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Variable Lebesgue norm estimates for BMO functions ^{*}

Mitsuo Izuki [†] and Yoshihiro Sawano [‡]

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Abstract

In this paper, we are going to obtain characterizations of the space $BMO(\mathbb{R}^n)$ through variable Lebesgue spaces.

1 Introduction

One of the most interesting problems on spaces with variable exponent is the boundedness of the Hardy–Littlewood maximal operator. The important sufficient conditions called “log-Hölder” have been obtained by Cruz-Uribe, Fiorenza, and Neugebauer [2] and Diening [3]. Under the conditions many results on spaces with variable exponent have been obtained now.

The aim of this paper is to obtain characterizations of $BMO(\mathbb{R}^n)$. Recently an attempt has been made to characterize $BMO(\mathbb{R}^n)$ through various function spaces. Throughout this paper $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function for a measurable set $S \subset \mathbb{R}^n$. All cubes are assumed to have their sides parallel to the coordinate axes. Given a function f and a measurable set S , f_S denotes the mean value of f on S , namely

$$f_S := \frac{1}{|S|} \int_S f(x) dx.$$

Definition 1.1. *The space $BMO(\mathbb{R}^n)$ consists of all measurable functions b satisfying*

$$\|b\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx < \infty, \quad (1)$$

where the supremum is taken over all cubes Q .

Recently, given a Banach function space X , we have been asking ourselves the following problem.

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Problem 1.2. *The norm $\|b\|_{BMO(\mathbb{R}^n)}$ is equivalent to*

$$\|b\|_X^* = \sup_{Q: \text{cube}} \frac{1}{\|\chi_Q\|_X} \|\chi_Q(b - b_Q)\|_X.$$

Here is a series of affirmative results concerning Problem 1.2.

1. $X = L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. This is well-known as the John–Nirenberg inequality (See Lemma 3.1 to follow).
2. X is a rearrangement invariant function space [7]. By rearrangement invariant we mean that the X -norm of a function f depends only upon the function $t \in (0, \infty) \mapsto |\{ |f| > t \}| \in (0, \infty)$.
3. X is a quasi-rearrangement invariant Banach function space with $p \leq p_Y \leq q_Y < \infty$ ([8]).

The aim of this paper is to show that this is the case even when X is not rearrangement invariant. First, we consider the case when X is a Morrey space.

Theorem 1.3. *Let $1 \leq q \leq p < \infty$. If we define the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ by*

$$\|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)} = \sup_{Q: \text{cube}} |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f(x)|^q dx \right)^{1/q},$$

then Problem 1.2 is true for $X = \mathcal{M}_q^p(\mathbb{R}^n)$.

The second (and main) spaces we take up in this paper are variable Lebesgue spaces. A measurable function $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty]$ is called a variable exponent. A variable exponent space showed up around 1990s [11]. After 2005 the theory which are fundamental in harmonic analysis is established very rapidly. For more details we refer to the recent book [5]. Here is a precise definition.

Definition 1.4. *Given a variable exponent $p(\cdot)$, one denotes*

$$\begin{aligned} \Omega_{\infty, p} &:= \{x \in \mathbb{R}^n : p(x) = \infty\} = p^{-1}(\infty) \\ \rho_p(f) &:= \int_{\mathbb{R}^n \setminus \Omega_{\infty, p}} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_{\infty, p})}. \end{aligned}$$

The variable Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \{f \text{ is measurable} : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \{ \lambda > 0 : \rho_p(f/\lambda) < \infty \}.$$

This is a special case of the theory developed by Luxemburg and Nakano [13, 14, 15]. We additionally set

$$p_- := \text{ess inf} \{p(x) : x \in \mathbb{R}^n\}, \quad p_+ := \text{ess sup} \{p(x) : x \in \mathbb{R}^n\}.$$

Theorem 1.5. *If a variable exponent $p(\cdot)$ satisfies $1 \leq p_- \leq p_+ < \infty$ and the estimates*

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq -\frac{C_1}{\log|x-y|} \quad \left(|x-y| \geq \frac{1}{2} \right)$$

and

$$\left| \frac{1}{p(x)} - \frac{1}{p(\infty)} \right| \leq \frac{C_2}{\log(e+|x|)} \quad (x \in \mathbb{R}^n)$$

holds for some $C_1, C_2, p(\infty) > 0$, then Problem 1.2 is true for $X = L^{p(\cdot)}(\mathbb{R}^n)$, that is,

$$C^{-1} \|b\|_{BMO(\mathbb{R}^n)} \leq \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{BMO(\mathbb{R}^n)}$$

holds for all $b \in BMO(\mathbb{R}^n)$.

Needless to say, $L^{p(\cdot)}(\mathbb{R}^n)$ is not rearrangement invariant. Examples in [17] show that $\mathcal{M}_q^p(\mathbb{R}^n)$ is rearrangement invariant only when $p = q$.

Theorem 1.3 is considerably easy to prove. Indeed, from the definition of the Morrey norm, we have

$$\begin{aligned} \frac{1}{\|\chi_Q\|_{L^q(\mathbb{R}^n)}} \|\chi_Q(b - b_Q)\|_{L^q(\mathbb{R}^n)} &\leq \frac{1}{\|\chi_Q\|_{\mathcal{M}_q^p(\mathbb{R}^n)}} \|\chi_Q(b - b_Q)\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \\ &\leq \frac{1}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}} \|\chi_Q(b - b_Q)\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

So the matters are reduced to the case when $X = L^p(\mathbb{R}^n)$.

However, a similar argument does not seem to work for Theorem 1.5. Especially the estimate which corresponds to

$$\frac{1}{\|\chi_Q\|_{L^q(\mathbb{R}^n)}} \|\chi_Q(b - b_Q)\|_{L^q(\mathbb{R}^n)} \leq \frac{1}{\|\chi_Q\|_{\mathcal{M}_q^p(\mathbb{R}^n)}} \|\chi_Q(b - b_Q)\|_{\mathcal{M}_q^p(\mathbb{R}^n)}$$

is hard to obtain.

We organize the remaining part of this paper as follows: Section 2 intends as an review of variable Lebesgue spaces. We prove Theorem 1.5 in Section 3. Section 4 contains another characterization of $BMO(\mathbb{R}^n)$ related to the variable exponent Lebesgue norms.

Finally we give a convention which we use throughout the rest of this paper. A symbol C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2 Some basic facts on variable Lebesgue spaces

Given a function $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x .

One of the key developments of the theory of variable Lebesgue spaces is that we obtained a good criterion of the boundedness of the Hardy–Littlewood maximal operators [3, 4, 5].

Definition 2.1. Let $r(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a measurable function.

1. The function $r(\cdot)$ is said to be locally log-Hölder continuous if

$$|r(x) - r(y)| \leq \frac{C}{-\log(|x - y|)} \quad (|x - y| \leq 1/2) \quad (2)$$

holds. The set LH_0 consists of all locally log-Hölder continuous functions.

2. The function $r(\cdot)$ is said to be log-Hölder continuous at infinity if there exists a constant $r(\infty)$ such that

$$|r(x) - r(\infty)| \leq \frac{C}{\log(e + |x|)}. \quad (3)$$

The set LH_∞ consists of all log-Hölder continuous at infinity functions.

3. Define $LH := LH_0 \cap LH_\infty$ and say that each function belonging to LH is globally log-Hölder continuous.

The next proposition is initially proved by Cruz-Uribe et al. [2], when $p_+ < \infty$. Later Cruz-Uribe et al. [1] and Diening et al. [5] have independently extended the result even to the case of $p_+ = \infty$.

Proposition 2.2. Suppose that a variable exponent $p(\cdot)$ satisfies $1 < p_- \leq p_+ \leq \infty$ and $1/p(\cdot) \in LH$. Then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, namely

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \quad (4)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

We note that $p(\cdot)$ always satisfies $p_- > 1$ whenever (4) is true ([5]). In the case of $p_- = 1$, the weak $(p(\cdot), p(\cdot))$ type inequality for M holds. The following has been also proved by Cruz-Uribe et al. [1].

Proposition 2.3. If a variable exponent $p(\cdot)$ satisfies $1 = p_- \leq p_+ \leq \infty$ and $1/p(\cdot) \in LH$, then we have that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\sup_{t>0} t \left\| \chi_{\{Mf(x)>t\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (5)$$

We will need the following two lemmas in order to get the main results.

Lemma 2.4. If a variable exponent $p(\cdot)$ satisfies the weak $(p(\cdot), p(\cdot))$ type inequality (5) for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then

$$\|f\|_Q \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all cubes Q .

Proof. Take $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and a cube Q arbitrarily. We may assume $|f|_Q > 0$. Let $t = |f|_Q/2$. Now that $|f|_Q \chi_Q(x) \leq M(f \chi_Q)(x)$, we obtain $M(f \chi_Q)(x) > t$ whenever $x \in Q$. Thus we have

$$\begin{aligned} |f|_Q \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq |f|_Q \left\| \chi_{\{M(f \chi_Q)(x) > t\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq |f|_Q \cdot C t^{-1} \|f \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= C \|f \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

□

Remark 2.5. Lerner [12] has proved the converse of Lemma 2.4, provided that $p(\cdot)$ is radial decreasing and satisfies $p_- > 1$.

The next lemma is due to Diening [4, Lemma 5.5].

Lemma 2.6. *If a variable exponent $p(\cdot)$ satisfies $1 < p_- \leq p_+ < \infty$ and M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a constant $0 < \delta_1 < 1$ such that for all $0 < \delta < \delta_1$, all families of pairwise disjoint cubes Y , all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $|f|_Q > 0$ ($Q \in Y$) and all $t_Q > 0$ ($Q \in Y$),*

$$\left\| \sum_{Q \in Y} t_Q \left| \frac{f}{|f|_Q} \right|^\delta \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \sum_{Q \in Y} t_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

In particular

$$\left\| f^\delta \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C |f|_Q^\delta \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \quad (6)$$

holds.

3 Main results

We describe some known facts before we state the main results.

Lemma 3.1. *If $1 \leq q < \infty$, then we have that for all $b \in BMO(\mathbb{R}^n)$,*

$$\|b\|_{BMO(\mathbb{R}^n)} \leq \sup_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^q dx \right)^{1/q} \leq C_0 q \|b\|_{BMO(\mathbb{R}^n)}, \quad (7)$$

where $C_0 > 0$ is a constant independent of q .

The left hand-side inequality of (7) directly follows from the Hölder inequality. The right one is a famous consequence of an application of the John–Nirenberg inequality (cf. [10]).

Proposition 3.2. *There exist two positive constants C_1, C_2 depending only on n such that for all $b \in BMO(\mathbb{R}^n)$, all cubes Q and all $t \geq 0$,*

$$|\{x \in Q : |b(x) - b_Q| > t\}| \leq C_1 |Q| \exp(-C_2 t / \|b\|_{BMO(\mathbb{R}^n)}).$$

Lemma 3.1 can additionally be generalized to the case of variable exponents. Now we are going to prove Theorem 1.5. Recall that we announced that we are going to prove;

If a variable exponent $p(\cdot)$ satisfies $1 < p_- \leq p_+ < \infty$ and M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then we have that for all $b \in BMO(\mathbb{R}^n)$,

$$C^{-1} \|b\|_{BMO(\mathbb{R}^n)} \leq \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{BMO(\mathbb{R}^n)} \quad (8)$$

The author [9] has initially proved Theorem ???. Later we will give another proof of it.

In view of Lemma 3.1, it may be a natural question to prove (8) for the case of $p_- = 1$. Now we shall prove Theorem 1.5.

Proof of Theorem 1.5. Take a cube Q and $b \in BMO(\mathbb{R}^n)$ arbitrarily. By virtue of Lemma 2.4 we have

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \cdot \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This gives us the left hand side inequality of the theorem. Next we shall prove the right hand side one. Let us fix a number r so that $rp_- > 1$ and write $u(\cdot) := rp(\cdot)$. Then the variable exponent $u(\cdot)$ satisfies $1 < u_-$ and $1/u(\cdot) \in LH$. Hence the boundedness of M on $L^{u(\cdot)}(\mathbb{R}^n)$ holds by Proposition 2.2. Using Lemma 2.6, we can take a constant $\delta \in (0, 1/r)$ so that

$$\|f^\delta \chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)} \leq C |f_Q|^\delta \|\chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Now we obtain

$$\begin{aligned} \|f^{r\delta} \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \|f^\delta \chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)}^r \\ &\leq C |f_Q|^{r\delta} \|\chi_Q\|_{L^{u(\cdot)}(\mathbb{R}^n)}^r \\ &= C |f_Q|^{r\delta} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (9)$$

If we put $f := |b - b_Q|^{1/(r\delta)}$ and apply Lemma 3.1 with $q = 1/(r\delta) > 1$, then we get

$$|f_Q|^{r\delta} = \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^{1/(r\delta)} dx \right)^{r\delta} \leq C \|b\|_{BMO(\mathbb{R}^n)}. \quad (10)$$

Combing (9) and (10) we obtain

$$\|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{BMO(\mathbb{R}^n)} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This leads us to the desired inequality and completes the proof. \square

Proof of Theorem ???. We have only to follow the same argument as the proof of Theorem 1.5 with $r = 1$. \square

4 Related inequalities

According to Lemma 3.1, we have

$$\left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^q dx \right)^{1/q} \leq C_0 q \|b\|_{BMO(\mathbb{R}^n)},$$

where $C_0 > 0$ is independent of $q \in [1, \infty)$. This can be rephrased as

$$\frac{1}{|Q|} \int_Q \left(\frac{|b(x) - b_Q|}{C_0 q \|b\|_{BMO(\mathbb{R}^n)}} \right)^q dx \leq 1$$

for all cubes Q . Observe that the estimate above is uniform over $1 \leq q < \infty$. Therefore, the following inequality seems to hold;

$$\frac{1}{|Q|} \int_Q \left(\frac{|b(x) - b_Q|}{C_0 p(x) \|b\|_{BMO(\mathbb{R}^n)}} \right)^{p(x)} dx \leq 1$$

Suppose that $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a variable exponent which is not necessarily continuous or bounded. Then define

$$\|b\|_{p(\cdot)}^\dagger = \sup_Q \left(\inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \left(\frac{|b(x) - b_Q|}{p(x)\lambda} \right)^{p(x)} dx \leq 1 \right\} \right)$$

for measurable functions b . Now we are going to prove;

Theorem 4.1. *If a variable exponent $p(\cdot)$ satisfies $p(x) < \infty$ for almost every $x \in \mathbb{R}^n$, then we have*

$$\|b\|_{p(\cdot)}^\dagger \leq C \|b\|_{BMO(\mathbb{R}^n)}.$$

Furthermore, if $p(\cdot)$ is bounded, then the norms $\|\cdot\|_{p(\cdot)}^\dagger$ and $\|\cdot\|_{BMO(\mathbb{R}^n)}$ are mutually equivalent.

Proof. According to the John-Nirenberg inequality, we have

$$\frac{1}{|Q|} \int_Q \left\{ \exp \left(\frac{\lambda |b(x) - b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}} \right) - 1 \right\} dx \leq 1$$

for some $\lambda > 0$. Since

$$\begin{aligned} \left(\frac{\lambda |b(x) - b_Q|}{3p(x) \|b\|_{BMO(\mathbb{R}^n)}} \right)^{p(x)} &= \left(\frac{1}{3p(x)} \right)^{p(x)} \left(\frac{\lambda |b(x) - b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}} \right)^{p(x)} \\ &\leq \min \left\{ \left(\frac{1}{[p(x)]} \right)^{[p(x)]}, \left(\frac{1}{[p(x)+1]} \right)^{[p(x)+1]} \right\} \left(\frac{\lambda |b(x) - b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}} \right)^{p(x)} \\ &\leq \exp \left(\frac{\lambda |b(x) - b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}} \right) - 1. \end{aligned}$$

Hence it follows that

$$\|b\|_{p(\cdot)}^\dagger \leq 3\lambda^{-1} \|b\|_{BMO(\mathbb{R}^n)}.$$

If $p(\cdot)$ is bounded, then

$$\begin{aligned}
\|b\|_{p(\cdot)}^\dagger &\geq \sup_Q \left(\inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \left(\frac{|b(x) - b_Q|}{p_+ \lambda} \right)^{p(x)} dx \leq 1 \right\} \right) \\
&= \sup_Q \left(\inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{|b(x) - b_Q|}{p_+ \lambda} \right)^{p(x)} \right\} dx \leq 1 \right\} \right) \\
&= \sup_Q \left(\inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|b(x) - b_Q|}{2p_+ \lambda} dx \leq 1 \right\} \right) \\
&= (2p_+)^{-1} \|b\|_{BMO(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, these norms are mutually equivalent. \square

Remark 4.2. Let Φ be a Young function. Namely, $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism which is convex. If we assume that $\Phi(t) \leq t^a$ ($t \geq 2$) for some $a > 1$ and define

$$\|b\|_\Phi^\dagger = \sup_Q \left(\inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|b(x) - b_Q|}{p(x)\lambda} \right) dx \leq 1 \right\} \right)$$

for measurable functions b , then $\|b\|_\Phi^\dagger$ is equivalent to $\|b\|_{BMO}$. Indeed, as we have shown in [16], the norm $\|b\|_\Phi^\dagger$ remains unchanged if we redefine $\Phi(t) = \Phi(2)(t/2)^a$ for $0 \leq t \leq 2$. Therefore, $\|b\|_\Phi^\dagger \leq C\|b\|_{BMO}$ by virtue of Lemma 3.1. The reverse inequality is also clear since we have $\Phi(t) \geq \Phi(1)t$ for $t \geq 1$.

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