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A SIMPLE PROOF OF HARDY'S INEQUALITY IN A LIMITING CASE

FUTOSHI TAKAHASHI

ABSTRACT. In this short note, we provide a simple proof of Hardy's inequality in a limiting case. In the proof we do not need any rearrangement technique or the one-dimensional argument.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^N with $0 \in \Omega$. The classical Hardy's inequality is of the form

$$\left(\frac{N-p}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx \leq \int_{\Omega} |\nabla u|^p dx \quad (1.1)$$

for all $u \in W_0^{1,p}(\Omega)$, where $N \geq 3$ and $1 < p < N$. See [3] for its simple proof which uses the Fundamental Theorem of Calculus and Hölder's inequality only. It is well known that the constant $\left(\frac{N-p}{p}\right)^p$ is optimal and never attained in $W_0^{1,p}(\Omega)$.

For $p = N$, the inequality (1.1) loses its sense and instead of (1.1) the inequality

$$\left(\frac{N-1}{N}\right)^N \int_{\Omega} \frac{|u(x)|^N}{|x|^N \left(\log \frac{Re}{|x|}\right)^N} dx \leq \int_{\Omega} |\nabla u|^N dx \quad (1.2)$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $R = \sup_{x \in \Omega} |x|$. Again, the constant $\left(\frac{N-1}{N}\right)^N$ is known to be optimal; see for example, [1], [2]. We call (1.2) as Hardy's inequality in a limiting case. Main aim of this short note is to provide a simple proof of Hardy's inequality in a limiting case. We do not need any rearrangement technique such as Polya-Szegö inequality for the spherical decreasing rearrangement, or a technical one-dimensional argument. Also our method can provide the sharper inequality treated in [8], [2], [9] and [7].

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Theorem 1.1. *Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$ with $0 \in \Omega$. Let $g : (1, +\infty) \rightarrow \mathbb{R}$ be a C^2 function with the properties that*

$$g'(s) < 0, \quad g''(s) > 0 \quad \text{for any } s > 1 \quad (1.3)$$

and there exists $C > 0$ such that

$$\frac{(-g'(s))^{2(N-1)}}{(g''(s))^{N-1}} \leq C \quad \text{for any } s > 1. \quad (1.4)$$

Put $R = \sup_{x \in \Omega} |x|$. Then the inequality

$$\begin{aligned} & \left(\frac{N-1}{N}\right)^N \int_{\Omega} \frac{|u(x)|^N}{|x|^N} \left(-g'\left(\log \frac{Re}{|x|}\right)\right)^{N-2} g''\left(\log \frac{Re}{|x|}\right) dx \\ & \leq \int_{\Omega} \frac{(-g'\left(\log \frac{Re}{|x|}\right))^{2(N-1)}}{(g''\left(\log \frac{Re}{|x|}\right))^{N-1}} \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \end{aligned} \quad (1.5)$$

holds true for any $u \in W_0^{1,N}(\Omega)$.

Corollary 1.2. *Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$ with $0 \in \Omega$. Then the inequality*

$$\left(\frac{N-1}{N}\right)^N \int_{\Omega} \frac{|u(x)|^N}{|x|^N \left(\log \frac{R}{|x|}\right)^N} dx \leq \int_{\Omega} \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \quad (1.6)$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $R = \sup_{x \in \Omega} |x|$.

Note that, different from the function $\frac{1}{|x|^N \left(\log \frac{Re}{|x|}\right)^N}$ appeared in (1.2), the function $\frac{1}{|x|^N \left(\log \frac{R}{|x|}\right)^N}$ in (1.6) becomes unbounded when $|x| \sim 0$ and also $|x| \sim R$.

Corollary 1.3. *Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$ with $0 \in \Omega$. Let $\alpha > 0$. Then the inequality*

$$\left(\frac{N-1}{N}\right)^N \alpha^N \int_{\Omega} \frac{|u(x)|^N}{|x|^{N-\alpha(N-1)}} dx \leq \int_{\Omega} |x|^{\alpha(N-1)} \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \quad (1.7)$$

holds for all $u \in W_0^{1,N}(\Omega)$.

If we take $\alpha = \frac{N}{N-1}$ in (1.7), we have

$$\int_{\Omega} |u(x)|^N dx \leq \int_{\Omega} |x|^N \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \left(\leq R^N \int_{\Omega} |\nabla u|^N dx \right),$$

thus (1.7) can be seen as a generalization of Poincaré's inequality for $u \in W_0^{1,N}(\Omega)$.

The proof of Theorem 1.1 relies on the divergence theorem and Hölder's inequality. Similar "simple" approaches have been proposed in [10], [4], and [5], mainly to derive (1.1). Their proof uses the identity

$$\operatorname{div} \left(\frac{x}{|x|^\lambda} \right) = \frac{N - \lambda}{|x|^\lambda} \quad \text{for } |x| \neq 0, \lambda \in \mathbb{R}.$$

On the other hand, the identity

$$\operatorname{div} \left(\frac{x}{|x|^N \left(\log \frac{R}{|x|} \right)^{N-1}} \right) = \frac{N - 1}{|x|^N \left(\log \frac{R}{|x|} \right)^N} \quad \text{for } |x| \neq 0, R,$$

will be the base of our proof. See the next section.

Another approach to the limiting case of Hardy's inequality, using the mean integral of a Schwarz symmetrization, has been done by Ioku; see [6]:Remark 1.4.

2. PROOF OF THEOREM.

In this section, we prove Theorem 1.1. For $\varepsilon > 0$ small, put

$$R_\varepsilon = \sup_{x \in \Omega} (|x|^2 + 2\varepsilon^2)^{1/2}, \quad X_\varepsilon(x) = \log \frac{R_\varepsilon e}{(|x|^2 + \varepsilon^2)^{1/2}}$$

and

$$\psi_\varepsilon(x) = g(X_\varepsilon(x)) \in C^2(\bar{\Omega}). \quad (2.1)$$

We calculate

$$\begin{aligned} |\nabla \psi_\varepsilon(x)|^{N-2} \nabla \psi_\varepsilon(x) &= (-g'(X_\varepsilon))^{N-1} \left(\frac{|x|^{N-2} x}{(|x|^2 + \varepsilon^2)^{N-1}} \right), \\ \Delta_N \psi_\varepsilon(x) &= \operatorname{div} (|\nabla \psi_\varepsilon(x)|^{N-2} \nabla \psi_\varepsilon(x)) \\ &= (N - 1)(-g'(X_\varepsilon))^{N-2} g''(X_\varepsilon) \frac{|x|^N}{(|x|^2 + \varepsilon^2)^N} + (-g'(X_\varepsilon))^{N-1} \frac{2(N - 1)\varepsilon^2 |x|^{N-2}}{(|x|^2 + \varepsilon^2)^N}, \end{aligned}$$

where we have used the assumption that $g'(s) < 0$. For $u \in W_0^{1,N}(\Omega)$, divergence theorem assures that

$$\int_{\Omega} |u|^N \Delta_N \psi_\varepsilon dx = - \int_{\Omega} \nabla (|u|^N) \cdot |\nabla \psi_\varepsilon|^{N-2} \nabla \psi_\varepsilon dx. \quad (2.2)$$

The RHS of (2.2) is estimated from above as

$$\begin{aligned}
|\text{RHS of (2.2)}| &= \left| N \int_{\Omega} |u|^{N-2} u \nabla u \cdot |\nabla \psi_{\varepsilon}|^{N-2} \nabla \psi_{\varepsilon} dx \right| \\
&= \left| N \int_{\Omega} |u|^{N-2} u (-g'(X_{\varepsilon}))^{N-1} \left(\frac{|x|^{N-2} x \cdot \nabla u}{(|x|^2 + \varepsilon^2)^{N-1}} \right) dx \right| \\
&\leq N \int_{\Omega} |u|^{N-1} (-g'(X_{\varepsilon}))^{N-1} \left(\frac{|x|^{N-2} |x \cdot \nabla u|}{(|x|^2 + \varepsilon^2)^{N-1}} \right) dx \\
&\leq N \left(\int_{\Omega} \frac{|u|^N |x|^N}{(|x|^2 + \varepsilon^2)^N} (-g'(X_{\varepsilon}))^{N-2} g''(X_{\varepsilon}) dx \right)^{\frac{N-1}{N}} \times \\
&\quad \times \left(\int_{\Omega} (-g'(X_{\varepsilon}))^{2(N-1)} (g''(X_{\varepsilon}))^{-(N-1)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \right)^{\frac{1}{N}},
\end{aligned}$$

where we have used Hölder's inequality. On the other hand, the LHS of (2.2) is estimated from below as

$$\begin{aligned}
|\text{LHS of (2.2)}| &= (N-1) \int_{\Omega} |u|^N \left\{ (-g'(X_{\varepsilon}))^{N-2} g''(X_{\varepsilon}) \frac{|x|^N}{(|x|^2 + \varepsilon^2)^N} + (-g'(X_{\varepsilon}))^{N-1} \frac{2\varepsilon^2 |x|^{N-2}}{(|x|^2 + \varepsilon^2)^N} \right\} dx \\
&\geq (N-1) \int_{\Omega} |u|^N (-g'(X_{\varepsilon}))^{N-2} g''(X_{\varepsilon}) \frac{|x|^N}{(|x|^2 + \varepsilon^2)^N} dx.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(N-1) \int_{\Omega} \frac{|u|^N |x|^N}{(|x|^2 + \varepsilon^2)^N} (-g'(X_{\varepsilon}))^{N-2} g''(X_{\varepsilon}) dx \\
\leq N \left(\int_{\Omega} \frac{|u|^N |x|^N}{(|x|^2 + \varepsilon^2)^N} (-g'(X_{\varepsilon}))^{N-2} g''(X_{\varepsilon}) dx \right)^{\frac{N-1}{N}} \times \\
\quad \times \left(\int_{\Omega} (-g'(X_{\varepsilon}))^{2(N-1)} (g''(X_{\varepsilon}))^{-(N-1)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \right)^{\frac{1}{N}},
\end{aligned}$$

which implies

$$\begin{aligned}
\left(\frac{N-1}{N} \right)^N \int_{\Omega} |u|^N (-g'(X_{\varepsilon}))^{N-2} g''(X_{\varepsilon}) \frac{|x|^N}{(|x|^2 + \varepsilon^2)^N} dx \\
\leq \int_{\Omega} \frac{(-g'(X_{\varepsilon}))^{2(N-1)}}{(g''(X_{\varepsilon}))^{N-1}} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx.
\end{aligned}$$

Finally, we let $\varepsilon \rightarrow 0$ in the both sides of the above inequality. Note that $R_{\varepsilon} \rightarrow R$ and $X_{\varepsilon} \rightarrow \log\left(\frac{R\varepsilon}{|x|}\right)$ a.e. $x \in \Omega$ as $\varepsilon \rightarrow 0$. By using Fatou's

lemma in the LHS and the Lebesgue dominated convergence theorem with the assumption (1.4) in the RHS, we conclude (1.5) holds. \square

Proof of Corollary 1.2, 1.3.

Proof. For the proof of Corollary 1.2, let $g(s) = -\log(s-1)$ for $s > 1$. Then we see $g'(s) = -\frac{1}{s-1} < 0$, $g''(s) = \frac{1}{(s-1)^2} > 0$, and

$$\frac{(-g'(s))^{2(N-1)}}{(g''(s))^{N-1}} = \frac{\left(\frac{1}{s-1}\right)^{2(N-1)}}{\left(\frac{1}{(s-1)^2}\right)^{N-1}} = 1 \quad \text{for any } s > 1.$$

Thus the assumptions (1.3),(1.4) are satisfied. Inserting

$$g' \left(\log \frac{Re}{|x|} \right) = \frac{-1}{\log \frac{R}{|x|}}, \quad g'' \left(\log \frac{Re}{|x|} \right) = \frac{1}{\left(\log \frac{R}{|x|}\right)^2}$$

into (1.5), we obtain (1.6).

For Corollary 1.3, take $g(s) = e^{-\alpha s}$ for $s > 1$. We easily see that

$$\frac{(-g'(s))^{2(N-1)}}{(g''(s))^{N-1}} = e^{-(N-1)\alpha s} \leq 1 \quad \text{for any } s > 1$$

and

$$g' \left(\log \frac{Re}{|x|} \right) = -\alpha \left(\frac{|x|}{Re} \right)^\alpha, \quad g'' \left(\log \frac{Re}{|x|} \right) = \alpha^2 \left(\frac{|x|}{Re} \right)^\alpha.$$

Using these and (1.5), we have (1.7). \square

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