ON THE GROWTH OF HYPERBOLIC 3-DIMENSIONAL GENERALIZED SIMPLEX REFLECTION GROUPS

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Abstract. We prove that the growth rates of three-dimensional generalized simplex reflection groups, i.e. three-dimensional non-compact hyperbolic Coxeter groups with four generators are always Perron numbers.

1. Introduction

A convex polyhedron P of finite volume in the n-dimensional hyperbolic space $\mathbb{H}^n$ is called a *Coxeter polyhedron* if its dihedral angles are submultiples of π. Any Coxeter polyhedron is a fundamental domain of the discrete group Γ generated by the set S consisting of the reflections with respects to its facets. We call (Γ, S) an *n-dimensional hyperbolic Coxeter group.* In particular when P is a (generalized) simplex of $\mathbb{H}^n$, (Γ, S) is also called a (*generalized*) *simplex reflection group* ([8]). In this situation we can define the *word length* $\ell_S(x)$ *of* $x \in \Gamma$ *with respect to* S by the smallest integer $n \geq 0$ for which there exist $s_1, s_2, \cdots, s_n \in S$ such that $x = s_1 s_2 \cdots s_n$. The *growth function* $f_S(t)$ of (Γ, S) is the formal power series $\sum_{k=0}^{\infty} a_k t^k$ where a_k is the number of elements $g \in \Gamma$ satisfying $\ell_S(g) = k$. It is known that the *growth rate* of (Γ, S), $\omega := \limsup_{k\to\infty} \sqrt[k]{a_k}$ is bigger than 1 ([3]) and less than or equal to the cardinality $|S|$ of S. By means of Cauchy-Hadamard formula, the *radius of convergence* R of $f_S(t)$ is the reciprocal of ω, i.e. $1/|S| \leq R < 1$. In practice the growth function $f_S(t)$ which is analytic on $|t| < R$ extends to a rational function $P(t)/Q(t)$ on $\mathbb{C}$ by analytic continuation where $P(t), Q(t) \in \mathbb{Z}[t]$ are relatively prime. There are formulas due to Solomon and Steinberg to calculate the rational function $P(t)/Q(t)$ from the Coxeter diagram of (Γ, S) ([10, 11]. See also [4]).

Theorem 1. (Solomon's formula)
The growth function $f_S(t)$ of an irreducible spherical Coxeter group (Γ, S) can be written as $f_S(t) = \prod_{i=1}^{k} [m_i + 1]$ where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of (Γ, S).

Theorem 2. (Steinberg's formula)
Let (Γ, S) be a hyperbolic Coxeter group. Let us denote the Coxeter subgroup

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of (Γ, S) *generated by the subset* $T \subseteq S$ *by* (Γ_T, T)*, and denote its growth function by* $f_T(t)$*. Set* $\mathcal{F} = \{T \subseteq S \,:\, \Gamma_T$ *is finite* $\}$*. Then*

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

In this case, $t = R$ is a pole of $f_S(t)$. Hence R is a real zero of the denominator $Q(t)$ closest to the origin $0 \in \mathbb{C}$ of all zeros of $Q(t)$. Solomon's formula implies that $P(0) = 1$. Hence $a_0 = 1$ means that $Q(0) = 1$. Therefore $\omega > 1$, the reciprocal of R, becomes a real algebraic integer whose conjugates have moduli less than or equal to the modulus of ω. If $t = R$ is the unique zero of $Q(t)$ with the smallest modulus, then $\omega > 1$ is a real algebraic integer whose conjugates have moduli less than the modulus of ω: such a real algebraic integer is called a *Perron number*.

For two and three-dimensional cocompact hyperbolic Coxeter groups, Cannon-Wagreich and Parry showed that the growth rates are Salem numbers ([1, 7]), where a real algebraic integer $\tau > 1$ is called a *Salem number* if τ^{-1} is an algebraic conjugate of τ and all algebraic conjugates of τ other than τ and τ^{-1} lie on the unit circle. From the definition, a Salem number is a Perron number.

Kellerhals and Perren calculated the growth functions of all four-dimensional cocompact hyperbolic Coxeter groups with at most 6 generators and showed that ω are not Salem numbers while they checked that ω are Perron numbers numerically. ([6]).

In the non-compact case, Floyd proved that the growth rates of two-dimensional non-compact hyperbolic Coxeter groups are *Pisot-Vijayaraghavan numbers*, where a real algebraic integer $\tau > 1$ is called a Pisot-Vijayaraghavan number if algebraic conjugates of τ other than τ lie in the unit disk ([2]). A Pisot-Vijayaraghavan number is also a Perron number by definition.

From these results for low-dimensional cases, Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter groups are always Perron numbers. In the present paper, we go to the next stage: three-dimensional non-compact hyperbolic Coxeter groups of finite covolume. We will show that the growth rate of a three-dimensional generalized simplex reflection group is a Perron number.

2. Denominators of growth functions

There are exactly 23 three-dimensional generalized simplex reflection groups ([5, 8]). By means of Steinberg's formula we can calculate growth functions of them.

Proposition 1. *The denominator polynomials* $Q(t)$ *of the growth functions* $f_S(t) = P(t)/Q(t)$ *of the 23 three-dimensional generalized simplex reflection groups* (Γ, S) *are as follows:*

- $(t-1)(3t^2 + t - 1)$
- $(t-1)(3t^3 + t^2 + t - 1)$

- $(t-1)(2t^4+3t^3+t^2-1)$
- $(t-1)(t^5+t^4+t-1)$
- $(t-1)(2t^5+t^4+t^2+t-1)$
- $(t-1)(3t^5+t^4+t^3+t^2+t-1)$
- $(t-1)(t^7+t^6+t^5+t^4+t^3-1)$
- $(t-1)(t^7+t^6+t^5+t^4-1)$
- $(t-1)(t^7+t^6+2t^5+2t^4+t^3+t^2-1)$
- $(t-1)(t^7+t^6+2t^5+t^4+t^3+t-1)$
- $(t-1)(t^8+2t^7+2t^6+3t^5+t^4+t^3-1)$
- $(t-1)(t^9+t^7+t^6+t^4+t^2+t-1)$
- $(t-1)(t^{13}+t^{12}+2t^{11}+2t^{10}+2t^9+2t^8+2t^7+2t^6+2t^5+t^4+t^3-1)$
- $(t-1)(t^2+t+1)(t^2+t-1)$
- $(t-1)(t^4+t^3+t^2+t+1)(t^2+t-1)$
- $(t-1)(t^3+t-1)$
- $(t-1)(t^4+t^3+t^2+t+1)(t^3+t-1)$
- $(t-1)(t^4+t^3+t^2+t-1)$
- $(t-1)(t^4+t^3+t^2+t+1)(t^4+t^3+t^2+t-1)$
- $(t-1)(t^5+t^4+t^2-1)$
- $(t-1)(t^5+t^3+t-1)$
- $(t-1)(t^6+t^5+t^4+t^3+t^2+t-1)$
- $(t-1)(t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t-1)$

We remark that the factor $(t-1)$ appears in every denominator of $f_S(t)$ because of the fact that $1/f_S(1)=\chi(\Gamma)=0$ in the odd-dimensional case due to a result of Serre ([9]).

3. Main result

Theorem 3. *The growth rate of a three-dimensional generalized simplex reflection group is a Perron number.*

In Table 1 below, we show the distributions of poles of $f_S(t)$ for a particular case of three-dimensional generalized simplex reflection groups.

By Proposition 1, the following lemma is sufficient to prove the theorem.

Lemma 1. *Consider the polynomial of degree $n \geq 2$*

$$g(t)=\sum_{k=1}^{n} a_k t^k - 1,$$

where a_k is a non-negative integer. We also assume that the greatest common divisor of $\{k \in \mathbb{N} \mid a_k \neq 0\}$ is 1. Then there is a real number r_0, $0<r_0<1$ which is the unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$.

Proof. Let us put $h(t)=\sum_{k=1}^{n} a_k t^k$. Note that $g(t)=0$ if and only if $h(t)=1$.

(Step1) Observe $h(0) = 0$, $h(1) > 1$, and $h(t)$ is strictly monotone increasing where t is in the open interval $(0, 1)$. From the intermediate value theorem, there exists the unique real number r_0 in $(0, 1)$ such that $h(r_0) = 1$.

(Step2) Suppose there exists a complex number z whose absolute value is less than r_0 and satisfying the condition $h(z) = 1$. Denote $z = re^{i\theta}$ where $0 < r < r_0$ and $0 \leqslant \theta < 2\pi$. Then

$$1 = |h(z)| = |\sum_{k=1}^{n} a_k (re^{i\theta})^k| \leq \sum_{k=1}^{n} |(a_k r^k) e^{ik\theta}| = \sum_{k=1}^{n} a_k r^k = h(r) < h(r_0) = 1,$$

which is a contradiction. Hence r_0 has the smallest absolute value of all zeros of $g(t)$.

(Step3) Consider a complex number z whose absolute value is equal to r_0. Set $z = r_0 e^{i\theta}$ and $0 \leqslant \theta < 2\pi$. Then $1 = \sum_{k=1}^{n} a_k r_0^k e^{ik\theta}$ implies

$$1 = \sum_{k=1}^{n} a_k r_0^k \cos k\theta \leq \sum_{k=1}^{n} a_k r_0^k = 1$$

Hence $\cos k\theta = 1$ for any $k \in \mathbb{N}$ with $a_k \neq 0$. The assumption that the greatest common divisor of $\{k \in \mathbb{N} \mid a_k \neq 0\}$ is 1 means that $\theta = 0$. Therefore $z = r_0$, and we conclude that r_0 is the unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$. $\square$

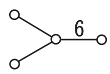
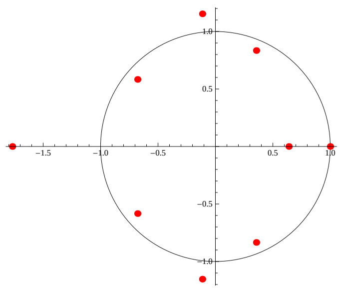
Coxeter diagram	
$f_S(t)$	$\frac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^8+2t^7+2t^6+3t^5+t^4+t^3-1)}$
poles of $f_S(t)$	

Table 1.

4. Remark

By Proposition 1, the next lemma shows that some growth rates of three-dimensional generalized simplex reflection groups are not only Perron numbers but also Pisot-Vijayaraghavan numbers (see Table 2 below).

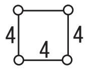
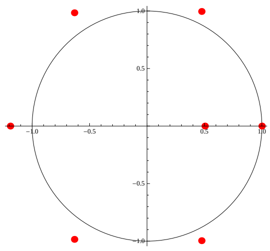
Coxeter diagram	
$f_S(t)$	$\dfrac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^6+t^5+t^4+t^3+t^2+t-1)}$
poles of $f_S(t)$	

Table 2.

Lemma 2. *For $n \geq 2$, the polynomial $g(t) = \sum_{k=1}^{n} t^k - 1$ has the unique zero in the unit disk $\{t \in \mathbb{C} \mid |t| < 1\}$ and does not have zeros on the unit circle $|t| = 1$.*

Proof. Define $h_1(t) = t^{n+1}$, $h_2(t) = -2t + 1$, and

$$h(t) = h_1(t) + h_2(t) = t^{n+1} - 2t + 1 = (t-1)g(t).$$

Then for any $1/2 < r < 1$ sufficiently close to 1, $h(r) < 0$. Any complex number t on the circle $\{t \in \mathbb{C} \mid |t| = r\}$ satisfies

$$|h_1(t)| = |t^{n+1}| = r^{n+1} < 2r - 1 \leq |2t - 1| = |h_2(t)|.$$

Because $h_2(t)$ has the unique zero $t = 1/2$ in the disk $|t| < r$, it follows from Rouché's theorem that $h(t)$ also has the unique zero in the disk $|t| < r$. Since this holds for any $r < 1$ sufficiently close to 1, it means that $h(t)$, hence $g(t)$ has the unique zero in the unit disk $|t| < 1$. Finally we show that $g(t)$ does not have zeros on the unit circle $|t| = 1$. Suppose there exists $\theta \in \mathbb{R}$ such that $g(e^{i\theta}) = 0$. Then $h(e^{i\theta}) = 0$ implies that $1 = |e^{i(n+1)\theta}| = |2e^{i\theta} - 1|$. Hence $e^{i\theta} = 1$, which contradicts to $g(1) \neq 0$. Therefore $g(t)$ has the unique zero in the unit disk $\{t \in \mathbb{C} \mid |t| < 1\}$ and does not have zeros on the unit circle $|t| = 1$. □

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