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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR NONLINEAR NEUMANN PROBLEMS WITH INDEFINITE COEFFICIENTS

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Abstract. We consider the following nonlinear Neumann boundary value problem:

$$\begin{cases} -\Delta u + u = a(x)|u|^{p-2}u & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. We suppose a and b are possibly sign-changing functions in $\overline{\Omega}$ and on $\partial\Omega$ respectively. Under some additional assumptions on a and b, we show that (1.1) has infinitely many solutions for sufficiently small $\lambda > 0$ if $1 < q < 2 < p \leq 2^*$. When $p = 2^*$, we use the concentration compactness argument to ensure the PS condition for the associated functional. We also consider a general problem including the supercritical case and obtain the existence of infinitely many solutions.

1. **Introduction.** In this paper we investigate the following nonlinear Neumann boundary value problem:

$$\begin{cases} -\Delta u + u = a(x)|u|^{p-2}u & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative. We suppose a and b are possibly sign-changing functions in $\overline{\Omega}$ and on $\partial\Omega$ respectively. Main purpose of this paper is to show the existence of infinitely many solutions for (1.1). To do that, we consider the associated functional which is defined on $H^1(\Omega)$ and continuously Frechét differentiable on that space:

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p}\int_\Omega a(x)|u|^p dx - \frac{\lambda}{q}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

Before state our results, we put a condition on b:

(B) there exist an open set $D \subset \mathbb{R}^N$ with $D \cap \partial\Omega \neq \emptyset$ and a positive constant $\delta > 0$ such that $b \geq \delta$ on $D \cap \partial\Omega$.

Now our first result is the following:

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Theorem 1.1. *Let $1 < q < 2 < p \leq 2^* = 2N/(N-2)$. Suppose $a \in C(\overline{\Omega})$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B). Then there exists $\Lambda > 0$ such that (1.1) has infinitely many solutions $(u_k) \subset H^1(\Omega)$ for every $0 < \lambda < \Lambda$. Moreover $\mathcal{F}(u_k) < 0$ and $\mathcal{F}(u_k) \to 0$ as $k \to \infty$.*

Remark 1.2. In fact, it is enough to choose $a \in L^\infty(\Omega)$ if $p < 2^*$.

Remark 1.3. If we assume $b \in C(\partial\Omega)$ and there exists $x_0 \in \partial\Omega$ such that $b(x_0) > 0$, then we also have the same conclusion as in Theorem 1.1.

In 1994, Ambrosetti, Brezis and Cerami([1]) considered the elliptic problem with the "convex-concave" nonlinearities. They obtained a variety of results related to the existence of (possibly multiple) solutions for the problems with Dirichlet boundary conditions. Recently some authors have begun to consider such problems with nonlinear Neumann boundary conditions. J.Garcia-Azorero, I.Peral and J.D.Rossi([6]) study problem (1.1) for the case $a \equiv 1$ and $b \equiv 1$. They obtain the Ambrosetti, Brezis and Cerami type results. Among other things, they prove that if $1 < q < 2 < p < 2^*$ and $\lambda > 0$ is sufficiently small, there exist infinitely many solutions for (1.1) with negative energies. Motivated by their result, we consider the general case with the variable coefficients a and b. Now our first question is "Under what conditions on a and b, can we get the existence of infinitely many solutions for (1.1) with negative energies?" One of the answers is as in Theorem 1.1. We emphasize that a and b may change its sign. Note that in Theorem 1.1, we also consider the critical case, i.e. $p = 2^*$ which is not considered in [6]. If p is critical, a typical difficulty occurs in proving PS conditions for $\mathcal{F}$ because of the lack of compactness of the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. We overcome this difficulty by applying the concentration compactness lemma by Lions([9]) and get Theorem 1.1. To the end of this paragraph, we add some results related to our motivation. In the pioneering work by H. Brezis and L. Nirenberg ([2]), they suggest that their problem with variable coefficients becomes more delicate. Following their suggestion, some results appear recently. First see [5] where the convex concave problems with indefinite coefficients (but for Dirichlet boundary conditions) are considered. See also [3] where a nonlinear Neumann boundary value problem with variable coefficients are discussed. In both works, the critical nonlinearities are treated and the existence of (mountain pass) solutions are proved under suitable conditions on their variable coefficients.

Next, we shall consider a more general problem:

$$\begin{cases} -\Delta u + u = f(x,u) & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u & \text{on } \ \partial\Omega, \end{cases} \tag{1.2}$$

where $f(x,u) : \Omega \times \mathbb{R} \mapsto \mathbb{R}$. In this case the associated functional becomes

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2 dx\right) - \int_\Omega F(x,u)dx - \frac{\lambda}{q}\int_{\partial\Omega} b(x)|u|^q d\sigma,$$

where $F(x,t) = \int_0^t f(x,s)ds$. As we shall know from the proof of Theorem 1.1, the concave term in (1.1) is essential for the existence of infinitely many solutions with negative energies. So we can generalize the convex term in (1.1). Now we naturally ask that "Under what conditions on $f(x,u)$, can we ensure the existence of infinitely many solutions with negative energies?" Here we put two conditions for f:

(F1) there exists $\sigma > 0$ such that $f(x,t)$ is a Carathéodory function on $\Omega \times [-\sigma, \sigma]$ and odd in t for all $x \in \Omega$ if $t \in [-\sigma, \sigma]$,

(F2) $f(x,t) = o(|t|)$ as $t \to 0$.

Here we say $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function on $\Omega \times [-\sigma, \sigma]$ (or in $\Omega \times \mathbb{R}$ respectively) if for almost all $x \in \Omega$, the function $t \mapsto f(x,t)$ is continuous on $[-\sigma, \sigma]$ (in $\mathbb{R}$), and for all $t \in [-\sigma, \sigma]$ ($t \in \mathbb{R}$) the function $x \mapsto f(x,t)$ is measurable in Ω. Using the argument in [12], we can obtain the following result for the general problem (1.2).

Theorem 1.4. *Suppose $1 < q < 2$ and f satisfies the conditions (F1) and (F2). Assume further, $b \in L^\infty(\partial\Omega)$ and satisfies the condition (B). Then (1.2) has infinitely many solutions $(u_k) \subset H^1(\Omega)$ for every $\lambda > 0$. Moreover $I(u_k) < 0$, $I(u_k) \to 0$ and $\|u_k\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$.*

Remark 1.5. Thanks to the conditions on $f(x,t)$, $F(x,t)$ is well-defined if $|t|$ is small enough. Consequently I is well-defined for functions in $H^1(\Omega) \cap L^\infty(\Omega)$ whose $L^\infty(\Omega)$ norms are sufficiently small.

Remark 1.6. We need no restriction for $\lambda > 0$ to be sufficiently small for the existence.

In view of Theorem 1.4, we obtain a conclusion for (1.1) including the supercritical case:

Corollary 1.7. *Let $1 < q < 2 < p < \infty$. We suppose $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B). Then (1.1) has infinitely many solutions $(u_k) \subset H^1(\Omega)$ for every $\lambda > 0$. Moreover $I(u_k) < 0$, $I(u_k) \to 0$ and $\|u_k\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$.*

This paper is organized as follows: In section 2, we give the proof of Theorem 1.1 for the subcritical case, i.e. $p < 2^*$. We use the variational method in [6]. By careful reading of the proof in [6] and considering the conditions on the coefficients a and b, we can get the result. Especially see the proof of Lemma 2.2. Next, in section 3, we give the proof of Theorem 1.1 for the critical case, i.e. $p = 2^*$. As we already say, the main difficulty arises in the proof of the PS conditions for $\mathcal{F}$. We shall show the proof of $L^{2^*}(\Omega)$ convergence for the PS sequences. This is the key of the proof of this section, see Lemma 3.1. Lastly in section 4, we consider the general problem (1.2) and give the proof of Theorem 1.4. Here we use the argument in [12].

As usual, we denote $H^1(\Omega)$ as the Sobolev space consisted by the all functions which belong to $L^2(\Omega)$ and its first weak derivative also belong to $L^2(\Omega)$. We write its norm as $\|\cdot\|_{H^1(\Omega)} = \left\{\int_\Omega \left(|\nabla \cdot|^2 + (\cdot)^2\right)\right\}^{\frac{1}{2}}$. Note also that we denote $H^{-1}(\Omega)$ as the dual space of $H^1(\Omega)$. We write the duality in $H^{-1}(\Omega)$ and $H^1(\Omega)$ as $\langle \cdot, \cdot \rangle$ and the norm of $H^{-1}(\Omega)$ is $\|\cdot\|_{H^{-1}(\Omega)} = \sup_{v \in H^1(\Omega), \|v\|_{H^1(\Omega)} \le 1} |\langle \cdot, v\rangle|$.

2. **The subcritical case.** In this section we consider the subcritical case. Let $1 < q < 2 < p < 2^*$. Here we use the variational method in [6] (see also [10]). First of all, since in general, $\mathcal{F}$ is not bounded from below, we perform the appropriate truncation for the functional $\mathcal{F}$. To do that, first notice that by the Sobolev embedding and the trace theorem,

$$\mathcal{F}(u) \ge \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{c_1}{p}\|u\|^p_{H^1(\Omega)} - \frac{\lambda c_2}{q}\|u\|^q_{H^1(\Omega)} = f_\lambda(\|u\|_{H^1(\Omega)})$$

where $f_\lambda(x) := \frac{1}{2}x^2 - \frac{c_1}{p}x^p - \frac{\lambda c_2}{q}x^q$. Take $\Lambda_0 > 0$ so small that $\max_{[0,\infty)} f_\lambda$ is positive for all $0 < \lambda < \Lambda_0$. Choose $0 < m < x_0 < x_1 < M$ so that $f(m) < 0 < f(x_0) < f(x_1) < f(M)$ where m and M are unique local minimum and maximum points of f respectively. Now consider a cut off function $\tau \in C^1(\mathbb{R})$ defined by

$$\tau(\xi) = \begin{cases} 1 & \text{if } 0 \le \xi < x_0, \\ 0 & \text{if } \xi > x_1, \end{cases}$$

$$0 \le \tau(\xi) \le 1 \quad \text{if } x_0 \le \xi \le x_1,$$

and define the C^1 functional on $H^1(\Omega)$:

$$\Phi(u) = \tau(\|u\|_{H^1(\Omega)}).$$

Finally we define the truncated functional:

$$\tilde{\mathcal{F}}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p}\int_\Omega a(x)\Phi(u)|u|^p dx - \frac{\lambda}{q}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

We can easily check that $\tilde{\mathcal{F}}$ is well-defined and continuously Fréchet differentiable on $H^1(\Omega)$. Notice also that $\tilde{\mathcal{F}} = \mathcal{F}$ on some neighborhood of u satisfying $\tilde{\mathcal{F}}(u) < 0$. In addition observe that $\tilde{\mathcal{F}}(u)$ is even in u and $\tilde{\mathcal{F}}(0) = 0$.

Now we can get the following lemma

Lemma 2.1. *$\tilde{\mathcal{F}}$ is bounded from below and satisfies the (PS)$_c$ condition if $c < 0$.*

Proof. Let us first show that $\tilde{\mathcal{F}}$ is bounded from below. In fact, by the definition of $\Phi(u)$, if $\|u\|_{H^1(\Omega)} < x_1$, $0 \le \Phi(u) \le 1$ and if $\|u\|_{H^1(\Omega)} > x_1$, $\Phi(u) = 0$. So $\Phi(u)\|u\|^p_{H^1(\Omega)} \le x_1^p$. Hence by the Sobolev embedding and the trace theorem,

$$\begin{aligned}\tilde{\mathcal{F}}(u) &\ge \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{c_1}{p}\Phi(u)\|u\|^p_{H^1(\Omega)} - \frac{\lambda c_2}{q}\|u\|^q_{H^1(\Omega)} \\ &\ge \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{c_1 x_1^p}{p} - \frac{\lambda c_2}{q}\|u\|^q_{H^1(\Omega)}.\end{aligned}$$

Since $q < 2$, $\tilde{\mathcal{F}}$ is bounded from below.

We next prove that $\tilde{\mathcal{F}}$ satisfies the (PS)$_c$ condition if $c < 0$. To do that, let (u_j) be a (PS)$_c$ sequence for $\tilde{\mathcal{F}}$ at level $c < 0$. By the property of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{F}}(u_j) = \mathcal{F}(u_j)$ for large j since $c < 0$. Therefore (u_j) is also a (PS)$_c$ sequence for $\mathcal{F}$, i.e., $\mathcal{F}(u_j) \to c$ and $\mathcal{F}'(u_j) \to 0$ in $H^{-1}(\Omega)$. Now we claim that (u_j) is bounded in $H^1(\Omega)$. Actually

$$\begin{aligned}c + 1 &\ge \mathcal{F}(u_j) - \frac{1}{p}\langle \mathcal{F}'(u_j), u_j\rangle + \frac{1}{p}\langle \mathcal{F}'(u_j), u_j\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u_j\|^p_{H^1(\Omega)} - \lambda\left(\frac{1}{q} - \frac{1}{p}\right) b_\infty \|u_j\|^q_{H^1(\Omega)} - \|u\|_{H^1(\Omega)}\end{aligned}$$

for large j, where $b_\infty := \|b\|_{L^\infty(\partial\Omega)}$. Since $1 < q < 2 < p$, (u_j) is bounded in $H^1(\Omega)$. Therefore we can assume there exists $u \in H^1(\Omega)$ such that $u_j \rightharpoonup u$ weakly in $H^1(\Omega)$. Moreover noting that $p < 2^*$ and $q < 2$, by the Rellich Theorem, we can also assume

$$\begin{aligned}&u_j \to u \text{ in } L^p(\Omega), \\ &u_j \to u \text{ in } L^q(\partial\Omega).\end{aligned} \tag{2.1}$$

Then we obtain

$$\begin{aligned}&|u_j|^{p-2}u_j \to |u|^{p-2}u \text{ in } H^{-1}(\Omega), \\ &|u_j|^{q-2}u_j \to |u|^{q-2}u \text{ in } H^{-1}(\Omega).\end{aligned}$$

By Lax-Milgram Theorem (or Lemma 2.1 in [6]), we conclude

$$u_j \to u \text{ in } H^1(\Omega).$$

This completes the proof. □

The condition (B) on the indefinite function b in Theorem 1.1 is essential for the following lemma:

Lemma 2.2. *Suppose $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B). Then for every $n \in \mathbb{N}$, there exist an n-dimensional subspace $E_n \subset H^1(\Omega)$, and positive constants $\rho > 0$ and $\varepsilon > 0$ such that*

$$\tilde{\mathcal{F}}(u) \leq -\varepsilon$$

for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$.

Proof. From the condition (B) on $b \in L^\infty(\partial\Omega)$, for every $n \in \mathbb{N}$, we can construct an n-dimensional subspace E_n in $\{u \in C^\infty(\overline{\Omega}) \mid u \equiv 0 \text{ on } \partial\Omega \setminus D\}$ such that if $u \in E_n$, then $u \equiv 0$ on $\partial\Omega$ if and only if $u = 0$. If we take a nonzero function $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$, by the Sobolev embedding, we get

$$\begin{aligned}\tilde{\mathcal{F}}(u) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p}\int_\Omega a(x)\Phi(u)|u|^p dx - \frac{\lambda}{q}\int_{\partial\Omega\cap D} b(x)|u|^q d\sigma \\ &\leq \frac{1}{2}\rho^2 + \frac{a_\infty c_3}{p}\rho^p - \frac{\lambda\delta}{q}\int_{\partial\Omega}|u|^q d\sigma,\end{aligned}$$

where $a_\infty := \|a\|_{L^\infty(\Omega)}$. Since E_n is finite dimensional, the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{L^q(\partial\Omega)}$ are equivalent. So we obtain

$$\tilde{\mathcal{F}}(u) \leq \frac{1}{2}\rho^2 + \frac{a_\infty c_3}{p}\rho^p - \frac{\lambda\delta c_4}{q}\rho^q.$$

Since $q < 2 < p$, there exist $\rho > 0$ and $\varepsilon > 0$ such that

$$\tilde{\mathcal{F}}(u) \leq -\varepsilon$$

for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$. □

Now we introduce a topological tool, the "genus"([8],[4], see also [10]). we give the following definition according to [10]:

Consider the class

$$\Sigma = \{A \subset H^1(\Omega) \setminus \{0\} \mid A \text{ is closed, } A = -A\}.$$

Then we define the genus, $\gamma : \Sigma \to \{0\} \cup \mathbb{N} \cup \{\infty\}$ so that

$$\gamma(A) = \min\{k \in \mathbb{N} \mid \text{there exists an odd map } \phi \in C(A, \mathbb{R}^k \setminus \{0\})\},$$

or if there exists no such a minimum, $\gamma(A) = \infty$. In addition we define $\gamma(\emptyset) = 0$. Consequently we get the following properties of the genus([10]). Let $A, B \in \Sigma$ then

1. Normalization: If $x \neq 0$, $\gamma(\{x\} \cup \{-x\}) = 1$.
2. Mapping property: If there exists an odd map $f \in C(A, B)$ then $\gamma(A) \leq \gamma(B)$.
3. Monotonicity property: If $A \subset B$, $\gamma(A) \leq \gamma(B)$.
4. Subadditivity: $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
5. Continuity property: If A is compact, then $\gamma(A) < \infty$ and there exists $d > 0$ such that $N_d = \{u \in H^1(\Omega) \mid \operatorname{dist}(u, A) \leq d\} \in \Sigma$ and $\gamma(N_d) = \gamma(A)$.

Here we prove the following lemma.

Lemma 2.3. *Let $n \in \mathbb{N}$. Assume there exist an n-dimensional subspace $E_n \subset H^1(\Omega)$ and positive constants $\rho > 0$ and $\varepsilon > 0$ such that $\tilde{\mathcal{F}}(u) \leq -\varepsilon$ for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$. Then*

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \geq n.$$

where $\tilde{\mathcal{F}}^c = \{u \in H^1(\Omega) \mid \tilde{\mathcal{F}}(u) \leq c\}$.

Proof. We define $S_{\rho,n} = \{u \in E_n \mid \|u\|_{H^1(\Omega)} = \rho\}$. From the assumption, we have $S_{\rho,n} \subset \tilde{\mathcal{F}}^{-\varepsilon}$. From the monotonicity of the genus, we conclude that

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \geq \gamma(S_{\rho,n}) = n.$$

□

Finally we prove the main theorem of this section.

Theorem 2.4. *Let*

$$\Sigma = \{A \subset H^1(\Omega) \setminus \{0\} \mid A \text{ is closed, } A = -A\},$$
$$\Sigma_k = \{A \in \Sigma \mid \gamma(A) \geq k\},$$

and put

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \tilde{\mathcal{F}}(u),$$

then c_k is a negative critical value of $\mathcal{F}$. Moreover if $c := c_k = c_{k+1} = \cdots = c_{k+r}$

$$\gamma(K_c) \geq r + 1$$

where $K_c = \{u \in H^1(\Omega) \mid \tilde{\mathcal{F}} = c,\ \tilde{\mathcal{F}}'(u) = 0\}$.

Proof. We first prove that c_k is negative. From Lemma 2.2 and 2.3, there exists $\varepsilon > 0$ such that $\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \geq k$. Since $\tilde{\mathcal{F}}$ is even and continuous, $\tilde{\mathcal{F}}^{-\varepsilon} \in \Sigma_k$. Hence $c_k \leq -\varepsilon < 0$. In addition from lemma 2.1, $c_k > -\infty$.

To complete the proof, let us assume that $c := c_k = c_{k+1} = \cdots = c_{k+r}$. As $\tilde{\mathcal{F}}$ is C^1, K_c is closed. Since $\tilde{\mathcal{F}}(u)$ is even in u, K_c is symmetric. In addition as $\tilde{F}(0) = 0$ and $c < 0$, $0 \notin K_c$. Hence $K_c \in \Sigma$. Now we assume $\gamma(K_c) \leq r$ to the contrary. As $\tilde{\mathcal{F}}$ satisfies $(\mathrm{PS})_c$ condition, K_c is compact. Hence from the continuity of the genus, there exists $d > 0$ such that $N_d(K_c) \in \Sigma_k$ and $\gamma(N_d(K_c)) \leq r$ where $N_d(K_c) = \{u \in H^1(\Omega) \mid \mathrm{dist}(u, K_c) \leq d\}$. Now we use the deformation theorem([10]). From that, there exists an odd homeomorphism $\eta(1, \cdot) : H^1(\Omega) \to H^1(\Omega)$ such that

$$\eta(1, \tilde{\mathcal{F}}^{c+\frac{\varepsilon}{2}} \setminus N_d(K_c)) \subset \tilde{\mathcal{F}}^{c-\frac{\varepsilon}{2}}.$$

By the definition of c_{k+r}, there exists $A \in \Sigma_k$ such that $A \subset \tilde{\mathcal{F}}^{c+\frac{\varepsilon}{2}}$. So we have

$$\eta(1, A \setminus N_d(K_c)) \subset \tilde{\mathcal{F}}^{c-\frac{\varepsilon}{2}}. \tag{2.2}$$

On the other hand, from the subadditivity of the genus,

$$\begin{aligned} \gamma(\overline{A \setminus N_d(K_c)}) &\geq \gamma(A) - \gamma(N_d(K_c)) \\ &\geq k. \end{aligned}$$

Noting that $\eta(1, \cdot)$ is an odd homeomorphism, we have $\eta(1, \overline{A \setminus N_d(K_c)}) \in \Sigma$. Moreover considering the mapping property of the genus, we get

$$\begin{aligned} \gamma(\eta(1, \overline{A \setminus N_d(K_c)})) &\geq \gamma(\overline{A \setminus N_d(K_c)}) \\ &\geq k. \end{aligned}$$

Then $\eta(1, \overline{A \setminus N_d(K_c)}) \in \Sigma_k$ and we conclude that

$$\sup_{u \in \eta(1, \overline{A \setminus N_d(K_c)})} \tilde{\mathcal{F}}(u) \geq c = c_k,$$

which contradicts (2.2). Therefore we have $\gamma(K_c) \geq r+1$. Hence, for every $k \in \mathbb{N}$, $K_{c_k} \neq \emptyset$. Let u_k be a critical point of $\tilde{\mathcal{F}}$ with $\tilde{\mathcal{F}}(u_k) = c_k$. Since $c_k < 0$, $\tilde{\mathcal{F}} = \mathcal{F}$ on some neighborhood of u_k. Consequently u_k is a critical point of $\mathcal{F}$. This finishes the proof. □

Now we can prove the following corollary using Theorem 2.4 and the deformation theorem. The argument is similar to the above one. Hence we leave it for readers.

Corollary 2.5. *Let c_k be as defined in Theorem 2.4. Then $c_k \to 0$.*

We give the proof of Theorem 1.1 for the subcritical case.

Proof of Theorem 1.1 for the subcritical case. We suppose $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B). Choose $\Lambda = \Lambda_0$ and take $0 < \lambda < \Lambda$ as in the first paragraph of this section. Then by Theorem 2.4, we have the negative critical values $c_1, c_2, \cdots$ of $\mathcal{F}$. In addition from Corollary 2.5 , $c_k \to 0$ as $k \to \infty$. Hence we conclude that the set $\{c_k \mid k \in \mathbb{N}\}$ has infinitely many distinct elements. This completes the proof. □

3. **The critical case.** In this section we prove Theorem 1.1 for the critical case, i.e. $p = 2^*$. Let $1 < q < 2$ and consider the functional:

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{2^*}\int_\Omega a(x)|u|^{2^*} dx - \frac{\lambda}{q}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

The organization of the proof is same with that for the subcritical case if once we ensure the strong $L^{2^*}(\Omega)$ convergence for PS sequences. We begin with the following lemma.

Lemma 3.1. *Assume $a \in C(\overline{\Omega})$ and $b \in L^\infty(\partial\Omega)$. Let $c < 0$ and $(u_j) \subset H^1(\Omega)$ be a (PS)$_c$ sequence for $\mathcal{F}$. Then there exists $\Lambda_1 > 0$ such that for every $0 < \lambda < \Lambda_1$, (u_j) strongly converges in $L^{2^*}(\Omega)$ up to subsequences.*

Proof. By the same argument in the proof of Lemma 2.1, we ensure that (u_j) is bounded in $H^1(\Omega)$. Hence we can assume there exists $u \in H^1(\Omega)$ such that $u_j \rightharpoonup u$ weakly in $H^1(\Omega)$. Further, by the Rellich Theorem, we can also assume that

$$\begin{aligned} &u_j \to u \text{ in } L^2(\Omega),\\ &u_j \to u \text{ in } L^q(\partial\Omega),\\ &u_j \to u \text{ a.e. on } \Omega. \end{aligned} \tag{3.1}$$

We now apply the concentration compactness lemma by Lions [9], see also [11]. By the result, we can assume there exist some at most countable set J, distinct points $(x_k)_{k\in J} \subset \overline{\Omega}$ and positive constants $(\nu_k)_{k\in J}$, $(\mu_k)_{k\in J}$ such that

$$\begin{aligned} &|\nabla u_j|^2 \rightharpoonup d\mu \geq |\nabla u|^2 + \sum_{k\in J} \mu_k \delta_{x_k},\\ &|u_j|^{2^*} \rightharpoonup d\nu = |u|^{2^*} + \sum_{k\in J} \nu_k \delta_{x_k} \end{aligned} \tag{3.2}$$

in the measure sense, where δ_x denotes the Dirac measure with mass 1 concentrated at $x \in \mathbb{R}^N$. In addition by the result in [6] (Lemma 7.3),

$$\begin{aligned} \mu_k &\geq S\nu_k^{\frac{2}{2^*}} \text{ if } x_k \in \Omega, \\ \mu_k &\geq \frac{S}{2^{\frac{2}{N}}}\nu_k^{\frac{2}{2^*}} \text{ if } x_k \in \partial\Omega, \end{aligned} \tag{3.3}$$

where

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

We show there exists $\Lambda_1 > 0$ such that $J = \emptyset$ for all $0 < \lambda < \Lambda_1$ if $c < 0$. To do that, assume $c < 0$ and take $0 < \lambda < \Lambda_1$ where $\Lambda_1 > 0$ is determined later. Now we suppose $J \neq \emptyset$ to the contrary. Fix $k \in J$ and introduce a cut off function $\phi \in C^\infty(\mathbb{R}^N)$ such that

$$\phi(x) = \begin{cases} 1 \text{ if } x \in B(x_k, \varepsilon), \\ 0 \text{ if } x \in B(x_k, 2\varepsilon)^c, \end{cases}$$
$$|\nabla\phi| \leq \frac{2}{\varepsilon}.$$

Since (u_j) is bounded in $H^1(\Omega)$, $u_j\phi$ is also bounded in $H^1(\Omega)$. In addition, as $\mathcal{F}'(u_j) \to 0$ in $H^{-1}(\Omega)$, we have

$$\langle \mathcal{F}'(u_j), u_j\phi \rangle \to 0.$$

Hence recalling (3.2) and noting $a \in C(\overline{\Omega})$, we get

$$\begin{aligned} 0 &= \lim_{j\to\infty} \langle \mathcal{F}'(u_j), u_j\phi \rangle \\ &= \lim_{j\to\infty} \left\{ \int_\Omega \nabla u_j \cdot \nabla(u_j\phi) dx + \int_\Omega u_j^2 \phi dx - \int_\Omega a(x)|u_j|^{2^*}\phi dx - \lambda \int_{\partial\Omega} b(x)|u_j|^q \phi d\sigma \right\} \\ &= \lim_{j\to\infty} \int_\Omega (\nabla u_j \cdot \nabla\phi) u_j + \int_{\overline{\Omega}} \phi d\mu + \int_\Omega u^2\phi dx - \int_{\overline{\Omega}} a(x)\phi d\nu - \lambda \int_{\partial\Omega} b(x)|u|^q \phi d\sigma \end{aligned} \tag{3.4}$$

where the third term and the fifth term in the last inequality are obtained by (3.1) and Vitali's convergence theorem. Here,

$$\begin{aligned} 0 &\leq \lim_{j\to\infty} \left| \int_\Omega (\nabla u_j \cdot \nabla\phi) u_j \right| \\ &\leq \lim_{j\to\infty} \left(\int_{\Omega\cap B(x_k,2\varepsilon)} |\nabla u_j|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega\cap B(x_k,2\varepsilon)} u^2 |\nabla\phi|^2 dx \right)^{\frac{1}{2}} \\ &\leq C_1 \left(\int_{\Omega\cap B(x_k,2\varepsilon)} |u|^{2^*} dx \right)^{\frac{1}{2^*}} \left(\int_{\Omega\cap B(x_k,2\varepsilon)} |\nabla\phi|^N \right)^{\frac{1}{N}} \\ &\leq C_2 \left(\int_{\Omega\cap B(x_k,2\varepsilon)} |u|^{2^*} dx \right)^{\frac{1}{2^*}} \to 0 \text{ as } \varepsilon \to 0, \end{aligned}$$

where for the second inequality we use the Schwartz inequality and (3.1), for the third inequality we use the fact that (u_j) is bounded in $H^1(\Omega)$ and Hölder inequality, and for the forth inequality we use the fact that $|\nabla\phi| \le 2/\varepsilon$. Taking $\varepsilon \to 0$ for (3.4), we obtain

$$\mu_k - a(x_k)\nu_k \le 0. \tag{3.5}$$

Since μ_k and ν_k are positive, we can assume $a(x_k) > 0$. Considering (3.3) and (3.5) together, we have

$$\nu_k \ge \frac{1}{2}\left(\frac{S}{a(x_k)}\right)^{\frac{N}{2}}.$$

So using this inequality and (3.2) again,

$$\begin{aligned} c &= \lim_{j\to\infty}\left\{\mathcal{F}(u_j) - \frac{1}{2}\langle \mathcal{F}'(u_j), u_j\rangle\right\} \\ &= \lim_{j\to\infty}\left\{\frac{1}{N}\int_\Omega a(x)|u_j|^{2^*}dx - \lambda\left(\frac{1}{q}-\frac{1}{2}\right)\int_{\partial\Omega} b(x)|u_j|^q d\sigma\right\} \\ &\ge \frac{1}{N}\int_\Omega a(x)|u|^{2^*}dx + \frac{a(x_k)}{2}\left(\frac{S}{a(x_k)}\right)^{\frac{N}{2}} - \lambda\left(\frac{1}{q}-\frac{1}{2}\right)\int_{\partial\Omega} b(x)|u|^q d\sigma, \end{aligned} \tag{3.6}$$

where last inequality is obtained by the assumption $a(x_k) > 0$. Now since u is a critical point of $\mathcal{F}$, we have $\langle \mathcal{F}'(u), u\rangle/N = 0$ and then

$$\frac{1}{N}\int_\Omega a(x)|u|^{2^*}dx = \frac{1}{N}\|u\|^2_{H^1(\Omega)} - \frac{1}{N}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

Substituting this equality into (3.6), noting $1 < q < 2 < q^* = 2(N-1)/(N-2)$ and using the trace theorem, we have

$$c \ge \frac{a(x_k)}{2}\left(\frac{S}{a(x_k)}\right)^{\frac{N}{2}} + \frac{1}{N}\|u\|^2_{H^1(\Omega)} - \lambda\left(\frac{1}{q}+\frac{1}{N}-\frac{1}{2}\right)C_3\|u\|^q_{H^1(\Omega)}.$$

If we consider

$$g(x) = \frac{1}{N}x^2 - \lambda\left(\frac{1}{q}+\frac{1}{N}-\frac{1}{2}\right)C_3 x^q$$

for $x > 0$, we have

$$g(x) \ge -\lambda^{\frac{2}{2-q}}\left\{C_3\left(\frac{1}{q}+\frac{1}{N}-\frac{1}{2}\right)\right\}^{\frac{2}{2-q}}\left(\frac{qN}{2}\right)^{\frac{q}{2-q}}\left(1-\frac{q}{2}\right) =: -\lambda^{\frac{2}{2-q}}K.$$

Hence noting $\frac{N}{2} - 1 > 0$ and above inequality, we get

$$c \ge \frac{\max_{x\in\overline{\Omega}} a(x)}{2}\left(\frac{S}{\max_{x\in\overline{\Omega}} a(x)}\right)^{\frac{N}{2}} - \lambda^{\frac{2}{2-q}}K$$

Now we can take $\Lambda_1 > 0$ so small that the right-hand side of the above inequality is greater than 0 for all $0 < \lambda < \Lambda_1$. This leads us to the contradiction since $c < 0$. Note that we can choose $\Lambda_1 > 0$ so that it does not depend on our choice of $k \in \mathbb{N}$. Hence we conclude $J = \emptyset$ for all $0 < \lambda < \Lambda_1$ if $c < 0$. Consequently by (3.2) again, we have

$$\int_\Omega |u_j|^{2^*}dx \to \int_\Omega |u|^{2^*}dx.$$

Using the Vitali's convergence theorem, we conclude that

$$u_j \to u \text{ in } L^{2^*}(\Omega).$$

This completes the proof. □

This lemma enable us to ensure $(\mathrm{PS})_c$ conditions for $\mathcal{F}$. Now we can prove Theorem 1.1 for the critical case by the same argument in section 2.

Proof of Theorem 1.1 for the critical case. We suppose $a \in C(\overline{\Omega})$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B). As we already say, the organization of the proof for the critical case is same with that for the subcritical case. So we give only a comment for the choice of $\Lambda > 0$. To perform the appropriate truncation for the functional $\mathcal{F}$, we first choose $\Lambda_0 > 0$ by the same argument with that in section 2. Next we take $\Lambda_1 > 0$ from Lemma 3.1. Then it is enough to select $\Lambda = \min\{\Lambda_0, \Lambda_1\}$. □

4. **A general case.** In this section, we consider a general case, i.e. problem (1.2). Here we note the associated functional:

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2 dx\right) - \int_\Omega F(x,u)dx - \int_{\partial\Omega} b(x)|u|^q d\sigma,$$

where $F(x,t) = \int_0^t f(x,s)ds$. Let $1 < q < 2$ as previous sections. Using the argument in [12], we shall prove Theorem 1.4. A similar result for a Dirichlet boundary value problem is found in [7]. To the beginning we put some properties for a modified function $\tilde{f}(x,u)$:

($\tilde{F}1$) $|\tilde{F}(x,u)| \le \frac{1}{4}u^2$ where $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds$,
($\tilde{F}2$) there exists $0 < \theta < (2-q)/2$ such that $\tilde{f}(x,u)u - q\tilde{F}(u) \le \theta u^2$,
($\tilde{F}3$) there exists $0 < a < \frac{\sigma}{2}$ such that $\tilde{f}(x,u) = f(x,u)$ if $|u| < a$.

Now we construct the modified function.

Lemma 4.1. *Let $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ satisfy the conditions (F1) and (F2) i.e.*

(F1) there exists $\sigma > 0$ such that $f(x,t)$ is a Carathéodory function on $\Omega \times [-\sigma,\sigma]$ and odd in t for all $x \in \Omega$ if $t \in [-\sigma,\sigma]$,
(F2) $f(x,t) = o(|t|)$ as $t \to 0$.

Then there exists a Carathéodory function $\tilde{f}(x,t)$ in $\Omega \times \mathbb{R}$ which is odd in t and satisfies the conditions ($\tilde{F}1$), ($\tilde{F}2$) and ($\tilde{F}3$).

Proof. For fixed $0 < \theta < (2-q)/2$, take $0 < \varepsilon < \frac{\theta}{14}$. From (F2) there exists $0 < a < \frac{\sigma}{2}$ such that $|f(x,u)u| \le \varepsilon u^2$ and $|F(x,u)| \le \varepsilon u^2$ if $|u| \le 2a$. Now define a cut off function $\rho \in C^1(\mathbb{R})$ such that $\rho(t) = 1$ if $|t| \le a$, $\rho(t) = 0$ if $|t| > 2a$ and $0 \le \rho \le 1$ otherwise. Further, we can assume $|\rho'(t)| \le 2/a$. Firstly, we define

$$\tilde{F}(x,u) = \rho(u)F(x,u) + (1-\rho(u))F_\infty(u)$$

where $F_\infty(u) = \beta u^2$ for some $0 < \beta < \frac{1}{16}\theta$. Then we have

$$|\tilde{F}(x,u)| \le \frac{1}{4}u^2. \tag{4.1}$$

Indeed, if $|u| \le 2a$, we get

$$\begin{aligned} |\tilde{F}(x,u)| &\le |F(x,u)| + F_\infty(x,u) \\ &\le (\varepsilon + \beta)u^2 \\ &\le \frac{1}{4}u^2, \end{aligned}$$

and if $|u| > 2a$, we obtain

$$\begin{aligned}|\tilde{F}(x,u)| &\leq F_\infty(x,u)\\ &\leq \frac{1}{4}u^2.\end{aligned}$$

Next we put

$$\tilde{f}(x,u) = \frac{\partial \tilde{F}}{\partial u}(x,u).$$

Then we get

$$\tilde{f}(x,u) = \rho'(u)F(x,u) + \rho(u)f(x,u) - \rho'(u)F_\infty(u) + (1-\rho(u))F'_\infty(u).$$

By (F1), clearly $\tilde{f}(x,u)$ is a Carathéodory function in $\Omega \times \mathbb{R}$, odd in u and

$$f(x,u) = \tilde{f}(x,u) \text{ if } |u| < a. \tag{4.2}$$

In addition, we put

$$\begin{aligned}\tilde{f}(x,u)u - q\tilde{F}(x,u) =& \left(\rho'(u)u - q\rho(u)\right)F(x,u) + \rho(u)f(x,u)u\\ &- \left(\rho'(u)u + q(1-\rho(u))\right)F_\infty(u) + (1-\rho(u))F'_\infty(u)u.\end{aligned}$$

Here we claim

$$\tilde{f}(x,u) - q\tilde{F}(x,u) \leq \theta u^2. \tag{4.3}$$

In fact, if $|u| \leq 2a$ we have

$$\begin{aligned}\tilde{f}(x,u) - q\tilde{F}(x,u) &\leq (7\varepsilon + 8\beta)u^2,\\ &\leq \theta u^2,\end{aligned}$$

and if $|u| > 2a$ we get

$$\begin{aligned}\tilde{f}(x,u) - q\tilde{F}(x,u) &\leq 4\beta u^2,\\ &\leq \theta u^2.\end{aligned}$$

(4.1), (4.2) and (4.3) conclude the proof. □

From now on, let $\tilde{f}(x,u)$ be the one constructed in Lemma 4.1. We consider the modified problem:

$$\begin{cases}-\Delta u + u = \tilde{f}(x,u) & \text{in } \ \Omega,\\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u & \text{on } \ \partial\Omega,\end{cases} \tag{4.4}$$

and the associated functional:

$$\tilde{I}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)dx - \int_\Omega \tilde{F}(x,u)dx - \lambda\int_{\partial\Omega} b(x)|u|^q d\sigma,$$

where $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds$. Noting the condition $(\tilde{F}1)$, we can easily check that $\tilde{I}$ is well-defined on $H^1(\Omega)$ and continuously Frechét differentiable on that space. Next, we show an important property of the modified functional.

Lemma 4.2. *$< \tilde{I}'(u,u) >= 0$ and $\tilde{I}(u) = 0$ if and only if $u = 0$.*

Proof. Suppose $< \tilde{I}'(u), u >= 0$ and $\tilde{I}(u) = 0$. Then we have

$$\begin{aligned}0 =& < \tilde{I}'(u), u >\\ =& \int_\Omega \left(|\nabla u|^2 + u^2\right)dx - \lambda\int_{\partial\Omega} b(x)|u|^q d\sigma - \int_\Omega \tilde{f}(x,u)u dx,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned} 0 &= qI(u) \\ &= \frac{q}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \lambda \int_{\partial\Omega} b(x)|u|^q d\sigma - q\int_\Omega \tilde{f}(x,u)udx. \end{aligned} \tag{4.6}$$

Substituting (4.6) from (4.5) and noting the condition ($\tilde{F}2$), we get

$$\begin{aligned} \left(\frac{2-q}{2}\right)\|u\|^2_{H^1(\Omega)} &= \int_\Omega \left(\tilde{f}(x,u) - q\tilde{F}(x,u)\right) dx \\ &\le \theta\|u\|^2_{H^1(\Omega)}. \end{aligned}$$

Hence $u = 0$. □

Considering the oddness of $\tilde{f}(x,u)$ and the condition ($\tilde{F}1$) on $\tilde{f}(x,u)$, we can check the following properties of $\tilde{I}$:

1. $\tilde{I}(u)$ is even in u,
2. $\tilde{I}(0) = 0$,
3. $\tilde{I}$ is bounded from below,
4. $\tilde{I}$ satisfies $(\mathrm{PS})_c$ conditions for $c \le 0$,
5. for every $n \in \mathbb{N}$, there exist an n-dimensional subspace $E_n \subset H^1(\Omega)$ and positive constants $\rho > 0$ and $\varepsilon > 0$ such that $\tilde{I}(u) \le -\varepsilon$ for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$.

Most part of the proof is analogous to the one in section 2. So we leave it to readers. The above properties of $\tilde{I}$ are enough to get the existence of infinitely many solutions $\{u_k\} \subset H^1(\Omega)$ for (1.2) with $\tilde{I}(u_k) < 0$ and $\tilde{I}(u_k) \to 0$ as $k \to \infty$ as in section 2. Finally we come to the proof of Theorem 1.4.

The proof of Theorem 1.4. First assume $b \in L^\infty(\partial\Omega)$ and satisfies the condition (B). Since $\tilde{I}(u_k) \to 0$ and $\tilde{I}'(u_k) = 0$, the sequence of solutions $\{u_k\}$ is $(\mathrm{PS})_0$ sequence for $\tilde{I}$. Then by the $(\mathrm{PS})_0$ condition for $\tilde{I}$, we can assume u_k converges to some function $u \in H^1(\Omega)$. We claim $u = 0$. In fact, from the continuity of $\tilde{I}$, $\tilde{I}(u) = 0$. Hence by Lemma 4.2, $u = 0$. Considering a priori estimate for the weak solutions for (4.4) (see [6] and references therein), we get for all $\beta \ge 1$, $u_k \in W^{1,\beta}(\Omega)$ and $u_k \to 0$ in $W^{1,\beta}(\Omega)$. Consequently, by the Morrey inequality, we get $\|u_k\|_{L^\infty(\Omega)} \le a$ for large $k \in \mathbb{N}$. Hence for each large $k \in \mathbb{N}$, recalling the condition ($\tilde{F}3$), we conclude that u_k is a weak solution of (1.1). This completes the proof. □

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