we associate a digraph D_A on n vertices $\{v_1, v_2, \ldots, v_n\}$ in such a way that (v_i, v_j) is an arc of D_A if and only if

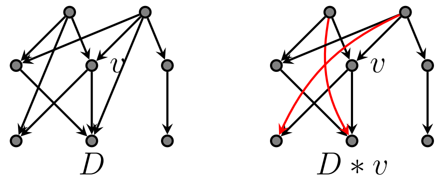

FIGURE 1. A local complementation.

$A^i_j = 1$. Equivalently D_A is the digraph whose adjacency matrix is A. In other words, v_j is an out-neighbor of v_i in D_A if and only if $A^i_j = 1$. Therefore,

$$N^+_{D_A}(v_i) = \{v_j \mid A^i_j = 1\} \quad \text{and} \quad \deg^+_{D_A}(v_i) = |\{j \mid A^i_j = 1\}|,$$
$$N^-_{D_A}(v_j) = \{v_i \mid A^i_j = 1\} \quad \text{and} \quad \deg^-_{D_A}(v_j) = |\{i \mid A^i_j = 1\}|,$$

and the statements (i) and (ii) in Lemma 2.2 can be translated as follows.

Lemma 4.1. *Let $A \in \mathfrak{B}(n)$. Then*

(i) $M(A)$ is orientable if and only if every vertex of D_A has an even out-degree.
(ii) $M(A)$ admits a symplectic form if and only if for each vertex v of D_A, there are even number of vertices (including v itself) having the same set of in-neighbors with v.

We claim that D_A is acyclic for each Bott matrix A in $\mathfrak{B}(n)$. If A is strictly upper triangular, this is obvious because $v_1, v_2, \ldots, v_n$ is an acyclic ordering. When A is conjugate to a strictly upper triangular matrix B by a permutation matrix of a permutation σ on $\{1, 2, \ldots, n\}$, then $A^i_j = B^{\sigma(i)}_{\sigma(j)}$ as observed in (2.2) and therefore D_A is isomorphic to D_B which is acyclic.

The mapping from $\mathfrak{B}(n)$ to the set of acyclic digraphs on fixed n vertices $\{v_1, v_2, \ldots, v_n\}$ is bijective. Therefore, the three operations introduced in Section 3 can be translated and visualized as operations on acyclic digraphs. It is easy to see that the operation (Op1) corresponds to the isomorphism of graphs.

In the following, we will discuss operations corresponding to (Op2) and (Op3). For sets X and Y, we denote $(X \setminus Y) \cup (Y \setminus X)$ by $X \Delta Y$.

Local complementation. Let D be a digraph. For $v \in V(D)$, we define $D * v$ to be the digraph with $V(D * v) = V(D)$ and

$$A(D * v) = A(D) \Delta \{(u, w) \in N^-_D(v) \times N^+_D(v)\}.$$

Namely, $(u, w) \in N^-_D(v) \times N^+_D(v)$ is removed from D if it is an arc of D and added to D otherwise. The operation to obtain $D * v$ from D is called the *local complementation* at v. See Figure 1

Note that $D * v * v = D$. If D is acyclic, then so is $D * v$.

Slide. For distinct vertices u, v of a digraph D with $N^-_D(u) = N^-_D(v)$ (possibly empty), we define $D \diamond uv$ to be the digraph with $V(D \diamond uv) = V(D)$ and

$$A(D \diamond uv) = A(D) \Delta \{(v, w) \mid w \in N^+_D(u)\}.$$

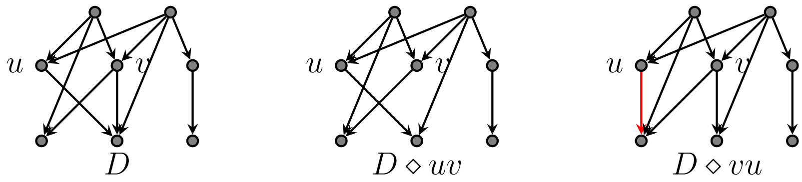

FIGURE 2. A slide.

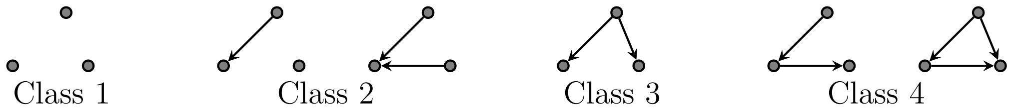

FIGURE 3. Four Bott equivalence classes of acyclic digraphs on 3 vertices.

Namely, when $N_D^-(u) = N_D^-(v)$, (v, w) for $w \in N_D^+(u)$ is removed from D if $w \in N_D^+(v)$ and added to D otherwise. We call it the *slide* on uv. See Figure 2. We were not able to find a literature on slides. Note that $D \diamond uv \diamond uv = D$ and $D \diamond uv \neq D \diamond vu$. If D is acyclic, then so is $D \diamond uv$.

The acyclic digraph D_A associated with a Bott matrix A in $\mathfrak{B}(n)$ has the canonical acyclic ordering $v_1, \dots, v_n$ of the vertices and one can easily check that

$$D_{\Phi_k(A)} = D_A * v_k \quad \text{and} \quad D_{\Phi^{\ell,m}(A)} = D_A \diamond v_\ell v_m.$$

This means that the operation Φ_k in (3.2) corresponds to the local complementation at v_k and the operation $\Phi^{\ell,m}$ in (3.3) corresponds to the slide on $v_\ell v_m$.

We say that two digraphs are *Bott equivalent* if one is transformed into an isomorphic copy of the other through successive application of local complementations and slides. The above observation shows that the correspondence $A \to D_A$ gives a bijective correspondence between Bott equivalence classes in $\mathfrak{B}(n)$ and Bott equivalence classes of acyclic digraphs on n vertices.

Example 4.2. There are 2 non-isomorphic acyclic digraphs on 2 vertices and they are not Bott equivalent. There are 6 non-isomorphic acyclic digraphs on 3 vertices and they are classified into four Bott equivalence classes listed in Figure 3. (Compare with Example 3.1.)

We list the number $\mathcal{D}_n$ of Bott equivalence classes of acyclic digraphs on n vertices in Table 1 up to $n = 8$. Note that $\mathcal{D}_n$ is in between $2^{(n-2)(n-3)/2}$ (see Example 3.2) and the number of non-isomorphic acyclic digraphs on n vertices counted by Robinson [24].

5. AFFINE DIFFEOMORPHISMS

In this section we associate an affine diffeomorphism between real Bott manifolds to each operation introduced in Section 3, and prove the implication (1) $\Rightarrow$ (2) in Theorem 1.1. We restate it for convenience as follows.

Proposition 5.1. *If Bott matrices A, B in $\mathfrak{B}(n)$ are Bott equivalent, then the associated real Bott manifolds $M(A)$ and $M(B)$ are affinely diffeomorphic.*

Proof. It suffices to find a group isomorphism $\phi\colon G(B)\to G(A)$ and a ϕ-equivariant affine automorphism $\tilde f$ of T^n which induces an affine diffeomorphism from $M(B)$ to $M(A)$. We may assume that $B=\Phi_P(A)$, $B=\Phi_k(A)$, or $B=\Phi^{\ell,m}(A)$.

The case of the operation (Op1). Suppose $B=\Phi_P(A)$ for a permutation matrix P of a permutation σ on $\{1,2,\dots,n\}$.

We define a group isomorphism $\phi_P\colon G(B)\to G(A)$ by

$$\phi_P(b_{\sigma(i)}):=a_i \tag{5.1}$$

and an affine automorphism $\tilde f_P$ of T^n by

$$\tilde f_P(z_1,\dots,z_n):=(z_{\sigma(1)},\dots,z_{\sigma(n)}).$$

Then it follows from (2.1) (applied to $b_{\sigma(i)}$) that the j-th component of $\tilde f_P(b_{\sigma(i)}(z))$ ($z\in T^n$) is $z_{\sigma(j)}(B^{\sigma(i)}_{\sigma(j)})$ for $j\neq i$ and $-z_{\sigma(i)}$ for $j=i$ while that of $a_i(\tilde f_P(z))$ is $z_{\sigma(j)}(A^i_j)$ for $j\neq i$ and $-z_{\sigma(i)}$ for $j=i$. Since $A^i_j=B^{\sigma(i)}_{\sigma(j)}$ by (3.1), this shows that $\tilde f_P$ is ϕ_P-equivariant.

It follows from Lemma 2.3 and (5.1) that the affine diffeomorphism $f_P\colon M(B)\to M(A)$ induced from $\tilde f_P$ satisfies

$$f_P^*(x_j)=y_{\sigma(j)}\quad\text{for all } j\in\{1,2,\dots,n\}. \tag{5.2}$$

The case of the operation (Op2). Suppose $B=\Phi_k(A)$. We define a group isomorphism $\phi_k\colon G(B)\to G(A)$ by

$$\phi_k(b_i):=a_ia_k^{A^i_k} \tag{5.3}$$

and an affine automorphism $\tilde f_k$ of T^n by

$$\tilde f_k(z_1,\dots,z_n):=(z_1,\dots,z_{k-1},\sqrt{-1}z_k,z_{k+1},\dots,z_n).$$

We shall check that $\tilde f_k$ is ϕ_k-equivariant, that is

$$\tilde f_k(b_i(z))=a_ia_k^{A^i_k}(\tilde f_k(z))\quad\text{for all } i\in\{1,2,\dots,n\}\text{ and } z\in T^n. \tag{5.4}$$

The identity is obvious when $i=k$ because $A^k_k=0$ and $B^k_j=A^k_j$ for every j by (3.2). Suppose $i\neq k$. Then the j-th component of the left hand side of (5.4) is given by

$$\begin{cases} z_j(B^i_j) & \text{for } j\neq i,k,\\ -z_i & \text{for } j=i,\\ \sqrt{-1}(z_k(B^i_k)) & \text{for } j=k,\end{cases}$$

while that of the right hand side of (5.4) is given by

$$\begin{cases} z_j(A^i_j+A^k_jA^i_k) & \text{for } j\neq i,k,\\ -z_i(A^k_iA^i_k) & \text{for } j=i,\\ (-1)^{A^i_k}(\sqrt{-1}z_k)(A^i_k) & \text{for } j=k.\end{cases}$$

Since $B^i_j=A^i_j+A^k_jA^i_k$ by (3.2), the j-th components above agree for $j\neq i,k$. They also agree for $j=i$ because either A^k_i or A^i_k is zero. We note that $B^i_k=A^i_k$ by

(3.2) because $A^k_k = 0$. Therefore the k-th components above are both $\sqrt{-1}z_k$ when $B^i_k = A^i_k = 0$ and $\sqrt{-1}\bar{z}_k$ when $B^i_k = A^i_k = 1$. Thus the j-th components above agree for every j.

It follows from Lemma 2.3 and (5.3) that the affine diffeomorphism $f_k \colon M(B) \to M(A)$ induced from $\tilde{f}_k$ satisfies

$$(f_k)^*(x_j) = y_j \quad \text{for } j \neq k, \qquad (f_k)^*(x_k) = y_k + \sum_{i=1}^{n} A^i_k y_i. \tag{5.5}$$

The case of the operation (Op3). Suppose that $B = \Phi^{\ell,m}(A)$. We define a group isomorphism $\phi^{\ell,m} \colon G(B) \to G(A)$ by

$$\phi^{\ell,m}(b_i) := \begin{cases} a_\ell a_m & \text{for } i = m, \\ a_i & \text{for } i \neq m, \end{cases} \tag{5.6}$$

and an affine automorphism $\tilde{f}^{\ell,m}$ of T^n by

$$\tilde{f}^{\ell,m}(z_1, \dots, z_n) := (z_1, \dots, z_{\ell-1}, z_\ell z_m, z_{\ell+1}, \dots, z_n).$$

We shall check that $\tilde{f}^{\ell,m}$ is $\phi^{\ell,m}$-equivariant. To simplify notation we abbreviate $\tilde{f}^{\ell,m}$ and $\phi^{\ell,m}$ as $\tilde{f}$ and ϕ respectively. What we prove is the identity

$$\tilde{f}(b_i(z)) = \phi(b_i)\tilde{f}(z) \quad \text{for all } i \in \{1, 2, \dots, n\} \text{ and } z \in T^n. \tag{5.7}$$

Assume $i \neq \ell, m$. Then the j-th component of the left hand side of (5.7) is given by

$$\tilde{f}(b_i(z))_j = \begin{cases} z_j(B^i_j) & \text{for } j \neq i, \ell, \\ -z_i & \text{for } j = i, \\ z_\ell(B^i_\ell) z_m(B^i_m) & \text{for } j = \ell, \end{cases}$$

while since $\phi(b_i) = a_i$ by (5.6), the j-th component of the right hand side of (5.7) is given by

$$(\phi(b_i)\tilde{f}(z))_j = \begin{cases} z_j(A^i_j) & \text{for } j \neq i, \ell, \\ -z_i & \text{for } j = i, \\ (z_\ell z_m)(A^i_\ell) & \text{for } j = \ell. \end{cases}$$

This shows that $\tilde{f}(b_i(z))_j = (\phi(b_i)\tilde{f}(z))_j$ for all j because $B^i = A^i$ by (3.3) and $A_\ell = A_m$ by the condition on A (hence $B^i_j = A^i_j$ for any j and $B^i_\ell = A^i_\ell = A^i_m = B^i_m$), proving (5.7) when $i \neq \ell, m$.

Assume $i = \ell$. Then

$$\tilde{f}(b_\ell(z))_j = \begin{cases} z_j(B^\ell_j) & \text{for } j \neq \ell, \\ -z_m(B^\ell_m) z_\ell & \text{for } j = \ell, \end{cases}$$

while since $\phi(b_\ell) = a_\ell$ by (5.6), we have

$$(\phi(b_\ell)\tilde{f}(z))_j = \begin{cases} z_j(A^\ell_j) & \text{for } j \neq \ell, \\ -z_\ell z_m & \text{for } j = \ell. \end{cases}$$

This shows that $\tilde{f}(b_\ell(z))_j = (\phi(b_\ell)\tilde{f}(z))_j$ for all j because $B^\ell_j = A^\ell_j$ for any j and $B^\ell_m = A^\ell_m = 0$ by (3.3) and (3.4), proving (5.7) when $i = \ell$.

Assume $i = m$. Then

$$\tilde{f}(b_m(z))_j = \begin{cases} z_j(B^m_j) & \text{for } j \neq \ell, m, \\ -z_m & \text{for } j = m, \\ -z_\ell(B^m_\ell)z_m & \text{for } j = \ell, \end{cases}$$

while since $\phi(b_m) = a_\ell a_m$ by (5.6), we have

$$(\phi(b_m)\tilde{f}(z))_j = \begin{cases} z_j(A^\ell_j + A^m_j) & \text{for } j \neq \ell, m, \\ -z_m(A^\ell_m) & \text{for } j = m, \\ (-z_\ell z_m)(A^m_\ell) & \text{for } j = \ell. \end{cases}$$

This shows that $\tilde{f}(b_m(z))_j = (\phi(b_m)\tilde{f}(z))_j$ for all j because $B^m_j = A^\ell_j + A^m_j$ for any j, $A^\ell_m = 0$, $B^m_\ell = A^\ell_\ell + A^m_\ell = 0$ and $A^m_\ell = 0$ by (3.3) and (3.4), proving (5.7) when $i = m$.

It follows from Lemma 2.3 and (5.6) that the affine diffeomorphism $f^{\ell,m} \colon M(B) \to M(A)$ induced from $\tilde{f}^{\ell,m}$ satisfies

$$(f^{\ell,m})^*(x_j) = \begin{cases} y_\ell + y_m & \text{for } j = \ell, \\ y_j & \text{for } j \neq \ell. \end{cases} \tag{5.8}$$ $\square$

6. Cohomology isomorphisms

In this section we prove the latter statement in Theorem 1.1 and the implication (3) ⇒ (1) at the same time, summarized in the following proposition.

Proposition 6.1. *Let A, B be Bott matrices in $\mathfrak{B}(n)$. Every isomorphism*

$$H^*(M(A); \mathbb{Z}/2) \to H^*(M(B); \mathbb{Z}/2)$$

is induced from a composition of affine diffeomorphisms corresponding to the three operations (Op1), (Op2) and (Op3), and if $H^*(M(A); \mathbb{Z}/2)$ and $H^*(M(B); \mathbb{Z}/2)$ are isomorphic as graded rings, then A and B are Bott equivalent.

By Proposition 5.1, we may assume through affine diffeomorphisms corresponding to (Op1) that our Bott matrices are strictly upper triangular. We introduce a notion and prepare a lemma. Remember that

$$H^*(M(A); \mathbb{Z}/2) = \mathbb{Z}/2[x_1, \dots, x_n]/(x_j^2 = x_j \sum_{i=1}^n A^i_j x_i \mid j = 1, \dots, n). \tag{6.1}$$

One easily sees that if A is strictly upper triangular, then products $x_{i_1} \dots x_{i_q}$ $(1 \leq i_1 < \cdots < i_q \leq n)$ form a basis of $H^q(M(A); \mathbb{Z}/2)$ as a vector space over $\mathbb{Z}/2$ so that the dimension of $H^q(M(A); \mathbb{Z}/2)$ is $\binom{n}{q}$ (see [18, Lemma 5.3]).

We set

$$\alpha_j = \sum_{i=1}^n A^i_j x_i \quad \text{for } j \in \{1, 2, \dots, n\}$$

where $\alpha_1 = 0$ since A is a strictly upper triangular matrix. Then the relations in (6.1) are written as

$$x_j^2 = \alpha_j x_j \quad \text{for } j \in \{1, 2, \dots, n\}. \tag{6.2}$$

Motivated by this identity we introduce the following notion.

Definition. We say that an element $\alpha \in H^1(M(A); \mathbb{Z}/2)$ is an *eigen-element* of $H^*(M(A); \mathbb{Z}/2)$ if there exists $x \in H^1(M(A); \mathbb{Z}/2)$ such that

$$x^2 = \alpha x, \quad x \neq 0, \quad \text{and} \quad x \neq \alpha.$$

The *eigen-space* of α, denoted by $\mathcal{E}_A(\alpha)$, is the set of all elements $x \in H^1(M(A); \mathbb{Z}/2)$ satisfying the equation

$$x^2 = \alpha x.$$

Clearly $\mathcal{E}_A(\alpha)$ is a vector subspace of $H^1(M(A); \mathbb{Z}/2)$. We also introduce a notation $\bar{\mathcal{E}}_A(\alpha)$ which is the quotient of $\mathcal{E}_A(\alpha)$ by the subspace spanned by α, and call it the *reduced eigen-space* of α.

Eigen-elements and (reduced) eigen-spaces are invariants preserved under graded ring isomorphisms. By (6.2), $\alpha_1, \alpha_2, \dots, \alpha_n$ are eigen-elements of $H^*(M(A); \mathbb{Z}/2)$ and the following lemma shows that these are the only eigen-elements.

Lemma 6.2. *Let A be a strictly upper triangular Bott matrix in $\mathfrak{B}(n)$. Let $\alpha_j = \sum_{i=1}^n A^i_j x_i$ for all $j = 1, 2, \dots, n$. If α is an eigen-element of $H^*(M(A); \mathbb{Z}/2)$, then $\alpha = \alpha_j$ for some j and the eigen-space $\mathcal{E}_A(\alpha)$ of α is generated by α and all x_i's with $\alpha_i = \alpha$.*

Proof. By definition there exists a non-zero element $x \in H^1(M(A); \mathbb{Z}/2)$ different from α such that $x^2 = \alpha x$. Since both x and $x + \alpha$ are non-zero, there exist i and j such that $x = x_i + p_i$ and $x + \alpha = x_j + q_j$ where p_i is a linear combination of $x_1, \dots, x_{i-1}$ and q_j is a linear combination of $x_1, \dots, x_{j-1}$. Then

$$\alpha = x_i + x_j + p_i + q_j \quad \text{and} \quad x_i x_j + x_i q_j + x_j p_i + p_i q_j = 0 \tag{6.3}$$

where the latter identity follows from $x(x + \alpha) = 0$. As remarked above, products $x_{i_1} x_{i_2}$ ($1 \leq i_1 < i_2 \leq n$) form a basis of $H^2(M(A); \mathbb{Z}/2)$, and therefore $i = j$ for the latter identity in (6.3) to hold. Then, since $x_j^2 = x_j \alpha_j$, it follows from the latter identity in (6.3) that $\alpha_j = q_j + p_i$ (and $p_i q_j = 0$). This together with the former identity in (6.3) shows that $\alpha = \alpha_j$, proving the former statement of the lemma.

We express a non-zero element $x \in \mathcal{E}_A(\alpha)$ as $\sum_{i=1}^n c_i x_i$ ($c_i \in \mathbb{Z}/2$) and let m be the maximum number among i's with $c_i \neq 0$. We call the number m the leading suffix of x.

Case 1. The case where x_m appears when we express α as a linear combination of $x_1, \dots, x_n$. We express $x(x + \alpha)$ as a linear combination of the basis elements $x_{i_1} x_{i_2}$ ($1 \leq i_1 < i_2 \leq n$). Since x_m appears in both x and α, it does not appear in $x + \alpha$. Therefore the term in $x(x + \alpha)$ which contains x_m is $x_m(x + \alpha)$ and it must vanish because $x(x + \alpha) = 0$. Therefore $x = \alpha$.

Case 2. The case where x_m does not appear in the linear expression of α. In this case, the term in $x(x + \alpha)$ which contains x_m is $x_m(x_m + \alpha) = x_m(\alpha_m + \alpha)$ since

$x_m^2 = \alpha_m x_m$, and it must vanish because $x(x+\alpha) = 0$ and x_m does not appear in the linear expression of α_m. Therefore $\alpha_m = \alpha$ and hence $x_m \in \mathcal{E}_A(\alpha)$. The sum $x + x_m$ is again an element of $\mathcal{E}_A(\alpha)$. If $x = x_m$, then we are done. Suppose $x \neq x_m$ (equivalently $x + x_m$ is non-zero). Then the leading suffix of $x + x_m$, say m_1, is strictly smaller than m and the same argument applied to $x + x_m$ shows that $\alpha_{m_1} = \alpha$ and $x + x_m + x_{m_1}$ is again an element of $\mathcal{E}_A(\alpha)$. Repeating this argument, x ends up with a linear combination of x_i's with $\alpha_i = \alpha$. □

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. As remarked before, we may assume that both A and B are strictly upper triangular. Let $x_1, x_2, \dots, x_n$ be the canonical basis of $H^*(M(A); \mathbb{Z}/2)$ and let $y_1, y_2, \dots, y_n$ be the canonical basis of $H^*(M(B); \mathbb{Z}/2)$. Let $\alpha_j = \sum_{i=1}^n A^i_j x_i$, $\beta_j = \sum_{i=1}^n B^i_j y_i$ for $j \in \{1, 2, \dots, n\}$.

Let $\varphi\colon H^*(M(A); \mathbb{Z}/2) \to H^*(M(B); \mathbb{Z}/2)$ be a graded ring isomorphism. It preserves the eigen-elements and (reduced) eigen-spaces. In the following we shall show that we can change φ into the identity map by composing isomorphisms induced from affine diffeomorphisms corresponding to the operations (Op1), (Op2) and (Op3).

Through an affine diffeomorphism corresponding to the operation (Op1) we may assume that $\varphi(\alpha_j) = \beta_j$ for all j because of (5.2). Then φ restricts to an isomorphism $\mathcal{E}_A(\alpha_j) \to \mathcal{E}_B(\beta_j)$ between eigen-spaces and induces an isomorphism $\bar{\mathcal{E}}_A(\alpha_j) \to \bar{\mathcal{E}}_B(\beta_j)$ between reduced eigen-spaces.

Let α, β be an eigen-element of $H^*(M(A); \mathbb{Z}/2)$ and $H^*(M(B); \mathbb{Z}/2)$, respectively. Suppose that $\varphi(\alpha) = \beta$. Let

$$J = \{j \mid \alpha_j = \alpha, j \in \{1, 2, \dots, n\}\}.$$

Let $\bar{x}_j$, $\bar{y}_j$ be the image of x_j in $\bar{\mathcal{E}}_A(\alpha)$ and the image of y_j in $\bar{\mathcal{E}}_B(\beta)$, respectively. By Lemma 6.2, $\{\bar{x}_j \mid j \in J\}$ is a basis of $\bar{\mathcal{E}}_A(\alpha)$ and $\{\bar{y}_j \mid j \in J\}$ is a basis of $\bar{\mathcal{E}}_B(\beta)$. Thus if we express

$$\varphi(\bar{x}_j) = \sum_{i \in J} F^i_j \bar{y}_i \quad \text{for } j \in J$$

with $F^i_j \in \mathbb{Z}/2$, then the matrix $F_J := (F^i_j)_{i,j \in J}$ is invertible.

Since $\alpha_j = \sum_{i=1}^n A^i_j x_i$, we have $A_\ell = A_m$ if and only if $\alpha_\ell = \alpha_m$. Therefore we can apply affine diffeomorphisms corresponding to the operation (Op3) to every pair of distinct ℓ, m in J. Let $A' = \Phi^{\ell,m}(A)$ and $f = f^{\ell,m}\colon M(A') \to M(A)$ be the affine diffeomorphism considered in the previous section. Let $x'_1, \dots, x'_n$ be the canonical generators of $H^*(M(A'); \mathbb{Z}/2)$. Then it follows from (5.8) that if we express

$$(\varphi \circ (f^{-1})^*)(\bar{x'_j}) = \sum_{i \in J} F'^i_j \bar{y}_i \quad \text{for } j \in J,$$

then the matrix $F'_J = (F'^i_j)_{i,j \in J}$ is obtained from F_J by adding m-th column to ℓ-th column. Similarly, an affine diffeomorphism corresponding to the operation (Op1) induces a permutation of columns of F_J by (5.2). Since F_J is an invertible binary matrix, one can change it to the identity matrix by permuting columns and adding a column to another column. Therefore, through a sequence of affine diffeomorphisms

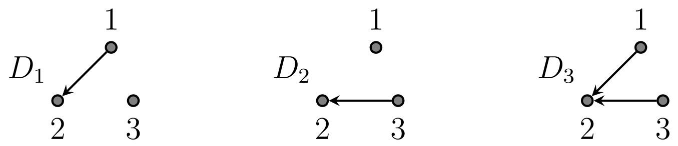

FIGURE 4. Three Bott equivalent acyclic digraphs D_1, D_2, and D_3.

corresponding to the operations (Op1) and (Op3), we may assume that F_J is the identity matrix. This can be done for each J so that we may assume that

$$\varphi(x_j) = y_j \text{ or } y_j + \beta_j \quad \text{for every } j \in \{1, 2, \ldots, n\}.$$

Finally, through an affine diffeomorphism corresponding to the operation (Op2), we may assume that $\varphi(x_j) = y_j$ for every j by (5.5). This means that after a successive application of the operations (Op1), (Op2) and (Op3), we reach $A = B$ because $\varphi(\alpha_j) = \beta_j$, $\alpha_j = \sum_{i=1}^n A_j^i x_i$ and $\beta_j = \sum_{i=1}^n B_j^i y_i$ for every $j \in \{1, 2, \ldots, n\}$, proving the proposition. □

7. Unique Decomposition of Real Bott manifolds

We say that a real Bott manifold is *indecomposable* if it is not diffeomorphic to a product of more than one real Bott manifolds. The purpose of this section is to provide a graph theoretical proof to Theorem 1.3 in Section 1. An algebraic proof can be found in [17].

The *disjoint union* $D_1 \oplus D_2$ of two digraphs D_1 and D_2 is a digraph on the disjoint union of $V(D_1)$ and $V(D_2)$ such that $A(D_1 \oplus D_2) = A(D_1) \cup A(D_2)$. Obviously, both D_1 and D_2 are acyclic if and only if so is $D_1 \oplus D_2$. A digraph is *connected* if it is connected as an undirected graph. An acyclic digraph D is *indecomposable* if all acyclic digraphs Bott equivalent to D are connected.

For example, in Figure 4, three acyclic digraphs D_1, D_2, and D_3 are shown. Obviously these digraphs are Bott equivalent. Notice that D_3 is connected but D_1 and D_2 are disconnected. Thus Bott equivalent graphs of a connected graph are not necessarily connected. Each component of D_1 and D_2 is indecomposable and yet $\{1, 2\} \neq \{2, 3\}$. Hence a decomposition of an acyclic digraph into indecomposable acyclic digraphs does not induce a unique partition of the vertex set.

Surprisingly the next theorem will show that a decomposition of an acyclic digraph into indecomposable acyclic digraphs is unique up to Bott equivalence. Note that a real Bott manifold $M(A)$ for a Bott matrix $A \in \mathfrak{B}(n)$ is indecomposable if and only if the acyclic digraph associated to A is indecomposable. Therefore Theorem 1.3 is equivalent to the following theorem.

Theorem 7.1. *Suppose that $D_1, D_2, \ldots, D_k$ and $H_1, H_2, \ldots, H_\ell$ are indecomposable acyclic digraphs. If $\bigoplus_{i=1}^k D_i$ is Bott equivalent to $\bigoplus_{j=1}^\ell H_j$, then $k = \ell$ and there is a permutation σ on $\{1, \ldots, k\}$ such that D_i is Bott equivalent to $H_{\sigma(i)}$ for each $i = 1, \ldots, k$.*

We prepare several lemmas before the proof of Theorem 7.1. A vertex of a digraph D is called a *root* if it has no in-neighbor. In other words, a vertex v of D is a root if and only if the column vector in A_D corresponding to v is zero. Let $L_0(D)$ be the set of roots of D.

Obviously local complementations do not change connected components. It is easy to check that slides at non-roots do not change connected components as well, because if we want to slide uv for non-roots u and v, then u and v must share a common in-neighbor. The only trouble that might arise is that slides at roots may change connected components as we have seen in Figure 4.

The following lemma is easy to check.

Lemma 7.2. *Let D be an acyclic digraph. Let u, v be distinct roots of D.*

(i) If $N_D^-(x) = N_D^-(y)$ for distinct $x, y \in V(D) \setminus L_0(D)$, then

$$D \diamond xy \diamond uv = D \diamond uv \diamond xy.$$

(ii) For each vertex x of D, we have

$$D * x \diamond uv = D \diamond uv * x.$$

By Lemma 7.2, one can push all slides on $L_0(D)$ to the left. So it suffices to consider only slides on $L_0(D)$ when we are concerned with the indecomposability of connected components of D. Clearly slides do not change the set of roots.

For subsets X and Y of $V(D)$, we denote by $[X, Y]_D$ the submatrix of A_D whose rows correspond to X and columns to Y, and by $\mathrm{row}[X, Y]_D$ the vector subspace of $(\mathbb{Z}/2)^{|Y|}$ generated by row vectors in $[X, Y]_D$. For a set X of vertices of D, we write $D \setminus X$ to denote the subgraph obtained by deleting vertices in X and all the edges incident with a vertex in X. For a vertex v of D, we simply write $D \setminus v$ for $D \setminus \{v\}$.

Lemma 7.3. *Let D and H be acyclic digraphs such that $V(D) = V(H)$. Then H can be obtained from D by applying slides only on $L_0(D)$ if and only if the following two conditions hold.*

(1) $D \setminus L_0(D) = H \setminus L_0(H)$.

(2) $L_0(D) = L_0(H)$ and $\mathrm{row}[L_0(D), V(D) \setminus L_0(D)]_D = \mathrm{row}[L_0(H), V(H) \setminus L_0(H)]_H$.

Proof. The forward implication is trivial, because slides on roots are row additions in the matrix $[L_0(D), V(D) \setminus L_0(D)]_D$.

To prove the converse, let us assume that (1) and (2) hold. Let $Y = V(D) \setminus L_0(D)$. If necessary, we can interchange two rows corresponding to x and y of $[L_0(D), Y]_D$ by replacing D with $D \diamond xy \diamond yx \diamond xy$. Hence, all elementary row operations can be obtained by slides on roots. By (2), $[L_0(D), Y]_D$ and $[L_0(H), Y]_H$ have the same reduced row echelon form. Therefore, $[L_0(H), Y]_H$ can be obtained from $[L_0(D), Y]_D$ by elementary row operations. This together with (1) implies the lemma. □

Lemma 7.4. *Let D be an acyclic digraph with at least two vertices. If D is indecomposable, then*

$$\mathrm{rank}[L_0(D), V(D) \setminus L_0(D)]_D = |L_0(D)|.$$

Proof. Since D is connected, D must have a non-root. If $\mathrm{rank}[L_0(D), V(D) \setminus L_0(D)]_D < |L_0(D)|$, then there is a set of row vectors whose sum is zero. So by applying slides, we obtain an acyclic digraph H having a vertex with no out-neighbors and therefore H is disconnected. This contracts to the assumption that D is indecomposable. $\square$

Lemma 7.5. *Let D be an indecomposable acyclic digraph. Let $Y = V(D) \setminus L_0(D)$. Let G be an acyclic digraph. If H is obtained from $D \oplus G$ by applying slides on $L_0(D \oplus G)$, then there exists a set X of roots of H such that $|X| = |L_0(D)|$ and $H[X \cup Y]$ is connected, where $H[X \cup Y] = H \setminus (V(H) \setminus (X \cup Y))$.*

Proof. Proof. If D has a single vertex, then this is trivial. So we may assume that D has non-roots. Let $Z = V(G) \setminus L_0(G)$. Since $H \setminus Z$ can be obtained from $(D \oplus G) \setminus Z$ by applying slides on $L_0((D \oplus G) \setminus Z) = L_0(D \oplus G)$, we deduce from Lemma 7.3 that

$$\mathrm{row}[L_0(D \oplus G), Y]_{D \oplus G} = \mathrm{row}[L_0(H), Y]_H.$$

By Lemma 7.4, $\mathrm{rank}[L_0(D), Y]_D = |L_0(D)|$. Therefore there exists a subset X of $L_0(H)$ such that $|X| = |L_0(D)|$ and the rows in $[X, Y]_H$ are linearly independent.

By considering an isomorphic copy of H, we may assume that $X = L_0(D)$. This implies that

$$\mathrm{row}[L_0(H), Y]_H = \mathrm{row}[X, Y]_H = \mathrm{row}[X, Y]_{H[X \cup Y]}.$$

Since $L_0(G)$ has no arcs to Y in $D \oplus G$,

$$\mathrm{row}[L_0(D \oplus G), Y]_{D \oplus G} = \mathrm{row}[X, Y]_D.$$

Therefore $\mathrm{row}[X, Y]_D = \mathrm{row}[X, Y]_{H[X \cup Y]}$.

Then D and $H[X \cup Y]$ satisfy (1) and (2) of Lemma 7.3 and therefore $H[X \cup Y]$ can be obtained from D by applying slides on $L_0(D)$. Since D is indecomposable, $H[X \cup Y]$ must be connected. $\square$

$\square$

Now, we complete the proof of Theorem 7.1.

Proof of Theorem 7.1. We claim that it is enough to consider the case when H is obtained from D through slides on $L_0(D)$. Suppose there is a sequence of local complementations and slides to apply to D to obtain H. By Lemma 7.2, we may assume that slides on $L_0(D)$ are done first. Let H' be the acyclic digraph obtained by applying all the slides on $L_0(D)$. Then H can be obtained from H' by applying slides on non-roots and local complementations. Since these operations do not change the connected components, H and H' must have the identical set of connected components up to Bott equivalence. By reversing these slides and local complementations, we can observe that each component of H' is indecomposable and therefore H' is the disjoint union of ℓ indecomposable acyclic digraphs. This proves the claim.

Then by Lemma 7.3,

$$\begin{aligned} L_0(D) &= L_0(H), \\ D \setminus L_0(D) &= H \setminus L_0(H), \\ \mathrm{row}[L_0(D), V(D) \setminus L_0(D)]_D &= \mathrm{row}[L_0(H), V(H) \setminus L_0(H)]_H. \end{aligned} \tag{7.1}$$

Let $A_i = [L_0(D_i), V(D_i) \setminus L_0(D_i)]_{D_i}$. Then

$$\operatorname{rank}[L_0(D), V(D) \setminus L_0(D)]_D = \operatorname{rank} \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix} = \sum_{i=1}^{k} \operatorname{rank} A_i.$$

Similarly if $B_i = [L_0(H_i), V(H_i) \setminus L_0(H_i)]_{H_i}$, then

$$\operatorname{rank}[L_0(H), V(H) \setminus L_0(H)]_H = \sum_{i=1}^{\ell} \operatorname{rank} B_i.$$

By (7.1), $\sum_{i=1}^{k} \operatorname{rank} A_i = \sum_{i=1}^{\ell} \operatorname{rank} B_i$. By Lemma 7.4, $\operatorname{rank} A_i = |L_0(D_i)|$ if D_i has at least two vertices and therefore $|L_0(D)| - \sum_{i=1}^{k} \operatorname{rank} A_i$ is the number of isolated vertices in D. By (7.1), $|L_0(D)| - \sum_{i=1}^{k} \operatorname{rank} A_i = |L_0(H)| - \sum_{i=1}^{\ell} \operatorname{rank} B_i$ and therefore D and H should have the same number of isolated vertices. Let s be the number of isolated vertices and we may assume that $D_1, \ldots, D_{k-s}$ and $H_1, \ldots, H_{\ell-s}$ have non-roots. Lemma 7.5 implies that there exists a function $\sigma : \{1, 2, \ldots, k-s\} \to \{1, 2, \ldots, \ell - s\}$ such that for each $i \in \{1, 2, \ldots, k - s\}$, $V(D_i) \setminus L_0(D_i)$ stays in one connected component H_j of H for some $j = \sigma(i)$ and moreover $|V(H_j)| \geq |V(D_i)|$. Similarly $V(H_j) \setminus L_0(H_j)$ stays in one connected component D_m of D for some m with $|V(D_m)| \geq |V(H_j)|$. Since $V(D_i) \setminus L_0(D_i) \subseteq V(D_m)$, we conclude that $i = m$, $|V(D_i)| = |V(H_j)|$, and $V(D_i) \setminus L_0(D_i) = V(H_j) \setminus L_0(H_j)$. We may assume that $V(D_i) = V(H_j)$ by permuting roots. Then it is easy to observe that D_i and H_j satisfy (1) and (2) of Lemma 7.3 from (7.1) and therefore D_i and $H_{\sigma(i)}$ are Bott equivalent. Clearly σ is injective because D_i and $H_{\sigma(i)}$ must share non-roots. Since $V(D) = V(H)$, σ should be bijective and $k = \ell$. $\square$

8. Numerical invariants of real Bott manifolds

In this section, we produce numerical invariants of real Bott manifolds $M(A)$ using the Bott matrix $A \in \mathfrak{B}(n)$ or its associated acyclic digraph D_A.

8.1. **Type.** Recall that $L_0(D)$ is the set of roots of D. For $k \geq 1$, we define $L_k(D)$ to be the set of vertices v of D such that a longest directed path ending at v has exactly k arcs in D. We call $L_k(D)$ for $k \geq 0$ the *k-th level set* of D and the sequence

$$(|L_0(D)|, |L_1(D)|, |L_2(D)|, \ldots, |L_{n-1}(D)|)$$

the *type* of D.

Proposition 8.1. *If D and H are Bott equivalent acyclic digraph, then $|L_k(D)| = |L_k(H)|$ for all nonnegative integer k. In particular, Bott equivalent acyclic digraphs have the identical type.*

Proof. It is enough to show that both local complementations and slides do not change $L_i(D)$. Let w be a vertex in $L_i(D)$. Then there is a longest directed path P_w in D ending at w with exactly i arcs.

Let us first show that $w \in L_i(D * v)$ for any $v \in V(D)$. It is enough to show that P_w is a path in $D * v$ as well, because, if so, then a longest path in $D * v$ ending at w will be a path in $(D * v) * v = D$ as well. Suppose that P_w is not a path in $D * v$. Then the local complementation at v must remove at least one arc (x, y) of P_w and therefore (x, v) and (v, y) are arcs of D. Since D is acyclic, v is not on P_w. Then by replacing the arc (x, y) by a path xvy in P_w, we can find a path longer than P_w in D, contradictory to the assumption that P_w is a longest path ending at w. This proves the claim that $L_i(D) = L_i(D * v)$ for all i.

Now let us prove that $w \in L_i(D \diamond uv)$ for $u \neq v \in V(D)$ with $N_D^-(u) = N_D^-(v)$. Again, it is enough to show that$D \diamond uv$ has a path of length i ending at w; because if $D \diamond uv$ has a longer path ending at w, then so does D by the fact that $(D \diamond uv) \diamond uv = D$. We may assume that the slide along uv removes at least one arc (x, y) of P_w. Then $v = x$ and both (v, y) and (u, y) are arcs of D. Since $N_D^-(u) = N_D^-(v)$, we can replace v by u in P_w to obtain a path of the same length in $D \diamond uv$. This completes the proof. □

The level sets of D_A for a Bott matrix $A \in \mathfrak{B}(n)$ can be described in terms of A as follows. We identify the i-th vertex v_i of D_A with i for $i \in \{1, 2, \ldots, n\}$. Then $L_0(D_A)$ can be identified with $L_0(A) := \{j \mid A_j = 0\}$. We define $A(1)$ to be the matrix obtained from A by removing all j-th columns and j-th rows for all $j \in L_0(A)$. Then $L_1(A) := \{j \mid A(1)_j = 0\}$ can be identified with $L_1(D_A)$. Inductively we define $A(k)$ for $k \geq 2$ to be the matrix obtained from $A(k-1)$ by removing all j-th columns and j-th rows for all j with $A(k-1)_j = 0$. Then $L_k(A) := \{j \mid A(k)_j = 0\}$ can be identified with $L_k(D_A)$.

The type of D_A can also be described in terms of $H^*(M(A); \mathbb{Z}/2)$ as follows. First note that $|L_0(D_A)|$ agrees with the dimension of the $\mathbb{Z}/2$-vector space

$$N(\mathcal{H}_0) = \{x \in \mathcal{H}_0^1 \mid x^2 = 0\}$$

where $\mathcal{H}_0 = H^*(M(A); \mathbb{Z}/2)$ and $\mathcal{H}_0^1$ denotes the degree one part of $\mathcal{H}_0$, i.e., $\mathcal{H}_0^1 = H^1(M(A); \mathbb{Z}/2)$. We define $\mathcal{H}_1$ to be the quotient graded ring of $\mathcal{H}_0$ by the ideal generated by $N(\mathcal{H}_0)$ and inductively define $\mathcal{H}_k$ for $k \geq 2$ to be the quotient graded ring of $\mathcal{H}_{k-1}$ by the ideal generated by $N(\mathcal{H}_{k-1})$. Then $|L_k(D_A)|$ agrees with $\dim N(\mathcal{H}_k)$.

8.2. **Rank.** The operations (Op1), (Op2) and (Op3) in Section 3 preserve the rank of a Bott matrix A in $\mathfrak{B}(n)$, denoted $\operatorname{rank} A$. Therefore the matrix rank is an invariant of Bott equivalent acyclic digraphs. Here is a geometrical meaning of $\operatorname{rank} A$.

Proposition 8.2. *Let A be a Bott matrix in $\mathfrak{B}(n)$. Then*

$$\dim_{\mathbb{Q}} H^1(M(A); \mathbb{Q}) \leq n - \operatorname{rank} A \quad \textit{and} \quad \sum_{i=0}^{n} \dim_{\mathbb{Q}} H^i(M(A); \mathbb{Q}) = 2^{n - \operatorname{rank} A}.$$

Proof. Remember that $M(A)$ is the quotient of T^n by the action of a finite group $G(A)$; see Section 2. Therefore it follows from [5, Theorem 2.4 in p.120] that

$$H^i(M(A); \mathbb{Q}) = H^i(T^n; \mathbb{Q})^{G(A)} \quad \text{for every } i,$$

where the right hand side above denotes the invariants of the induced $G(A)$-action on $H^*(T^n;\mathbb{Q})$. Then it is shown in [14, Lemma 2.1] that

$$\dim_{\mathbb{Q}} H^i(M(A);\mathbb{Q}) = \left|\{J \subseteq \{1,\dots,n\} \mid |J| = i, \sum_{j\in J} A_j = 0\}\right|. \tag{8.1}$$

In particular, $\dim_{\mathbb{Q}} H^1(M(A);\mathbb{Q})$ agrees with the number of zero column vectors in A which is less than or equal to $n - \operatorname{rank} A$, proving the inequality in the proposition.

It also follows from (8.1) that

$$\sum_{i=0}^{n} \dim_{\mathbb{Q}} H^i(M(A);\mathbb{Q}) = |X|,$$

where $X = \{J \subseteq \{1,\dots,n\} \mid \sum_{j\in J} A_j = 0\}$. Since an element of X corresponds to a vector in the null space of A whose dimension is $n - \operatorname{rank} A$, we have $|X| = 2^{n-\operatorname{rank} A}$, proving the equality in the proposition. □

In 1985, Halperin [13] conjectured that if a compact torus T^k of dimension k acts on a finite dimensional topological space X *almost freely*, then

$$\sum_{i=0}^{\dim X} \dim_{\mathbb{Q}} H^i(X;\mathbb{Q}) \geq 2^k.$$

This conjecture is called the *toral rank conjecture* or the *Halperin-Carlsson conjecture*. No counterexamples and some partial affirmative answers are known, see [1] for example. We show that the toral rank conjecture holds for real Bott manifolds.

Theorem 8.3. *Let A be a Bott matrix in $\mathfrak{B}(n)$. If $M(A)$ admits an effective topological action of a torus T^k of dimension k, then*

$$\sum_{i=0}^{n} \dim_{\mathbb{Q}} H^i(M(A);\mathbb{Q}) \geq 2^k.$$

Proof. Choose any point $p \in M(A)$ and consider a map $f_p\colon T^k \to M(A)$ defined by $f_p(t) := tp$. Let $\pi_1(X)$ be the fundamental group of a topological space X and let $Z(\pi_1(X))$ be the center of $\pi_1(X)$. Since $M(A)$ is an aspherical manifold, f_p induces an injective homomorphism $\pi_1(T^k) \to Z(\pi_1(M(A)))$ (which implies that the action of T^k on $M(A)$ is almost free, i.e., any isotropy subgroup is finite), see [9]. Kamishima and Nazra [16, Proposition 2.4] have shown that the intersection of $Z(\pi_1(M(A)))$ and the commutator subgroup $[\pi_1(M(A)), \pi_1(M(A))]$ of $\pi_1(M(A))$ is a trivial subgroup of $\pi_1(M(A))$. Hence, the natural map

$$\pi_1(T^k) \to Z(\pi_1(M(A))) \to \pi_1(M(A))/[\pi_1(M(A)), \pi_1(M(A))] = H_1(M(A);\mathbb{Z})$$

is injective. It follows that

$$k \leq \dim_{\mathbb{Q}} H_1(M(A);\mathbb{Q}) = \dim_{\mathbb{Q}} H^1(M(A);\mathbb{Q}).$$

Hence the theorem follows from Proposition 8.2. □

8.3. **Odd out-degree vertices.** The *odd height* of an acyclic digraph D is the length of a longest directed path ending at a vertex of odd out-degree. In other words, it is the maximum k such that $L_k(D)$ contains a vertex of odd out-degree. If D has no vertex of odd out-degree, then we define the odd height of D to be ∞.

Proposition 8.4. *Bott equivalent acyclic digraphs have the same odd height.*

Proof. Let D be an acyclic digraph and let $v \in V(D)$. Let $x \neq y \in V(D)$ with $N_D^-(x) = N_D^-(y)$. Then

$$\deg^+_{D*v}(w) \equiv \begin{cases} \deg^+_D(w) + \deg^+_D(v) & \pmod 2 \quad \text{if } w \in N_D^-(v), \\ \deg^+_D(w) & \pmod 2 \quad \text{otherwise,} \end{cases} \tag{8.2}$$

$$\deg^+_{D\diamond xy}(w) \equiv \begin{cases} \deg^+_D(w) + \deg^+_D(x) & \pmod 2 \quad \text{if } w = y, \\ \deg^+_D(w) & \pmod 2 \quad \text{otherwise.} \end{cases} \tag{8.3}$$

Therefore, if every vertex of D has even out-degree, then a local complementation and a slide do not create a vertex of odd out-degree. So we may assume that D has a vertex of odd out-degree. Let k be the odd height of D and let w be a vertex of odd out-degree in $L_k(D)$.

Let us first consider $D * v$ for a vertex $v \in L_i(D)$. If $w \in N_D^-(v)$, then $\deg^+_D(v)$ is even as $i > k$; so $\deg^+_{D*v}(w)$ is odd by (8.2). If $w \notin N_D^-(v)$, then $\deg^+_{D*v}(w)$ is again odd by (8.2). This proves that the odd height of $D * v$ is at least the odd height of D. Since $(D * v) * v = D$, we conclude that D and $D * v$ have the same odd height.

Now we claim that $L_k(D \diamond xy)$ has a vertex of odd out-degree in $D \diamond xy$. Suppose that $\deg^+_{D\diamond xy}(w)$ is even. Then $w = y$ and $\deg^+_D(x)$ is odd by (8.3). We note that $x \in L_k(D)$ because x and y are in the same level set of D and $w = y$ is in $L_k(D)$. Therefore $x \in L_k(D)$ and $\deg^+_D(x)$ is odd, proving the claim. Since $(D \diamond xy) \diamond xy = D$, we conclude that D and $D \diamond xy$ have the same odd height. □

By Lemma 4.1 (i), for a Bott matrix A in $\mathfrak{B}(n)$, $M(A)$ is orientable if and only if the out-degree of every vertex of D_A is even. In other words, $M(A)$ is orientable if and only if the odd height of D_A is ∞. Hence, the notion of odd height may be thought of as a refinement of the orientability of real Bott manifolds.

8.4. **Sibling classes.** For $x, y \in V(D)$, we say that $x \sim_D y$ if $N_D^-(x) = N_D^-(y)$. Then $\sim_D$ is an equivalence relation on $V(D)$ and we call an equivalence class a *sibling class* of D. If $x \sim_D y$, then x and y are in the same level set; so each level $L_i(D)$ is partitioned into sibling classes. We note that a sibling class of D_A for a Bott matrix $A \in \mathfrak{B}(n)$ corresponds to a maximal set of identical columns of A.

Proposition 8.5. *Sibling classes are invariant under Bott equivalence.*

Proof. Let x and y be vertices in the same sibling class of an acyclic digraph D. Since $N_D^-(x) = N_D^-(y)$, for every $w \in V(D)$, both (w, x) and (w, y) are arcs of D or neither (w, x) nor (w, y) are arcs of D.

Firstly let us consider the case $D * v$ for $v \in V(D)$. If both (v, x) and (v, y) are arcs of D, then one easily sees that $N^-_{D*v}(x) = N^-_{D*v}(y)$. If both (v, x) and (v, y) are not arcs of D, then the set of in-neighbors of x and y remains unchanged under the local complementation at v. Therefore $N^-_{D*v}(x) = N^-_{D*v}(y)$ in any case.

Secondly let us consider the case $D \diamond uv$ for $u \neq v \in V(D)$ with $N^-_D(u) = N^-_D(v)$. If both (u,x) and (u,y) are not arcs of D, then the set of in-neighbors of x and y remains unchanged under the slide on uv. Suppose that both (u,x) and (u,y) are arcs of D. Then, (v,x) (resp. (v,y)) is an arc of $D \diamond uv$ if and only if (v,x) (resp. (v,y)) is not an arc of D. Therefore $N_{D\diamond uv}(x) = N_{D\diamond uv}(y)$ in any case. □

By Lemma 4.1 (ii), for a Bott matrix $A \in \mathfrak{B}(n)$, $M(A)$ admits a symplectic form if and only if the cardinality of every sibling class of D_A is even. Hence, the notion of sibling class can be seen as a refinement of the symplecticity of real Bott manifolds. It is easy to see that if the cardinality of each sibling class of D is even, then the odd height of D must be ∞. This is obvious from the topological viewpoint because every symplectic manifold is orientable.

8.5. **Cut-rank.** Recall that, for subsets X and Y of the vertex set of a digraph D, we write $[X,Y]_D$ to denote the submatrix of the adjacency matrix of D whose rows correspond to X and columns correspond to Y. Let $\rho_D(X) = \operatorname{rank}[X, V(D)\setminus X]_D$. This function $\rho_D\colon 2^{V(D)} \to \mathbb{Z}$ is called the *cut-rank* function of D.

The cut-rank function naturally appeared when studying properties of local complementations. In 1985, Bouchet [3] studied the cut-rank function on undirected graphs together with local complementation. There are motivations based on matroid theory. In fact, the cut-rank function of a undirected bipartite graph is the Tutte connectivity function of a binary matroid having that graph as a fundamental graph, see Oum [21]. Moreover local complementations of undirected graphs can be discovered when trying to find appropriate graph operations to describe matroid minors. This allowed generalizations of theorems on binary matroids to undirected graphs. For digraphs, in 1987, Bouchet [4] studied the cut-rank function of directed graphs and proved that local complementation on directed graphs preserves the cut-rank function. The name "cut-rank" was first introduced in Oum and Seymour [22].

For our application, the cut-rank function is not preserved under sliding. But slides can be applied only to the siblings, and therefore we can prove that the cut-rank function on a union of level sets is preserved as follows.

Proposition 8.6. *Let D, H be Bott equivalent acyclic digraphs on n vertices. Then,*

(i) $\rho_D(\cup_{j\in J} L_j(D)) = \rho_H(\cup_{j\in J} L_i(H))$ *for every subset J of* $\{0,1,2,\dots,n-1\}$,

(ii) $\operatorname{rank}[L_j(D), L_{j+1}(D)]_D = \operatorname{rank}[L_j(H), L_{j+1}(H)]_H$ *for all* $j \in \{0,1,2,\dots,n-2\}$.

Proof. It is enough to prove when $H = D * v$ or $H = D \diamond uw$. By Proposition 8.1, $L_j(D) = L_j(H)$ for each j.

(i) For a subset J of $\{0,1,2,\dots,n-1\}$, let $X = \cup_{j\in J} L_j(D)$ and $Y = V(D)\setminus X$. Let $M = [X,Y]_D$ and $M' = [X,Y]_H$. Then $\rho_D(X) = \operatorname{rank} M$ and $\rho_H(X) = \operatorname{rank} M'$.

Let us first consider the case when $H = D * v$ for a vertex v of D. Then either $v \in X$ or $v \in Y$. If $v \in X$, then M has a row indexed by v and M' is obtained from M by adding the row of v to all rows indexed by in-neighbors of v in X. If $v \in Y$, then M has a column indexed by v and M' is obtained from M by adding

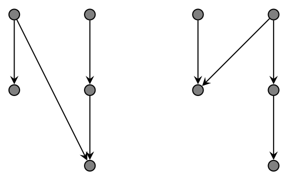

FIGURE 5. Theses acyclic digraphs are not Bott equivalent but have the identical set of invariants discussed in this paper.

the column of v to all columns indexed by out-neighbors of v in Y. Thus, in both cases, $\operatorname{rank} M = \operatorname{rank} M'$.

Now let us consider the case when $H = D \diamond uw$ for two vertices u, w having the same set of in-neighbors. Since u and w have the same set of in-neighbors, they belong to the same level. Therefore either $\{u, w\} \subseteq X$ or $\{u, w\} \subseteq Y$. If $\{u, w\} \subseteq Y$, then $M' = M$. If $\{u, w\} \subseteq X$, then M' is obtained from M by adding the row of u to the row of w. So $\operatorname{rank} M = \operatorname{rank} M'$. This completes the proof of (i).

(ii) Since D is acyclic, there is no arc from $L_a(D)$ to $L_b(D)$ if $a > b$. Therefore

$$\operatorname{rank}[L_j(D), L_{j+1}(D)]_D = \rho_D(L_j(D) \cup L_{j+2}(D) \cup L_{j+3}(D) \cup \cdots \cup L_{n-1}(D)),$$

and by (i), we have (ii). $\square$

We remark that the invariants discussed in Section 8 completely classify all Bott equivalence classes up to 4 vertices but not on 5 vertices. One can easily check that two acyclic digraphs in Figure 5 are not Bott equivalent but have the same set of invariants.

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