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COHOMOLOGY OF TORIC ORIGAMI MANIFOLDS WITH ACYCLIC PROPER FACES

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Abstract. Toric origami manifolds are generalizations of symplectic toric manifolds, where the origami symplectic form, in contrast to the usual symplectic form, is allowed to degenerate in a good controllable way. It is widely known that symplectic toric manifolds are encoded by Delzant polytopes. The cohomology and equivariant cohomology rings of a symplectic toric manifold can be described in terms of the corresponding polytope. One can obtain a similar description for the cohomology of a toric origami manifold M in terms of the orbit space M/T when M is orientable and the orbit space M/T is contractible; this was done by Holm and Pires in [9]. Generally, the orbit space of a toric origami manifold need not be contractible. In this paper we study the topology of orientable toric origami manifolds for the wider class of examples: we require that any proper face of the orbit space is acyclic, while the orbit space itself may be arbitrary. Furthermore, we give a general description of the equivariant cohomology ring of torus manifolds with locally standard torus action in the case when proper faces of the orbit space are acyclic and the free part of the action is a trivial torus bundle.

Introduction

Origami manifolds appeared in differential geometry recently as a generalization of symplectic manifolds [5]. Toric origami manifolds are in turn generalizations of symplectic toric manifolds. Recall that a symplectic toric manifold is a compact connected symplectic manifold of dimension $2n$ with an effective Hamiltonian action of a compact n-dimensional torus T. A famous result of Delzant [7] describes a bijective correspondence between symplectic toric manifolds and simple convex polytopes, called Delzant polytopes. The polytope associated to a symplectic toric manifold M is the image of the moment map on M.

A folded symplectic form on a $2n$-dimensional manifold M is a closed 2-form ω whose top power ω^n vanishes transversally on a subset W and whose restriction to points in W has maximal rank. Then W is a codimension-one submanifold of M, called the fold. The maximality of the restriction of ω to W implies the existence of a line field on W and ω is called an origami form if the line field is the vertical bundle of some principal S^1-fibration $W \to X$. The notions of a Hamiltonian action and a moment map are defined similarly to the symplectic case and a toric origami manifold is defined to be a compact connected origami manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of a torus T. Similarly to Delzant's theorem for symplectic toric manifolds, toric origami manifolds bijectively correspond to special combinatorial structures, called origami templates, via moment maps [5].

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An origami template is a collection of Delzant polytopes with some additional gluing data encoded by a template graph G.

To describe the cohomology ring and T-equivariant cohomology ring of toric origami manifold M in terms of the corresponding origami template is a problem of certain interest. In this paper we present some partial results concerning this problem.

While in general toric origami manifolds can be non-orientable, in the paper we restrict to the orientable case. Under this assumption the action of T on a toric origami manifold M is locally standard, so the orbit space M/T is a manifold with corners. One can describe the orbit space M/T as a result of gluing polytopes of the origami template. This shows that M/T is homotopy equivalent to the template graph G. Any proper face of M/T is homotopy equivalent to some subgraph of G. Thus a toric origami manifold has the property that the orbit space and all its faces are homotopy equivalent to wedges of circles or contractible.

If G is a tree, then M/T and all its faces are contractible. Then a general result of [12] applies. It gives a description similar to toric varieties (or quasitoric manifolds): $H_T^*(M) \cong \mathbb{Z}[M/T]$ and $H^*(M) \cong \mathbb{Z}[M/T]/(\theta_1, \dots, \theta_n)$. Here $\mathbb{Z}[M/T]$ is a face ring of the manifold with corners M/T, and $(\theta_1, \dots, \theta_n)$ is an ideal generated by the linear system of parameters, defined by the characteristic map on M/T. This case is discussed in detail in [9]. If G has cycles, even the Betti numbers of M remain unknown in general, unless $\dim M = 4$ (this case was described by Holm and Pires in [10]).

In this paper we study the cohomology of an orientable toric origami manifold M in the case when M/T is itself arbitrary, but every proper face of M/T is acyclic. A different approach to this task, based on the spectral sequence of the filtration by orbit types, is proposed in a more general situation in [1]. For toric origami manifolds the calculation of Betti numbers in this paper gives the same answer but a simpler proof.

The paper is organized as follows. In Section 1 we recall necessary definitions and properties of toric origami manifolds and origami templates. In Section 2 we describe the procedure which simplifies a given toric origami manifold step-by-step, and give an inductive formula for Betti numbers. In Section 3 we give more convenient formulas, expressing Betti numbers of M in terms of the first Betti number of M/T and the face numbers of the dual simplicial poset. Section 4 is devoted to an equivariant cohomology. While toric origami manifolds serve as a motivating example, we describe the equivariant cohomology ring in a more general setting. In Section 5 we describe the properties of the Serre spectral sequence of the fibration $\pi\colon ET \times_T M \to BT$ for a toric origami manifold M. The restriction homomorphism $\iota^*\colon H_T^*(M) \to H^*(M)$ induces a graded ring homomorphism $\bar{\iota}^*\colon H_T^*(M)/(\pi^*(H^2(BT))) \to H^*(M)$. In Section 6 we use Schenzel's theorem and the calculations of previous sections to show that $\bar{\iota}^*$ is an isomorphism except in degrees 2, 4 and $2n-1$, injective in degrees 2 and $2n-1$, and surjective in degree 4; we also find the ranks of the kernels and cokernels in these exceptional cases. Since $\bar{\iota}^*$ is a ring homomorphism, these considerations describe the product structure on the most part of $H^*(M)$, except for the cokernel of $\bar{\iota}^*$ in degree 2. Section 7 illustrates our considerations in the 4-dimensional case. In section 8 we give a geometrical description of the cokernel of $\bar{\iota}^*$ in degree 2 and suggest a partial description of the cohomology multiplication for these extra elements. The discussion of Section 9 shows which part of the results can be generalized to the case of non-acyclic faces.

1. Toric origami manifolds

In this section we recall the definitions and properties of toric origami manifolds and origami templates. Details can be found in [5], [13] or [9].

A folded symplectic form on a $2n$-dimensional manifold M is a closed 2-form ω whose top power ω^n vanishes transversally on a subset W and whose restriction to points in W has maximal rank. Then W is a codimension-one submanifold of M and is called the fold. If W is empty, ω is a genuine symplectic form. The pair (M, ω) is called a folded symplectic manifold. Since the restriction of ω to W has maximal rank, it has a one-dimensional kernel at each point of W. This determines a line field on W called the null foliation. If the null foliation is the vertical bundle of some principal S^1-fibration $W \to X$ over a compact base X, then the folded symplectic form ω is called an origami form and the pair (M, ω) is called an origami manifold. The action of a torus T on an origami manifold (M, ω) is Hamiltonian if it admits a moment map $\mu\colon M \to \mathfrak{t}^*$ to the dual Lie algebra of the torus, which satisfies the conditions: (1) μ is equivariant with respect to the given action of T on M and the coadjoint action of T on the vector space $\mathfrak{t}^*$ (this action is trivial for the torus); (2) μ collects Hamiltonian functions, that is, $d\langle \mu, V\rangle = \imath_{V^\#}\omega$ for any $V \in \mathfrak{t}$, where $V^\#$ is the vector field on M generated by V.

Definition. A toric origami manifold (M, ω, T, μ), abbreviated as M, is a compact connected origami manifold (M, ω) equipped with an effective Hamiltonian action of a torus T with $\dim T = \frac{1}{2}\dim M$ and with a choice of a corresponding moment map μ.

When the fold W is empty, a toric origami manifold is a symplectic toric manifold. A theorem of Delzant [7] says that symplectic toric manifolds are classified by their moment images called Delzant polytopes. Recall that a Delzant polytope in $\mathbb{R}^n$ is a simple convex polytope, whose normal fan is smooth (with respect to some given lattice $\mathbb{Z}^n \subset \mathbb{R}^n$). In other words, all normal vectors to facets of P have rational coordinates, and, whenever facets $F_1, \dots, F_n$ meet in a vertex of P, the primitive normal vectors $\nu(F_1), \dots, \nu(F_n)$ form a basis of the lattice $\mathbb{Z}^n$. Let $\mathcal{D}_n$ denote the set of all Delzant polytopes in $\mathbb{R}^n$ (w.r.t. a given lattice) and $\mathcal{F}_n$ be the set of all their facets.

The moment data of a toric origami manifold can be encoded into an origami template (G, Ψ_V, Ψ_E), where

- G is a connected graph (loops and multiple edges are allowed) with the vertex set V and edge set E;
- $\Psi_V\colon V \to \mathcal{D}_n$;
- $\Psi_E\colon E \to \mathcal{F}_n$;

subject to the following conditions:

- If $e \in E$ is an edge of G with endpoints $v_1, v_2 \in V$, then $\Psi_E(e)$ is a facet of both polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$, and these polytopes coincide near $\Psi_E(e)$ (this means there exists an open neighborhood U of $\Psi_E(e)$ in $\mathbb{R}^n$ such that $U \cap \Psi_V(v_1) = U \cap \Psi_V(v_2)$).
- If $e_1, e_2 \in E$ are two edges of G adjacent to $v \in V$, then $\Psi_E(e_1)$ and $\Psi_E(e_2)$ are disjoint facets of $\Psi(v)$.

The facets of the form $\Psi_E(e)$ for $e \in E$ are called the fold facets of the origami template.

The following is a generalization of the theorem by Delzant to toric origami manifolds.

Theorem 1.1 ([5])**.** *Assigning the moment data of a toric origami manifold induces a one-to-one correspondence*

$$\{\textit{toric origami manifolds}\} \leftrightsquigarrow \{\textit{origami templates}\}$$

up to equivariant origami symplectomorphism on the left-hand side, and affine equivalence on the right-hand side.

Denote by $|(G, \Psi_V, \Psi_E)|$ the topological space constructed from the disjoint union $\bigsqcup_{v \in V} \Psi_V(v)$ by identifying facets $\Psi_E(e) \subset \Psi_V(v_1)$ and $\Psi_E(e) \subset \Psi_V(v_2)$ for any edge $e \in E$ with endpoints v_1, v_2.

An origami template (G, Ψ_V, Ψ_E) is called coörientable if the graph G has no loops (this means all edges have different endpoints). Then the corresponding toric origami manifold is also called coörientable. If M is orientable, then M is coörientable [9]. If M is coörientable, then the action of T^n on M is locally standard [9, lemma 5.1]. We review the definition of locally standard action in Section 4.

Let (G, Ψ_V, Ψ_E) be an origami template and M the associated toric origami manifold which is supposed to be orientable in the following. The topological space $|(G, \Psi_V, \Psi_E)|$ is a manifold with corners with the face structure induced from the face structures on polytopes $\Psi_V(v)$, and $|(G, \Psi_V, \Psi_E)|$ is homeomorphic to M/T as a manifold with corners. The space $|(G, \Psi_V, \Psi_E)|$ has the same homotopy type as the graph G, thus $M/T \cong |(G, \Psi_V, \Psi_E)|$ is either contractible or homotopy equivalent to a wedge of circles.

Under the correspondence of Theorem 1.1 the fold facets of the origami template correspond to the connected components of the fold W of M. If $F = \Psi_E(e)$ is a fold facet of the template (G, Ψ_V, Ψ_E), then the corresponding component $Z = \mu^{-1}(F)$ of the fold $W \subset M$ is a principal S^1-bundle over a compact space B. The space B is a $(2n-2)$-dimensional symplectic toric manifold corresponding to the Delzant polytope F. In the following we also call the connected components Z of the fold W the "folds" by abuse of terminology.

2. Betti numbers of toric origami manifolds

Let M be an orientable toric origami manifold of dimension $2n$ with a fold Z. Let F be the corresponding folded facet in the origami template of M and let B be the symplectic toric manifold corresponding to F. The normal line bundle of Z to M is trivial so that an invariant closed tubular neighborhood of Z in M can be identified with $Z \times [-1, 1]$. We set

$$\tilde{M} := M - \mathrm{Int}(Z \times [-1, 1]).$$

This has two boundary components which are copies of Z. We close $\tilde{M}$ by gluing two copies of the disk bundle associated to the principal S^1-bundle $Z \to B$ along their boundaries. The resulting closed manifold (possibly disconnected), denoted M', is again a toric origami manifold.

Let G be the graph associated to the origami template of M and let $b_1(G)$ be its first Betti number. We assume that $b_1(G) \geq 1$. A folded facet in the origami template of M corresponds to an edge of G. We choose an edge e in a (non-trivial) cycle of G and let F, Z and B be respectively the folded facet, the fold and the symplectic toric manifold corresponding to the edge e. Then M' is connected and the graph G' associated to M' is nothing but the graph G with the edge e removed, so $b_1(G') = b_1(G) - 1$.

Two copies of B lie in M' as closed submanifolds, denoted B_+ and B_-. Let N_+ (resp. N_-) be an invariant closed tubular neighborhood of B_+ (resp. B_-) and Z_+ (resp. Z_-) be the boundary of N_+ (resp. N_-). Note that $M' - \mathrm{Int}(N_+ \cup N_-)$ can naturally be identified with $\tilde{M}$, so that

$$\tilde{M} = M' - \mathrm{Int}(N_+ \cup N_-) = M - \mathrm{Int}(Z \times [-1,1])$$

and

$$M' = \tilde{M} \cup (N_+ \cup N_-), \qquad \tilde{M} \cap (N_+ \cup N_-) = Z_+ \cup Z_-, \tag{2.1}$$

$$M = \tilde{M} \cup (Z \times [-1,1]), \qquad \tilde{M} \cap (Z \times [-1,1]) = Z_+ \cup Z_-. \tag{2.2}$$

Remark. It follows from (2.1) and (2.2) that

$$\chi(M') = \chi(\tilde{M}) + 2\chi(B), \quad \chi(M) = \chi(\tilde{M})$$

and hence $\chi(M') = \chi(M) + 2\chi(B)$. Note that this formula holds without the acyclicity assumption (made later) on proper faces of M/T.

We shall investigate relations among the Betti numbers of $M, M', \tilde{M}, Z$ and B. The spaces $\tilde{M}$ and Z are auxiliary ones and our aim is to find relations among the Betti numbers of M, M' and B. In the following, all cohomology groups and Betti numbers are taken with $\mathbb{Z}$ coefficients unless otherwise stated but the reader will find that the same argument works over any field.

Lemma 2.1. *$b_{2i}(Z) - b_{2i-1}(Z) = b_{2i}(B) - b_{2i-2}(B)$ for any i.*

Proof. Since $\pi\colon Z \to B$ is a principal S^1-bundle and $H^{odd}(B) = 0$, the Gysin exact sequence for the principal S^1-bundle splits into a short exact

$$0 \to H^{2i-1}(Z) \to H^{2i-2}(B) \to H^{2i}(B) \xrightarrow{\pi^*} H^{2i}(Z) \to 0 \quad \text{for any } i \tag{2.3}$$

and this implies the lemma. □

Lemma 2.2. *$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M') - b_{2i-1}(M') - 2b_{2i-2}(B)$ for any i.*

Proof. We consider the Mayer-Vietoris exact sequence in cohomology for the triple $(M', \tilde{M}, N_+ \cup N_-)$:

$$\begin{aligned}
&\to H^{2i-2}(M') \to H^{2i-2}(\tilde{M}) \oplus H^{2i-2}(N_+ \cup N_-) \to H^{2i-2}(Z_+ \cup Z_-)\\
&\xrightarrow{\delta^{2i-2}} H^{2i-1}(M') \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(N_+ \cup N_-) \to H^{2i-1}(Z_+ \cup Z_-)\\
&\xrightarrow{\delta^{2i-1}} H^{2i}(M') \quad \to H^{2i}(\tilde{M}) \oplus H^{2i}(N_+ \cup N_-) \qquad \to H^{2i}(Z_+ \cup Z_-)\\
&\xrightarrow{\delta^{2i}} H^{2i+1}(M') \to
\end{aligned}$$

Since the inclusions $B = B_\pm \mapsto N_\pm$ are homotopy equivalences and $Z_\pm = Z$, the restriction homomorphism $H^q(N_+ \cup N_-) \to H^q(Z_+ \cup Z_-)$ above can be replaced by $\pi^* \oplus \pi^*\colon H^q(B) \oplus H^q(B) \to H^q(Z) \oplus H^q(Z)$ which is surjective when q is even by (2.3). Therefore, δ^{2i-2} and δ^{2i} in the exact sequence above are trivial. It follows that

$$\begin{aligned}
&b_{2i-1}(M') - b_{2i-1}(\tilde{M}) - 2b_{2i-1}(B) + 2b_{2i-1}(Z)\\
&-b_{2i}(M') + b_{2i}(\tilde{M}) + 2b_{2i}(B) - 2b_{2i}(Z) = 0.
\end{aligned}$$

Here $b_{2i-1}(B) = 0$ because B is a symplectic toric manifold, and $2b_{2i-1}(Z) + 2b_{2i}(B) - 2b_{2i}(Z) = 2b_{2i-2}(B)$ by Lemma 2.1. Using these identities, the identity above reduces to the identity in the lemma. □

Next we consider the Mayer-Vietoris exact sequence in cohomology for the triple $(M, \tilde{M}, Z \times [-1,1])$:

$$\begin{aligned}
&\to H^{2i-2}(M) \to H^{2i-2}(\tilde{M}) \oplus H^{2i-2}(Z \times [-1,1]) \to H^{2i-2}(Z_+ \cup Z_-)\\
&\to H^{2i-1}(M) \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(Z \times [-1,1]) \to H^{2i-1}(Z_+ \cup Z_-)\\
&\to H^{2i}(M) \quad \to H^{2i}(\tilde{M}) \oplus H^{2i}(Z \times [-1,1]) \qquad \to H^{2i}(Z_+ \cup Z_-) \to
\end{aligned}$$

We make the following assumption:

($*$) The restriction map $H^{2j}(\tilde{M})\oplus H^{2j}(Z\times[-1,1]) \to H^{2j}(Z_+\cup Z_-)$ in the Mayer-Vietoris sequence above is surjective for $j\geq 1$.

Note that the restriction map above is not surjective when $j=0$ and we will see in Lemma 2.5 below that the assumption ($*$) is satisfied when every proper face of M/T is acyclic.

Lemma 2.3. *Suppose that the assumption* ($*$) *is satisfied. Then*

$$\begin{aligned} &b_2(\tilde{M})-b_1(\tilde{M})=b_2(M)-b_1(M)+b_2(B),\\ &b_{2i}(\tilde{M})-b_{2i-1}(\tilde{M})=b_{2i}(M)-b_{2i-1}(M)+b_{2i}(B)-b_{2i-2}(B) \quad \textit{for } i\geq 2. \end{aligned}$$

Proof. By the assumption ($*$), the Mayer-Vietoris exact sequence for the triple $(M,\tilde{M},Z\times[-1,1])$ splits into short exact sequences:

$$\begin{aligned} 0 \to &H^0(M)\to H^0(\tilde{M})\oplus H^0(Z\times[-1,1])\to H^0(Z_+\cup Z_-)\\ \to &H^1(M)\to H^1(\tilde{M})\oplus H^1(Z\times[-1,1])\to H^1(Z_+\cup Z_-)\\ \to &H^2(M)\to H^2(\tilde{M})\oplus H^2(Z\times[-1,1])\to H^2(Z_+\cup Z_-)\to 0 \end{aligned}$$

and for $i\geq 2$

$$\begin{aligned} 0 \to &H^{2i-1}(M)\to H^{2i-1}(\tilde{M})\oplus H^{2i-1}(Z\times[-1,1])\to H^{2i-1}(Z_+\cup Z_-)\\ \to &H^{2i}(M) \quad \to H^{2i}(\tilde{M})\oplus H^{2i}(Z\times[-1,1]) \qquad \to H^{2i}(Z_+\cup Z_-)\to 0. \end{aligned}$$

The former short exact sequence above yields

$$b_2(\tilde{M})-b_1(\tilde{M})=b_2(M)-b_1(M)+b_2(Z)-b_1(Z)+1$$

while the latter above yields

$$b_{2i}(\tilde{M})-b_{2i-1}(\tilde{M})=b_{2i}(M)-b_{2i-1}(M)+b_{2i}(Z)-b_{2i-1}(Z) \quad \text{for } i\geq 2.$$

Here $b_{2i}(Z)-b_{2i-1}(Z)=b_{2i}(B)-b_{2i-2}(B)$ for any i by Lemma 2.1, so our lemma follows. □

Lemma 2.4. *Suppose that the assumption* ($*$) *is satisfied and $n\geq 2$. Then*

$$\begin{aligned} &b_1(M')=b_1(M)-1, \quad b_2(M')=b_2(M)+b_2(B)+1,\\ &b_{2i+1}(M')=b_{2i+1}(M) \quad \textit{for } 1\leq i\leq n-2. \end{aligned}$$

Proof. It follows from Lemma 2.2 and Lemma 2.3 that

$$b_{2i}(M')-b_{2i-1}(M')=b_{2i}(M)-b_{2i-1}(M)+b_{2i}(B)+b_{2i-2}(B) \quad \text{for } i\geq 2. \tag{2.4}$$

Take $i=n$ in (2.4) and use Poincaré duality. Then we obtain

$$b_0(M')-b_1(M')=b_0(M)-b_1(M)+b_0(B)$$

which reduces to the first identity in the lemma. This together with the first identity in Lemma 2.3 implies the second identity in the lemma.

Similarly, take $i=n-1(\geq 2)$ in (2.4) and use Poincaré duality. Then we obtain

$$b_2(M')-b_3(M')=b_2(M)-b_3(M)+b_0(B)+b_2(B).$$

This together with the second identity in the lemma implies $b_3(M')=b_3(M)$.

Take i to be $n-i$ in (2.4) (so $2\leq i\leq n-2$) and use Poincaré duality. Then we obtain

$$b_{2i}(M')-b_{2i+1}(M')=b_{2i}(M)-b_{2i+1}(M)+b_{2i-2}(B)+b_{2i}(B).$$

This together with (2.4) implies

$$b_{2i+1}(M')-b_{2i-1}(M')=b_{2i+1}(M)-b_{2i-1}(M) \quad \text{for } 2\leq i\leq n-2.$$

Since we know $b_3(M')=b_3(M)$, this implies the last identity in the lemma. □

The following is a key lemma.

Lemma 2.5. *Suppose that every proper face of M/T is acyclic. Then the homomorphism $H^{2j}(\tilde{M}) \to H^{2j}(Z_+ \cup Z_-)$ induced from the inclusion is surjective for $j \geq 1$, in particular, the assumption $(*)$ is satisfied.*

Proof. Since $B_+ \cup B_-$ is a deformation retract of $N_+ \cup N_-$, the following diagram is commutative:

$$\begin{array}{ccc} H^{2j}(M') & \longrightarrow & H^{2j}(B_+ \cup B_-) \\ \downarrow & & \downarrow \pi_\pm^* \\ H^{2j}(\tilde{M}) & \longrightarrow & H^{2j}(Z_+ \cup Z_-) \end{array}$$

where $\pi_\pm \colon Z_+ \cup Z_- \to B_+ \cup B_-$ is the projection and the other homomorphisms are induced from the inclusions. By (2.3) $\pi_\pm^*$ is surjective, so it suffices to show that the homomorphism $H^{2j}(M') \to H^{2j}(B_+ \cup B_-)$ is surjective for $j \geq 1$.

The inverse image of a codimension j face of M'/T by the quotient map $M' \to M'/T$ is a codimension $2j$ closed orientable submanifold of M' and defines an element of $H_{2n-2j}(M')$ so that its Poincaré dual yields an element of $H^{2j}(M')$. The same is true for $B = B_+$ or B_-. Note that $H^{2j}(B)$ is additively generated by τ_K's where K runs over all codimension j faces of $F = B/T$.

Set $F_\pm = B_\pm/T$, which are copies of the folded facet $F = B/T$. Let K_+ be any codimension j face of F_+. Then there is a codimension j face L of M'/T such that $K_+ = L \cap F_+$. We note that $L \cap F_- = \emptyset$. Indeed, if $L \cap F_- \neq \emptyset$, then $L \cap F_-$ must be a codimension j face of F_-, say H_-. If H_- is the copy K_- of K_+, then L will create a codimension j non-acyclic face of M/T which contradicts the acyclicity assumption on proper faces of M/T. Therefore, $H_- \neq K_-$. However, $F_\pm$ are respectively facets of some Delzant polytopes, say $P_\pm$, and the neighborhood of F_+ in P_+ is same as that of F_- in P_- by definition of an origami template (although P_+ and P_- may not be isomorphic). Let $\bar{H}$ and $\bar{K}$ be the codimension j faces of P_- such that $\bar{H} \cap F = H_-$ and $\bar{K} \cap F = K_-$. Since $H_- \neq K_-$, the normal cones of $\bar{H}$ and $\bar{K}$ are different. However, these normal cones must agree with that of L because $L \cap F_+ = K_+$ and $L \cap F_- = H_-$ and the neighborhood of F_+ in P_+ is same as that of F_- in P_-. This is a contradiction.

The codimension j face L of M'/T associates an element $\tau_L \in H^{2j}(M')$. Since $L \cap F_+ = K_+$ and $L \cap F_- = \emptyset$, the restriction of τ_L to $H^{2j}(B_+ \cup B_-) = H^{2j}(B_+) \oplus H^{2j}(B_-)$ is $(\tau_{K_+}, 0)$, where $\tau_{K_+} \in H^{2j}(B_+)$ is associated to K_+. Since $H^{2j}(B_+)$ is additively generated by τ_{K_+}'s where K_+ runs over all codimension j faces of F_+, this shows that for any element $(x_+, 0) \in H^{2j}(B_+) \oplus H^{2j}(B_-) = H^{2j}(B \cup B_-)$, there is an element $y_+ \in H^{2j}(M')$ whose restriction image is $(x_+, 0)$. The same is true for any element $(0, x_-) \in H^{2j}(B_+) \oplus H^{2j}(B_-)$. This implies the lemma. □

Finally we obtain the following.

Theorem 2.6. *Let M be an orientable toric origami manifold of dimension $2n$ $(n \geq 2)$ such that every proper face of M/T is acyclic. Then*

$$b_{2i+1}(M) = 0 \quad \text{for } 1 \leq i \leq n-2. \tag{2.5}$$

Moreover, if M' and B are as above, then

(2.6)
$$b_1(M') = b_1(M) - 1 \quad (\text{hence } b_{2n-1}(M') = b_{2n-1}(M) - 1 \text{ by Poincaré duality}),$$
$$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \text{for } 1 \leq i \leq n-1.$$

Finally, $H^*(M)$ is torsion free.

Proof. We have $b_1(M') = b_1(M) - 1$ by Lemma 2.4. Therefore, if $b_1(M) = 1$, then $b_1(M') = 0$, that is, the graph associated to M' is acyclic and hence $b_{odd}(M') = 0$ by [9] (or [12]). This together with Lemma 2.4 shows that $b_{2i+1}(M) = 0$ for $1 \le i \le n-2$ when $b_1(M) = 1$. If $b_1(M) = 2$, then $b_1(M') = 1$ so that $b_{2i+1}(M') = 0$ for $1 \le i \le n-2$ by the observation just made and hence $b_{2i+1}(M) = 0$ for $1 \le i \le n-2$ by Lemma 2.4. Repeating this argument, we see (2.5).

(2.6) follows from Lemma 2.4 and (2.4) together with the fact $b_{2i+1}(M) = 0$ for $1 \le i \le n-2$.

Arguments developed in this section work with $\mathbb{Z}/p$-coefficients for any prime p and (2.5) and (2.6) hold for Betti numbers with $\mathbb{Z}/p$-coefficients, so the Betti numbers of M with $\mathbb{Z}$-coefficients agree with the Betti numbers of M with $\mathbb{Z}/p$-coefficients for any prime p. This implies that $H^*(M)$ has no torsion. □

As for $H^1(M)$, we have a clear geometrical picture.

Proposition 2.7. *Let M be an orientable toric origami manifold of dimension $2n$ ($n \ge 2$) such that every proper face of M/T is acyclic. Let $Z_1, \dots, Z_{b_1}$ be folds in M such that the graph associated to the origami template of M with the b_1 edges corresponding to $Z_1, \dots, Z_{b_1}$ removed is a tree. Then $Z_1, \dots, Z_{b_1}$ freely generate $H_{2n-1}(M)$, equivalently, their Poincaré duals $z_1, \dots, z_{b_1}$ freely generate $H^1(M)$. Furthermore, all the products generated by $z_1, \dots, z_{b_1}$ are trivial because $Z_1, \dots, Z_{b_1}$ are disjoint and the normal bundle of Z_j is trivial for each j.*

Proof. We will prove the proposition by induction on b_1. Let Z, M' be as before. Since $b_1(M') = b_1 - 1$, there are folds $Z_1, \dots, Z_{b_1-1}$ in M' such that $Z_1, \dots, Z_{b_1-1}$ freely generate $H_{2n-1}(M')$ by induction assumption. The folds $Z_1, \dots, Z_{b_1-1}$ are naturally embedded in M and we will prove that these folds together with Z freely generate $H_{2n-1}(M)$.

We consider the Mayer-Vietoris exact sequence for a triple $(M, \tilde{M}, Z \times [-1,1])$:

$$\begin{aligned} 0 \to H_{2n}(M) &\xrightarrow{\partial_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{2*}} H_{2n-1}(\tilde{M}) \oplus H_{2n-1}(Z \times [-1,1]) \\ &\to H_{2n-1}(M) \xrightarrow{\partial_*} H_{2n-2}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{2*}} H_{2n-2}(\tilde{M}) \oplus H_{2n-2}(Z \times [-1,1]) \end{aligned}$$

where ι_1 and ι_2 are the inclusions. Since $\iota_1^*\colon H^{2n-2}(\tilde{M}) \to H^{2n-2}(Z_+ \cup Z_-)$ is surjective by Lemma 2.5, $\iota_{1*}\colon H_{2n-2}(Z_+ \cup Z_-) \to H_{2n-2}(\tilde{M})$ is injective when tensored with $\mathbb{Q}$. However, $H^*(Z)$ has no torsion in odd degrees because $H^{2i-1}(Z)$ is a subgroup of $H^{2i-2}(B)$ for any i by (2.3) and $H^*(B)$ is torsion free. Therefore, $H_*(Z)$ has no torsion in even degrees. Therefore, $\iota_{1*}\colon H_{2n-2}(Z_+ \cup Z_-) \to H_{2n-2}(\tilde{M})$ is injective without tensoring with $\mathbb{Q}$ and hence the above exact sequence reduces to this short exact sequence:

$$\begin{aligned} 0 \to H_{2n}(M) &\xrightarrow{\partial_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{2*}} H_{2n-1}(\tilde{M}) \oplus H_{2n-1}(Z \times [-1,1]) \\ &\to H_{2n-1}(M) \to 0. \end{aligned}$$

Noting $\partial_*([M]) = [Z_+] - [Z_-]$ and $\iota_{2*}([Z_\pm]) = [Z]$, one sees that the above short exact sequence implies an isomorphism

$$\iota_*\colon H_{2n-1}(\tilde{M}) \cong H_{2n-1}(M) \tag{2.7}$$

where $\iota\colon \tilde{M} \to M$ is the inclusion map.

We now consider the Mayer-Vietoris exact sequence for a triple $(M', \tilde{M}, N_+ \cup N_-)$:

$$\begin{aligned} 0 \to H_{2n}(M') &\xrightarrow{\partial'_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{3*}} H_{2n-1}(\tilde{M}) \oplus H_{2n-1}(N_+ \cup N_-) \\ &\to H_{2n-1}(M') \xrightarrow{\partial'_*} H_{2n-2}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{3*}} H_{2n-2}(\tilde{M}) \oplus H_{2n-2}(N_+ \cup N_-) \end{aligned}$$

where ι_3 is the inclusion map. Here $H_{2n-1}(N_+\cup N_-)=H_{2n-1}(B_+\cup B_-)=0$ and $\iota_{1*}\colon H_{2n-2}(Z_+\cup Z_-)\to H_{2n-2}(\tilde M)$ is injective as observed above. Therefore, the above exact sequence reduces to this short exact sequence:

$$0\to H_{2n}(M')\xrightarrow{\partial'_*} H_{2n-1}(Z_+\cup Z_-)\xrightarrow{\iota_{1*}} H_{2n-1}(\tilde M)\xrightarrow{\iota_*} H_{2n-1}(M')\to 0.$$

Here $\partial_*([M])=[Z_+]-[Z_-]$ and $H_{2n-1}(M')$ is freely generated by $Z_1,\dots,Z_{b_1-1}$ by induction assumption. Therefore, the above short exact sequence implies that

$$H_{2n-1}(\tilde M) \text{ is freely generated by } Z_1,\dots,Z_{b_1-1} \text{ and } Z_+ \text{ (or } Z_-).$$

This together with (2.7) completes the induction step and proves the lemma. □

3. Relations between Betti numbers and face numbers

Let M be an orientable toric origami manifold of dimension $2n$ ($n\ge 2$) such that every proper face of M/T is acyclic. In this section we will describe $b_{2i}(M)$ in terms of the face numbers of M/T and $b_1(M)$. Let $\mathcal{P}$ be the simplicial poset dual to $\partial(M/T)$. As usual, we define

$$\begin{aligned} f_i &= \text{ the number of } (n-1-i)\text{-faces of } M/T \\ &= \text{ the number of } i\text{-simplices in } \mathcal{P} \quad \text{for } i=0,1,\dots,n-1 \end{aligned}$$

and the h-vector $(h_0,h_1,\dots,h_n)$ by

$$\sum_{i=0}^{n} h_i t^{n-i}=(t-1)^n+\sum_{i=0}^{n-1} f_i(t-1)^{n-1-i}. \tag{3.1}$$

Theorem 3.1. *Let M be an orientable toric origami manifold of dimension $2n$ such that every proper face of M/T is acyclic. Let b_j be the j-th Betti number of M and $(h_0,h_1,\dots,h_n)$ be the h-vector of M/T. Then*

$$\sum_{i=0}^{n} b_{2i}t^i=\sum_{i=0}^{n} h_i t^i+b_1(1+t^n-(1-t)^n),$$

in other words, $b_0=h_0=1$ and

$$\begin{aligned} b_{2i}&=h_i-(-1)^i\binom{n}{i}b_1 \quad \textit{for } 1\le i\le n-1,\\ b_{2n}&=h_n+(1-(-1)^n)b_1. \end{aligned}$$

Remark. We have $h_n=(-1)^n+\sum_{i=0}^{n}(-1)^{n-1-i}f_i$ by (3.1) and $\chi(\partial(M/T))=\sum_{i=0}^{n-1}(-1)^i f_i$ because every proper face of M/T is acyclic. Therefore, $h_n=(-1)^n-(-1)^n\chi(\partial(M/T))$. Since $b_{2n}=1$, it follows from the last identity in Theorem 3.1 that

$$\chi(\partial(M/T))-\chi(S^{n-1})=((-1)^n-1)b_1.$$

Moreover, since $b_{2i}=b_{2n-2i}$, we have

$$\begin{aligned} h_{n-i}-h_i&=(-1)^i((-1)^n-1)b_1\binom{n}{i}\\ &=(-1)^i(\chi(\partial(M/T)-\chi(S^{n-1}))\binom{n}{i} \quad \text{for } 0\le i\le n. \end{aligned}$$

These are generalized Dehn-Sommerville relations for $\partial(M/T)$ (or for the simplicial poset $\mathcal{P}$), see [18, p. 74] or [3, Theorem 7.44].

We will use the notation in Section 2 freely. For a manifold Q of dimension n with corners (or faces), we define the f-polynomial and h-polynomial of Q by

$$f_Q(t) = t^n + \sum_{i=0}^{n-1} f_i(Q)t^{n-1-i}, \qquad h_Q(t) = f_Q(t-1)$$

as usual.

Lemma 3.2. $h_{M'/T}(t) = h_{M/T}(t) + (t+1)h_F(t) - (t-1)^n$. *Therefore*

$$t^n h_{M'/T}(t^{-1}) = t^n h_{M/T}(t^{-1}) + (1+t)t^{n-1}h_F(t^{-1}) - (1-t)^n.$$

Proof. In the proof of Lemma 2.5 we observed that no facet of M'/T intersects with both F_+ and F_-. This means that no face of M'/T intersects with both F_+ and F_- because any face of M'/T is contained in some facet of M'/T. Noting this fact, one can find that

$$f_i(M'/T) = f_i(M/T) + 2f_{i-1}(F) + f_i(F) \quad \text{for } 0 \le i \le n-1$$

where F is the folded facet and $f_{n-1}(F) = 0$. Therefore,

$$\begin{aligned} f_{M'/T}(t) &= t^n + \sum_{i=0}^{n-1} f_i(M'/T)t^{n-1-i} \\ &= t^n + \sum_{i=0}^{n-1} f_i(M'/T)t^i + 2\sum_{i=0}^{n-1} f_{i-1}(F)t^{n-1-i} + \sum_{i=0}^{n-2} f_i(F)t^{n-1-i} \\ &= f_{M/T}(t) + 2f_F(t) + tf_F(t) - t^n. \end{aligned}$$

Replacing t by $t-1$ in the identity above, we obtain the former identity in the lemma. Replacing t by t^{-1} in the former identity and multiplying the resulting identity by t^n, we obtain the latter identity. □

Proof of Theorem 3.1. Since $\sum_{i=0}^n h_i(M/T)t^i = t^n h_{M/T}(t^{-1})$, Theorem 3.1 is equivalent to

$$\sum_{i=0}^{n} b_{2i}(M)t^i = t^n h_{M/T}(t^{-1}) + b_1(M)(1 + t^n - (1-t)^n). \tag{3.2}$$

We shall prove (3.2) by induction on $b_1(M)$. The identity (3.2) is well-known when $b_1(M) = 0$. Suppose that $k = b_1(M)$ is a positive integer and the identity (3.2) holds for M' with $b_1(M') = k-1$. Then

$$\begin{aligned} &\sum_{i=0}^{n} b_{2i}(M)t^i \\ =&1 + t^n + \sum_{i=1}^{n-1}(b_{2i}(M') - b_{2i}(B) - b_{2i-2}(B))t^i \quad \text{(by Theorem 2.6)} \\ =&\sum_{i=0}^{n} b_{2i}(M')t^i - (1+t)\sum_{i=0}^{n-1} b_{2i}(B)t^i + 1 + t^n \\ =&t^n h_{M'/T}(t^{-1}) + b_1(M')(1 + t^n - (1-t)^n) - (1+t)t^{n-1}h_F(t^{-1}) + 1 + t^n \\ &\qquad\qquad\qquad\qquad \text{(by (3.2) applied to } M') \\ =&t^n h_{M/T}(t^{-1}) + b_1(M)(1 + t^n - (1-t)^n) \\ &\qquad\qquad \text{(by Lemma 3.2 and } b_1(M') = b_1(M) - 1), \end{aligned}$$

proving (3.2) for M. This completes the induction step and the proof of Theorem 3.1. □

4. Equivariant cohomology and face ring

A torus manifold M of dimension $2n$ is an orientable connected closed smooth manifold with an effective smooth action of an n-dimensional torus T having a fixed point ([8]). An orientable toric origami manifold with acyclic proper faces in the orbit space has a fixed point, so it is a torus manifold. The action of T on M is called *locally standard* if every point of M has a T-invariant open neighborhood equivariantly diffeomorphic to a T-invariant open set of a faithful representation space of T. Then the orbit space M/T is a nice manifold with corners. The torus action on an orientable toric origami manifold is locally standard. In this section, we study the equivariant cohomology of a locally standard torus manifold with acyclic proper faces of the orbit space.

We review some facts from [12]. Let Q be a nice manifold with corners (or faces) of dimension n. Let $\Bbbk$ be a ground commutative ring with unit. The face ring $\Bbbk[Q]$ of Q is a graded ring defined by

$$\Bbbk[Q] := \Bbbk[v_F : \ F \text{ a face}]/I_Q$$

where $\deg v_F = 2 \operatorname{codim} F$ and I_Q is the ideal generated by all elements

$$v_G v_H - v_{G\vee H} \sum_{E \in G\cap H} v_E$$

where $G \vee H$ is a unique minimal face of Q that contains both G and H. The dual poset of the face poset of Q is a simplicial poset of dimension $n-1$ and its face ring over $\Bbbk$ (see [18, p.113]) agrees with $\Bbbk[Q]$. For any vertex $p \in Q$ the restriction map s_p is defined as the quotient map

$$s_p \colon \Bbbk[Q] \to \Bbbk[Q]/(v_F : \ p \notin F)$$

and it is proved in [12, Proposition 5.5] that the image $s_p(\Bbbk[Q])$ is the polynomial ring $\Bbbk[v_{Q_{i_1}}, \dots, v_{Q_{i_n}}]$ where $Q_{i_1}, \dots, Q_{i_n}$ are the n different facets containing p.

Lemma 4.1 (Lemma 5.6 in [12])**.** *If every face of Q has a vertex, then the sum $s = \oplus_p s_p$ of restriction maps over all vertices $p \in Q$ is a monomorphism from $\Bbbk[Q]$ to the sum of polynomial rings.*

In particular, $\Bbbk[Q]$ has no nonzero nilpotent element if every face of Q has a vertex. It is not difficult to see that every face of Q has a vertex if every proper face of Q is acyclic.

Let M be a locally standard torus manifold. Then the orbit space M/T is a nice manifold with corners. Let $q\colon M \to M/T$ be the quotient map. Note that $M^\circ := M - q^{-1}(\partial(M/T))$ is the T-free part. The projection $ET \times M \to M$ induces a map $\bar{q}\colon ET\times_T M \to M/T$. Similarly we have a map $\bar{q}^\circ : ET\times_T M^\circ \to M^\circ/T$. The exact sequence of the equivariant cohomology groups for a pair (M, M°) together with the maps $\bar{q}$ and $\bar{q^\circ}$ produces the following commutative diagram:

$$\begin{array}{ccccc} H^*_T(M, M^\circ) & \xrightarrow{\eta^*} & H^*_T(M) & \xrightarrow{\iota^*} & H^*_T(M^\circ) \\ & & \uparrow{\scriptstyle \bar{q}^*} & & \uparrow{\scriptstyle (\bar{q}^\circ)^*} \\ & & H^*(M/T) & \xrightarrow{\bar{\iota}^*} & H^*(M^\circ/T) \end{array}$$

where η, ι and $\bar{\iota}$ are the inclusions. Since the T-action on M° is free and $\bar{\iota}\colon M^\circ/T \to M/T$ is a homotopy equivalence, we have graded ring isomorphisms

$$H^*_T(M^\circ) \xrightarrow{((\bar{q}^\circ)^*)^{-1}} H^*(M^\circ/T) \xrightarrow{(\bar{\iota}^*)^{-1}} H^*(M/T) \tag{4.1}$$

and the composition $\rho := \bar{q}^* \circ (\bar{\iota}^*)^{-1} \circ ((\bar{q^\circ})^*)^{-1}$, which is a graded ring homomorphism, gives the right inverse of ι^*, so the exact sequence above splits. Therefore,

η^* and $\bar{q}^*$ are both injective and

$$H_T^*(M) = \eta^*(H_T^*(M, M^\circ)) \oplus \rho(H_T^*(M^\circ)) \quad \text{as graded groups.} \tag{4.2}$$

Note that both factors at the right hand side above are graded subrings of $H_T^*(M)$ because η^* and ρ are both graded ring homomorphisms.

Let $\mathcal{P}$ be the poset dual to the face poset of M/T as before. Then $\mathbb{Z}[\mathcal{P}] = \mathbb{Z}[M/T]$ by definition.

Proposition 4.2. *Suppose every proper face of the orbit space M/T is acyclic, and the free part of the action gives a trivial principal bundle $M^\circ \to M^\circ/T$. Then $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}] \oplus \tilde{H}^*(M/T)$ as graded rings.*

Proof. Let R be the cone of $\partial(M/T)$ and let $M_R = M_R(\Lambda)$ be the T-space $R \times T/\sim$ where we use the characteristic function Λ obtained from M for the identification $\sim$. Let M_R° be the T-free part of M_R. Since the free part of the action on M is trivial, we have $M - M^\circ = M_R - M_R^\circ$. Hence,

$$H_T^*(M, M^\circ) \cong H_T^*(M_R, M_R^\circ) \quad \text{as graded rings} \tag{4.3}$$

by excision. Since $H_T^*(M_R^\circ) \cong H^*(M_R^\circ/T) \cong H^*(R)$ and R is a cone, $H_T^*(M_R^\circ)$ is isomorphic to the cohomology of a point. Therefore,

$$H_T^*(M_R, M_R^\circ) \cong H_T^*(M_R) \quad \text{as graded rings in positive degrees.} \tag{4.4}$$

On the other hand, the dual decomposition on the geometric realization $|\mathcal{P}|$ of $\mathcal{P}$ defines a face structure on the cone P of $\mathcal{P}$. Let $M_P = M_P(\Lambda)$ be the T-space $P \times T/\sim$ defined as before. Then a similar argument to that in [6, Theorem 4.8] shows that

$$H_T^*(M_P) \cong \mathbb{Z}[\mathcal{P}] \quad \text{as graded rings} \tag{4.5}$$

(this is mentioned as Proposition 5.13 in [12]). Since every face of P is a cone, one can construct a face preserving degree one map from R to P and it induces an equivariant map $f\colon M_R \to M_P$. Then a similar argument to the proof of Theorem 8.3 in [12] shows that f induces a graded ring isomorphism

$$f^*\colon H_T^*(M_P) \xrightarrow{\cong} H_T^*(M_R) \tag{4.6}$$

since every proper face of R is acyclic. It follows from (4.3), (4.4), (4.5) and (4.6) that

$$H_T^*(M, M^\circ) \cong \mathbb{Z}[\mathcal{P}] \quad \text{as graded rings in positive degrees.} \tag{4.7}$$

Thus, by (4.1) and (4.2) it suffices to prove that the cup product of any $a \in \eta^*(H_T^*(M, M^\circ))$ and any $b \in \rho(\tilde{H}_T^*(M^\circ))$ is trivial. Since $\iota^*(a) = 0$ (as $\iota^* \circ \eta^* = 0$), we have $\iota^*(a \cup b) = \iota(a) \cup \iota(b) = 0$ and hence $a \cup b$ lies in $\eta^*(H_T^*(M, M^\circ))$. Since $\rho(H_T^*(M^\circ)) \cong H^*(M/T)$ as graded rings by (4.1) and $H^m(M/T) = 0$ for a sufficiently large m, $(a \cup b)^m = \pm a^m \cup b^m = 0$. However, we know that $a \cup b \in \eta^*(H_T^*(M, M^\circ))$ and $\eta^*(H_T^*(M, M^\circ)) \cong \mathbb{Z}[\mathcal{P}]$ in positive degrees by (4.7). Since $\mathbb{Z}[\mathcal{P}]$ has no nonzero nilpotent element as remarked before, $(a \cup b)^m = 0$ implies $a \cup b = 0$. □

As discussed in [12, Section 6], there is a homomorphism

$$\varphi\colon \mathbb{Z}[\mathcal{P}] = \mathbb{Z}[M/T] \to \hat{H}_T^*(M) := H_T^*(M)/H^*(BT)\text{-torsions.} \tag{4.8}$$

In fact, φ is defined as follows. For a codimension k face F of M/T, $q^{-1}(F) =: M_F$ is a connected closed T-invariant submanifold of M of codimension $2k$, and φ assigns $v_F \in \mathbb{Z}[M/T]$ to the equivariant Poincaré dual $\tau_F \in H_T^{2k}(M)$ of M_F. One can see that φ followed by the restriction map to $H_T^*(M^T)$ can be identified with the map s in Lemma 4.1. Therefore, φ is injective if every face of Q has a vertex as mentioned in [12, Lemma 6.4].

Proposition 4.3. *If a torus manifold M is locally standard, every proper face of M/T is acyclic and the free part of action gives a trivial principal bundle, then the $H^*(BT)$-torsion submodule of $H_T^*(M)$ agrees with $\bar{q}^*(\tilde{H}^*(M/T))$, where $\bar{q}\colon ET \times_T M \to M/T$ is the map mentioned before.*

Proof. First we prove that all elements in $\bar{q}^*(\tilde{H}^*(M/T))$ are $H^*(BT)$-torsions. We consider the following commutative diagram:

$$\begin{array}{ccc} H_T^*(M) & \xrightarrow{\psi^*} & H_T^*(M^T) \\ {\scriptstyle \bar{q}^*}\uparrow & & \uparrow \\ H^*(M/T) & \xrightarrow{\bar{\psi}^*} & H^*(M^T) \end{array}$$

where the horizontal maps ψ^* and $\bar{\psi}^*$ are restrictions to M^T and the right vertical map is the restriction of $\bar{q}^*$ to M^T. Since M^T is isolated, $\bar{\psi}^*(\tilde{H}^*(M/T)) = 0$. This together with the commutativity of the above diagram shows that $\bar{q}^*(\tilde{H}^*(M/T))$ maps to zero by ψ^*. This means that $\bar{q}^*(\tilde{H}^*(M/T))$ are $H^*(BT)$-torsions because the kernel of ψ^* are $H^*(BT)$-torsions by the Localization Theorem in equivariant cohomology.

On the other hand, since every face of M/T has a vertex, the map φ in (4.8) is injective as remarked above; so it follows from Proposition 4.2 that there is no other $H^*(BT)$-torsion elements. □

We conclude this section with observation on the orientability of M/T.

Lemma 4.4. *Let M be a closed smooth manifold of dimension $2n$ with a locally standard smooth action of the n-dimensional torus T. Then M/T is orientable if and only if so is M.*

Proof. Since M/T is a manifold with corners and M°/T is its interior, M/T is orientable if and only if so is M°/T. On the other hand, M is orientable if and only if so is M°. Indeed, since the complement of M° in M is the union of finitely many codimension-two submanifolds, the inclusion $\iota\colon M^\circ \to M$ induces an epimorphism on their fundamental groups and hence on their first homology groups with $\mathbb{Z}/2$-coefficients. Then it induces a monomorphism $\iota^*\colon H^1(M;\mathbb{Z}/2) \to H^1(M^\circ;\mathbb{Z}/2)$ because $H^1(X;\mathbb{Z}/2) = \mathrm{Hom}(H_1(X;\mathbb{Z}/2);\mathbb{Z}/2)$. Since $\iota^*(w_1(M)) = w_1(M^\circ)$ and ι^* is injective, $w_1(M) = 0$ if and only if $w_1(M^\circ) = 0$. This means that M is orientable if and only if so is M°.

Thus, it suffices to prove that M°/T is orientable if and only if so is M°. But, since M°/T can be regarded as the quotient of an iterated free S^1-action, it suffices to prove the following general fact: for a principal S^1-bundle $\pi\colon E \to B$ where E and B are both smooth manifolds, B is orientable if and only if so is E. First we note that the tangent bundle of E is isomorphic to the Whitney sum of the tangent bundle along the fiber T_fE and the pullback of the tangent bundle of B by π. Since the free S^1-action on E yields a nowhere zero vector field along the fibers, the line bundle τ_fE is trivial. Therefore

$$w_1(E) = \pi^*(w_1(B)). \tag{4.9}$$

We consider the Gysin exact sequence for our S^1-bundle:

$$\to H^{-1}(B;\mathbb{Z}/2) \to H^1(B;\mathbb{Z}/2) \xrightarrow{\pi^*} H^1(E,\mathbb{Z}/2) \to H^0(B;\mathbb{Z}/2) \to .$$

Since $H^{-1}(B;\mathbb{Z}/2) = 0$, the exact sequence above tells us that $\pi^*\colon H^1(B;\mathbb{Z}/2) \to H^1(E;\mathbb{Z}/2)$ is injective. This together with (4.9) shows that $w_1(E) = 0$ if and only if $w_1(B) = 0$, proving the desired fact. □

5. Serre spectral sequence

Let M be an orientable toric origami manifold M of dimension $2n$ such that every proper face of M/T is acyclic. Note that M°/T is homotopy equivalent to a graph, hence does not admit nontrivial torus bundles. Thus the free part of the action gives a trivial principal bundle $M^\circ \to M^\circ/T$, and we may apply the results of the previous section.

We consider the Serre spectral sequence of the fibration $\pi\colon ET\times_T M \to BT$. Since BT is simply connected and both $H^*(BT)$ and $H^*(M)$ are torsion free by Theorem 2.6, the E_2-terms are given as follows:

$$E_2^{p,q} = H^p(BT; H^q(M)) = H^p(BT)\otimes H^q(M).$$

Since $H^{odd}(BT) = 0$ and $H^{2i+1}(M) = 0$ for $1 \le i \le n-2$ by Theorem 2.6,

(5.1) $\quad E_2^{p,q}$ with $p+q$ odd vanishes unless p is even and $q = 1$ or $2n-1$.

We have differentials

$$\to E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q-r+1} \to$$

and

$$E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}.$$

We will often abbreviate $d_r^{p,q}$ as d_r when p and q are clear in the context. Since

$$d_r(u\cup v) = d_r u \cup v + (-1)^{p+q} u\cup d_r v \quad \text{for } u\in E_r^{p,q} \text{ and } v \in E_r^{p',q'}$$

and d_r is trivial on $E_r^{p,0}$ and $E_r^{p,0} = 0$ when p is odd,

(5.2) $\quad d_r$ is an $H^*(BT)$-module map.

Note that

(5.3) $\quad E_r^{p,q} = E_\infty^{p,q} \quad$ if $p < r$ and $q+1 < r$

since $E_r^{a,b} = 0$ if either $a < 0$ or $b < 0$.

There is a filtration of subgroups

$$H_T^m(M) = \mathcal{F}^{0,m} \supset \mathcal{F}^{1,m-1} \supset \cdots \supset \mathcal{F}^{m-1,1} \supset \mathcal{F}^{m,0} \supset \mathcal{F}^{m+1,-1} = \{0\}$$

such that

(5.4) $\quad \mathcal{F}^{p,m-p}/\mathcal{F}^{p+1,m-p-1} = E_\infty^{p,m-p} \quad$ for $p = 0,1,\dots,m$.

There are two edge homomorphisms. One edge homomorphism

$$H^p(BT) = E_2^{p,0} \to E_3^{p,0} \to \cdots \to E_\infty^{p,0} \subset H_T^p(M)$$

agrees with $\pi^*\colon H^*(BT)\to H_T^*(M)$. Since $M^T \ne \emptyset$, one can construct a cross section of the fibration $\pi\colon ET\times_T M\to BT$ using a fixed point in M^T; so π^* is injective and hence

(5.5) $\quad d_r\colon E_r^{p-r,r-1} \to E_r^{p,0}$ is trivial for any $r\ge 2$ and $p \ge 0$,

which is equivalent to $E_2^{p,0} = E_\infty^{p,0}$. The other edge homomorphism

$$H_T^q(M) \twoheadrightarrow E_\infty^{0,q} \subset \cdots \subset E_3^{0,q} \subset E_2^{0,q} = H^q(M)$$

agrees with the restriction homomorphism $\iota^*\colon H_T^q(M)\to H^q(M)$. Therefore, ι^* is surjective if and only if the differential $d_r\colon E_r^{0,q} \to E_r^{r,q-r+1}$ is trivial for any $r \ge 2$.

We shall observe the restriction homomorphism $\iota^*\colon H_T^q(M)\to H^q(M)$. Since M/T is homotopy equivalent to the wedge of $b_1(M)$ circles, $H_T^q(M)$ vanishes unless q is 1 or even by Proposition 4.2 while $H^q(M)$ vanishes unless q is $1, 2n-1$ or even in between 0 and $2n$ by Theorem 2.6.

Lemma 5.1. *$\iota^*\colon H_T^1(M) \to H^1(M)$ is an isomorphism (so $H^1(M)\cong H^1(M/T)$ by Proposition 4.2).*

Proof. By (5.5),

$$d_2\colon E_2^{0,1} = H^1(M) \to E_2^{2,0} = H^2(BT)$$

is trivial. Therefore $E_2^{0,1} = E_\infty^{0,1}$. On the other hand, $E_\infty^{1,0} = E_2^{1,0} = H^1(BT) = 0$. These imply the lemma. □

Since $H_T^{2n-1}(M) = 0$, $\iota^*\colon H_T^{2n-1}(M) \to H^{2n-1}(M)$ cannot be surjective unless $H^{2n-1}(M) = 0$.

Lemma 5.2. *$\iota^*\colon H_T^{2j}(M) \to H^{2j}(M)$ is surjective except for $j = 1$ and the rank of the cokernel of ι^* for $j = 1$ is $nb_1(M)$.*

Proof. Since $\dim M = 2n$, we may assume $1 \le j \le n$.

First we treat the case where $j = 1$. Since $H_T^3(M) = 0$, $E_\infty^{2,1} = 0$ by (5.4) and $E_\infty^{2,1} = E_3^{2,1}$ by (5.3). This together with (5.5) implies that

$$d_2\colon H^2(M) = E_2^{0,2} \to E_2^{2,1} = H^2(BT) \otimes H^1(M) \quad \text{is surjective.} \tag{5.6}$$

Moreover $d_3\colon E_3^{0,2} = \ker d_2 \to E_3^{3,0}$ is trivial since $E_3^{3,0} = 0$. Therefore, $E_3^{0,2} = E_\infty^{0,2}$ by (5.3). Since $E_\infty^{0,2}$ is the image of $\iota^*\colon H_T^2(M) \to H^2(M)$, the rank of $H^2(M)/\iota^*(H_T^2(M))$ is $nb_1(M)$ by (5.6).

Suppose that $2 \le j \le n-1$. We need to observe that the differentials

$$d_r\colon E_r^{0,2j} \to E_r^{r,2j-r+1}$$

are all trivial. In fact, the target group $E_r^{r,2j-r+1}$ vanishes. This follows from (5.1) unless $r = 2j$. As for the case $r = 2j$, we note that

$$d_2\colon E_2^{p,2} \to E_2^{p+2,1} \quad \text{is surjective for } p \ge 0, \tag{5.7}$$

which follows from (5.2) and (5.6). Therefore $E_3^{p+2,1} = 0$ for $p \ge 0$, in particular $E_r^{r,2j-r+1} = 0$ for $r = 2j$ because $j \ge 2$. Therefore $\iota^*\colon H_T^{2j}(M) \to H^{2j}(M)$ is surjective for $1 \le j \le n-2$.

The remaining case $j = n$ can be proved directly, namely without using the Serre spectral sequence. Let x be a T-fixed point of M and let $\varphi\colon x \to M$ be the inclusion map. Since M is orientable and φ is T-equivariant, the equivariant Gysin homomorphism $\varphi_!\colon H_T^0(x) \to H_T^{2n}(M)$ can be defined and $\varphi_!(1) \in H_T^{2n}(M)$ restricts to the ordinary Gysin image of $1 \in H^0(x)$, that is the cofundamental class of M. This implies the surjectivity of $\iota^*\colon H_T^{2n}(M) \to H^{2n}(M)$ because $H^{2n}(M)$ is an infinite cyclic group generated by the cofundamental class. □

6. Towards the ring structure

Let $\pi\colon ET \times_T M \to BT$ be the projection. Since $\pi^*(H^2(BT))$ maps to zero by the restriction homomorphism $\iota^*\colon H_T^*(M) \to H^*(M)$, ι^* induces a graded ring homomorphism

$$\bar{\iota}^*\colon H_T^*(M)/(\pi^*(H^2(BT))) \to H^*(M) \tag{6.1}$$

which is surjective except in degrees 2 and $2n-1$ by Lemma 5.2 (and an isomorphism in degree 1 by Lemma 5.1). Here $(\pi^*(H^2(BT)))$ denotes the ideal in $H_T^*(M)$ generated by $\pi^*(H^2(BT))$. The purpose of this section is to prove the following.

Proposition 6.1. *The $\bar{\iota}^*$ in (6.1) is an isomorphism except in degrees 2, 4 and $2n-1$. Moreover, the rank of the cokernel of $\bar{\iota}^*$ in degree 2 is $nb_1(M)$ and the rank of the kernel of $\bar{\iota}^*$ in degree 4 is $\binom{n}{2}b_1(M)$.*

The rest of this section is devoted to the proof of Proposition 6.1. We recall the following result, which was proved by Schenzel ([17], [18, p.73]) for Buchsbaum simplicial complexes and generalized to Buchsbaum simplicial posets by Novik-Swartz ([14, Proposition 6.3]).

Theorem 6.2 (Schenzel, Novik-Swartz)**.** *Let Δ be a Buchsbaum simplicial poset of dimension $n-1$ and let $\theta_1, \dots, \theta_n \in \Bbbk[\Delta]_1$ be a linear system of parameters. Then*

$$\begin{aligned} F(\Bbbk[\Delta]/(\theta_1, \dots, \theta_n), t) =& (1-t)^n F(\Bbbk[\Delta], t) \\ &+ \sum_{j=1}^{n} \binom{n}{j} \Big(\sum_{i=-1}^{j-2} (-1)^{j-i} \dim_{\Bbbk} \tilde{H}_i(\Delta) \Big) t^j \end{aligned}$$

where $\Bbbk$ is a field.

As is well-known,

$$(1-t)^n F(\Bbbk[\Delta], t) = \sum_{i=0}^{n} h_i t^i$$

and following [14], we define h'_i for $i = 0, 1, \dots, n$ by

$$F(\Bbbk[\Delta]/(\theta_1, \dots, \theta_n), t) = \sum_{i=0}^{n} h'_i t^i.$$

Remark. Novik-Swartz [14] introduced

$$h''_i := h'_i - \binom{n}{j} \dim_{\Bbbk} \tilde{H}_{j-1}(\Delta) = h_j + \binom{n}{j} \Big(\sum_{i=-1}^{j-1} (-1)^{j-i} \dim_{\Bbbk} \tilde{H}_i(\Delta) \Big)$$

for $1 \leq i \leq n-1$ and showed that $h''_j \geq 0$ and $h''_{n-j} = h''_j$ for $1 \leq j \leq n-1$.

We apply Theorem 6.2 to our simplicial poset $\mathcal{P}$ which is dual to the face poset of $\partial(M/T)$. For that we need to know the homology of the geometric realization $|\mathcal{P}|$ of $\mathcal{P}$. First we show that $|\mathcal{P}|$ has the same homological features as $\partial(M/T)$.

Lemma 6.3. *The simplicial poset $\mathcal{P}$ is Buchsbaum, and $|\mathcal{P}|$ has the same homology as $\partial(M/T)$.*

Proof. We give a sketch of the proof. Details can be found in [1, Lemma 3.14]. There is a dual face structure on $|\mathcal{P}|$, and there exists a face preserving map $g\colon \partial(M/T) \to |\mathcal{P}|$ mentioned in the proof of Proposition 4.2. Let F be a proper face of M/T and F' the corresponding face of $|\mathcal{P}|$. By induction on $\dim F$ we can show that g induces the isomorphisms $g_*\colon H_*(\partial F) \xrightarrow{\cong} H_*(\partial F')$, $g_*\colon H_*(F) \xrightarrow{\cong} H_*(F')$, and $g_*\colon H_*(F, \partial F) \xrightarrow{\cong} H_*(F', \partial F')$. Since F is an acyclic orientable manifold with boundary, we deduce by Poincaré-Lefschetz duality that $H_*(F', \partial F') \cong H_*(F, \partial F)$ vanishes except in degree $\dim F$. Note that F' is a cone over $\partial F'$ and $\partial F'$ is homeomorphic to the link of a nonempty simplex of $\mathcal{P}$. Thus the links of nonempty simplices of $\mathcal{P}$ are homology spheres, and $\mathcal{P}$ is Buchsbaum [14, Prop.6.2]. Finally, g induces an isomorphism of spectral sequences corresponding to skeletal filtrations of $\partial(M/T)$ and $|\mathcal{P}|$, thus induces an isomorphism $g_*\colon H_*(\partial(M/T)) \xrightarrow{\cong} H_*(|\mathcal{P}|)$. □

Lemma 6.4. *$|\mathcal{P}|$ has the same homology as $S^{n-1} \sharp b_1 (S^1 \times S^{n-2})$ (the connected sum of S^{n-1} and b_1 copies of $S^1 \times S^{n-2}$).*

Proof. By Lemma 6.3 we may prove that $\partial(M/T)$ has the same homology groups as $S^{n-1} \sharp b_1 (S^1 \times S^{n-2})$. Since M/T is homotopy equivalent to a wedge of circles, $H^i(M/T) = 0$ for $i \geq 2$ and hence the homology exact sequence of the pair $(M/T, \partial(M/T))$ shows that

$$H_{i+1}(M/T, \partial(M/T)) \cong H_i(\partial(M/T)) \quad \text{for } i \geq 2.$$

On the other hand, M/T is orientable by Lemma 4.4 and hence

$$H_{i+1}(M/T, \partial(M/T)) \cong H^{n-i-1}(M/T)$$

by Poincaré–Lefschetz duality, and $H^{n-i-1}(M/T)=0$ for $n-i-1\geq 2$. These show that

$$H_i(\partial(M/T))=0 \quad \text{for } 2\leq i\leq n-3.$$

Thus it remains to study $H_i(\partial(M/T))$ for $i=0,1,n-2,n-1$ but since $\partial(M/T)$ is orientable (because so is M/T), it suffices to show

$$H_i(\partial(M/T))\cong H_i(S^{n-1}\sharp b_1(S^1\times S^{n-2})) \quad \text{for } i=0,1. \tag{6.2}$$

When $n\geq 3$, $S^{n-1}\sharp b_1(S^1\times S^{n-2})$ is connected, so (6.2) holds for $i=0$ and $n\geq 3$. Suppose that $n\geq 4$. Then $H^{n-2}(M/T)=H^{n-1}(M/T)=0$, so the cohomology exact sequence for the pair $(M/T,\partial(M/T))$ shows that $H^{n-2}(\partial(M/T))\cong H^{n-1}(M/T,\partial(M/T))$ and hence $H_1(\partial(M/T))\cong H_1(M/T)$ by Poincaré–Lefschetz duality. Since M/T is homotopy equivalent to a wedge of b_1 circles, this proves (6.2) for $i=1$ and $n\geq 4$. When $n=3$, $H_1(M/T,\partial(M/T))\cong H^2(X)=0$. We also know $H_2(X)=0$. The homology exact sequence for the pair $(M/T,\partial(M/T))$ yields a short exact sequence

$$0\to H_2(M/T,\partial(M/T))\to H_1(\partial(M/T))\to H_1(M/T)\to 0.$$

Here $H_2(M/T,\partial(M/T))\cong H^1(M/T)$ by Poincaré–Lefschetz duality. Since M/T is homotopy equivalent to a wedge of b_1 circles, this implies (6.2) for $i=1$ and $n=3$.

It remains to prove (6.2) when $n=2$. We use induction on b_1. The assertion is true when $b_1=0$. Suppose that $b_1=b_1(M/T)\geq 1$. We cut M/T along a fold so that $b_1(M'/T)=b_1(M/T)-1$, where M' is the toric origami manifold obtained from the cut, see Section 2. Then $b_0(\partial(M'/T))=b_0(\partial(M/T))-1$. Since (6.2) holds for $\partial(M'/T)$ by induction assumption, this observation shows that (6.2) holds for $\partial(M/T)$. □

Lemma 6.5. *For $n\geq 2$, we have*

$$\sum_{i=0}^{n} h_i' t^i=\sum_{i=0}^{n} b_{2i}t^i-nb_1t+\binom{n}{2}b_1t^2.$$

Proof. By Lemma 6.4, for $n\geq 4$, we have

$$\dim \tilde{H}_i(\mathcal{P})=\begin{cases} b_1 & \text{if } i=1,\ n-2,\\ 1 & \text{if } i=n-1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{i=-1}^{j-2}(-1)^{j-i}\dim \tilde{H}_i(\mathcal{P})=\begin{cases} 0 & \text{if } j=1,2,\\ (-1)^{j-1}b_1 & \text{if } 3\leq j\leq n-1,\\ ((-1)^{n-1}+1)b_1 & \text{if } j=n.\end{cases}$$

Then, it follows from Theorem 6.2 that

$$\begin{aligned}\sum_{i=0}^{n} h_i' t^i &=\sum_{i=0}^{n} h_i t^i+\sum_{j=3}^{n-1}(-1)^{j-1}b_1\binom{n}{j}t^j+((-1)^{n-1}+1)b_1t^n\\ &=\sum_{i=0}^{n} h_i t^i-b_1(1-t)^n+b_1(1-nt+\binom{n}{2}t^2)+b_1t^n\\ &=\sum_{i=0}^{n} h_i t^i+b_1(1+t^n-(1-t)^n)-nb_1t+\binom{n}{2}b_1t^2\\ &=\sum_{i=0}^{n} b_{2i}t^i-nb_1t+\binom{n}{2}b_1t^2\end{aligned}$$

where we used Theorem 3.1 at the last identity. This proves the lemma when $n \geq 4$. When $n = 3$,

$$\dim \tilde{H}_i(\mathcal{P}) = \begin{cases} 2b_1 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and the same argument as above shows that the lemma still holds for $n = 3$. When $n = 2$,

$$\dim \tilde{H}_i(\mathcal{P}) = \begin{cases} b_1 & \text{if } i = 0, \\ b_1 + 1 & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the same holds in this case too. □

Remark. One can check that

$$\sum_{i=1}^{n-1} h_i'' t^i = \sum_{i=1}^{n-1} b_{2i} t^i - nb_1(t + t^{n-1}).$$

Therefore, $h_i'' = h_i''(\mathcal{P})$ is not necessarily equal to $b_{2i} = b_{2i}(M)$ although both are symmetric. This is not surprising because h_i'' depends only on the boundary of M/T. It would be interesting to ask whether $h_i''(\mathcal{P}) \leq b_{2i}(M)$ for any M such that the face poset of $\partial(M/T)$ is dual to $\mathcal{P}$ and whether the equality can be attained for some such M (M may depend on i).

Now we prove Proposition 6.1. At first we suppose that $\Bbbk$ is a field. By Proposition 4.2 we have $\mathbb{Z}[\mathcal{P}] = H_T^{even}(M)$. The images of ring generators of $H^*(BT;\Bbbk)$ by π^* provide an h.s.o.p. $\theta_1, \dots, \theta_n$ in $H_T^{even}(M;\Bbbk) = \Bbbk[\mathcal{P}]$. This fact simply follows from the characterization of homogeneous systems of parameters in face rings given by [4, Th.5.4]. Thus we have

$$F(H_T^{even}(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk))), t) = \sum_{i=0}^{n} b_{2i}(M) t^i - nb_1 t + \binom{n}{2} b_1 t^2 \tag{6.3}$$

by Lemma 6.5. Moreover, the graded ring homomorphism in (6.1)

$$\bar{\iota}^* \colon \Bbbk[\mathcal{P}]/(\theta_1, \dots, \theta_n) = H_T^{even}(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk))) \to H^{even}(M;\Bbbk) \tag{6.4}$$

is surjective except in degree 2 as remarked at the beginning of this section. Therefore, the identity (6.3) implies that $\bar{\iota}^*$ in (6.4) is an isomorphism except in degrees 2 and 4. Finally, the rank of the cokernel of $\bar{\iota}^*$ in degree 2 is $nb_1(M)$ by Lemma 5.2 and the rank of the kernel of $\bar{\iota}^*$ in degree 4 is $\binom{n}{2} b_1$ by (6.3), proving Proposition 6.1 over fields. Now we explain the case $\Bbbk = \mathbb{Z}$.

The map $\pi^* \colon H^*(BT;\Bbbk) \to H_T^*(M;\Bbbk)$ coincides with the map $\pi^* \colon H^*(BT;\mathbb{Z}) \to H_T^*(M;\mathbb{Z})$ tensored with $\Bbbk$, since both $H^*(BT;\mathbb{Z})$ and $H_T^*(M;\mathbb{Z})$ are $\mathbb{Z}$-torsion free. In particular, the ideals $(\pi^*(H^2(BT;\Bbbk)))$ and $(\pi^*(H^2(BT;\mathbb{Z}))\otimes\Bbbk) = (\pi^*(H^2(BT;\mathbb{Z})))\otimes \Bbbk$ coincide in $H_T^*(M;\Bbbk) \cong H_T^*(M;\mathbb{Z}) \otimes \Bbbk$. Consider the exact sequence

$$(\pi^*(H^2(BT;\mathbb{Z}))) \to H_T^*(M;\mathbb{Z}) \to H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z}))) \to 0$$

The functor $- \otimes \Bbbk$ is right exact, thus the sequence

$$(\pi^*(H^2(BT;\mathbb{Z}))) \otimes \Bbbk \to H_T^*(M;\mathbb{Z}) \otimes \Bbbk \to H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z}))) \otimes \Bbbk \to 0$$

is exact. These considerations show that

$$H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z}))) \otimes \Bbbk \cong H_T^*(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk)))$$

Finally, the map

$$\bar{\iota}^* \colon H_T^*(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk))) \to H^*(M,\Bbbk)$$

coincides (up to isomorphism) with the map

$$\bar{\iota}^*\colon H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z}))) \to H^*(M,\mathbb{Z}),$$

tensored with $\Bbbk$. The statement of Proposition 6.1 holds for any field thus holds for $\mathbb{Z}$.

We conclude this section with some observation on the kernel of $\bar{\iota}^*$ in degree 4 from the viewpoint of the Serre spectral sequence. Recall

$$H_T^4(M) = \mathcal{F}^{0,4} \supset \mathcal{F}^{1,3} \supset \mathcal{F}^{2,2} \supset \mathcal{F}^{3,1} \supset \mathcal{F}^{4,0} \supset \mathcal{F}^{5,-1} = 0$$

where $\mathcal{F}^{p,q}/\mathcal{F}^{p+1,q-1} = E_\infty^{p,q}$. Since $E_2^{p,q} = H^p(BT)\otimes H^q(X)$, $E_\infty^{p,q} = 0$ for p odd. Therefore,

$$\operatorname{rank} H_T^4(M) = \operatorname{rank} E_\infty^{0,4} + \operatorname{rank} E_\infty^{2,2} + \operatorname{rank} E_\infty^{4,0},$$

where we know $E_\infty^{0,4} = E_2^{0,4} = H^4(M)$ and $E_\infty^{4,0} = E_2^{4,0} = H^4(BT)$. As for $E_\infty^{2,2}$, we recall that

$$d_2\colon E_2^{p,2} \to E_2^{p+2,1} \quad \text{is surjective for any } p \geq 0$$

by (5.7). Therefore, noting $H^3(M) = 0$, one sees $E_3^{2,2} = E_\infty^{2,2}$. It follows that

$$\operatorname{rank} E_\infty^{2,2} = \operatorname{rank} E_2^{2,2} - \operatorname{rank} E_2^{4,1} = nb_2 - \binom{n+1}{2} b_1.$$

On the other hand, $\operatorname{rank} E_\infty^{0,2} = b_2 - nb_1$ and there is a product map

$$\varphi\colon E_\infty^{0,2}\otimes E_\infty^{2,0} \to E_\infty^{2,2}.$$

The image of this map lies in the ideal $(\pi^*(H^2(BT))$ and the rank of the cokernel of this map is

$$nb_2 - \binom{n+1}{2} b_1 - n(b_2 - nb_1) = \binom{n}{2} b_1.$$

Therefore

$$\operatorname{rank} E_\infty^{0,4} + \operatorname{rank}\operatorname{coker}\varphi = b_4 + \binom{n}{2} b_1$$

which agrees with the coefficient of t^2 in $F(H_T^{even}(M)/(\pi^*(H^2(BT))), t)$ by (6.3). This suggests that the cokernel of φ would correspond to the kernel of $\bar{\iota}^*$ in degree 4.

7. 4-dimensional case

In this section, we explicitly describe the kernel of $\bar{\iota}^*$ in degree 4 when $n = 2$, that is, when M is of dimension 4. In this case, $\partial(M/T)$ is the union of $b_1 + 1$ closed polygonal curves.

First we recall the case when $b_1 = 0$. In this case, $H_T^{even}(M) = H_T^*(M)$. Let $\partial(M/T)$ be an m-gon and $v_1, \dots, v_m$ be the primitive edge vectors in the multi-fan of M, where v_i and v_{i+1} spans a 2-dimensional cone for any $i = 1, 2, \dots, m$ (see [13]). Note that $v_i \in H_2(BT)$ and we understand $v_{m+1} = v_1$ and $v_0 = v_m$ in this section. Since v_j, v_{j+1} is a basis of $H_2(BT)$ for any j, we have $\det(v_j, v_{j+1}) = \pm 1$.

Let $\tau_i \in H_T^2(M)$ be the equivariant Poincaré dual to the characteristic submanifold corresponding to v_i. Then we have

$$\pi^*(u) = \sum_{i=1}^m \langle u, v_i\rangle \tau_i \quad \text{for any } u \in H^2(BT), \tag{7.1}$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between cohomology and homology, (see [11] for example). We multiply both sides in (7.1) by τ_i. Then, since $\tau_i\tau_j = 0$ if v_i and v_j do not span a 2-dimensional cone, (7.1) turns into

$$0 = \langle u, v_{i-1}\rangle \tau_{i-1}\tau_i + \langle u, v_i\rangle \tau_i^2 + \langle u, v_{i+1}\rangle \tau_i\tau_{i+1} \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{7.2}$$

If we take u with $\langle u, v_i\rangle = 1$, then (7.2) shows that τ_i^2 can be expressed as a linear combination of $\tau_{i-1}\tau_i$ and $\tau_i\tau_{i+1}$. If we take $u = \det(v_i, \ )$, where u is regarded as an element of $H^2(BT)$ because $H^2(BT) = \mathrm{Hom}(H_2(BT), \mathbb{Z})$, then (7.2) reduces to

$$\det(v_{i-1}, v_i)\tau_{i-1}\tau_i = \det(v_i, v_{i+1})\tau_i\tau_{i+1} \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{7.3}$$

Finally we note that $\tau_i\tau_{i+1}$ maps to the cofundamental class of M up to sign. We denote by $\mu \in H_T^4(M)$ the element (either $\tau_{i-1}\tau_i$ or $-\tau_{i-1}\tau_i$) which maps to the cofundamental class of M.

When $b_1 \geq 1$, the above argument works for each component of $\partial(M/T)$. In fact, according to [11], (7.1) holds in $H_T^*(M)$ modulo $H^*(BT)$-torsion but in our case there is no $H^*(BT)$-torsion in $H_T^{even}(M)$ by Proposition 4.3. Suppose that $\partial(M/T)$ consists of m_j-gons for $j = 1, 2, \dots, b_1 + 1$. To each m_j-gon, we have the class $\mu_j \in H_T^4(M)$ (mentioned above as μ). Since μ_j maps to the cofundamental class of M, $\mu_i - \mu_j$ $(i \neq j)$ maps to zero in $H^4(M)$; so it is in the kernel of $\bar{\iota}^*$. The subgroup of $H_T^{even}(M)/(\pi^*(H^2(BT)))$ in degree 4 generated by $\mu_i - \mu_j$ $(i \neq j)$ has the desired rank b_1.

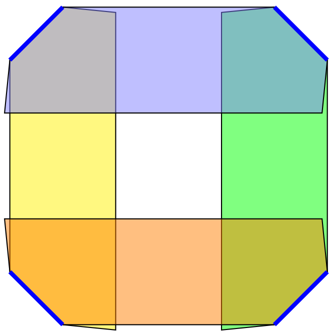

FIGURE 1. The origami template with four polygons

Example 7.1. Take the 4-dimensional toric origami manifold M corresponding to the origami template shown on fig. 1 (Example 3.15 of [5]). Topologically M/T is homeomorphic to $S^1 \times [0, 1]$ and the boundary of M/T as a manifold with corners consists of two closed polygonal curves, each having 4 segments. The multi-fan of M is the union of two copies of the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the product torus action. Indeed, if v_1, v_2 are primitive edge vectors in the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ which spans a 2-dimensional cone, then the other primitive edge vectors $v_3, \dots, v_8$ in the multi-fan of M are

$$v_3 = -v_1, \quad v_4 = -v_2, \quad \text{and} \quad v_i = v_{i-4} \quad \text{for } i = 5, \dots, 8$$

and the 2-dimensional cones in the multi-fan are

$$\angle v_1v_2, \quad \angle v_2v_3, \quad \angle v_3v_4, \quad \angle v_4v_1,$$
$$\angle v_5v_6, \quad \angle v_6v_7, \quad \angle v_7v_8, \quad \angle v_8v_5,$$

where $\angle vv'$ denotes the 2-dimensional cone spanned by vectors v, v'. Note that

$$\tau_i\tau_j = 0 \text{ if } v_i, v_j \text{ do not span a 2-dimensional cone.} \tag{7.4}$$

We have

$$\pi^*(u) = \sum_{i=1}^{8} \langle u, v_i\rangle \tau_i \quad \text{for any } u \in H^2(BT). \tag{7.5}$$

Let v_1^*, v_2^* be the dual basis of v_1, v_2. Taking $u = v_1^*$ or v_2^* , we see that

$$\tau_1 + \tau_5 = \tau_3 + \tau_7, \quad \tau_2 + \tau_6 = \tau_4 + \tau_8 \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{7.6}$$

Since we applied (7.5) to the basis v_1^*, v_2^* of $H^2(BT)$, there is no other essentially new linear relation among τ_i's.

Now, multiply the equations (7.6) by τ_i and use (7.4). Then we obtain

$$\begin{aligned}
&\tau_i^2 = 0 \quad \text{for any } i,\\
&(\mu_1 :=)\tau_1\tau_2 = \tau_2\tau_3 = \tau_3\tau_4 = \tau_4\tau_1,\\
&(\mu_2 :=)\tau_5\tau_6 = \tau_6\tau_7 = \tau_7\tau_8 = \tau_8\tau_5 \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))).
\end{aligned}$$

Our argument shows that these together with (7.4) are the only degree two relations among τ_i's in $H_T^*(M)/(\pi^*(H^2(BT)))$. The kernel of

$$\bar{\iota}^*\colon H_T^{even}(M;\mathbb{Q})/(\pi^*(H^2(BT;\mathbb{Q}))) \to H^{even}(M;\mathbb{Q})$$

in degree 4 is spanned by $\mu_1 - \mu_2$.

8. On the cokernel of $\bar{\iota}^*$ in degree 2

In this section we describe the elements of $H^2(M)$ that do not lie in the image of the map (6.4). In fact, we describe geometrically the homology $(2n-2)$-cycles, which are Poincare dual to these elements. A very similar technique was used in [15] to calculate the homology of 4-dimensional torus manifolds, whose orbit spaces are polygons with holes. In contrast to [15] we do not introduce particular cell structures on M, because this approach becomes more complicated for higher dimensions.

Denote the orbit space M/T by Q, so Q is a manifold with corners and acyclic proper faces, and Q is homotopy equivalent to a wedge of b_1 circles. Also let $q\colon M \to Q$ denote the projection to the orbit space, and Γ_i be the characteristic subgroup, i.e. the stabilizer of orbits in $F_i^\circ \subset Q$. For a any face G of Q denote by Γ_G the stabilizer subgroup of orbits $x \in G^\circ$. Thus $\Gamma_G = \prod_i \Gamma_i \subset T$, where the product is taken over all i such that $G \subseteq F_i$. The origami manifold M is homeomorphic to the model

$$Q \times T/\sim$$

where $(x_1,t_1) \sim (x_2,t_2)$ if $x_1 = x_2 \in G^\circ$ and $t_1t_2^{-1} \in \Gamma_G$ for some face $G \subset Q$. This fact is a consequence of a general result of the work [19]. In the following we identify M with the model $Q \times T/\sim$.

Consider a homology cycle $\sigma \in H_{n-1}(Q,\partial Q)$. Note that σ is Poincare–Lefschetz dual to some element of $H^1(Q) \cong H^1(\bigvee_{b_1} S^1) \cong \mathbb{Z}^{b_1}$. Let σ be represented by a pseudomanifold $\xi\colon (L,\partial L) \to (Q,\partial Q)$, where $\dim L = n-1$, and let $[L] \in H_{n-1}(L,\partial L)$ denote the fundamental cycle, so that $\xi_*([L]) = \sigma$. We assume that $\xi(L \setminus \partial L) \subset Q \setminus \partial Q$. Moreover, since every face of ∂Q is acyclic, we may assume that $\xi(\partial L)$ is contained in $\partial Q^{(n-2)}$, — the codimension 2 skeleton of Q. A pseudomanifold $(L,\partial L)$ defines a collection of $(2n-2)$-cycles in homology of M, one for each codimension-one subtorus of T, by the following construction.

Construction. First fix a coordinate splitting of the torus, $T = \prod_{i\in[n]} T_i^1$ in which the orientation of each T_i^1 is arbitrary but fixed. For each $j \in [n]$ consider the subtorus $T_{\widehat{j}} = T^{[n]\setminus j} = \prod_{i\in[n]\setminus j} T_i^1$, and let $\kappa\colon T_{\widehat{j}} \to T$ be the inclusion map. Given a pseudomanifold $(L,\partial L)$ as in the previous paragraph, consider the space $L \times T_{\widehat{j}}$ and the quotient construction $(L \times T_{\widehat{j}})/\sim_*$, where the identification $\sim_*$ is naturally induced from $\sim$ by the map ξ. Since $\xi(\partial L) \subset \partial Q^{(n-2)}$, the space $(\partial L \times T_{\widehat{j}})/\sim_*$

has dimension at most $2n-4$. Thus $(L \times T_{\widehat{j}})/\sim_*$ has the fundamental cycle $V_{L,j}$. Indeed, there is a diagram:

(8.1)

$$\begin{array}{ccccccc}
 & & & & H_{n-1}(L,\partial L)\otimes H_{n-1}(T_{\widehat{j}}) & & \\
 & & & & \downarrow \cong \text{ (Kunneth)} & & \\
0 & & & & H_{2n-2}(L\times T_{\widehat{j}}, \partial L\times T_{\widehat{j}}) & & 0\\
\| & & & & \downarrow \cong \text{ (excision)} & & \|\\
H_{2n-2}(\frac{\partial L\times T_{\widehat{j}}}{\sim_*}) & \longrightarrow & H_{2n-2}(\frac{L\times T_{\widehat{j}}}{\sim_*}) & \xrightarrow{\cong} & H_{2n-2}(\frac{L\times T_{\widehat{j}}}{\sim_*}, \frac{\partial L\times T_{\widehat{j}}}{\sim_*}) & \longrightarrow & H_{2n-3}(\frac{\partial L\times T_{\widehat{j}}}{\sim_*})
\end{array}$$

Let $T_{\widehat{j}}$ be oriented by the splitting $T \cong T_{\widehat{j}} \times T^1_j$. Given such an orientation, there exists the distinguished generator $\Omega_j \in H_{n-1}(T_{\widehat{j}})$. Then the fundamental cycle $V_{L,j} \in H_{2n-2}((L \times T_{\widehat{j}})/\sim_*)$ is defined as the image of $[L] \otimes \Omega_j \in H_{n-1}(L, \partial L) \otimes H_{n-1}(T_{\widehat{j}})$ under the isomorphisms of diagram (8.1). The induced map

$$\zeta_{L,j} \colon (L \times T_{\widehat{j}})/\sim_* \to (Q \times T_{\widehat{j}})/\sim \hookrightarrow (Q \times T)/\sim = M$$

determines the element $x_{L,j} = (\zeta_{L,j})_*(V_{L,j}) \in H_{2n-2}(M)$.

Proposition 8.1. *Let $\sigma_1, \dots, \sigma_{b_1}$ be a basis of $H_{n-1}(Q, \partial Q)$, and let $L_1, \dots, L_{b_1}$ be pseudomanifolds representing these cycles and satisfying the restrictions stated above. Consider the set of homology classes $\{x_{L,j}\} \subset H_{2n-2}(M)$, where L runs over the set $\{L_1, \dots, L_{b_1}\}$ and j runs over $[n]$. Then the set of Poincare dual classes of $x_{L,j}$ is a basis of the cokernel $H^2(M)/\iota^*(H^2_T(M))$.*

Proof. Consider disjoint circles $S_1, \dots, S_{b_1} \subset Q^\circ$, whose corresponding homology classes $[S_1], \dots, [S_{b_1}] \in H_1(Q)$ are dual to $\sigma_1, \dots, \sigma_{b_1}$. Thus for the intersection numbers we have $[S_i] \cap \sigma_k = \delta_{ik}$. Consider 2-dimensional submanifolds of the form $S_i \times T^1_l \subset M$, where $l \in [n]$. They lie in $q^{-1}(Q^\circ) \subset M$. Let $[S_i \times T^1_l] \in H_2(M)$ be the homology classes represented by these submanifolds. Then

$$[S_i \times T^1_l] \cap x_{L_k,j} = \delta_{ik}\delta_{lj}, \tag{8.2}$$

since all the intersections lie in $q^{-1}(Q^\circ) = Q^\circ \times T$.

The equivariant cycles of M sit in $q^{-1}(\partial Q)$. Thus the intersection of $S_i \times T^1_l \subset q^{-1}(Q^\circ)$ with any equivariant cycle is empty. Nondegenerate pairing (8.2) shows that the set $\{x_{L,j}\}$ is linearly independent modulo equivariant cycles. Its cardinality is precisely nb_1 and the statement follows from Lemma 5.2. □

Remark. The element $x_{L,j} \in H_{2n-2}(M)$ depends on the representing pseudomanifold L, not only on its homology class in $H_{n-1}(Q, \partial Q)$. The classes corresponding to different representing pseudomanifolds are connected by linear relations involving characteristic submanifolds. We describe these relations next.

At first let us introduce orientations on the objects under consideration. We fix an orientation of the orbit space Q. This defines an orientation of each facet (F_i is oriented by $TF_i \oplus \nu \cong TQ$, where the inward normal vector of the normal bundle ν is set to be positive). Since the torus T is oriented, we have a distinguished orientation of $M = Q \times T/\sim$. Recall that Γ_i is the characteristic subgroup corresponding to a facet $F_i \subset Q$. Since the action is locally standard, Γ_i is a 1-dimensional connected subgroup of T. Let us fix orientations of all characteristic subgroups (this choice of orientations is usually called an omniorientation). Then every Γ_i can be written as

$$\Gamma_i = \{(t^{\lambda_{i,1}}, \dots, t^{\lambda_{i,n}}) \in T \mid t \in T^1\}, \tag{8.3}$$

where $(\lambda_{i,1}, \dots, \lambda_{i,n}) \in \mathbb{Z}^n$ is a uniquely determined primitive integral vector.

Let us orient every quotient torus T/Γ_i by the following construction. For each Γ_i choose a codimension 1 subtorus $\Upsilon_i \subset T$ such that the product map $\Upsilon_i \times \Gamma_i \to T$ is an isomorphism. The orientations of T and Γ_i induce an orientation of Υ_i. The quotient map $T \to T/\Gamma_i$ induces an isomorphism between Υ_i and T/Γ_i providing the quotient group with an orientation. The orientation of T/Γ_i defined this way does not depend on the choice of the auxiliary subgroup Υ_i.

Finally, the orientations on F_i and T/Γ_i give an orientation of the characteristic submanifold $M_i = q^{-1}(F)$. This follows from the fact that M_i contains an open dense subset $q^{-1}(F_i^\circ) = F_i^\circ \times (T/\Gamma_i)$.

Construction. Let F_i be a facet of Q, and $[F_i] \in H_{n-1}(F_i, \partial F_i)$ its fundamental cycle. The cycles $[F_i]$ form a basis of $H_{n-1}(\partial Q, \partial Q^{(n-2)}) = \bigoplus_{\text{facets}} H_{n-1}(F_i, \partial F_i)$. Let $\xi_\varepsilon\colon (L_\varepsilon, \partial L_\varepsilon) \to (Q, \partial Q)$, $\varepsilon = 1, 2$, be two pseudomanifolds representing the same element $\sigma \in H_{n-1}(Q, \partial Q)$. Then there exists a pseudomanifold $(N, \partial N)$ of dimension n and a map $\eta\colon N \to Q$ such that L_1, L_2 are disjoint submanifolds of ∂N, $\eta|_{L_\varepsilon} = \xi_\varepsilon$ for $\varepsilon = 1, 2$, and $\eta(\partial N \setminus (L_1^\circ \sqcup L_2^\circ)) \subset \partial Q$ (this follows from the geometrical definition of homology, see [16, App. A.2]). The skeletal stratification of Q induces a stratification on N. The restriction of the map η sends $\partial N^{(n-2)}$ to $\partial Q^{(n-2)}$. Let δ be the connecting homomorphism

$$\delta\colon H_n(N, \partial N) \to H_{n-1}(\partial N, \partial N^{(n-2)})$$

in the long exact sequence of the triple $(N, \partial N, \partial N^{(n-2)})$. Consider the sequence of homomorphisms

$$\begin{aligned} &H_n(N, \partial N) \xrightarrow{\delta} H_{n-1}(\partial N, \partial N^{(n-2)}) \cong \\ &H_{n-1}(L_1, \partial L_1) \oplus H_{n-1}(L_2, \partial L_2) \oplus H_{n-1}(\partial N \setminus (L_1^\circ \cup L_2^\circ), \partial N^{(n-2)}) \xrightarrow{\mathrm{id} \oplus \mathrm{id} \oplus \eta_*} \\ &\qquad H_{n-1}(L_1, \partial L_1) \oplus H_{n-1}(L_2, \partial L_2) \oplus H_{n-1}(\partial Q, \partial Q^{(n-2)}). \end{aligned}$$

This sequence of homomorphisms sends the fundamental cycle $[N] \in H_n(N, \partial N)$ to the element

$$\Big([L_1], -[L_2], \sum\nolimits_i \alpha_i [F_i]\Big) \tag{8.4}$$

of the group $H_{n-1}(L_1, \partial L_1) \oplus H_{n-1}(L_2, \partial L_2) \oplus H_{n-1}(\partial Q, \partial Q^{(n-2)})$, for some coefficients $\alpha_i \in \mathbb{Z}$.

Proposition 8.2. *If L_1, L_2 are two pseudomanifolds representing a class $\sigma \in H_{n-1}(Q, \partial Q)$, and $j \in [n]$, then*

$$x_{L_1,j} - x_{L_2,j} + \sum_{\text{facets}} \alpha_i \lambda_{i,j} [M_i] = 0 \quad \text{in } H_{2n-2}(M). \tag{8.5}$$

Here M_i is the characteristic submanifold of M corresponding to F_i, the numbers α_i are given by (8.4)*, and the numbers $\lambda_{i,j}$ are given by* (8.3)*.*

Proof. Choose a relative pseudomanifold bordism N between L_1 and L_2 and consider the space $(N \times T_{\widehat{j}})/\sim_*$. Here $\sim_*$ is the equivalence relation induced from $\sim$ by the map η. We have a map $(\eta \times \kappa)/\sim\colon (N \times T_{\widehat{j}})/\sim_* \to M$. By the diagram chase, similar to (8.1), the space $(N \times T_{\widehat{j}})/\sim_*$ is a $(2n-1)$-pseudomanifold with boundary. Its boundary represents the element (8.5). Thus this element vanishes in homology. We only need to prove the following technical lemma.

Lemma 8.3. *Let F_i be a facet, Γ_i be its characteristic subgroup encoded by the vector $(\lambda_{i,1}, \ldots, \lambda_{i,1}) \in \mathbb{Z}^n$, $j \in [n]$. Let $\Omega_j \in H_{n-1}(T_{\widehat{j}})$ and $\Phi_i \in H_{n-1}(T/\Gamma_i)$ be the fundamental classes (in the orientations introduced previously). Then the composite map $T_{\widehat{j}} \hookrightarrow T \twoheadrightarrow T/\Gamma_i$ sends Ω_j to $\lambda_{i,j}\Phi_i$.*

Proof. Let $\{\mathbf{e}_s \mid s \in [n]\}$ be the positive basis of $H_1(T)$ corresponding to the splitting $T = \prod_s T^1_s$, and let $\{\mathbf{f}_r \mid r \in [n]\}$ be any positive basis of $H_1(T)$ such that $\mathbf{f}_n = (\lambda_{i,1}, \ldots, \lambda_{i,n})$. Thus $\Omega_j = (-1)^{n-j}\mathbf{e}_1 \wedge \ldots \wedge \widehat{\mathbf{e}_j} \wedge \ldots \wedge \mathbf{e}_n$. Let D be the matrix of basis change, $\mathbf{e}_s = \sum_{r=1}^n D^r_s \mathbf{f}_r$, and let $C = D^{-1}$. The element Ω_j maps to

$$(-1)^{n-j} \det(D^r_s)_{\substack{r\in\{1,\ldots,n-1\}\\ s\in\{1,\ldots,\hat{j},\ldots,n\}}}$$

which is equal to the element C^j_n by Cramer's rule. C^j_n is the j-th coordinate of $\mathbf{f}_n$ in the basis $\{\mathbf{e}_s\}$. Thus, by construction, $C^j_n = \lambda_{i,j}$. □

Proposition proved. □

Proposition 8.2 gives the idea how to describe the multiplication in $H^*(M)$. Equivalently, we need to describe the intersections of cycles in $H_*(M)$. Intersections of equivariant cycles are known — they are encoded by the face ring of Q. To describe the intersections of additional cycles $x_{L,j}$ sometimes we can do the following:

(1) Let M_F be the face submanifold of M, corresponding to the face $F \subset Q$. If $F \cap \partial L = \varnothing$, then $[M_F] \cap x_{L,j} = 0$ in the homology of M. Otherwise, in many cases we can choose a different representative L' of the same homology class as L with the property $\partial L' \cap F = 0$. Then, by proposition 8.2, $[M_F] \cap x_{L,j} = [M_F] \cap x_{L',j} + [M_F] \cap \sum_{\text{facets}} \alpha_i\lambda_{i,j}[M_i] = \sum_{\text{facets}} \alpha_i\lambda_{i,j}[M_F] \cap [M_i]$ which can be computed using relations in $\Bbbk[Q]/(\theta_1, \ldots, \theta_n)$.

(2) To compute the intersection of two elements of the form x_{L_1,j_1} and x_{L_2,j_2} sometimes we can use the same trick: find a pseudomanifold L'_1 which does not intersect L_2 and replace x_{L_1,j_1} by $x_{L'_1,j_1} + \sum_i \alpha_i\lambda_{i,j_1}[M_i]$. Then the intersection $x_{L'_1,j_1} \cap x_{L_2,j_2}$ vanishes and intersections of x_{L_2,j_2} with $[M_i]$ are computed using (1).

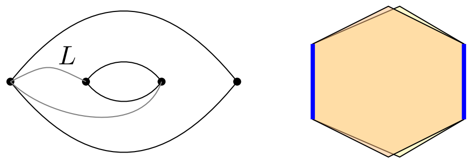

FIGURE 2. Manifold with corners Q for which the products of extra elements cannot be calculated using linear relations of Proposition 8.2

Remark. This general idea may not work in particular cases. Figure 2 provides an example of Q such that every pseudomanifold L with $\partial L \subset \partial Q^{(0)}$, representing the generator of $H_1(Q, \partial Q)$, intersects every facet of Q. Unfortunately, such situations may appear as realizations of origami templates. The picture on the right shows an origami template, whose geometric realization is the manifold with corners shown on the left.

9. Some observation on non-acyclic cases

The face acyclicity condition we assumed so far is not preserved under taking the product with a symplectic toric manifold N, but every face of codimension $\geq \frac{1}{2}\dim N + 1$ is acyclic. Motivated by this observation, we will make the following assumption on our toric origami manifold M of dimension $2n$:

every face of M/T of codimension $\geq r$ is acyclic for some integer r.

Note that $r = 1$ in the previous sections. Under the above assumption, the arguments in Section 2 work to some extent in a straightforward way. The main point is that Lemma 2.5 can be generalized as follows.

Lemma 9.1. *The homomorphism $H^{2j}(\tilde{M}) \to H^{2j}(Z_+ \cup Z_-)$ induced from the inclusion is surjective for $j \geq r$.*

Using this lemma, we see that Lemma 2.3 turns into the following.

Lemma 9.2.

$$\sum_{i=1}^{r}(b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M})) = \sum_{i=1}^{r}(b_{2i}(M) - b_{2i-1}(M)) + b_{2r}(B)$$

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) - b_{2i-1}(M) + b_{2i}(B) - b_{2i-2}(B) \quad \textit{for } i \geq r+1.$$

Combining Lemma 9.2 with Lemma 2.2, Lemma 2.4 turns into the following.

Lemma 9.3.

$$b_1(M') = b_1(M) - 1, \quad b_{2r}(M') = b_{2r}(M) + b_{2r-2}(B) + b_{2r}(B),$$
$$b_{2i+1}(M') = b_{2i+1}(M) \quad \textit{for } r \leq i \leq n-r-1.$$

Finally, Theorem 2.6 is generalized as follows.

Theorem 9.4. *Let M be an orientable toric origami manifold of dimension $2n$ $(n \geq 2)$ such that every face of M/T of codimension $\geq r$ is acyclic. Then*

$$b_{2i+1}(M) = 0 \quad \textit{for } r \leq i \leq n-r-1.$$

Moreover, if M' and B are as above, then

$$b_1(M') = b_1(M) - 1 \quad (\textit{hence } b_{2n-1}(M') = b_{2n-1}(M) - 1 \textit{ by Poincaré duality}),$$
$$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \textit{for } r \leq i \leq n-r.$$

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