# Global solutions to a chemotaxis system with non-diffusive memory

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**Abstract**

In this article, a existence theorem of global solutions with small initial data belonging to $L^1 \cap L^p, \ (n < p \le \infty)$ for a chemotaxis system are given on the whole space $\mathbb{R}^n, \ n \ge 3$. In the case $p = \infty$, our global solution is integrable with respect to the space variable on some time interval, and then conserves the mass for a short time, at least. The system consists of a chemotaxis equation with a logarithmic term and an ordinary equation without diffusion term.

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## 1 Introduction

This paper concerns global solutions to the system

$$(\mathrm{E}_\lambda) \quad \begin{cases} \partial_t u = \Delta u - \nabla \cdot \left( u \dfrac{\nabla v}{v} \right), \quad x \in \mathbb{R}^n, \quad t > 0 \\ \partial_t v = u v^\lambda, \quad x \in \mathbb{R}^n, \quad t > 0 \quad \text{and} \quad \lambda \in \mathbb{R} \\ u(0,x) = a(x) \ge 0 \quad v(0,x) = b(x) \ge 0, \quad x \in \mathbb{R}^n. \end{cases}$$

Here, $u$ is the unknown cell density of the chemotactic species and $v$ is the unknown density of non-diffusive chemical substance, which is produced by the species. This system is a particular case of Keller-Segel system [8] and related to the dynamics of self-reinforced random walks [14] [16], and also used as haptotaxis and angiogenesis models. One of interesting features of the system is the absence of diffusion term in the second equation. There are many papers which studied the classical Keller-Segel system with diffusion term in the second equation. Levine and Sleeman [10] investigated finite time blow-up phenomena for the system $(\mathrm{E}_1)$ in one dimensional case. Additional properties for the solutions of $(\mathrm{E}_1)$ have been obtained in [12]. In smooth bounded domains in $\mathbb{R}^n$ with $\lambda \le 1$, Rascle [15] and Yang, Chen and Liu [20] showed the existence of global solutions for the system. Corrias, Perthame and Zaag [5], [6] studied the same topics in a general system, which do not cover the case $\lambda = 1$. In [17], asymptotic behavior of radial symmetric solutions to $(\mathrm{E}_\lambda)$ is studied. When $\lambda \in [0,1)$ and $n = 1$, Kang, Stevens and Velázquez proved that for some initial data the corresponding solutions $u$ tends to Dirac mass as $t \to \infty$ in [7].

Through the transformation

$$z = \frac{v^{1-\lambda}}{1-\lambda} \quad \text{with} \quad \theta = \frac{1}{1-\lambda} \in \mathbb{R},$$

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the system ($\mathrm{E}_\lambda$) becomes

$$(\tilde{\mathrm{E}}_\theta)\quad \begin{cases} \partial_t u = \Delta u - \theta\nabla\cdot\left(u\dfrac{\nabla z}{z}\right), & x\in\mathbb{R}^n, \quad t>0 \\ \partial_t z = u, \quad x\in\mathbb{R}^n, \quad t>0 \\ u(0,x)=a(x) \quad z(0,x)=c(x)=\theta b(x)^{1-\lambda}, & x\in\mathbb{R}^n. \end{cases}$$

This system is invariant with respect to the scaling

$$(u(t,x), z(t,x)) \mapsto (\mu^{\alpha+2}u(\mu^2 t,\mu x), \mu^\alpha z(\mu^2 t, \mu x))$$

for all $\alpha\in\mathbb{R}$. In this article, we give a global existence theorem of solutions to the system ($\tilde{\mathrm{E}}_\theta$) with the special initial data $c\equiv 1$, which corresponds to the case $\lambda<1$ in ($\mathrm{E}_\lambda$), in the sense of mild solutions, more precisely, we construct solutions to the integral equations;

$$(\text{I.E.})\quad \begin{cases} u(t)=e^{t\Delta}a-\theta B[u](t), \text{ where } B[u](t)=\displaystyle\int_0^t e^{(t-\tau)\Delta}\nabla\left(u(\tau)\frac{\nabla z(\tau)}{z(\tau)}\right)d\tau \\ z(t)=1+\displaystyle\int_0^t u(\tau)d\tau. \end{cases}$$

As a consequence of the absence of diffusion term in the second equation, the regularity of $z$ and $\nabla z$ with respect to space variable is not better than that of 1 and $\nabla u$ respectively. For the initial data $c\equiv -1$ that corresponds to another case $\lambda>1$ in ($E_\lambda$), the same results as in below hold. D. Li, K. Li and Zhao [11] treated with the case $\lambda=1$ in which the system is changed to a hyperbolic-parabolic system through the transform $V=-\frac{\nabla v}{v}$, and constructed local and global solutions in Sobolev spaces with positive smoothness. Very recently, Ahn and Kang [1] proved the local existence of solutions to ($E_\lambda$) with $\lambda>0$, and the non-existence of self-similar solutions.

Our main result reads as follows

**Theorem 1.1** (Small data global existence). *Let $n\geq 3$, $n/(n-1)<q<n<r<p\leq\infty$ and $\theta\in\mathbb{R}$. There exists $\delta=\delta(n,p,q,r,|\theta|)>0$ such that if $\|a\|_{L^1\cap L^p}\leq\delta$, then there exists a global solution $u\in L^1(0,\infty;L^\infty(\mathbb{R}^n))$ of* (I.E.) *satisfying $\|u\|_{X^1_\infty}+\|u\|_{X^2_\infty}+\|u\|_{X^3_\infty}\lesssim\|a\|_{L^1\cap L^p}$ where*

$$\|u\|_{X^1_t}=\int_0^t\|u(\tau)\|_{L^\infty}d\tau,$$

$$\|u\|_{X^2_t}=\sup_{\tau<t}\left\|\int_0^\tau\nabla u(\sigma)d\sigma\right\|_{L^r} \text{ and } \|u\|_{X^3_t}=\sup_{\tau<t}\left\|\int_0^\tau\nabla u(\sigma)d\sigma\right\|_{L^q}.$$

*Moreover, with $U_\infty(t)=\int_0^t\|u(\tau)\|_{L^\infty}d\tau$ and $U(t,x)=\int_0^t u(\tau,x)d\tau$, one has*

$$U_\infty\in C([0,\infty)) \quad\text{and}\quad \nabla U\in C([0,\infty);L^r\cap L^q). \tag{1}$$

**Remark 1.1.** *The smallness assumption on the initial data $a$ is used to guarantee*

$$\sup_{t>0}\left\|\frac{1}{z_0(t)}\right\|_{L^\infty}\lesssim 1, \quad\text{where}\quad z_0(t)=1+\int_0^t e^{\tau\Delta}a\,d\tau.$$

*Indeed, $\displaystyle\int_0^\infty\|e^{\tau\Delta}a\|_{L^\infty}d\tau\approx\|a\|_{\dot B^{-2}_{\infty,1}}$ and $L^1\cap L^p\hookrightarrow\dot B^{-2}_{\infty,1}$. We then have*

$$|z_0(t,x)|\geq 1-\int_0^\infty|e^{\tau\Delta}a(x)|d\tau\geq 1/2.$$

*And, this property is preserved by the evolution of the system and $\displaystyle\sup_{t>0}\left\|\frac{1}{z(t)}\right\|_{L^\infty}\lesssim 1$.*

Because the first equation of $(\tilde{E}_\theta)$ is a divergence form, there formally holds the conservation of mass;

$$\int_{\mathbb{R}^n} u(t)dx = \int_{\mathbb{R}^n} a dx \quad (t \geq 0). \tag{2}$$

However, we do not know whether or not the global solution, constructed in Theorem 1.1, is in $L^1(\mathbb{R}^n)$. The following theorem ensures that the solution belongs to $L^1(\mathbb{R}^n)$ in some time interval, when $p = \infty$ in Theorem 1.1. As a consequence, we can make sure that our global solution constructed in Theorem 1.1 fulfills the conservation of mass (2) for a short time, at least.

**Theorem 1.2** (Large data local existence)**.** *Let $n \geq 1,\ 0 < s < 1$ and $\theta \in \mathbb{R}$.*

(i): *There exists a small constant $C = C(n, s, |\theta|) \in (0, 1)$ such that for any $a \in L^\infty$, we can find a local solution $u \in L^\infty((0, T) \times \mathbb{R}^n)$ with $T = \dfrac{C}{\|a\|_{L^\infty}}$ satisfying*

$$\|u\|_{Y_T^1} + \|u\|_{Y_T^2} + \|u\|_{Y_T^3} + \|u\|_{Y_T^4} \lesssim \|a\|_{L^\infty}$$

*where for $t > 0$*

$$\|u\|_{Y_t^1} = \sup_{\tau<t} \|u(\tau)\|_{L^\infty}, \ \|u\|_{Y_t^2} = \sup_{\tau<t} t^{s/2}\|u(t)\|_{\dot{B}^s_{\infty,\infty}},$$

$$\|u\|_{Y_t^3} = \sup_{\tau<t} \tau^{1/2}\|\nabla u(\tau)\|_{L^\infty} \quad \text{and} \quad \|u\|_{Y_t^4} = \sup_{\tau<t} \tau^{(1+s)/2}\|\nabla u(\tau)\|_{\dot{B}^s_{\infty,\infty}}.$$

(ii): *In addition, if the initial data $a$ also belongs to $L^1(\mathbb{R}^n)$, then the solution $u$, constructed in* (i)*, is in $C([0, T]; L^1)$ where $T = \dfrac{C}{\|a\|_{L^1\cap L^\infty}}$ and $C$ depending on $n$, $s$ and $|\theta|$, and fulfills*

$$\|u\|_{Z_t} = \sup_{\tau<t} \|u(\tau)\|_{L^1} \lesssim \|a\|_{L^1}.$$

**Remark 1.2.**

1. *The condition on $T$ guarantees the lower bound of $1/z$. Indeed, for all $t$ and $x$*

$$|z_0(t, x)| \geq 1 - \int_0^t \|e^{\tau\Delta} a\|_{L^\infty} d\tau \geq 1 - C > 0.$$

2. *Ahn and Kang [1] proved the local existence under the assumption $a, c(= z(0)) \in L^\infty \cap W^{1,p}$ with $p > n$ and certain lower bound of $c = z(0)$.*

Uniqueness of solutions in Theorem 1.2 is valid as follows;

**Theorem 1.3** (Uniqueness)**.** *Let $n \geq 1,\ 0 < s < 1$ and $T \in (0, \infty)$.*

(i): *If $u$ and $v$ are solutions in the class $C((0, T); L^\infty) \cap \cap_{j=2}^4 Y_T^j$ with the same initial data $a \in L^\infty$, then $u(t, x) = v(t, x)$ for all $t \in (0, T)$ and a.e. $x \in \mathbb{R}^n$.*

(ii): *If $u$ and $v$ are solutions in the class $C([0, T); L^1) \cap \cap_{j=1}^4 Y_T^j$ with the same initial data $a \in L^1 \cap L^\infty$, then $u(t, x) = v(t, x)$ for all $t \in (0, T)$ and a.e. $x \in \mathbb{R}^n$.*

This paper is organized as follows. In next section, we recall the definition and equivalence norms of (homogeneous) Besov space and establish our basic estimates. Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively. In Section 5, the proof of Theorem 1.3 is provided.

## 2 Preliminaries

Throughout this paper we use the following notations. $\mathcal{S}$ and $\mathcal{S}'$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. $A \lesssim B$ means $A \leq cB$ with a positive constant $c$. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. $C = C(a, b, c)$ means $C$ depends on $a, b$ and $c$.

Let us recall the definition of Besov spaces. We fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying supp $\varphi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\sum_{j\in\mathbb{Z}} \varphi\left(\frac{\xi}{2^j}\right) = 1$ for $\xi \in \mathbb{R}^n \backslash \{0\}$, and then $\varphi_j(D)f = \mathcal{F}^{-1}\left[\varphi\left(\frac{\cdot}{2^j}\right)\hat{f}(\cdot)\right]$.

**Definition 2.1.** *Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Besov space $\dot{B}^s_{p,q}$ is defined to be the space of $f \in \mathcal{S}'$ modulo polynomials such that*

$$\|f\|_{\dot{B}^s_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\varphi_j(D) f\|^q_{L^p} \right)^{1/q}, \quad q < \infty,$$
$$\|f\|_{\dot{B}^s_{p,\infty}} = \sup_{j \in \mathbb{Z}} 2^{js} \|\varphi_j(D) f\|_{L^p}.$$

It is well-known that $\dot{B}^s_{\infty,\infty}$ with $0 < s \notin \mathbb{N}$ coincides with Hölder space $\dot{C}^s$, i.e.,

$$\|f\|_{\dot{B}^s_{\infty,\infty}} \approx \|f\|_{\dot{C}^s} = \sum_{|\alpha|=[s]} \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^{s-[s]}},$$

where $[s]$ means the integer part of $s$, see for example [4]. Triebel [19] showed that Besov spaces are also characterized by means of heat semigroups $e^{t\Delta}$; with a non-negative integer $m > s/2$

$$\|f\|_{\dot{B}^s_{p,q}} \approx \left( \int_0^\infty \left( t^{m-s/2} \|(-\Delta)^m e^{t\Delta} f\|_{L^p} \right)^q \frac{dt}{t} \right)^{1/q}, \quad q < \infty,$$
$$\|f\|_{\dot{B}^s_{p,\infty}} \approx \sup_{t > \infty} t^{m-s/2} \|(-\Delta)^m e^{t\Delta} f\|_{L^p}. \tag{3}$$

The following lemma is our basic estimate for the proof of Theorem 1.1.

**Lemma 2.1** (Estimates for the non-linear term)**.** *Let $n \ge 1$. For vector functions $F = (f_1, \cdots, f_n)$, we have the following.*

(i): *For $1 \le p < n < q \le \infty$,*

$$\int_0^\infty \left\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot F(\tau) d\tau \right\|_{L^\infty} dt \lesssim \int_0^\infty \|F(\tau)\|_{\dot{B}^{-1}_{\infty,1}} d\tau \lesssim \int_0^\infty \|F(\tau)\|_{L^p \cap L^q} d\tau.$$

(ii): *For all $r \in (1, \infty)$*

$$\left\| \int_0^T \nabla \int_0^t e^{(t-\tau)\Delta} \nabla \cdot F(\tau) d\tau dt \right\|_{L^r} \lesssim \int_0^T \|F(\tau)\|_{L^r} d\tau.$$

*Proof.* (i): Changing order of integrals, we see that the right hand side is bounded by

$$\int_0^\infty \int_0^\infty \left\| e^{t\Delta} \nabla \cdot F(\tau) \right\|_{L^\infty} dt d\tau \approx \int_0^\infty \|F(\tau)\|_{\dot{B}^{-1}_{\infty,1}} d\tau$$

and then from the embedding $L^p \cap L^q \hookrightarrow \dot{B}^{-1}_{\infty,1}$, the desired inequality is verified.

(ii): We also change order of integrals and have that the right hand side is bounded by

$$\int_0^T \left\| \nabla \int_0^{T-\tau} e^{t\Delta} \nabla \cdot F(\tau) dt \right\|_{L^r} d\tau.$$

Though the boundedness of Riesz transform on $L^r(\mathbb{R}^n)$ and the duality, the $L^r$ norm above is dominated by

$$\left| \int_{\mathbb{R}^n} \left( \int_0^{T-\tau} e^{t\Delta} F(\tau) dt \right) \Delta G dx \right|$$

with some $G = (g_1, \cdots, g_n)$ and each $g_j \in \mathcal{S}$ satisfying $\|g_j\|_{L^{r'}} \le 1$. The proof is completed as follows;

$$\int_0^T \left| \int_{\mathbb{R}^n} F(\tau) \int_0^{T-\tau} \partial_t e^{t\Delta} G dt dx \right| d\tau = \int_0^T \left| \int_{\mathbb{R}^n} F(\tau) \left( e^{(T-\tau)\Delta} G - G \right) dx \right| d\tau$$
$$\lesssim \int_0^T \|F(\tau)\|_{L^r} d\tau \|G\|_{L^{r'}}.$$

□

The following estimates for products are verified by paraproduct formula due to Bony [2] and is applied in the proof of Theorem 1.2.

**Lemma 2.2.** *For $s > 0$,*

$$\|fg\|_{\dot{B}^s_{\infty,\infty}} \lesssim \|f\|_{\dot{B}^s_{\infty,\infty}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{B}^s_{\infty,\infty}}.$$

Since it is not hard to prove this, we omit the proof.

Decay estimates of the heat semigroup $e^{t\Delta}$ on Besov spaces are used in the proof.

**Lemma 2.3** ([9])**.** *If $\beta < \alpha$, it follows*

$$\|e^{t\Delta} f\|_{\dot{B}^\beta_{\infty,1}} \lesssim t^{-(\alpha-\beta)/2} \|f\|_{\dot{B}^\alpha_{\infty,\infty}}.$$

See [9] for the proof.

The following estimate is useful to control the norm $Y_t^4$. For the We give a proof based on a method due to Meyer [13], which involves the real interpolation theory.

**Lemma 2.4.** *For any $\alpha \in \mathbb{R}$, there holds*

$$\left\| \int_0^t e^{(t-\tau)\Delta} \nabla f(\tau) d\tau \right\|_{\dot{B}^{\alpha+1}_{\infty,\infty}} \lesssim \sup_{\tau < t} \|f(\tau)\|_{\dot{B}^\alpha_{\infty,\infty}}.$$

*Proof.* We define $\tilde{f}(\tau) = f(\tau)\chi_{(0,t)}(\tau)$ where $\chi_I$ is the characteristic function of $I \subset \mathbb{R}$, and decompose

$$\int_0^t e^{(t-\tau)\Delta} \nabla f(\tau) d\tau = \int_0^\infty e^{\tau\Delta} \nabla \tilde{f}(t-\tau) d\tau = I_A + II_A,$$

where $A > 0$ and

$$I_A = \int_0^A \nabla^\theta e^{\tau\Delta} \nabla^{1-\theta} \tilde{f}(t-\tau) d\tau, \; II_A = \int_A^\infty \nabla^{\tilde{\theta}} e^{\tau\Delta} \nabla^{1-\tilde{\theta}} \tilde{f}(t-\tau) d\tau.$$

Here, $0 < \theta, \tilde{\theta} < 1$. From Lemma 2.3, one has that

$$\|I_A\|_{\dot{B}^{\alpha-1/2}_{\infty,1}} \lesssim \int_0^A \tau^{-3/4} \|\tilde{f}(t-\tau)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}} d\tau \lesssim A^{1/4} \sup_{\tau<t} \|f(\tau)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}$$

$$\|II_A\|_{\dot{B}^{\alpha+1/2}_{\infty,1}} \lesssim \int_A^\infty \tau^{-5/4} \|\tilde{f}(t-\tau)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}} d\tau \lesssim A^{-1/4} \sup_{\tau<t} \|f(\tau)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}.$$

Applying the well-known property of Besov spaces, [18] or [3];

$$\dot{B}^\alpha_{\infty,\infty} = \left(\dot{B}^{\alpha-1/2}_{\infty,\infty}, \dot{B}^{\alpha+1/2}_{\infty,\infty}\right)_{1/2,\infty},$$

we end the proof as follows;

$$\begin{aligned} \left\| \int_0^t e^{(t-\tau)\Delta} \nabla f(\tau) d\tau \right\|_{\dot{B}^{\alpha+1}_{\infty,\infty}} &\lesssim \sup_{\lambda>0} \lambda^{-1/2} \left( \|I_{A_\lambda}\|_{\dot{B}^{\alpha-1/2}_{\infty,1}} + \lambda \|II_{A_\lambda}\|_{\dot{B}^{\alpha+1/2}_{\infty,1}} \right) \\ &\lesssim \sup_{\tau<t} \|f(\tau)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}. \end{aligned}$$

□

## 3 Proof of Theorem 1.1

Let

$$\|u\|_{X_t} = \|u\|_{X_t^1} + \|u\|_{X_t^2} + \|u\|_{X_t^3}.$$

The assumption on exponents ensures the following linear estimates.

**Lemma 3.1** (Linear estimates)**.**

$$\|e^{\cdot\Delta}a\|_{X_\infty} \lesssim \|a\|_{\dot{B}^{-2}_{\infty,1}} + \|a\|_{\dot{B}^{-1}_{r,1}} + \|a\|_{\dot{B}^{-1}_{q,1}} \lesssim \|a\|_{L^1\cap L^p}.$$

*Proof.* The first inequality follows from (3). From

$$\|a\|_{\dot{B}^{-2}_{\infty,1}} \lesssim \sum_{j\le 0} 2^{j(n-2)}\|a\|_{L^1} + \sum_{j\ge 1} 2^{j(n/p-2)}\|a\|_{L^p}$$
$$\|a\|_{\dot{B}^{-1}_{l,1}} \lesssim \sum_{j\le 0} 2^{j(n/l'-1)}\|a\|_{L^1} + \sum_{j\ge 1} 2^{-j}\|a\|_{L^l},\ (l = r, q) \quad \text{and}$$

the interpolation inequality $\|a\|_{L^r\cap L^q} \le \|a\|_{L^1\cap L^p}$, the second inequality is showed. □

With lemmas 2.1 and 3.1, we construct solutions by successive approximation;

$$u_0(t) = e^{t\Delta}a \quad \text{and} \quad u_{m+1}(t) = u_0(t) - B[u_m](t)$$

where

$$z_m(t) = 1 + \int_0^t u_m(\tau)d\tau.$$

We begin the proof of Theorem 1.1 with showing that $\{u_m\}_m$ is a bounded sequence in $X_\infty$.

From Lemma 3.1, $\|u_0\|_{X_\infty} \le C_X\|a\|_{L^1\cap L^p}$ where the constant $C_X$ depends on $n, p, q$ and $r$. Taking $\delta$ so that

$$\delta \le \frac{1}{4C_X}, \tag{4}$$

we see that $\sup_{t>0}\left\|\frac{1}{z_0(t)}\right\|_{L^\infty} \le 4/3$. Assuming $\|u_m\|_{X_\infty} \le 2C_X\|a\|_{L^1\cap L^p}$, we shall check $\|u_{m+1}\|_{X_\infty} \le 2C_X\|a\|_{L^1\cap L^p}$. First, the assumption implies $\sup_{t>0}\left\|\frac{1}{z_m(t)}\right\|_{L^\infty} \le 2$. Hence, applying Lemma 2.1 one can obtain

$$\begin{cases} \|B[u_{m+1}]\|_{X^1_\infty} \lesssim \|u_m\|_{X^1_\infty}\left(\|u_m\|_{X^2_\infty} + \|u_m\|_{X^3_\infty}\right) \\ \|B[u_{m+1}]\|_{X^2_\infty} \lesssim \|u_m\|_{X^1_\infty}\|u_m\|_{X^2_\infty} \\ \|B[u_{m+1}]\|_{X^3_\infty} \lesssim \|u_m\|_{X^1_\infty}\|u_m\|_{X^3_\infty}. \end{cases}$$

Combining this with Lemma 3.1, one has

$$\|u_{m+1}\|_{X_\infty} \le C_X\|a\|_{L^1\cap L^p} + C_X^*|\theta|\|u_m\|^2_{X_\infty}$$

with the constant $C_X^*$ depending on $n, p, q$ and $r$. Then, taking small $\delta$ so that

$$\delta \le \frac{1}{4C_XC_X^*|\theta|} \tag{5}$$

we see $\|u_{m+1}\|_{X_\infty} \le 2C_X\|a\|_{L^1\cap L^p}$. Therefore, $\{u_m\}_m$ is bounded in $X_\infty$. For simplicity, let $K_X = \sup_{m\in\mathbb{N}\cup\{0\}} \|u_m\|_{X_\infty} \le 2C_X\|a\|_{L^1\cap L^p}$.

To show that $\{u_m\}_m$ is also a Cauchy sequence in $X_\infty$, we divide $u_{m+1}(t) - u_m(t)$ into three parts;

$$u_{m+1}(t) - u_m(t) = \theta\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(A_m(\tau) + B_m(\tau) + C_m(\tau))\,d\tau$$

where

$$\begin{cases} A_m(\tau) = -\dfrac{u_m(\tau) - u_{m-1}(\tau)}{z_{m-1}(\tau)} \nabla z_{m-1}(\tau) \\ B_m(\tau) = \dfrac{u_m(\tau)\left(z_m(\tau) - z_{m-1}(\tau)\right)}{z_{m-1}(\tau) z_m(\tau)} \nabla z_{m-1}(\tau) \\ C_m(\tau) = -\dfrac{u_m(\tau)}{z_m(\tau)} \nabla\left(z_m(\tau) - z_{m-1}(\tau)\right). \end{cases}$$

From Lemma 2.1, it follows

$$\|u_{m+1} - u_m\|_{X^1_\infty} \lesssim |\theta| \int_0^\infty \|A_m(\tau) + B_m(\tau) + C_m(\tau)\|_{L^q \cap L^r} d\tau.$$

and

$$\begin{cases} \|A_m(\tau)\|_{L^q\cap L^r} \lesssim \|u_m(\tau) - u_{m-1}(\tau)\|_{L^\infty}\left(\|u_{m-1}\|_{X^2_\infty} + \|u_{m-1}\|_{X^3_\infty}\right) \\ \|B_m(\tau)\|_{L^q\cap L^r} \lesssim \|u_m(\tau)\|_{L^\infty}\|u_m - u_{m-1}\|_{X^1_\infty}\left(\|u_{m-1}\|_{X^2_\infty} + \|u_{m-1}\|_{X^3_\infty}\right) \\ \|C_m(\tau)\|_{L^q\cap L^r} \lesssim \|u_m(\tau)\|_{L^\infty}\left(\|u_m - u_{m-1}\|_{X^2_\infty} + \|u_m - u_{m-1}\|_{X^3_\infty}\right), \end{cases}$$

which imply

$$\|u_{m+1} - u_m\|_{X^1_\infty} \lesssim |\theta| K_X(1 + K_X)\|u_m - u_{m-1}\|_{X_\infty}.$$

The similar argument yields

$$\|u_{m+1} - u_m\|_{X_\infty} \lesssim |\theta| K_X\left(1 + K_X\right)\|u_m - u_{m-1}\|_{X_\infty}.$$

Since $K \le 1/2$, one obtains $\|u_{m+1} - u_m\|_{X_\infty} \le C^{**}_X |\theta| K \|u_m - u_{m-1}\|_{X_\infty}$ where the constant $C^{**}_X$ depends on $n, p, q$ and $r$. Taking small $\delta$ so that

$$\delta \le \frac{1}{4 C_X C^{**}_X |\theta|}, \tag{6}$$

we see then $\{u_m\}_m$ is a Cauchy sequence in $X_\infty$.

To end this proof, we provide a proof of the continuity of $U_\infty$ and $\nabla U$. For $\tilde{t} > t \ge 0$, since $\displaystyle\int_t^{\tilde{t}} \|u(\tau)\|_{L^\infty} d\tau = \int_t^{\tilde{t}} \|e^{\tau\Delta} a\|_{L^\infty} d\tau - |\theta|(\mathrm{I} + \mathrm{II})$ where

$$\begin{aligned} \mathrm{I} &\le \int_0^t \int_{t-\sigma}^{\tilde{t}-\sigma} \left\| e^{\tau\Delta}\nabla\left(u(\sigma)\frac{\nabla z(\sigma)}{z(\sigma)}\right)\right\|_{L^\infty} d\tau d\sigma \quad \text{and} \\ \mathrm{II} &\le \int_t^{\tilde{t}} \int_0^{\tilde{t}-\sigma} \left\| e^{\tau\Delta}\nabla\left(u(\sigma)\frac{\nabla z(\sigma)}{z(\sigma)}\right)\right\|_{L^\infty} d\tau d\sigma, \end{aligned}$$

Lebesgue's dominated convergence theorem ensures

$$U_\infty \in C([0,\infty); L^\infty) \text{ and } U_\infty(t) \to 0 \text{ as } t \searrow 0.$$

The continuity $\nabla U \in C([0,\infty); L^\infty \cap L^q)$ and the convergence $\nabla U(t) \to 0$ in $L^\infty \cap L^q$ as $t \searrow 0$ are also verified by the same argument above. The proof is completed.

**Remark 3.1.** *For any* $\alpha \in (0, 1 - n/r)$,

$$\begin{aligned} \int_0^\infty \|u(t)\|_{\dot{B}^\alpha_{\infty,\infty}} dt &\lesssim \|a\|_{\dot{B}^{\alpha-2}_{\infty,1}} + |\theta|\|u\|_{X_1}\left(\|u\|_{X_2} + \|u\|_{X_3}\right) \\ &\lesssim \|a\|_{L^1\cap L^p} + |\theta|\|u\|^2_X. \end{aligned}$$

# 4 Proof of Theorem 2

Let

$$\|u\|_{Y_t} = \|u\|_{Y_t^1} + \|u\|_{Y_t^2} + \|u\|_{Y_t^3} + \|u\|_{Y_t^4}.$$

(i): We verify that $\{u_m\}_m$ is also a Cauchy sequence in $Y_T$ with some $T$. We have $\|u_0\|_{Y_T} \le C_Y\|a\|_{L^\infty}$. By Lemma 2.2 and

$$\left\|\frac{1}{z_m(t)}\right\|_{\dot B^s_{\infty,\infty}} \approx \left\|\frac{1}{z_m(t)}\right\|_{\dot C^s} \lesssim \int_0^t \|u_m(\tau)\|_{\dot B^s_{\infty,\infty}} d\tau, \tag{7}$$

one can show that if $\sup_{t<T}\left\|\frac{1}{z_m(t)}\right\|_{L^\infty} \lesssim 1$,

$$\begin{cases} \|B[u_{m+1]}\|_{Y_T^1} \lesssim T\|u_m\|_{Y_T^1}\|u_m\|_{Y_T^3} \\ \|B[u_{m+1}]\|_{Y_T^2} \lesssim T\|u_m\|_{Y_T^2}\|u_m\|_{Y_T^3} + T\|u_m\|_{Y_T^1}\left(T\|u_m\|_{Y_T^2}\|u_m\|_{Y_T^3} + \|u_m\|_{Y_T^4}\right) \\ \|B[u_{m+1}]\|_{Y_T^3} \lesssim T\|u_m\|_{Y_T^2}\|u_m\|_{Y_T^3} + T\|u_m\|_{Y_T^1}\left(T\|u_m\|_{Y_T^2}\|u_m\|_{Y_T^3} + \|u_m\|_{Y_T^4}\right) \\ \|B[u_{m+1}]\|_{Y_T^4} \lesssim T\|u_m\|_{Y_T^2}\|u_m\|_{Y_T^3} + T\|u_m\|_{Y_T^1}\left(T\|u_m\|_{Y_T^2}\|u_m\|_{Y_T^3} + \|u_m\|_{Y_T^4}\right), \end{cases}$$

which imply

$$\|u_{m+1}\|_{Y_T} \le C_Y\|a\|_{L^\infty} + C_Y^*|\theta|T\left(1 + T\|u_m\|_{Y_T}\right)\|u_m\|_{Y_T}^2.$$

Here, Lemma 2.2 and (7) have been applied for $Y_T^2$, and Lemma 2.4 and (7) for $Y_T^3$. Therefore, taking

$$T \le \frac{1}{2C_Y\|a\|_{L^\infty}} \quad \text{and} \quad T \le \frac{1}{8C_Y C_Y^*|\theta|\|a\|_{L^\infty}} \tag{8}$$

and assuming $\|u_m\|_{Y_T} \le 2C_Y\|a\|_{L^\infty}$, we have $\|u_{m+1}\|_{Y_T} \le 2C_Y\|a\|_{L^\infty}$. Indeed, from this assumption, $|z_m(t,x)| \ge 1/2$. Let $K_Y(T) = \sup_{m\in\{0\}\cup\mathbb{N}} \|u_m\|_{Y_T} \le 2C_Y\|a\|_{L^\infty}$.

And then, we have

$$\begin{cases} \|A_m(t) + B_m(t) + C_m(t)\|_{L^\infty} \lesssim t^{1/2}K_Y(t)\left(1 + tK_Y(t)\right)\|u_m - u_{m-1}\|_{Y_t} \\ \|A_m(t) + B_m(t) + C_m(t)\|_{\dot B^s_{\infty,\infty}} \lesssim t^{(1-s)/2}K_Y(t)\left(1 + tK_Y(t)\right)^2\|u_m - u_{m-1}\|_{Y_t}. \end{cases}$$

Here, Lemma 2.2 and

$$\left\|\frac{1}{z_{m-1}(t)z_m(t)}\right\|_{\dot B^s_{\infty,\infty}} \lesssim \left\|\frac{1}{z_{m-1}(t)}\right\|_{\dot B^s_{\infty,\infty}} + \left\|\frac{1}{z_m(t)}\right\|_{\dot B^s_{\infty,\infty}} \lesssim t^{1-s/2}K_Y(t)$$

have been used in the second estimate. Then, we have

$$\|u_{m+1} - u_m\|_{Y_T} \le C_Y^{**}|\theta|K_Y(T)T\|u_m - u_{m-1}\|_{Y_T}.$$

If

$$T \le \frac{1}{10C_Y C_Y^{**}|\theta|\|a\|_{L^\infty}}, \tag{9}$$

$\{u_m\}_m$ is a Cauchy sequence in $Y_T$.

(ii): We shall show $\{u_m\}_m$ is also a Cauchy sequence in $C([0,T];L^1(\mathbb{R}^n))$ with some $T$. Let $m \in \mathbb{N}\cup\{0\}$ and we assume $\|u_m\|_{Z_T} \le 2\|a\|_{L^1}$. Of course, $\|u_0\|_{Z_T} \le \|a\|_{L^1}$. Because if $T$ fulfills (8) one has

$$\|u_{m+1}\|_{Z_T} \le \|a\|_{L^1} + C_Y^{**}|\theta|T\|u_m\|_{Z_T}\|u_m\|_{Y_T^3} \tag{10}$$

with a constant $C_Y^{**}$ depending on $n$ only, taking

$$T \le \frac{1}{2C_Y C_Y^{**}|\theta|\|a\|_{L^\infty}}, \tag{11}$$

we then have $\|u_{m+1}\|_{Z_T} \le 2\|a\|_{L^1}$. Hence, we write $K_Z(T) = \sup_{m\in\mathbb{N}\cup\{0\}} \|u_m\|_{Z_T} \le 2\|a\|_{L^1}$ with $T$ fulfills (8) and (11). To show $u_m \in C([0,\infty); L^1(\mathbb{R}^n))$ we write that for small $\varepsilon > 0$

$$
\begin{aligned}
u_{m+1}(t+\varepsilon) - u_{m+1}(t) = {} & e^{t\Delta}\left(e^{\varepsilon\Delta}a - a\right) \\
& + \theta \int_0^t \nabla e^{(t-\tau)\Delta}\left(e^{\varepsilon\Delta} - I_d\right)\left(u_m(\tau)\frac{\nabla z_m(\tau)}{z_m(\tau)}\right) d\tau \\
& - \theta \int_t^{t+\varepsilon} e^{(t+\varepsilon-\tau)\Delta}\nabla\left(u_m(\tau)\frac{\nabla z_m(\tau)}{z_m(\tau)}\right) d\tau.
\end{aligned}
$$

Combining this with Lebesgue's dominated convergence theorem ensures that the right continuity of $u_{m+1}$ in the topology of $L^1$. On the other hand, since

$$
\begin{aligned}
u_{m+1}(t) - u_{m+1}(t-\varepsilon) = {} & e^{(t-\varepsilon)\Delta}\left(e^{\varepsilon\Delta}a - a\right) \\
& + \theta \int_0^{t-\varepsilon} \nabla e^{(t-\varepsilon-\tau)\Delta}\left(e^{\varepsilon\Delta} - I_d\right)\left(u_m(\tau)\frac{\nabla z_m(\tau)}{z_m(\tau)}\right) d\tau \\
& - \theta \int_{t-\varepsilon}^{t} e^{(t-\tau)\Delta}\nabla\left(u_m(\tau)\frac{\nabla z_m(\tau)}{z_m(\tau)}\right) d\tau,
\end{aligned}
$$

we have the left continuity of $u_{m+1}$ from the Lebesgue's theorem again.

Next, we observe

$$
\|u_{m+1} - u_m\|_{Z_T} \lesssim |\theta| \sup_{t<T} \int_0^t (t-\tau)^{-1/2} \|A_m(\tau) + B_m(\tau) + C_m(\tau)\|_{L^1}\, d\tau
$$

and

$$
\begin{cases}
\|A_m(\tau)\|_{L^1} \lesssim \tau^{1/2}\|u_m - u_{m-1}\|_{Z_T}\|u_{m-1}\|_{Y_T^3} \\
\|B_m(\tau)\|_{L^1} \lesssim \tau^{3/2}\|u_m\|_{Z_T}\|u_{m-1}\|_{Y_T^3}\|u_m - u_{m-1}\|_{Y_T^1} \\
\|C_m(\tau)\|_{L^1} \lesssim \tau^{1/2}\|u_m\|_{Z_T}\|u_m - u_{m-1}\|_{Y_T^3}
\end{cases}
$$

and then for $T$ satisfying (11) we have

$$
\begin{aligned}
\|u_{m+1} - u_m\|_{Z_T} \le {} & C_Z|\theta|K_Y(T)T\|u_m - u_{m-1}\|_{Z_T} \\
& + C_Z|\theta|K_Y(T)K_Z(T)T^2\|u_m - u_{m-1}\|_{Y_T} + C_Z|\theta|K_Z(T)T\|u_m - u_{m-1}\|_{Y_T}.
\end{aligned}
$$

where the constant $C_Z$ depends on $n$ only. Here, taking

$$
T \le \frac{1}{10C_Y C_Z(1+|\theta|)\|a\|_{L^\infty}} \quad \text{and} \quad T \le \frac{1}{10C_Z(1+|\theta|)\|a\|_{L^1}}, \tag{12}
$$

and combining these with (9), we see

$$
\|u_{m+1} - u_m\|_{Y_T\cap Z_T} \le 4/5\|u_m - u_{m-1}\|_{Y_T\cap Z_T},
$$

which means that $\{u_m\}_m$ is a Cauchy sequence in $Y_T \cap Z_T$.

## 5 Proof of Theorem 1.3

We give the proof in the case (i), only. Let $w(t) = u(t) - v(t)$. By using estimates in the proof of Theorem 1.2, we can see that there exists small $t \in (0,T)$ so that

$$
\|w\|_{Y_t} \le 1/2\|w\|_{Y_t}.
$$

Let $T^* = \sup\{t \in (0,T); u(t,x) = v(t,x) \text{ a.e. } x \in \mathbb{R}^n\}$ and we assume that $T^* < T$, otherwise the proof is completed. Because $u(T^*) = v(T^*) \in L^\infty$, the above argument ensures that there exists $t_* > 0$ so that $u(t.x) = v(t,x)$ for $t \in [0, T^* + t_*]$ and a.e. $x \in \mathbb{R}^n$. This contradicts the definition of $T^*$ and then we have $T^* = T$.

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