ON EXTENDIBILITY OF BERS ISOMORPHISM

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ABSTRACT. Let S be a closed Riemann surface of genus $g(\geqq 2)$ and set $\dot{S} = S \setminus \{\widehat{z}_0\}$. Then we have the composed map $\varphi \circ r$ of a map $r : T(S) \times U \to F(S)$ and the Bers isomorphism $\varphi : F(S) \to T(\dot{S})$, where $F(S)$ is the Bers fiber space of S, $T(X)$ is the Teichmüller space of X and U is the upper haf-plane.

The purpose of this paper is to show the map $\varphi \circ r : T(S) \times U \to T(\dot{S})$. has a continuous extension to some subset of the boundary $T(S) \times \partial U$.

1. INTRODUCTION

Let S be a closed Riemann surface of genus $g(\geqq 2)$. Consider any pair (R, f) of a closed Riemann surface R of genus g and a quasiconformal map $f : S \to R$. Two pairs (R_1, f_1) and (R_2, f_2) are said to be *equivalent* if $f_2 \circ f_1^{-1} : R_1 \to R_2$ is homotopic to a biholomorphic map $h : R_1 \to R_2$. Let $[R, f]$ be the equivalence class of such a pair (R, f). We set

$$T(S) = \{[R, f] \mid f : S \to R : \mathrm{qc}\}$$

and call $T(S)$ the *Teichmüller space* of S.

It is known that S can be represented as U/G where U is the upper half-plane and G is a torsion free Fuchsian group.

Let $L_\infty(U, G)_1$ be the space of measurable function μ on U satisfying

(1) $\|\mu\|_\infty = \sup_{z \in U} |\mu(z)| < 1$,

(2) $(\mu \circ g)\dfrac{\overline{g}'}{g}$ for all $g \in G$.

For any $\mu \in L_\infty(U, G)_1$, there is a unique quasiconformal map w of U onto U satisfying normalization conditions $w(0) = 0, w(1) = 1$ and $w(\infty) = \infty$. Let $Q(G)$ be the be the set of all normalized quasiconformal map w such that wGw^{-1} is also Fushsian. We wite $w = w_\mu$. Two maps $w_1, w_2 \in Q(G)$ are said to be *equivalent* if $w_1 = w_2$ on the real axis $\mathbb{R}$. Let $[w]$ be the equivalence class of $w \in Q(G)$. We set

$$T(G) = \{[w] \mid w \in Q(G)\}$$

and call $T(G)$ the *Teichmüller space* of G.

Then we have a canonical bijection

$$T(G) \ni [w_\mu] \mapsto [U/G_\mu, f_\mu] \in T(S) \tag{1.1}$$

where $G_\mu = w_\mu G w_\mu^{-1}$ and f_μ is the map induced by $w_\mu : U \to U$. Throughout this paper, we always identify $T(G)$ with $T(S)$ via the bijection (1.1).

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For any $\mu \in L_\infty(U,G)_1$, there is a unique quasiconformal map w of $\hat{\mathbb{C}}$ with $w(0)=0, w(1)=1, w(\infty)=\infty$, such that w satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ on U, and is conformal on the lower half-plane L. We write $w = w^\mu$.

The *Bers fiber space* $F(G)$ over $T(G)$ is defined by

$$F(G) = \{([w_\mu], z) \in T(G) \times \hat{\mathbb{C}} \mid [w_\mu] \in T(G),\ z \in w^\mu(U)\}.$$

Take a point $z_0 \in U$ and denote the set of all points $g(z_0)$, $g \in G$, by A. Let

$$v : U \to U - A$$

be a holomorphic universal covering map and define

$$\dot{G} = \{h \in \text{Aut } U \mid v \circ h = g \circ v \text{ for some } g \in G\ \}.$$

We see that $U/\dot{G} = U/G - \{\pi(z_0)\}$, where $\pi : U \to S = U/G$ is the natural projection. And set $\dot{S} = U/\dot{G}$. By Lemma 6.3 of Bers[1], every point in $F(G)$ is represented as a point $([w_\mu], w^\mu(z_0))$ for some $\mu \in L_\infty(U,G)_1$. For $\mu \in L_\infty(U,G)_1$, we define $\nu \in L_\infty(U,\dot{G})_1$ by

$$\mu(v(z))\frac{\overline{v'(z)}}{v'(z)} = \nu(z).$$

Hence we have a map $\varphi : F(G) \to T(\dot{G})$ by

$$([w_\mu], w^\mu(z_0)) \mapsto [w_\nu].$$

Then the important Bers isomorphism thorem (Theorem 9 of [1]) asserts that φ is a biholomorphic bijection map. Moreover we define a map $r : T(G) \times U \to F(G)$ by

$$([w_\mu], w_\mu(z_0)) \mapsto ([w_\mu], w^\mu(z_0)).$$

By Lemma 6.4 of [1], this map r is a real analytic bijection.

Via the bijection (1.1), the Bers fiber space $F(S)$ over $T(S)$ is defined by

$$F(S) = \{([R_\mu, f_\mu], z) \in T(S) \times \hat{\mathbb{C}} \mid [R_\mu, f_\mu] \in T(S),\ z \in w^\mu(U)\}.$$

Similarly, we have the isomorphism $F(S) \to T(\dot{S})$ and the real analytic bijection $T(\dot{S}) \times U \to F(S)$, and we denote them by the same symbols φ and r, respectively.

The Teichmüller space $T(S)$ can be regarded canonically as a bounded domain of a complex Banach space $B_2(L,G)$ in the following way: let $B_2(L,G)$ consist of all holomorphic functions ϕ defined on L such that

$$\phi(g(z))g'(z)^2 = \phi(z) \text{ for } g \in G \text{ and } z \in L$$

and

$$\|\phi\|_\infty = \sup_{z \in L} |(\text{Im}z)^2 \phi(z)| < \infty.$$

For any $\mu \in L_\infty(U,G)_1$, we denote by ϕ^μ the Schwarzian derivative of w^μ in L, that is,

$$\phi^\mu = \{w^\mu, z\} = \frac{(w^\mu)'''(z)}{(w^\mu)'(z)} - \frac{3}{2}\left(\frac{(w^\mu)''(z)}{(w^\mu)'(z)}\right)^2.$$

If $\mu \in L_\infty(U,G)_1$, then $\phi^\mu \in B_2(L,G)$ and the *Bers embedding* $T(S) \ni [R_\mu, f_\mu] \mapsto \phi^\mu \in B_2(L,G)$ is a biholomorphic bijection of $T(S)$ onto a holomorphically bounded domain in $B_2(L,G)$. From now on, we will identify $T(S)$ with its canonical image in $B_2(L,G)$.

Similarly, we define the Bers embedding of $T(\dot{S})$ into $B_2(L, \dot{G})$. Since $F(S)$ is a domain of $B_2(L, G) \times \hat{\mathbb{C}}$ and $T(\dot{S})$ is a bounded domain in $B_2(L, \dot{G})$, we define the topological boundaries of them naturally. Let $\overline{F(G)}$ denote the closure of $F(G)$.

Zhang [13] proved the Bers isomorphism φ cannot be continuously extended to $\overline{F(S)}$ if the dimension of $T(S)$ is greater than zero. Then we have the following question: is there a subset of $\overline{F(S)} - F(S)$ to which φ can be continuously extended ?

To consider this question, we will use results of Leininger, Mj and Schleimer about the curve complexes of S and of $\dot{S}$ in [7]. To do this, first we compose the isomorphism $\varphi : F(S) \to T(\dot{S})$ and the map $r : T(S) \times U \to F(S)$, then we obtain new map $\varphi \circ r : T(S) \times U \to T(\dot{S})$.

On the other hand, Leininger, Mj and Schleimer defined a map $\Phi : \mathcal{C}(S) \times U \to \mathcal{C}(\dot{S})$, where $\mathcal{C}(S)$ and $\mathcal{C}(\dot{S})$ are the curve complexes of S and of $\dot{S}$, respectively. (For definitions and more details, see §3). Let $\mathbb{A}$ be a subset of ∂U consisting of all points filling S. Then they proved that the map $\Phi(v, \cdot)$ can be continuously extended to $\{v\} \times \mathbb{A}$ for any $v \in \mathcal{C}(S)$.

To use their results, we define a map $\mathcal{E} : T(S) \to \mathcal{C}(S)$ by sending p to a simple closed curve on S of the minimal extremal length Ext_p (similarly, define $\dot{\mathcal{E}} : T(\dot{S}) \to \mathcal{C}(\dot{S})$) then we conisder the following diagram

$$\begin{array}{ccc} T(S) \times U & \xrightarrow{\phi \circ r} & T(\dot{S}) \\ {\scriptstyle \mathcal{E} \times id}\big\downarrow & & {\scriptstyle \dot{\mathcal{E}}}\big\downarrow \\ \mathcal{C}(S) \times U & \xrightarrow{\Phi} & \mathcal{C}(\dot{S}) \end{array}$$

Our main theorem is as follows:

Theorem 4.1 *The map $\varphi \circ r : T(S) \times U \to T(\dot{S})$ has a limit in $\{p_0\} \times \mathbb{A}$ for any point $p_0 \in T(S)$.*

2. Gromov-hyperbolic spaces

In this section, we shall give the boundary at infinity of hyperbolic space. For details, see Klarreich [6].

Let (Δ, d) be a metric space. If Δ is equipped with a basepoint 0, we define the *Gromov product* $\langle x|y \rangle$ of points x and y in Δ by

$$\langle x|y \rangle = \langle x|y \rangle_0 = \frac{1}{2}\{d(x, 0) + d(y, 0) - d(x, y)\}.$$

For $\delta \geqq 0$, the metric space Δ is said to be *δ-hyperbolic* if

$$\langle x|y \rangle \geqq \min\{\langle x|z \rangle, \langle y|z \rangle\} - \delta$$

holds for every $x, y, z \in \Delta$ and for every choice of basepoint. We say that Δ is *hyperbolic in the sense of Gromov* if Δ is δ-hyperbolic for some $\delta \geqq 0$.

If Δ is a hyperbolic space, we can define a boundary of Δ in the following way: We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of points in Δ *converges at infinity* if it satisfies $\lim_{m,n\to\infty} \langle x_m|x_n \rangle = \infty$. Given two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ that converge at infinity, they are called to be *equivalent* if $\lim_{m,n\to\infty} \langle x_m|y_n \rangle = \infty$. Since Δ is a hyperbolic, we see that this is an equivalence relation ($\sim$). We set

$$\partial_\infty \Delta = \{\{x_n\}_{n=1}^{\infty} \mid \{x_n\}_{n=1}^{\infty} \text{converges at infinity}\}/\sim$$

and call $\partial_\infty \Delta$ the *boundary at infinity* of Δ. If $\xi \in \partial_\infty \Delta$, then we say that a sequence of points in Δ *converges to* ξ if the sequence belongs to the equivalence class ξ. We set

$$\overline{\Delta} = \Delta \cup \partial_\infty \Delta.$$

3. Leininger, Mj and Schleimer's work

3.1. **Curve Complex.** Let $S = U/G$ be a closed Riemann surface of genus $g(\geqq 2)$ and $\pi : U \to S$ be the natural projection. We take a point z_0 in U and set $\widehat{z}_0 = \pi(z_0)$. Put $\dot{S} = S \setminus \{\widehat{z}_0\}$.

We begin to define the curve complex $\mathcal{C}(S)$ of S in the following way: the vertices of $\mathcal{C}(S)$ are homotopy classes of non-peripheral simple closed curves on S. Two curves are connected by an edge if they can be realized disjointly on S, and in general a collection of curves spans a simplex if the curves can be realized disjointly on S. Similarly, we may define $\mathcal{C}(\dot{S})$.

We turn $\mathcal{C}(S)$(resp $\mathcal{C}(\dot{S})$) into a metric space by specifying that each edge has length 1, and define the distance $d_{\mathcal{C}(S)}$(resp $d_{\mathcal{C}(\dot{S})}$) by taking shortest paths.

Theorem 3.1 (Masur and Minsky [9], Theorem 1.1)**.** *The spaces $\mathcal{C}(S)$ and $\mathcal{C}(\dot{S})$ are δ-hypebolic for some $\delta > 0$.*

We put $\overline{\mathcal{C}}(S) = \mathcal{C}(S) \cup \partial_\infty \mathcal{C}(S)$ and $\overline{\mathcal{C}}(\dot{S}) = \mathcal{C}(\dot{S}) \cup \partial_\infty \mathcal{C}(\dot{S})$, respectively.

3.2. **Definition of Φ.** Denote by $\mathrm{Diff}^+(S)$ the group of all orientation preserving diffeomorphisms of S onto itself. Let $\mathrm{Diff}_0(S)$ be a group which consists of all elements in $\mathrm{Diff}^+(S)$ isotopic to the identity map id.

We define the evaluation map

$$\mathrm{ev} : \mathrm{Diff}^+(S) \to S$$

by $\mathrm{ev}(f) = f(\widehat{z}_0)$. A theorem of Earle and Eells asserts that $\mathrm{Diff}_0(S)$ is contractible. Hence, for the map $\mathrm{ev}|\mathrm{Diff}_0(S)$, there is a unique lift

$$\widetilde{\mathrm{ev}} : \mathrm{Diff}_0(S) \to U$$

under the condition that $\widetilde{\mathrm{ev}}(id) = z_0$.

Next, we will define a map $\widetilde{\Phi} : \mathcal{C}(S) \times \mathrm{Diff}_0(S) \to \mathcal{C}(\dot{S})$. To give an idea of the definition of $\widetilde{\Phi}$, we consider the case of $\mathcal{C}^0(S) \times \mathrm{Diff}_0(S)$. Take a point $(v, f) \in \mathcal{C}^0(S) \times \mathrm{Diff}_0(S)$. Then there is an isotopy $f_t,\ t \in [0,1]$, between $f_0 = id$ and $f_1 = f$. Setting $C(t) = f_t(\widehat{z}_0)$ for every $t \in [0,1]$, we have a path C from $\widehat{z}_0$ to $f(\widehat{z}_0)$ on S. Move a point in S from $f(\widehat{z}_0)$ to $\widehat{z}_0$ along C and drag v back along the moving point. Then we obtain new simple closed curve on $\dot{S}$ and denote the curve by $f^{-1}(v)$. Thus we define $\widetilde{\Phi}(v, f) = f^{-1}(v)$.

However, when $f(\widehat{z}_0) \in v$, we can not define $\widetilde{\Phi}(v, f)$ as above. We solve this problem in the following way: Now choose $\{\epsilon(v)\}_{v \in \mathcal{C}^0(S)} \subset \mathbb{R}_{>0}$ so that the $\epsilon(v)$-neighborhood $N(v) = N_{\epsilon(v)}$ of v has the following properties:

(i) $N(v)$ is homeomorphic to $S^1 \times [0,1]$
(ii) $N(v_1) \cap N(v_2) = \emptyset$ if $v_1 \cap v_2 = \emptyset$.

Let $N^\circ(v)$ be the interior of $N(v)$ and $v^\pm$ the boundary components of $N(v)$. For instance, we may take $\epsilon(v)$ as the half of the width of the collar neighborhood of the geodesic representative of v. Notice that $\epsilon(v)$ is depending only on the length of the geodesic representative of v (cf. [4]).

If $v \subset \mathcal{C}(S)$ is a simplex with vertices $\{v_0, v_1, \cdots, v_k\}$, then we consider the barycentric coordinates for points in v:

$$\{\sum_{j=0}^{k} s_j v_j \mid \sum_{j=0}^{k} s_j = 1 \text{ and } s_j \geq 0, \text{ for } j = 0, 1, \cdots, k\}$$

For a point (v, f) with v a vertex of $\mathcal{C}(S)$, we can define $\widetilde{\Phi}$ in the following way: If $f(\widehat{z}_0) \notin N^\circ(v)$, then we define

$$\widetilde{\Phi}(v, f) = f^{-1}(v)$$

as above.

If $f(\widehat{z}_0) \in N^\circ(v)$, then $f^{-1}(v^+)$ and $f^{-1}(v^-)$ are not isotopic in $\dot{S}$. We set

$$t = \frac{d(v^+, f(\widehat{z}_0))}{2\epsilon(v)},$$

where $d(v^+, f(\widehat{z}_0))$ is the distance inside $N(v)$ from $f(\widehat{z}_0)$ to v^+. Then we define

$$\widetilde{\Phi}(v, f) = tf^{-1}(v^+) + (1-t)f^{-1}(v^-)$$

in barycentric coordinates on the edge $[f^{-1}(v^+), f^{-1}(v^-)]$.

In general, for a point $(x, f) \in \mathcal{C}(S) \times \mathrm{Diff}_0(S)$ with $x = \sum_{j=0}^{k} s_j v_j$, we define $\widetilde{\Phi}(x, f)$ as follows: If $f(\widehat{z}_0) \notin \bigcup_{j=0}^{k} N^\circ(v_j)$, then we define

$$\widetilde{\Phi}(x, f) = \sum_j s_j f^{-1}(v_j).$$

If $f(\widehat{z}_0) \in N^\circ(v_i)$ for exactly one i, we set

$$t = \frac{d(v^+, f(\widehat{z}_0))}{2\epsilon(v_i)},$$

and define

$$\widetilde{\Phi}(x, f) = s_i(tf^{-1}(v_i^+) + (1-t)f^{-1}(v_i^-)) + \sum_{j \neq i} s_j f^{-1}(v_j).$$

Finally, by Proposition 2.2 in [7], if $\widetilde{\mathrm{ev}}(f_1) = \widetilde{\mathrm{ev}}(f_2)$ in U, then we see that $\widetilde{\Phi}(x, f_1) = \widetilde{\Phi}(x, f_2)$. From this, we have a map $\Phi : \mathcal{C}(S) \times U \to \mathcal{C}(\dot{S})$ satisfying $\widetilde{\Phi} = \Phi \circ (id \times \widetilde{\mathrm{ev}})$.

3.3. **Properties of Φ.** A subsurface of S is said to be an *essential* if it is either a component of the complement of a geodesic multicurve in S, the annular neighborhood $N(v)$ of some geodesic $v \in \mathcal{C}^0(S)$, or else S.

If a point $x \in \partial U$ has the following properties,

(i) for every geodesic ray $r \subset U$ ending at x and for every $v \in \mathcal{C}^0(S)$ which nontrivially intersects an essential subsurface Y, we have $\pi(r) \cap v \neq \emptyset$ and
(ii) there is a geodesic ray $r \subset U$ ending at x such that $\pi(r) \subset Y$,

we call such a point x a *filling point* for Y (or simply, *x fills Y*). We set

$$\mathbb{A} = \{x \in \partial U \mid x \text{ fills } S\}.$$

Next, we take a geodesic ℓ in U whose projection $\pi(\ell)$ is a non-simple closed geodesic. Let $\{\ell_n\}_{n=1}^{\infty}$ be a set of all pairwise distinct $\pi_1(S)$-translates of ℓ such that

$$H(\ell_1) \supset H(\ell_2) \supset \cdots,$$

where $H(\ell_k)$ is the half space bounded by ℓ_k. We denote the closure of $H(\ell_k)$ in $U \cup \partial U$ by $\overline{H(\ell_k)}$. Since ℓ are all distinct and $\pi_1(S)$ acts properly discontinuously on U, we see that

$$\bigcap_{n=1}^{\infty} \overline{H(\ell_n)} = \{x\}$$

for some $x \in \partial U$.

We have the following results.

Proposition 3.1 ([7], Proposition 3.4)**.** *If $\{\ell_n\}_{n=1}^{\infty}$ is a sequence nesting down to a point $x \in \mathbb{A}$, then for any choice of basepoint $u_0 \in \mathcal{C}(\dot{S})$,*

$$d_{\mathcal{C}(\dot{S})}(\Phi(\mathcal{C}(S) \times H(\ell_n)), u_0) \to \infty$$

as $n \to \infty$.

Theorem 3.2 ([7], Theorem 3.5)**.** *For any $v \in \mathcal{C}(S)$, the map*

$$\Phi(v, \cdot) : U \to \mathcal{C}(\dot{S})$$

can be continuously extended to

$$\overline{\Phi}(v, \cdot) : U \cup \mathbb{A} \to \overline{\mathcal{C}}(\dot{S}).$$

4. Main Theorem

Let α be a nontrivial simple closed curve on a Riemann surface R. Denote by $\mathrm{Mod}(A)$ the modulus of an annulus in R whose core curve is homotopic in R to α. We define the extremal length $\mathrm{Ext}(\alpha)$ of α on R by

$$\mathrm{Ext}_R(\alpha) = \inf_A 1/\mathrm{Mod}(A),$$

where the infimum is over all annuli $A \subset R$ whose core curve is homotopic in R to α.

Given any point $p = (R, f) \in T(S)$ and a nontrivial simple closed curve γ on S, we define the extremal length $\mathrm{Ext}_p(\gamma)$ by

$$\mathrm{Ext}_p(\gamma) = \mathrm{Ext}_R(f(\gamma)).$$

Then there is a natural map $\mathcal{E} : T(S) \to \mathcal{C}(S)$ which sends any $p \in T(S)$ to an element of $\mathcal{C}^0(S)$ of minimal Ext_p, Similarly, we define a map $\dot{\mathcal{E}} : T(\dot{S}) \to \mathcal{C}(\dot{S})$.

By virtue of Bers' theorem and Maskit's comparizon theorem, there is a constant E_0 depending only on the topology of S such that

$$\mathrm{Ext}_{p_0}(\mathcal{E}(p_0)) \leq E_0 \tag{4.1}$$

([2] and [8]). Henceforth, we fix such E_0 and we may suppose that such E_0 is available for simple closed curves on both S and $\dot{S}$.

Theorem 4.1. *The map $\varphi \circ r : T(S) \times U \to T(\dot{S})$ has a limit in $\{p_0\} \times \mathbb{A}$ for any point $p_0 \in T(S)$.*

Proof.

We may assume that p_0 is the base point (S, id) of $T(S)$. Let $\{(p_m, z_m)\}_{m=1}^{\infty}$ be any sequence in $T(S) \times U$ converging to $(p_0, z_\infty) \in T(S) \times \mathbb{A}$. We set $(\xi_m, z_m) = (\mathcal{E} \times id)(p_m, z_m)$ and $q_m = \varphi \circ r(p_m, z_m)$. Moreover, put

$$\delta_m = \Phi(\xi_m, z_m)$$

and $\gamma_m = \dot{\mathcal{E}}(q_m)$.

By filling at the puncture $\widehat{z}_0$ of $\dot{S}$, for each m there is an element $\gamma_{0,m} \in \mathcal{C}(S)$ such that

$$\gamma_m = \Phi(\gamma_{0,m}, z_m).$$

We first check the following lemma.

Lemma 4.1. $\lim_{m\to\infty} \delta_m = \lim_{n\to\infty} \gamma_n$ *in* $\partial_\infty \mathcal{C}(\dot{S})$*, that is,*

$$\lim_{m,n\to\infty} \langle \delta_m | \gamma_n \rangle_0 = \infty. \tag{4.2}$$

Proof. To show this, we begin with the following two claims.

Claim 1. $d_{\mathcal{C}(\dot{S})}(\delta_m, 0) \to \infty$ *and* $d_{\mathcal{C}(\dot{S})}(\gamma_m, 0) \to \infty$ *as* $m \to \infty$*.*

Proof of Claim 1. Let $\{\ell_n\}_{n=1}^{\infty}$ be a sequence nesting down to the point $z_\infty \in \mathbb{A}$. Then there is a sequence of half spaces $\{H(\ell_n)\}_{n=1}^{\infty}$ having following properties

$$H(\ell_1) \supset H(\ell_2) \supset \cdots$$

and

$$\bigcap_{n=1}^{\infty} \overline{H(\ell_n)} = \{z_\infty\}.$$

For a sufficiently large number N_0, there is a number n_0 such that z_m $(m = n_0,\ n_0 + 1, n_0 + 2, \cdots)$ are all contained in $H(\ell_{N_0})$. For each m, there is a number N_m such that z_m is contained in $H(\ell_{N_m})$ but not in $H(\ell_{N_m+1})$. From $\delta_m = \Phi(\xi_m, z_m)$ and $\gamma_m = \Phi(\gamma_{0,m}, z_m)$, we see

$$\delta_m \in \Phi(\mathcal{C}(S) \times H(\ell_{N_m})),$$
$$\gamma_m \in \Phi(\mathcal{C}(S) \times H(\ell_{N_m})).$$

Since Theorem 3.1 shows that

$$d_{\mathcal{C}(\dot{S})}(\Phi(\mathcal{C}(S) \times H(\ell_m)), 0) \to \infty \quad (m \to \infty),$$

we have $d_{\mathcal{C}(\dot{S})}(\delta_m, 0) \to \infty$ and $d_{\mathcal{C}(\dot{S})}(\gamma_m, 0) \to \infty$ as $m \to \infty$, as desired.

Claim 2. $d_{\mathcal{C}(\dot{S})}(\delta_m, \gamma_m) = O(1)$ *as* $m \to \infty$*.*

Proof of Claim 2. To clarify the argument, we first assume that $p_m = p_0$ for all m.

Take $f_m \in \mathrm{Diff}_0(S)$ with $(id \times \widetilde{\mathrm{ev}})(\xi, f_m) = (\xi, z_m)$. Let $N(\xi)$ as §3.2. Since $\xi = \mathcal{E}(p_0)$ and (4.1), we have

$$\mathrm{Mod}(N(\xi)) \geq 1/E_1 \tag{4.3}$$

where $E_1 > 0$ is a constant depending only on the topology of S.

Suppose first that $\widehat{z}_m = f_m(\widehat{z}_0) \notin N^\circ(\xi)$. Then, by definition, δ_m is homotopic to $f_m^{-1}(\xi)$ on $\dot{S}$. By the assumption, the interior of the annulus $N(\xi)$ is embedded in $S - \{z_m\}$. Therefore, by (4.3), we have

$$\mathrm{Ext}_{q_m}(\delta_m) \leq 1/\mathrm{Mod}(N(\xi)) \leq E_1.$$

Meanwhile, $\mathrm{Ext}_{q_m}(\gamma_m) \leqq E_0$ because $\gamma_m = \dot{\mathcal{E}}(q_m)$. Thus by Minsky and Masur's lemma [9] and Minsky's lemma [10], we get

$$d_{\mathcal{C}(\dot{S})}(\gamma_m, \delta_m) \leqq 2i(\gamma_m, \delta_m) + 1 \leq 2(E_1 E_0)^{1/2} + 1,$$

which is what we desired.

Suppose $\widehat{z}_m \in N^\circ(\xi)$. Let ξ^* be the core geodesic of $N(\xi)$. Take a conformal (not isometric) coordinates

$$h_m : \xi^* \times [-\epsilon(\xi), \epsilon(\xi)] \to N(\xi)$$

such that $\xi^* \times \{0\}$ maps to the core geodesic of $N(\xi)$ and for each t, $\xi^* \times \{t\}$ is sent to the equidistant circle to the core geodesic. Let $t_m \in [-\epsilon(\xi), \epsilon(\xi)]$ such that $\widehat{z}_m \in h_m(\xi^* \times \{t_m\})$. Then, by definition,

$$\delta_m = \left(1 + \frac{t_m}{2\epsilon(\xi)}\right) f_m^{-1}(\xi^+) + \left(1 - \frac{t_m}{2\epsilon(\xi)}\right) f_m^{-1}(\xi^-)$$

where $\xi^\pm$ is the components of $\partial N(\xi)$. Henceforth, we suppose $t_m > 0$. The case $t_m \geqq 0$ can be dealt with the same manner.

Let A_m be the component of $N(\xi) \setminus h_m(\xi^* \times \{t_m\})$ which containing ξ^*. Since h_m is conformal,

$$\mathrm{Mod}(A_m) \geqq (\mathrm{Mod} N(\xi))/2.$$

and the core of A_m is homotopic to ξ^- in $S - \{\widehat{z}_m\}$. Therefore,

$$\mathrm{Ext}_{q_m}(\xi^-) \leqq 2E_1,$$

where we recognize ξ^- as a simple closed curve on $S - \{\widehat{z}_m\}$. Therefore, we have

$$\begin{aligned} d_{\mathcal{C}(\dot{S})}(f_m^{-1}(\xi^-), \gamma_m) &\leqq 2i(f_m^{-1}(\xi^-), \gamma_m) + 1 \\ &\leqq 2\mathrm{Ext}_{q_m}(\xi^-)^{1/2}\mathrm{Ext}_{q_m}(\gamma_m)^{1/2} + 1 \\ &\leqq 2\sqrt{2}(E_1 E_0)^{1/2} + 1. \end{aligned}$$

Thus we deduce

$$\begin{aligned} d_{\mathcal{C}(\dot{S})}(\gamma_m, \delta_m) &\leqq d_{\mathcal{C}(\dot{S})}(\gamma_m, f_m^{-1}(\xi^-)) + d_{\mathcal{C}(\dot{S})}(f_m^{-1}(\xi^-), \delta_m) \\ &\leqq 2\sqrt{2}(E_1 E_0)^{1/2} + 2, \end{aligned}$$

which implies Claim 2 holds when $p_m = p_0$ for all m.

We next deal with the general case. Let S_m be the underlying Riemann surface for p_m. Let $w_m \in Q_{norm}$ be a quasiconformal deformation from p_0 to p_m, and $G_m = w_m G w_m^{-1}$. We let $\widehat{z}'_m \in S_m$ be the projection of z_m via the covering projection $\mathbb{H} \to \mathbb{H}/G_m = S_m$. Let $N_m(\xi_m) \subset S_m$ be the collar neighborhood of the geodesic representative of ξ_m on S_m. Since $\xi_m = \mathcal{E}(p_m)$, the modulus of $N_m(\xi_m)$ is bounded by a constant independent of m. By the same argument as above, we can find an essential subannulus B_m in $N_m(\xi_m) \setminus \{\widehat{z}'_m\}$ such that the core of B_m is homotopic to ξ_m on S_m and the modulus of B_m is uniformly bounded above and below.

Let $\eta_m \in \mathcal{C}(\dot{S})$ be the element corresponding to the core of B_m. Since $\gamma_m = \dot{\mathcal{E}}(q_m)$ and the argument above, the extremal lengths of γ_m and η_m on q_m is uniformly bounded above. Therefore, by Minsky's inequality, we have

$$d_{\mathcal{C}(\dot{S})}(\eta_m, \gamma_m) = O(1)$$

for all m. On the other hand, Since η_m is the core of an essential subannulus B_m of $N_m(\xi_m)$, η_m is homotopic to one of the components of $\partial N_m(\xi_m)$ in $S_m - \{\widehat{z}'_m\}$. Hence, by the definition of δ_m, we get

$$d_{\mathcal{C}(\dot{S})}(\eta_m, \delta_m) = O(1).$$

Therefore, we conclude that

$$d_{\mathcal{C}(\dot{S})}(\delta_m, \gamma_m) \leqq d_{\mathcal{C}(\dot{S})}(\delta, \eta_m) + d_{\mathcal{C}(\dot{S})}(\eta_m, \delta_m) = O(1),$$

which is what we desired.

We now check that the equation (4.2) holds. From two claims above, we get

$$\lim_{m\to\infty} \langle \delta_m | \gamma_m \rangle = \infty.$$

Since $\mathcal{C}(\dot{S})$ is δ-hyperbolic,

$$\langle \delta_m | \gamma_n \rangle \geqq \min\{\langle \delta_m | \gamma_m \rangle, \langle \gamma_m | \gamma_n \rangle\} - \delta$$

holds. Therefore we conclude $\lim_{m,n\to\infty} \langle \delta_m | \gamma_n \rangle = \infty$. Namely,

$$\lim_{m\to\infty} \Phi \circ (\mathcal{E} \times id)(p_m, z_m) = \lim_{n\to\infty} \dot{\mathcal{E}} \circ (\varphi \circ r)(p_n, z_n), \tag{4.4}$$

holds, which implies Lemma 4.1. ■

We now return to the proof of Theorem 4.1. Since (4.4) holds for any sequence $\{(p_m, z_m)\}_{m=1}^{\infty}$ in $T(S) \times U$ converging to $(p_0, z_\infty) \in T(S) \times \mathbb{A}$, from now we may consider the case of $p_m = p_0$ for every $m \geqq 1$. For a sequence $\{(p_0, z_m)\}_{m=1}^{\infty}$ converging to $(p_0, z_\infty) \in \{p_0\} \times \mathbb{A}$, we assume $\{\varphi \circ r(p_0, z_m)\}$ converges to q_∞. Then by using (4.4), we obtain

$$\begin{aligned} \lim_{m\to\infty} \dot{\mathcal{E}} \circ (\varphi \circ r)(p_0, z_m) &= \lim_{m\to\infty} \Phi \circ (\mathcal{E} \times id)(p_0, z_m) \\ &= \lim_{m\to\infty} \Phi(\xi, z_m), \end{aligned}$$

where $\xi = \mathcal{E}(p_0) \in \mathcal{C}(S)$. Theorem 3.2 shows that there is a γ_∞ in $\partial_\infty \mathcal{C}(\dot{S})$ such that

$$\lim_{m\to\infty} \Phi(\xi, z_m) = \gamma_\infty.$$

By Klarreich's work of [6], we can identify $\partial_\infty \mathcal{C}(\dot{S})$ with the space of ending lamination $\mathcal{EL}(\dot{S})$. Thus γ_∞ is an ending lamination.

Put $q_m = \varphi \circ r(p_0, z_m)$. We regard $\{q_m\}_{m=1}^{\infty}$ as the sequence in a Bers slice $T(\dot{S}) \times \{q_0\}$. For each pair (q_m, q_0), there is a unique quasifuchsian group Γ_m up to conjugation such that $\Omega(\Gamma_m)/\Gamma_m = \dot{S}_{q_m} \cup \dot{S}_{q_0}$, where $\Omega(\Gamma_m)$ is the region of discontinuity of Γ_m and the symbol $\dot{S}_q$ means the Riemann surface corresponding to $q \in T(\dot{S})$. Since $\{q_m\}_{m=1}^{\infty}$ converges to q_∞, by using Ending lamination theorem for surface groups of [3], there is a unique Kleinian group Γ_∞ up to conjugation such that $\{\Gamma_m\}_{m=1}^{\infty}$ converges to Γ_∞ algebraically. This implies that the sequence $\{q_m\}_{m=1}^{\infty}$ converges to q_∞ without depending on the choice of a convergent sequence to (p_0, z_∞). This shows $\varphi \circ r$ has a limit in $\{p_0\} \times \mathbb{A}$. ■

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