TORSIONS IN THE COHOMOLOGY OF TORUS ORBIFOLDS

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Abstract. We study torsions in the integral cohomology of a certain family of $2n$-dimensional orbifolds X with actions of the n-dimensional compact torus. Compact simplicial toric varieties are in our family. For a prime number p, we find a necessary condition for the integral cohomology of X to have no p-torsion. Then we prove that the necessary condition is sufficient in some cases. We also give an example of X which shows that the necessary condition is not sufficient in general.

Introduction

A toric variety is a normal complex algebraic variety of complex dimension n with an algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit. A toric variety is not necessarily compact and may have singularity. The famous theorem of Danilov-Jurkiewicz gives an explicit description of the integral cohomology ring of a compact smooth toric variety in terms of the associated fan. It in particular says that the integral cohomology groups are torsion-free and concentrated in even degrees.

The analogous result holds for a compact simplicial toric variety X (simplicial means that X is an orbifold) but with rational coefficients. S. Fischli and A. Jordan studied the integral cohomology groups $H^*(X)$ in their dissertations [5], [9] using spectral sequences. Their results give an explicit computation of $H^k(X)$ and $H^{2n-k}(X)$ for $k \leq 3$ under some conditions. Based on their results, M. Franz developed Maple package torhom [6] to compute those cohomology groups. One can see that $H^*(X)$ has torsion in general while it has no torsion when X is a weighted projective space ([10]). Therefore we are naturally led to ask when $H^*(X)$ has torsion or no torsion.

The orbit space Q of a compact simplicial toric variety X by the restricted action of the n-dimensional compact torus T is a nice manifold with corners (sometimes called a manifold with faces). All faces of Q (even Q itself) are contractible and Q is often homeomorphic to a simple polytope as manifolds with corners. MacPherson showed that X is homeomorphic to the quotient space $(Q \times T)/\sim$ under some equivalence relation $\sim$ defined using the primitive vectors in the one-dimensional cones in the fan of X (see [7]). The one-dimensional cones correspond to the facets of Q so that one can think of the primitive vectors as a map

$$v\colon \{Q_1, Q_2, \dots, Q_m\} \to \mathbb{Z}^n \qquad (Q_i\text{'s are facets of } Q).$$

The map v satisfies some linear independence condition and a map satisfying the condition is called a *characteristic function* on Q (see Definition in Section 1). Note that

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there are many characteristic functions which do not arise from compact simplicial toric varieties.

Bahri-Sarkar-Song [1] consider the quotient space $X(Q, v) = (Q \times T)/\sim$. Although they restrict their concern to Q being a simple polytope, the characteristic function v used to define the equivalence relation $\sim$ is arbitrary; so the quotient space do not necessarily arise from a compact simplicial toric variety. They give a sufficient condition for $H^*(X(Q, v))$ to be torsion-free in terms of Q and v.

In this paper, we also consider the quotient space $X = X(Q, v) = (Q \times T)/\sim$ where v is arbitrary as above but our Q is a connected nice manifold with corners and not necessarily a simple polytope. When Q has a vertex (equivalently X has a T-fixed point), our X is a torus orbifold in the sense of [8]. We give an explicit description of $H^k(X)$ and $H^{2n-k}(X)$ for $k \leq 2$ under some condition on Q. Motivated by the explicit description of $H^{2n-1}(X)$, we introduce a positive integer $\mu(Q_I)$ depending on the characteristic function v for each $Q_I = \bigcap_{i \in I} Q_i$, where I is a subset of $\{1, \dots, m\}$ and we understand $Q_I = Q$ when $I = \emptyset$ and $\mu(Q_I) = 1$ when $Q_I = \emptyset$. The $\mu(Q_I)$'s are all one when X has no singularity. Here is a summary of our results, which follows from Propositions 5.1, 7.1, 7.2 and 7.4.

Theorem. *Let p be a prime number and suppose that every face of Q (even Q itself) is acyclic with $\mathbb{Z}/p$-coefficient. If $H^*(X(Q, v))$ has no p-torsion, then $\mu(Q_I)$ is coprime to p for every Q_I. The converse holds when the face poset of Q is isomorphic to the face poset of one of the following:*

1. *the suspension Σ^n of the $(n-1)$-simplex Δ^{n-1}, i.e. Σ^n is obtained from $\Delta^{n-1} \times [-1, 1]$ by collapsing $\Delta^{n-1} \times \{1\}$ and $\Delta^{n-1} \times \{-1\}$ to a point respectively,*
2. *Δ^n,*
3. *$\Delta^{n-1} \times [-1, 1]$.*

Remark. The n-simplex Δ^n and the prism $\Delta^{n-1} \times [-1, 1]$ can be obtained from the suspension Σ^n by performing a vertex cut once and twice respectively. So, the reader might think that the converse mentioned in the theorem above would hold for Q obtained from Σ^n by performing a vertex cut repeatedly. However, we will see in Section 8 that this is not true for Q obtained from Σ^3 by performing a vertex cut four times.

The paper is organized as follows. In Section 1 we set up notations. In Section 2 we compute $H^{2n-k}(X)$ $(k \leq 2)$ for the quotient space $X = (Q \times T)/\sim$ using the idea in Yeroshkin's paper [14]. Namely, we delete a small neighborhood of the singular set in X to obtain a smooth manifold and investigate the relation of the cohomology groups between X and the smooth manifold. In Section 3 we show that the quotient map $X \to Q$ induces an isomorphism on their fundamental groups when Q has a vertex. In Section 4 we apply the results in Sections 2 and 3 to the case when $n = 2$ and 3. In Section 5 we introduce $\mu(Q_I)$ and find a necessary condition for $H^*(X)$ to have no p-torsion. In Section 6 we recall Theorem on Elementary Divisors and deduce two facts used in Section 7. In Section 7 we prove that the necessary condition obtained in Section 5 is sufficient for Q mentioned in the theorem above. Section 8 gives an example mentioned in the remark above. In the appendix we will observe that a result of Fischli or Jordan on $H^{2n-1}(X)$ and the torsion part of $H^{2n-2}(X)$ agrees with our Proposition 2.2 when X is a compact simplicial toric variety.

1. Setting and notation

In this section, we set up some notations and give some remarks. Let Q be a connected manifold with corners of dimension n (see [4, p.180] for the precise definition of a manifold with corners) Then faces are defined and a codimension-one face is called a facet. We assume that Q is *nice*, which means that every codimension-k face is a connected component of intersections of k facets. The teardrop, which is homeomorphic to the 2-disk, is a manifold with corners but not nice (see [4, p.181]). A simple polytope is a nice manifold with corners and any intersection of faces is connected unless it is empty. However, intersections of faces of a nice manifold with corners are not necessarily connected. For instance, a 2-gon, that is the suspension Σ^2 in the theorem in the Introduction, is a nice manifold with corners but the intersection of the two facets consists of two vertices.

Let S^1 be the unit circle group of the complex numbers $\mathbb{C}$ and T be an n-dimensional connected compact abelian Lie group. As is well-known, T is isomorphic to $(S^1)^n$. We set

$$N := \mathrm{Hom}(S^1, T) \cong \mathbb{Z}^n.$$

Let Q have m facets and we denote them by $Q_1, \dots, Q_m$.

Definition. A function $v\colon \{Q_1, \dots, Q_m\} \to N$ is called a *characteristic function on* Q if it satisfies the following two conditions:

(1) $v(Q_i)$ is primitive for each $i \in [m] := \{1, \dots, m\}$ and
(2) whenever $Q_I = \bigcap_{i\in I} Q_i$ is nonempty for $I \subset [m]$, $v(Q_i)$'s $(i \in I)$ are linearly independent over $\mathbb{Q}$.

We call $v(Q_i)$'s the *characteristic vectors* and abbreviate $v(Q_i)$ as v_i. We denote by $\hat{N}$ the sublattice of N generated by $v_1, \dots, v_m$. Condition (2) above implies that when Q has a vertex, $\operatorname{rank} \hat{N} = n$. It also implies that when $Q_I \neq \emptyset$, the toral subgroup of T generated by $v_i(S^1)$'s $(i \in I)$, denoted by T_I, is of dimension $|I|$ where $|I|$ is the cardinality of I.

To the pair (Q, v) we associate a quotient space

$$X(Q, v) := (Q \times T)/\!\sim$$

with the equivalence relation $\sim$ on the product $Q \times T$ defined by

$$(q, t) \sim (q', t') \text{ if and only if } q = q' \text{ and } t^{-1}t' \in T_I$$

where I is the subset of $[m]$ such that Q_I is the smallest face of Q containing $q = q'$. The space $X(Q, v)$ has a T-action induced from the natural T-action on $Q \times T$. The orbit space of $X(Q, v)$ by the T-action is Q and the quotient map

$$\pi\colon X(Q, v) \to Q = X(Q, v)/T$$

is induced from the projection map $Q \times T \to Q$. A T-fixed point in $X(Q, v)$ corresponds to a vertex of Q, so $X(Q, v)$ has a T-fixed point if and only if Q has a vertex.

If v_i's $(i \in I)$ are a part of a basis of N for every I with $Q_I \neq \emptyset$, then $X(Q, v)$ is a manifold but otherwise $X(Q, v)$ is an orbifold. The singularity of $X(Q, v)$ lies in the union of $\pi^{-1}(Q_I)$ over all I with $|I| \geq 2$.

As mentioned in the Introduction, if X is a compact simplicial toric variety of complex dimension n so that X has an algebraic action of $(\mathbb{C}^*)^n$ having a dense orbit, then the orbit space Q of X by the compact n-dimensional subtorus T of $(\mathbb{C}^*)^n$ is a nice manifold with corners and X is homeomorphic to $X(Q,v)$ where v_i's are primitive edge vectors of the fan associated to X. Moreover, faces of Q (even Q itself) are all contractible, which follows from the existence of the residual action of $(\mathbb{C}^*)^n/T$ on $Q = X/T$.

2. $H^{2n-k}(X(Q,v))$ for $k \le 2$

In this section, we abbreviate $X(Q,v)$ as X and all (co)homology groups will be taken with $\mathbb{Z}$-coefficients unless otherwise stated. When $n=1$, Q is a closed interval if Q has a vertex and a circle otherwise, and X is homeomorphic to S^2 or a torus accordingly. We will assume $n \ge 2$ in this section. Remember that $\pi\colon X \to Q$ is the quotient map.

Let $Q^{(n-2)}$ be the union of Q_I over all I with $|I| \ge 2$ and we assume $Q^{(n-2)} \neq \emptyset$. The singular set of X lies in $\pi^{-1}(Q^{(n-2)})$ as remarked in Section 1. Let Q' be a "small closed tubular neighborhood" of $Q^{(n-2)}$ of Q and set $X' := \pi^{-1}(Q')$.

Lemma 2.1. *$H^{2n-k}(X) \cong H_k(X \backslash \operatorname{Int} X')$ for $k \le 2$.*

Proof. Note that $H^r(X') = 0$ for $r \ge 2n-3$ because X' is homotopy equivalent to $\pi^{-1}(Q^{(n-2)})$ and $\dim \pi^{-1}(Q^{(n-2)}) = 2n-4$. Therefore, the exact sequence in cohomology for the pair (X, X') yields an isomorphism

$$H^{2n-k}(X, X') \cong H^{2n-k}(X) \quad \text{for } k \le 2. \tag{2.1}$$

On the other hand,

$$\begin{aligned} H^{2n-k}(X, X') &\cong H^{2n-k}(X \backslash \operatorname{Int} X', \partial X') \quad \text{by excision} \\ &\cong H_k(X \backslash \operatorname{Int} X') \quad \text{by Poincaré-Lefschetz duality.} \end{aligned} \tag{2.2}$$

(Note that $X \backslash \operatorname{Int} X'$ is a manifold with boundary $\partial X'$.) The lemma follows from (2.1) and (2.2). $\square$

Proposition 2.2. *$H^{2n}(X) \cong \mathbb{Z}$ and $H^{2n-1}(X) \cong H_1(Q) \oplus N/\hat{N}$. If $H_1(Q_i) = 0$ for every i, then*

$$H^{2n-2}(X) \cong \mathbb{Z}^{m - \operatorname{rank} \hat{N}} \oplus H_2(Q) \oplus \big(H_1(Q) \otimes H_1(T)\big) \oplus \big(\wedge^2 N / \hat{N} \wedge N\big).$$

Remark. When Q has a vertex, $\operatorname{rank} \hat{N} = n$ as remarked in Section 1. Moreover, when Q has a vertex and $n=2$, the last term $\wedge^2 N/\hat{N} \wedge N$ above is zero. Indeed, since we may assume $N = \mathbb{Z}^2$ and $\hat{N} = \langle e_1, a e_2 \rangle$ with some integer a, $\hat{N} \wedge N = \langle e_1 \wedge e_2 \rangle = \wedge^2 N$, where $\{e_1, e_2\}$ denotes the standard base of $\mathbb{Z}^2$.

Proof. The statement for $H^{2n}(X)$ follows immediately from Lemma 2.1.

We shall prove the statement for $H^{2n-1}(X)$. Let $Q^0 := (\operatorname{Int} Q) \cap (Q \backslash Q')$ and Q^1 be the intersection of $(Q \backslash Q')$ and a small open neighborhood of ∂Q in Q. Since

$$\pi^{-1}(Q^0) \simeq Q \times T, \quad \pi^{-1}(Q^1) \simeq \coprod_{i=1}^m \big(Q_i \times T/v_i(S^1)\big),$$

$$\pi^{-1}(Q^0) \cap \pi^{-1}(Q^1) \simeq \coprod_{i=1}^m (Q_i \times T), \quad \pi^{-1}(Q^0 \cup Q^1) = X \backslash X',$$

the Mayer-Vietoris exact sequence in homology for the triple $(X\backslash X', \pi^{-1}(Q^0), \pi^{-1}(Q^1))$ yields the following exact sequence:

$$\begin{aligned}
&\bigoplus_{i=1}^m H_2(Q_i\times T) \xrightarrow{f_2} H_2(Q\times T)\oplus \bigoplus_{i=1}^m H_2(Q_i\times T/v_i(S^1)) \to H_2(X\backslash X') \\
&\to \bigoplus_{i=1}^m H_1(Q_i\times T) \xrightarrow{f_1} H_1(Q\times T)\oplus \bigoplus_{i=1}^m H_1(Q_i\times T/v_i(S^1)) \to H_1(X\backslash X') \\
&\to \bigoplus_{i=1}^m H_0(Q_i\times T) \xrightarrow{f_0} H_0(Q\times T)\oplus \bigoplus_{i=1}^m H_0(Q_i\times T/v_i(S^1)).
\end{aligned} \tag{2.3}$$

As is easily seen, f_0 is injective; so

$$H_1(X\backslash X') \cong \operatorname{coker} f_1. \tag{2.4}$$

We write f_1 as (ψ_1, φ_1) according to the decomposition of the target space. Since

$$\varphi_1\colon \bigoplus_{i=1}^m H_1(Q_i\times T) \to \bigoplus_{i=1}^m H_1(Q_i\times T/v_i(S^1)),$$

which is f_1 composed with the projection on the second factor, is surjective, one has

$$\operatorname{coker} f_1 \cong H_1(Q\times T)/\psi_1(\ker\varphi_1). \tag{2.5}$$

Since $H_1(Y\times T) = H_1(Y)\oplus H_1(T)$ for any topological space Y, elements in $\ker\varphi_1$ are of the form $(c_1v_1,\dots,c_mv_m)$ with integers c_i, where $H_1(T)$ is identified with $N = \operatorname{Hom}(S^1,(S^1)^n)$ in a natural way. It follows that

$$H_1(Q\times T)/\psi_1(\ker\varphi_1) \cong H_1(Q)\oplus N/\hat{N}. \tag{2.6}$$

The statement for $H^{2n-1}(X)$ in the proposition follows from (2.4), (2.5), (2.6) and Lemma 2.1.

The computation of $H^{2n-2}(X)$ is similar to that of $H^{2n-1}(X)$. We write f_2 as (ψ_2,φ_2) similarly to f_1. As is easily seen, $\ker f_1$ is a free abelian group of rank $m - \operatorname{rank}\hat{N}$; so it follows from (2.3) that

$$H_2(X\backslash X') \cong \mathbb{Z}^{m-\operatorname{rank}\hat{N}}\oplus \operatorname{coker} f_2. \tag{2.7}$$

Similarly to φ_1, the map

$$\varphi_2\colon \bigoplus_{i=1}^m H_2(Q_i\times T) \to \bigoplus_{i=1}^m H_2(Q_i\times T/v_i(S^1)) \tag{2.8}$$

is surjective; so

$$\operatorname{coker} f_2 \cong H_2(Q\times T)/\psi_2(\ker\varphi_2). \tag{2.9}$$

Here,

$$H_2(Y\times T) = H_2(Y)\oplus\big(H_1(Y)\otimes H_1(T)\big)\oplus H_2(T) \tag{2.10}$$

for any topological space Y by the Künneth formula. Therefore, since $H_1(Q_i)=0$ by assumption, it follows from (2.8) and (2.10) that $\ker\varphi_2$ is contained in $\bigoplus_{i=1}^m H_2(T)$. We note that $H_2(T)$ and $H_2(T/v_i(S^1))$ can be identified with $\wedge^2 N$ and $\wedge^2(N/\langle v_i\rangle)$ respectively and the kernel of the projection $\wedge^2 N\to\wedge^2(N/\langle v_i\rangle)$ is $\langle v_i\rangle\wedge N$. Therefore

$$\operatorname{coker} f_2 \cong H_2(Q)\oplus\big(H_1(Q)\otimes H_1(T)\big)\oplus\big(\wedge^2 N/\hat{N}\wedge N\big)$$

This together with (2.7) and (2.9) proves the statement for $H^{2n-2}(X)$ in the proposition. $\square$

3. Fundamental groups

For a subset I of $[m]$, we define

$$T_I^m := \{(h_1, \ldots, h_m) \in T^m \mid h_j = 1 \quad (\forall j \notin I)\}.$$

and consider a space

$$\mathcal{Z}_Q := (Q \times T^m)/{\sim_e}$$

where $\sim_e$ is the equivalence relation on the product $Q \times T^m$ defined by

$$(q, s) \sim_e (q', s') \text{ if and only if } q = q' \text{ and } t^{-1}t' \in T_I^m$$

and I is the subset of $[m]$ such that Q_I is the smallest face of Q containing $q = q'$. One can check that $\mathcal{Z}_Q$ is a manifold.

Lemma 3.1. *The projection map $\kappa\colon \mathcal{Z}_Q \to Q$ induces an isomorphism $\kappa_*\colon \pi_1(\mathcal{Z}_Q) \cong \pi_1(Q)$ on the fundamental groups.*

Remark. When Q is a simple polytope, $\mathcal{Z}_Q$ is called a *moment-angle manifold* and known to be 2-connected (see [3]).

Proof. An open tubular neighborhood of Q_i in Q can be identified with $Q_i \times \mathbb{R}_{\geq 0}$. Then $\kappa^{-1}(Q_i \times \{1\}) \to \kappa^{-1}(Q_i)$ is a principal S^1-bundle and the total space E_i of the associated complex line bundle can be identified with an open tubular neighborhood of $Z_i := \kappa^{-1}(Q_i)$ in $\mathcal{Z}_Q$. Therefore, if a continuous map $f\colon S^1 \to \mathcal{Z}_Q$ meets Z_i, then we slightly push f in the fiber direction of E_i so that the deformed f does not meet Z_i. Applying this deformation to f for every i, we see that f is homotopic to a continuous map whose image lies in $\kappa^{-1}(\operatorname{Int} Q) = \operatorname{Int} Q \times T^m$. This means that the inclusion map $\iota\colon \operatorname{Int} Q \times T^m \to \mathcal{Z}_Q$ induces an epimorphism

$$\iota_*\colon \pi_1(\operatorname{Int} Q \times T^m) = \pi_1(\operatorname{Int} Q) \times \pi_1(T^m) \to \pi_1(\mathcal{Z}_Q).$$

Since $\operatorname{Int} Q$ is homotopy equivalent to Q, we may replace $\operatorname{Int} Q$ by Q above and we have a sequence

$$\pi_1(Q) \times \pi_1(T^m) \xrightarrow{\iota_*} \pi_1(\mathcal{Z}_Q) \xrightarrow{\kappa_*} \pi_1(Q), \tag{3.1}$$

where the composition $\kappa_* \circ \iota_*$ agrees with the projection on the first factor, so that the kernel of ι_* is contained in the second factor $\pi_1(T^m)$.

Let S_i be the i-th S^1-factor of T^m and choose a point $q_i \in (Q_i \times \{1\}) \cap \operatorname{Int} Q$. Then $\iota(\{q_i\} \times S_i)$ is a fiber of the principal S^1-bundle $\kappa^{-1}(Q_i \times \{1\}) \to Z_i = \kappa^{-1}(Q_i)$, so it shrinks to a point in Z_i. Therefore $\pi_1(T^m)$ is in the kernel of the epimorphism ι_* and this implies the lemma. $\square$

We recall a result from Bredon's book [2].

Lemma 3.2. [2, Corollary 6.3 in p.91]. *If X is arcwise connected G-space, G compact Lie, and if there is an orbit which is connected (e.g., G connected or $X^G \neq \emptyset$), then the quotient map $X \to X/G$ induces an epimorphism on their fundamental groups.*

The characteristic map $v\colon \{Q_1, \ldots, Q_m\} \to \operatorname{Hom}(S^1, T)$ defines a homomorphism $T^m \to T$, denoted v again. Note that $v(T^m)$ is a subtorus of T of dimension $\operatorname{rank} \hat{N}$, in particular, v is surjective if and only if $\operatorname{rank} \hat{N} = \operatorname{rank} N$ (this is the case when Q has a vertex). The product map $id \times v \colon Q \times T^m \to Q \times T$ induces a continuous map

$$V \colon \mathcal{Z}_Q \to X = X(Q, v).$$

Proposition 3.3. *If Q has a vertex, then $\pi_* \colon \pi_1(X) \cong \pi_1(Q)$.*

Proof. We have a sequence

$$\kappa_* = \pi_* \circ V_* \colon \pi_1(\mathcal{Z}_Q) \xrightarrow{V_*} \pi_1(X) \xrightarrow{\pi_*} \pi_1(Q).$$

Since κ_* is an isomorphism by Lemma 3.1, it suffices to prove that V_* is surjective.

Since Q has a vertex, $\operatorname{rank} \hat{N} = \operatorname{rank} N$ and the homomorphism $v \colon T^m \to T$ is surjective. Then one can see that

$$X = \mathcal{Z}_Q / \ker v.$$

Since $\hat{N}$ is a sublattice of N of finite index, there is a finite covering homomorphism $\rho \colon \hat{T} \to T$ corresponding to $\hat{N}$, where $\hat{T}$ is also a compact connected abelian Lie group of dimension n (precisely speaking, $\rho_*(\pi_1(\hat{T})) = \hat{N}$ when N is regarded as $\pi_1(T)$) and the characteristic function v uniquely determines a characteristic function $\hat{v} \colon \{Q_1, \dots, Q_m\} \to \operatorname{Hom}(S^1, \hat{T})$ such that $\rho_*(\hat{v}(Q_i)) = v(Q_i)$ for any i. Then we have

$$\hat{X} := X(Q, \hat{v}) = (Q \times \hat{T})/\sim$$

and $\hat{v}$ induces a homomorphism $T^m \to \hat{T}$, denoted $\hat{v}$ again similarly to v, and $\hat{X} = \mathcal{Z}_Q / \ker \hat{v}$. Moreover, we have $X = \hat{X} / \ker \rho$. Namely, the quotient map $V \colon \mathcal{Z}_Q \to X$ factors as the composition of two quotient maps

$$\mathcal{Z}_Q \xrightarrow{\alpha} \mathcal{Z}_Q / \ker \hat{v} = \hat{X} \xrightarrow{\beta} \hat{X} / \ker \rho = X.$$

Theorem on Elementary Divisors (see Section 6) implies that since $\hat{v}(Q_i)$'s span $\hat{N}$, the homomorphism $\hat{v} \colon T^m \to \hat{T}$ composed with a suitable automorphism of T^m can be viewed as a projection map if we take a suitable identification of $\hat{T}$ with T^n ; so $\ker \hat{v}$ is connected and hence $\alpha_* \colon \pi_1(\mathcal{Z}_Q) \to \pi_1(\hat{X})$ is surjective by Lemma 3.2. The action of $\hat{T}$ on $\hat{X}$ has a fixed point since Q has a vertex and $\ker \rho$ is contained in $\hat{T}$, so the action of $\ker \rho$ on $\hat{X}$ has a fixed point. Therefore $\beta_* \colon \pi_1(\hat{X}) \to \pi_1(X)$ is also surjective again by Lemma 3.2. □

Corollary 3.4. *If Q has a vertex and $H_1(Q) = H_2(Q) = 0$, then $H^1(X) = 0$ and $H^2(X) \cong \mathbb{Z}^{m-n}$.*

Proof. By Proposition 3.3, $\pi_1(X) \cong \pi_1(Q)$ and hence $H_1(X) \cong H_1(Q)$. Therefore $H_1(X) = 0$ since $H_1(Q) = 0$ by assumption and hence $H^1(X) = 0$ and $H^2(X)$ has no torsion by the universal coefficient theorem. On the other hand, since X is an orbifold, Poincaré duality holds with $\mathbb{Q}$-coefficients. Therefore the rank of $H^2(X)$ is equal to that of $H^{2n-2}(X)$, that is $m - n$ by Proposition 2.2 and its subsequent remark. □

4. Low dimensional cases

A nice manifold with corners Q is called *face-acyclic* ([11]) if every face of Q (even Q itself) is acyclic. We shall apply the previous results when Q is face-acyclic and $n = \dim Q$ is 2 or 3. The following corollary follows from Proposition 2.2 and Corollary 3.4.

Corollary 4.1. *Suppose that Q is face-acyclic and* $\dim Q = 2$*, that is, Q is an m-gon* $(m \geq 2)$*. Then we have*

$$H^j(X) \cong \begin{cases} \mathbb{Z} & (j = 0, 4) \\ \mathbb{Z}^{m-2} & (j = 2) \\ N/\hat{N} & (j = 3) \\ 0 & (\text{otherwise}). \end{cases}$$

Example. Let a be a positive integer. Take Q to be a 2-simplex, $N = \mathbb{Z}^2$ and

$$v_1 = (2a, 1),\ v_2 = (0, 1),\ v_3 = (-a, -1).$$

Then $\hat{N} = \langle ae_1, e_2 \rangle$ and $N/\hat{N} \cong \mathbb{Z}/a$. The space X is not a weighted projective space when $a \geq 2$ since it has torsion in cohomology, where $\{e_1, e_2\}$ denotes the standard base of $\mathbb{Z}^2$ as before.

Corollary 4.2. *Suppose that Q is face-acyclic and* $\dim Q = 3$*. Then*

$$H^j(X) \cong \begin{cases} \mathbb{Z} & (j = 0, 6) \\ \mathbb{Z}^{m-3} & (j = 2) \\ 0 \textit{ or some torsion group} & (j = 3) \\ \mathbb{Z}^{m-3} \oplus \wedge^2 N/(\hat{N} \wedge N) & (j = 4) \\ N/\hat{N} & (j = 5) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Since Q is face-acyclic, one can easily see that Q must have a vertex; so all the statements except for $j = 3$ follows from Proposition 2.2 and Corollary 3.4. In order to prove the statement for $j = 3$, it suffices to show $H^3(X; \mathbb{Q}) = 0$ and this is equivalent to showing that the euler characteristic of X is $2m - 4$ (note that we know the rank of $H^j(X)$ except for $j = 3$).

Since Q is face-acyclic and of dimension 3, the boundary of Q is a 2-sphere, every 2-face of Q is a 2-disk and the number of 2-faces is m by definition. Let V be the number of vertices of Q. Then the number of edges of Q is $3V/2$ and hence we obtain an identity $V - 3V/2 + m = 2$ by Euler's formula, which implies $V = 2m - 4$. On the other hand, it is known that the euler characteristic of X is equal to that of the T-fixed point set X^T (see [2, Theorem 10.9 in p.163]). In our case X^T is isolated and corresponds to the vertices of Q. Therefore, the euler characteristic of X is equal to V, that is $2m - 4$. □

Example. It happens that $\hat{N} \wedge N = \wedge^2 N$ even if $\hat{N} \neq N$. For instance, take Q to be a 3-simplex, $N = \mathbb{Z}^3$ and

$$v_1 = (0, 0, 1),\ v_2 = (2, 0, 1),\ v_3 = (0, 1, 1),\ v_4 = (-2, -1, -1).$$

Then

$$\hat{N} = \langle 2e_1, e_2, e_3 \rangle, \quad \hat{N} \wedge N = \langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 \rangle = \wedge^2 N,$$

where $\{e_1, e_2, e_3\}$ denotes the standard base of $\mathbb{Z}^3$.

Corollary 4.2 says that if $\hat{N} = N$, then $H^j(X)$ has no torsion except $j = 3$. However, $H^3(X)$ can be nontrivial (so, a nontrivial torsion group) when $\hat{N} = N$. We shall give such an example below. One can also find many such examples using Maple package torhom.

Example. Let a be a positive integer and take the following five primitive vectors in $\mathbb{Z}^3$:

$$\begin{aligned} &v_+ = (0,0,1),\\ &v_1 = (2a,1,0),\ v_2 = (0,1,0),\ v_3 = (-a,-1,0),\\ &v_- = (1,0,-1). \end{aligned}$$

Then $\hat{N} = N$. We consider the complete simplicial fan Δ having the following six 3-dimensional cones

$$\angle v_+v_1v_2,\ \angle v_+v_1v_3,\ \angle v_+v_2v_3,\ \angle v_-v_1v_2,\ \angle v_-v_1v_3,\ \angle v_-v_2v_3$$

where $\angle v_\epsilon v_i v_j$ ($\epsilon \in \{+,-\}$, $i,j \in \{1,2,3\}$) denotes the cone spanned by v_ϵ, v_i and v_j. Let X be the compact simplicial toric variety associated to the fan Δ. Let ρ be the projection of $\mathbb{R}^3$ on the line $\mathbb{R}$ corresponding to the last coordinates of $\mathbb{R}^3$. Then the vectors v_1, v_2, v_3 are in the kernel of ρ and $\rho(v_\pm)$ are primitive vectors and determine the complete 1-dimensional fan. This means that we have a fibration $F \to X \to \mathbb{C}P^1$ where the fiber F is the compact simplicial toric variety associated to the fan obtained by projecting the fan Δ on the plane $\mathbb{R}^2$ corresponding to the first two coordinates of $\mathbb{R}^3$. The E_2-terms of the Serre spectral sequence of the fibration are

$$E_2^{p,q} = H^p(\mathbb{C}P^1; H^q(F))$$

and $E_2^{p,q} = 0$ unless $p = 0,2$ and $q = 0,2,3,4$ by Corollary 4.1. Therefore all the differentials except

$$d_2^{0,3}\colon E_2^{0,3} \to E_2^{2,2} \quad \text{and} \quad d_2^{0,4}\colon E_2^{0,4} \to E_2^{2,3}$$

are trivial. Here, $E_2^{0,3} = H^0(\mathbb{C}P^1; H^3(F)) = H^3(F)$ is trivial or a torsion group by Corollary 4.1 while $E_2^{2,2} = H^2(\mathbb{C}P^1; H^2(F)) = H^2(F)$ is a free abelian group again by Corollary 4.1, so $d_2^{0,3}$ must be trivial. Therefore $E_2^{0,3} = E_\infty^{0,3}$. Since $E_2^{p,q}$ with $p+q = 3$ vanishes unless $(p,q) = (0,3)$, we obtain an isomorphism $H^3(X) \cong H^3(F)$. Here $H^3(F) \cong \mathbb{Z}/a$ again by Corollary 4.1 (see Example after Corollary 4.1) and hence we have $H^3(X) \cong \mathbb{Z}/a$. On the other hand, since $\hat{N} = N$ as remarked above, $H^j(X)$ has no torsion for $j \neq 3$ by Corollary 4.2.

5. A necessary condition for no p-torsion

Let I be a subset of $[m]$ with $Q_I \neq \emptyset$. Although Q_I is not necessarily connected, we understand that Q_I stands for a connected component of Q_I in this section for notational convenience. Then the characteristic function v associates a characteristic function v_I on Q_I as follows. Since v_i's ($i \in I$) are linearly independent over $\mathbb{Q}$, they span a $|I|$-dimensional linear subspace of $N \otimes \mathbb{R}$ and its intersection with N is a rank $|I|$ sublattice of N, denoted N_I. Then $N(I) := N/N_I$ is a free abelian group of rank $n - |I|$ and we denote the projection map from N to $N(I)$ by π_I. If $Q_I \cap Q_j$ for $j \in [m]\backslash I$ is nonempty, then its connected components are facets of Q_I, and any facet of Q_I is of this form. The element $\pi_I(v_j) \in N(I)$ is not necessarily primitive and we define $v_I(Q_I \cap Q_j)$ to be the primitive vector in $N(I)$ which has the same direction as $\pi_I(v_j)$, where $Q_I \cap Q_j$ also stands for a connected component of $Q_I \cap Q_j$. Then one can see that v_I is a characteristic function on Q_I. Similarly to $\hat{N}$, one can define a sublattice $\hat{N}(I)$ of $N(I)$ using v_I. We allow $I = \emptyset$ and understand $Q_\emptyset = Q$, $N(\emptyset) = N$

and $\hat{N}(\emptyset) = \hat{N}$. We define

$$\mu(Q_I) := \begin{cases} |N(I)/\hat{N}(I)| & \text{when } Q_I \neq \emptyset, \\ 1 & \text{when } Q_I = \emptyset. \end{cases}$$

Here $|N(I)/\hat{N}(I)|$ is not necessarily finite. For instance, take $Q = S^1 \times [-1, 1]$ and assign characteristic vectors $(1, 0)$ and $(-1, 0)$ to the facets $S^1 \times \{1\}$ and $S^1 \times \{-1\}$ respectively. Then $N/\hat{N}$ is an infinite cyclic group and hence $|N(I)/\hat{N}(I)|$ is infinite for $I = \emptyset$. One can easily construct a similar example such that $|N(I)/\hat{N}(I)|$ is infinite for some $I \neq \emptyset$.

Remark. When $|I| = n$, $N(I) = \{0\}$; so $\mu(Q_I) = 1$. When $|I| = n - 1$, $N(I)$ is of rank one and $\hat{N}(I)$ is generated by a primitive vector; so $\hat{N}(I) = N(I)$ and hence $\mu(Q_I) = 1$ in this case too. Another case which ensures $\mu(Q_I) = 1$ is the following. Let q be a vertex of Q. Then there is a subset J of $[m]$ with $|J| = n$ such that $q \in Q_J$. If $\{v_j\}_{j \in J}$ is a base of N, then $\mu(Q_I) = 1$ for every subset I of J, which easily follows from the definition of $\mu(Q_I)$.

We note that for a prime number p, $H^*(X(Q, v); \mathbb{Z})$ has no p-torsion if and only if $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$, which follows from the universal coefficient theorem (see [12, Corollary 56.4]).

Proposition 5.1. *If $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$, then $H_1(Q_I; \mathbb{Z}/p) = 0$ and $\mu(Q_I)$ is finite and coprime to p for every I.*

Proof. We abbreviate $X(Q, v)$ as X as before. Since $H^{odd}(X; \mathbb{Z}/p) = 0$, we have $H^{odd}(X^G; \mathbb{Z}/p) = 0$ for every p-subgroup G of T_I by repeated use of [2, Theorem 2.2 in pp.376-377]. For a positive integer k, let G_k be the p-subgroup of T_I consisting of all elements of order at most p^k. Then $G_k \subset G_{k'}$ for $k \leq k'$ and the union $\bigcup_{k=1}^{\infty} G_k$ is dense in T_I. Therefore $X^{G_k} = X^{T_I}$ if k is sufficiently large.[1] Since $X_I = \pi^{-1}(Q_I)$ is a connected component of X^{T_I}, this shows that $H^{odd}(X_I; \mathbb{Z}/p) = 0$. But $H^{2(n-|I|)-1}(X_I)$ is isomorphic to $H_1(Q_I) \oplus N(I)/\hat{N}(I)$ by Proposition 2.2 and hence the universal coefficient theorem implies the proposition. □

When $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$, Proposition 5.1 gives a constraint on the topology of Q_I, that is $H_1(Q_I; \mathbb{Z}/p) = 0$. It is proved in [11] that if $X(Q, v)$ is a manifold and $H^{odd}(X(Q, v); \mathbb{Z}) = 0$, then Q is face-acyclic. This implies that there will be more constraints on the topology of Q_I when $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$, to be more precise, we expect that Q is *face p-acyclic* which means that (every component of) Q_I is acyclic with $\mathbb{Z}/p$-coefficients for every I. Therefore, in order to consider the converse of Proposition 5.1, it would be appropriate to assume that Q is face p-acyclic. We will prove in Section 7 that the converse holds in some cases while we will see in Section 8 that the converse does not hold in general.

[1]Detailed explanation about this assertion. Since the set of isotropy groups of X is finite, there is a positive integer r such that $X^{G_k} = X^{G_r}$ for every $k \geq r$. Since G_r is a subgroup of T_I, we have $X^{G_r} \supset X^{T_I}$. We shall prove the opposite inclusion. Let $x \in X^{G_r}$. The isotropy subgroup T_x at x contains G_k for every $k \geq r$ because $X^{G_k} = X^{G_r}$ but since T_x is a closed subgroup of T, T_x must contain the closure of $\cup_{k=r}^{\infty} G_k$, that is T_I. Therefore $x \in X^{T_I}$ and hence $X^{G_r} = X^{T_I}$.

6. Theorem on Elementary Divisors

We recall Theorem on Elementary Divisors which plays a role in the next section, see [13] for the details.

Theorem 6.1 (Theorem on Elementary Divisors)**.** *Let N' be a submodule of rank n' in $N=\mathbb{Z}^n$. Then there are bases $\{u'_1,\dots,u'_{n'}\}$ of N' and $\{u_1,\dots,u_n\}$ of N such that $u'_i=\epsilon_i u_i$ with some integer ϵ_i for $i=1,2,\dots,n'$ and $\epsilon_1|\epsilon_2|\dots|\epsilon_{n'}$. Moreover if $A=(a_1,\dots,a_k)$ is an $n\times k$ integer matrix whose column vectors $a_1,\dots,a_k$ generate N' and*

$$\delta_i := \gcd\{\det B \mid B \text{ is an } i\times i \text{ submatrix of } A\},$$

then $\delta_i=\delta_{i-1}\epsilon_i$ for $i=1,2,\dots,n'$. In particular, if $n'=n$, then $\delta_n=|N/N'|$.

We deduce two facts from Theorem 6.1.

Lemma 6.2. *Let A be an $n\times n$ integer matrix of rank n and $\tilde{A}\colon \mathbb{R}^n/\mathbb{Z}^n\to\mathbb{R}^n/\mathbb{Z}^n$ be the epimorphism induced from A. Then $\ker\tilde{A}\cong\operatorname{coker}A$.*

Proof. By Theorem 6.1 we may think of A as the diagonal matrix with diagonal entries $\epsilon_1,\dots,\epsilon_n$. Then one easily sees that $\ker\tilde{A}$ and $\operatorname{coker}A$ are both isomorphic to $\prod_{i=1}^n\mathbb{Z}/\epsilon_i$, proving the lemma. □

Let $a_1,\dots,a_{n+1}$ be elements of $\mathbb{Z}^n$ which generate a sublattice $\langle a_1,\dots,a_{n+1}\rangle$ of rank n and set $d_i:=|\det(a_j)_{j\neq i}|$ for $i\in[n+1]$. It follows from Theorem 6.1 that

$$\delta_n=\gcd(d_1,\dots,d_{n+1})=|\mathbb{Z}^n/\langle a_1,\dots,a_{n+1}\rangle|. \tag{6.1}$$

Suppose that a_{n+1} is primitive. Let $\bar{a}_k$ $(k\neq n+1)$ be the projection image of a_k on $\mathbb{Z}^n/\langle a_{n+1}\rangle$ and let a'_k be the primitive vector in the quotient lattice $\mathbb{Z}^n/\langle a_{n+1}\rangle$ which has the same direction as $\bar{a}_k$ when $\bar{a}_k$ is nonzero and a'_k be the zero vector when so is $\bar{a}_k$. Set $d'_j:=\det(a'_1,\dots,\widehat{a'_j},\dots,a'_n)$. With this understood we have the following.

Lemma 6.3. $\gcd(d_1,\dots,d_n)\big|d_{n+1}$*, in other words,* $\gcd(d_1,\dots,d_n)=\gcd(d_1,\dots,d_{n+1})$*. Moreover,* $\gcd(d'_1,\dots,d'_n)\big|\gcd(d_1,\dots,d_{n+1})$*.*

Proof. Since a_{n+1} is primitive, we may assume that $a_{n+1}=(0,\dots,0,1)^T$ by Theorem 6.1. We have

$$d_{n+1}=|\det(a_1,\dots,a_n)|=\Big|\sum_{j=1}^n a_j^n\tilde{a}_j^n\Big| \tag{6.2}$$

where a_j^n is the (n,j) entry of the matrix $(a_1,\dots,a_n)$ and $\tilde{a}_j^n$ is its cofactor. Since $a_{n+1}=(0,\dots,0,1)^T$, $\tilde{a}_j^n$ agrees with $d_j=|\det(a_1,\dots,\widehat{a_j},\dots,a_{n+1})|$ up to sign. Therefore $\tilde{a}_j^n$ is divisible by $\gcd(d_1,\dots,d_n)$ for every j and this together with (6.2) implies the former statement in the lemma.

Since $a_{n+1}=(0,\dots,0,1)^T$, $\mathbb{Z}^n/\langle a_{n+1}\rangle$ can naturally be identified with $\mathbb{Z}^{n-1}$ and we have

$$d_j=|\det(a_1,\dots,\widehat{a_j},\dots,a_{n+1})|=|\det(\bar{a}_1,\dots,\widehat{\bar{a}_j},\dots,\bar{a}_n)| \quad \text{for } j=1,2,\dots,n$$

where $\bar{a}_k$ $(k=1,2,\dots,n)$ be the projection image of a_k on $\mathbb{Z}^n/\langle a_{n+1}\rangle=\mathbb{Z}^{n-1}$. Since $\bar{a}_k$ is a positive scalar multiple of a'_k, $d'_j=|\det(a'_1,\dots,\widehat{a'_j},\dots,a'_n)|$ divides the latter term above. This together with the former statement in the lemma implies the latter statement in the lemma. . □

7. Converse of Proposition 5.1 in three cases

In this section we show that the converse of Proposition 5.1 holds when Q is face p-acyclic and has the same face poset as one of the following:

Case 1: the suspension Σ^n of an $(n-1)$-simplex Δ^{n-1} (see the Introduction),
Case 2: the n-simplex Δ^n,
Case 3: the prism $\Delta^{n-1}\times[-1,1]$.

Let q be a vertex of Q. Then q lies in Q_I for some $I\subset[m]$ with $|I|=n$. We set

$$d_Q(q) := |\det(v_i)_{i\in I}|$$

where $v_i=v(Q_i)$ as before.

Case 1. In this case Q has two vertices, say q and q', and $d_Q(q)=d_Q(q')=\mu(Q)$.

Proposition 7.1. *Suppose that Q is face p-acyclic, has the face poset as Σ^n and $\mu(Q)$ is coprime to p. Then $X(Q,v)$ has the same cohomology as S^{2n} with $\mathbb{Z}/p$-coefficients, in particular $H^{odd}(X(Q,v);\mathbb{Z}/p)=0$.*

Proof. Let $T^n=(S^1)^n$. Then $\mathrm{Hom}(S^1,T^n)$ is naturally isomorphic to $\mathbb{Z}^n$ and we identify them. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Z}^n$ and $e\colon\{Q_1,\dots,Q_n\}\to\mathbb{Z}^n=\mathrm{Hom}(S^1,T^n)$ be the characteristic function assigning e_i to Q_i. Then we have a T^n-space $X(Q,e)$ which is actually a manifold because $\{e_i\}_{i=1}^n$ is a basis of $\mathbb{Z}^n$.

The characteristic vectors $v_i\in N=\mathrm{Hom}(S^1,T)$ define an epimorphism $\tilde v\colon T^n\to T$ sending $(h_1,\dots,h_n)$ to $\prod_{i=1}^n v_i(h_i)$. One can see that the surjective map from $Q\times T^n$ to $Q\times T$ sending (q,t) to $(q,\tilde v(t))$ descends to a $\tilde v$-equivariant map from $X(Q,e)$ to $X(Q,v)$ and further descends to a homeomorphism

$$X(Q,e)/\ker\tilde v\approx X(Q,v).$$

Here $|\ker\tilde v|=|N/\hat N|$ by Lemma 6.2 and it is coprime to p by assumption. Moreover, since $\ker\tilde v$ is a subgroup of the connected group T^n acting on $X(Q,e)$, the induced action of $\ker\tilde v$ on $H^*(X(Q,e);\mathbb{Z}/p)$ is trivial. Therefore we have

$$H^*(X(Q,e)/\ker\tilde v;\mathbb{Z}/p)\cong H^*(X(Q,e);\mathbb{Z}/p)$$

(see [2, Theorem 2.4 in p.120]) and hence it suffices to prove that $X(Q,e)$ has the same cohomology as S^{2n} with $\mathbb{Z}/p$-coefficients.

Since Q has the same face poset as Σ^n and every face of Σ^n is contractible, there is a face preserving map $f\colon Q\to\Sigma^n$ which induces an isomorphism on the face posets. Since Q is face p-acyclic, f induces an isomorphism on cohomology with $\mathbb{Z}/p$-coefficients at each face. Similarly to the definition of e, one has a characteristic function on Σ^n, also denoted by e. Then the map from $Q\times T^n$ to $\Sigma^n\times T^n$ sending (q,t) to $(f(q),t)$ descends to a map

$$X(Q,e)\to X(\Sigma^n,e)$$

which induces an isomorphism on cohomology with $\mathbb{Z}/p$-coefficients. Since $X(\Sigma^n,e)$ is homeomorphic to S^{2n}, this proves the desired result. □

Case 2. Since Q has the same face poset as the n-simplex Δ^n, Q has $n+1$ facets $Q_1,\dots,Q_{n+1}$ and $n+1$ vertices $q_1,\dots,q_{n+1}$. We number them in such a way that q_i is the unique vertex not contained in Q_i. It follows from (6.1) and Lemma 6.3 that

$$\begin{aligned}&\mu(Q)=\gcd(d_Q(q_1),\dots,d_Q(q_{n+1}))=\gcd(d_Q(q_1),\dots,\widehat{d_Q(q_i)},\dots,d_Q(q_{n+1})),\\&\mu(Q_i)|\mu(Q)\quad\text{for any } i\in[n+1].\end{aligned}\tag{7.1}$$

Proposition 7.2. *Suppose that Q is face p-acyclic, has the same face poset as Δ^n and $\mu(Q)$ is coprime to p. Then $H^{odd}(X(Q,v);\mathbb{Z}/p)=0$.*

Proof. We abbreviate $X(Q,v)$ as X. We prove the proposition by induction on n. When $n=1$, Q is a closed interval and X is homeomorphic to S^2; so the proposition holds in this case. We assume that the proposition holds for any face p-acyclic $(n-1)$-dimensional manifold with corners satisfying the assumption in the proposition. For every i, Q_i has the same face poset as Δ^{n-1} and $\mu(Q_i)|\mu(Q)$ by (7.1), so $H^{odd}(X_i;\mathbb{Z}/p)=0$ by the induction assumption. On the other hand, since $\mu(Q)=\gcd(d_Q(q_1),\dots,d_Q(q_{n+1}))$ is coprime to p by assumption, $d_Q(q_i)$ is coprime to p for some i. For such i, Q/Q_i is face p-acyclic, has the same face poset as Σ^n and $\mu(Q/Q_i)=d_Q(q_i)$ is coprime to p, so $H^{odd}(X/X_i;\mathbb{Z}/p)=0$ by Proposition 7.1. These together with the exact sequence

$$\to H^{odd}(X/X_i;\mathbb{Z}/p)\to H^{odd}(X;\mathbb{Z}/p)\to H^{odd}(X_i;\mathbb{Z}/p)\to$$

show $H^{odd}(X;\mathbb{Z}/p)=0$. □

Case 3. We denote the facets of Q corresponding to $\Delta^{n-1}\times\{\pm1\}$ by $Q_\pm$ and the others by $Q_1,\dots,Q_n$. Accordingly, we abbreviate the characteristic vectors $v(Q_\pm)$ as $v_\pm$ and $v(Q_i)$ as v_i. We denote the vertices in Q_ϵ by $q_1^\epsilon,\dots,q_n^\epsilon$ for $\epsilon=\pm$ in such a way that q_i^ϵ is not contained in Q_i.

Lemma 7.3. *If $\mu(Q)$ is coprime to p and either $\mu(Q_+)$ or $\mu(Q_-)$ is coprime to p, then there is a vertex q of Q such that $d_Q(q)$ is coprime to p.*

Proof. We may assume that $\mu(Q_+)$ is coprime to p. We may also assume that $v_+=(0,\dots,0,1)^T$ by Theorem 6.1 through some identification of N with $\mathbb{Z}^n$. Suppose that

$$p\big|d_Q(q)\text{ for all vertices }q\text{ of }Q\tag{7.2}$$

and we will deduce a contradiction in the following.

By Lemma 6.3, $\det(v_1,\dots,v_n)$ is divisible by $\gcd(d_Q(q_1^\epsilon),\dots,d_Q(q_n^\epsilon))$, so it follows from (7.2) that

$$p\big|\det(v_1,\dots,v_n).\tag{7.3}$$

We write $v_i=(v_i^1,\dots,v_i^n)^T\in\mathbb{Z}^n$ for $i=1,2,\dots,n$.

Claim 1. There is an $i\in[n]$ such that $p\big|v_i^j$ for all $j\neq n$.

Proof. Since $v_+=(0,\dots,0,1)^T$, we naturally identify the quotient lattice $\mathbb{Z}^n/\langle v_+\rangle$ with $\mathbb{Z}^{n-1}$ and then the projection image $\bar v_i$ of v_i on the quotient lattice $\mathbb{Z}^{n-1}$ is $(v_i^1,\dots,v_i^{n-1})$. Set $s_i=\gcd(v_i^1,\dots,v_i^{n-1})$. Then $\bar v_i/s_i=:v_i'$ is primitive. Since $d_Q(q)$ is assumed to be divisible by p for all vertices q of Q, we have

$$p\big|\det(v_{i_1},\dots,v_{i_{n-1}},v_+)\quad\text{for every subset }\{i_1,\dots,i_{n-1}\}\text{ of }[n].\tag{7.4}$$

Here, since $v_+=(0,\dots,0,1)^T$, we have

$$\det(v_{i_1},\dots,v_{i_{n-1}},v_+)=\det(\bar v_{i_1},\dots,\bar v_{i_{n-1}})=\Big(\prod_{k=1}^{n-1}s_{i_k}\Big)\det(v'_{i_1},\dots,v'_{i_{n-1}}).\tag{7.5}$$

Now suppose that s_i is not divisible by p for any i. Then it follows from (7.4) and (7.5) that $p\big|\det(v'_{i_1},\dots,v'_{i_{n-1}})$ for every subset $\{i_1,\dots,i_{n-1}\}$ of $[n]$. Since $\mu(Q_+)$ agrees with the greatest common divisor of all $\det(v'_{i_1},\dots,v'_{i_{n-1}})$ by (6.1), this shows that $p\big|\mu(Q_+)$ which contradicts the assumption that $\mu(Q_+)$ is coprime to p. Therefore $p\big|s_i$ for some i, proving the claim.

Claim 2. $p\big|\det(v_{i_1},\dots,v_{i_{n-2}},v_-,v_+)$ for every subset $\{i_1,\dots,i_{n-2}\}$ of $[n]$.

Proof. Since $v_+=(0,\dots,0,1)^T$, we have

$$\det(v_{i_1},\dots,v_{i_{n-2}},v_-,v_+)=\det(\bar v_{i_1},\dots,\bar v_{i_{n-2}},\bar v_-) \tag{7.6}$$

where $\bar v_-=(v_-^1,\dots,v_-^{n-1})^T$ is the projection image of v_- on the quotient $\mathbb{Z}^n/\langle v_+\rangle=\mathbb{Z}^{n-1}$. We shall observe that the right hand side in (7.6) is divisible by p. Without loss of generality we may assume that the i in Claim 1 is n, so that $p\big|v_n^j$ for all $j\neq n$. We consider two cases.

Case a. The case where $n\in\{i_1,\dots,i_{n-2}\}$. Since $\bar v_n=(v_n^1,\dots,v_n^{n-1})^T$ and $p\big|v_n^j$ for all $j\neq n$, the right hand side in (7.6) is divisible by p.

Case b. The case where $n\notin\{i_1,\dots,i_{n-2}\}$. In this case, we consider the expansion of $\det(v_{i_1},\dots,v_{i_{n-2}},v_-,v_n)$ with respect to the last column. Since $v_n=(v_n^1,\dots,v_n^n)^T$ and $p\big|v_n^j$ for all $j\neq n$, we have

$$\det(v_{i_1},\dots,v_{i_{n-2}},v_-,v_n)\equiv v_n^n\det(\bar v_{i_1},\dots,\bar v_{i_{n-2}},\bar v_-)\pmod p. \tag{7.7}$$

Here the left hand side above is $d_Q(q)$ for $q=(\bigcap_{k=1}^{n-2}Q_{i_k})\cap Q_-\cap Q_n$, so it is divisible by p by (7.2). Moreover, v_n^n is not divisible by p because otherwise every entry of v_n is divisible by p and this contradicts v_n being primitive. It follows from (7.7) that the right hand side in (7.6) is divisible by p in this case, too.

This completes the proof of the claim.

Now (7.2), (7.3) and Claim 2 show that all $n\times n$ minors of $(v_1,\dots,v_n,v_-,v_+)$ are divisible by p and hence $p\big|\mu(Q)(=|N/\hat N|)$ by Theorem 6.1. This contradicts the assumption that $\mu(Q)$ is coprime to p, proving the lemma. □

Proposition 7.4. *Suppose that Q is face p-acyclic, has the same face poset as $\Delta^{n-1}\times[-1,1]$ and $\mu(Q)$, $\mu(Q_\pm)$ are coprime to p. Then $H^{odd}(X(Q,v);\mathbb{Z}/p)=0$.*

Proof. We abbreviate $X(Q,v)$ as X and denote by X_ϵ ($\epsilon=+$ or $-$) the inverse image of Q_ϵ by the quotient map $\pi\colon X\to Q$. Since Q_ϵ is face p-acyclic, has the same face poset as Δ^{n-1} and $\mu(Q_\epsilon)$ is coprime to p by assumption, we have

$$H^{odd}(X_\epsilon;\mathbb{Z}/p)=0 \tag{7.8}$$

by Proposition 7.2.

By Lemma 7.3 there is a vertex q of Q such that $d_Q(q)$ is coprime to p. Without loss of generality we may assume $q=q_n^-$, i.e. $d_Q(q_n^-)$ is coprime to p. Since we have (7.8) and the exact sequence

$$\to H^{odd}(X/X_+;\mathbb{Z}/p)\to H^{odd}(X;\mathbb{Z}/p)\to H^{odd}(X_+;\mathbb{Z}/p)\to,$$

it suffices to prove

$$H^{odd}(X/X_+;\mathbb{Z}/p)=0. \tag{7.9}$$

We consider two cases.

Case a. The case where $\det(v_1,\dots,v_n)\neq 0$. In this case, the characteristic function v on Q induces a characteristic function on Q/Q_+, denoted v^+, and $X/X_+=X(Q/Q_+,v^+)$. We note that Q/Q_+ is face p-acyclic and has the same face poset as Δ^n. Moreover, since q_n^- is a vertex of Q/Q_+ and $d_{Q/Q_+}(q_n^-)=d_Q(q_n^-)$ is coprime to p, $\mu(Q/Q_+)$ is coprime to p. Therefore, (7.9) follows from Proposition 7.2.

Case b. The case where $\det(v_1,\dots,v_n)=0$.

Claim. There is a vertex q of Q_n such that $d_{Q_n}(q)$ is coprime to p, so $\mu(Q_n)$ is coprime to p.

Proof. Write $v_i=(v_i^1,\dots,v_i^n)^T$ and $v_-=(v_-^1,\dots,v_-^n)^T$. Since v_n is primitive, we may assume $v_n=(0,\dots,0,1)^T$ by Theorem 6.1. Denote by $\bar v_i$ and $\bar v_-$ the projection images of v_i and v_- on $\mathbb{Z}^n/\langle v_n\rangle$ and by v_i' and v_-' the primitive vectors which have the same directions as $\bar v_i$ and $\bar v_-$ respectively. Then

$$d_{Q_n}(q_i^-)=|\det(v_1',\dots,\widehat{v_i'},\dots,v_{n-1}',v_-')|$$

by definition and hence

$$d_{Q_n}(q_i^-)\,\big|\,\det(\bar v_1,\dots,\widehat{\bar v_i},\dots,\bar v_{n-1},\bar v_-). \tag{7.10}$$

On the other hand, since $v_n=(0,\dots,0,1)^T$, we have

$$\det(v_1,\dots,v_n)=\det(\bar v_1,\dots,\bar v_{n-1})$$

and the left hand side above is zero by assumption. It follows that

$$\begin{aligned} d_Q(q_n^-)=&\det(v_1,\dots,v_{n-1},v_-)\\ =&v_-^n\det(\bar v_1,\dots,\bar v_{n-1})+\sum_{j=1}^{n-1}v_j^n(-1)^{n-j}\det(\bar v_1,\dots,\widehat{\bar v_j},\dots,\bar v_{n-1},\bar v_-)\\ =&\sum_{j=1}^{n-1}v_j^n(-1)^{n-j}\det(\bar v_1,\dots,\widehat{\bar v_j},\dots,\bar v_{n-1},\bar v_-)\end{aligned}$$

where the second identity above is the expansion of $\det(v_1,\dots,v_{n-1},v_-)$ with respect to the nth row. By (7.10) $\gcd(d_{Q_n}(q_1^-),\dots,d_{Q_n}(q_{n-1}^-))$ divides the last term above. Since $d_Q(q_n^-)$ is coprime to p, this means that $d_{Q_n}(q_i^-)$ is coprime to p for some i, proving the claim.

Now we shall prove (7.9) by induction on the dimension n of Q. When $n=1$, Q is a closed interval, X is S^2 and X_+ is a point; so (7.9) holds in this case. We assume $n\ge 2$ in the following. Let X_n be the inverse image of Q_n by the quotient map $\pi\colon X\to Q$. The face poset of Q_n is the same as that of $\Delta^{n-2}\times[-1,1]$ and Q_n is face p-acyclic. The facets corresponding to $\Delta^{n-2}\times\{\pm1\}$ are $Q_n\cap Q_\pm$ and $\mu(Q_n\cap Q_\pm)$ are coprime to p by (7.1) because $\mu(Q_\pm)$ are coprime to p by assumption. By the claim above $\mu(Q_n)$ is also coprime to p. Therefore

$$H^{odd}(X_n/(X_n\cap X_+);\mathbb{Z}/p)=0 \tag{7.11}$$

by the induction assumption.

The quotient $Q/(Q_n\cup Q_+):=\tilde Q$ is face p-acyclic and $\tilde Q$ has the same face poset as Σ^n. The characteristic function v on Q induces a characteristic function on $\tilde Q$, denoted $\tilde v$, because q_n^- is a vertex of $\tilde Q$ and $d_{\tilde Q}(q_n^-)=d_Q(q_n^-)$ is coprime to p, in particular nonzero. The quotient space $X_n/(X_n\cap X_+)$ is a subspace of X/X_+ and

$$(X/X_+)\Big/\big(X_n/(X_n\cap X_+)\big)=X(\tilde Q,\tilde v). \tag{7.12}$$

Since $d_{\tilde{Q}}(q_n^-) = \mu(\tilde{Q})$ is coprime to p, $H^{odd}(X(\tilde{Q}, \tilde{v}); \mathbb{Z}/p) = 0$ by Proposition 7.1. This together with (7.12), (7.11) and the exact sequence

$$\begin{aligned} &\to H^{odd}((X/X_+)\Big/(X_n/(X_n \cap X_+)); \mathbb{Z}/p) \to H^{odd}(X/X_+; \mathbb{Z}/p) \\ &\to H^{odd}(X_n/(X_n \cap X_+); \mathbb{Z}/p) \to \end{aligned}$$

implies (7.9). □

8. Example

In this section we shall give an example of a compact simplicial toric variety showing that the converse of Proposition 5.1 does not hold in general.

Let Q be the 3-dimensional simple polytope with the 7 facets $Q_+,\ Q_-, Q_1, ..., Q_5$, where Q_4 and Q_5 are triangles obtained by cutting two vertices of a prism, shown in Figure 1 below. The polytope Q can be obtained from Σ^3 by performing a vertex cut four times.

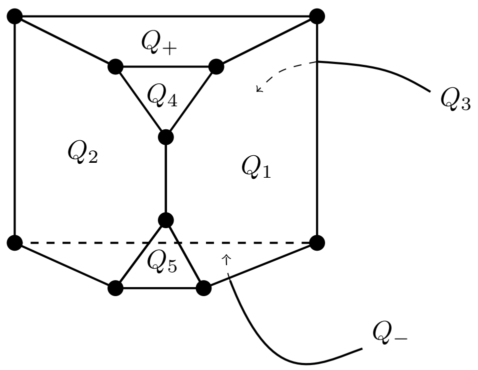

Figure 1

Let d be a positive integer. To the 7 facets $Q_1, ..., Q_5,\ Q_+,\ Q_-$, we respectively assign the following vectors

$$\begin{array}{lll} v_1 = (1,0,0) & v_2 = (-1,d,-d) & v_3 = (-1,-d,0) \\ v_4 = (0,1,0) & v_5 = (d,1-d,-d) & \\ v_+ = (0,0,1) & v_- = (1,-1,-1), & \end{array}$$

giving a characteristic function v on Q. There are ten vertices in Q. At each vertex, there are exactly three facets meeting and the determinant of the three vectors assigned to the facets is nonzero, indeed their absolute values are as follows:

$$\begin{array}{ll} |\det(v_1,v_4,v_+)| = |\det(v_2,v_4,v_+)| = |\det(v_1,v_5,v_-)| = 1 & \\ |\det(v_1,v_2,v_4)| = |\det(v_1,v_3,v_+)| = |\det(v_1,v_3,v_-)| = d & \\ |\det(v_1,v_2,v_5)| = d(2d-1) & |\det(v_2,v_5,v_-)| = d+1 \\ |\det(v_2,v_3,v_-)| = d(d+3) & |\det(v_2,v_3,v_+)| = 2d. \end{array}$$

(Precisely speaking, the vectors are regarded as column vectors here by taking transpose.) Therefore, at each vertex, the cone spanned by the three vectors is 3-dimensional and has the origin as the apex. One can also check that

$$\begin{aligned} v_4 &= (v_1+v_2+dv_+)/d & v_5 &= \big((d+1)v_1+v_2+d(2d-1)v_-\big)/2d \\ v_+ &= -(2v_1+v_2+v_3)/d & v_- &= ((d+3)v_1+v_2+2v_3)/d. \end{aligned}$$

Since d is a positive integer, this shows that $-v_+$ is in the cone $\angle v_1v_2v_3$ and v_4 is in the cone $\angle v_1v_2v_+$ while v_- is in the cone $\angle v_1v_2v_3$ and v_5 is in the cone $\angle v_1v_2v_-$ (see Figure 2), where $\angle uvw$ denotes the cone spanned by vectors u, v, w. This implies that the ten 3-dimensional cones have no overlap and cover the entire $\mathbb{R}^3$, giving a complete simplicial fan, so that the quotient space $X = X(Q, v)$ is homeomorphic to a compact simplicial toric variety.

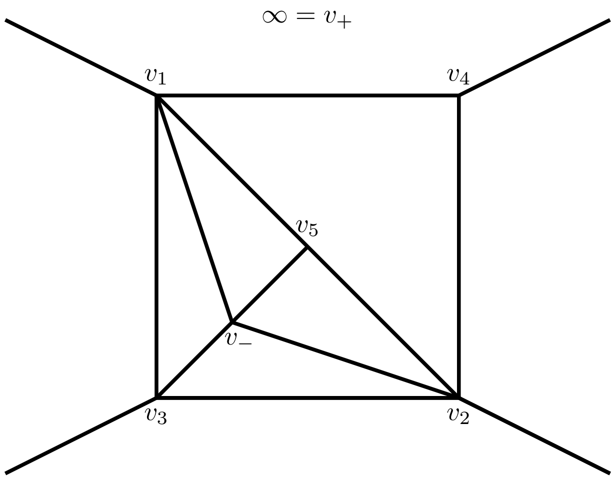

FIGURE 2

We shall check that $\mu(Q_I) = 1$ for each face Q_I of Q, where $\mu(Q_I)$ is defined in Section 5. As remarked in Section 5, $\mu(Q_I) = 1$ when $|I| = 2$ or 3. Clearly $\hat{N} = N(= \mathbb{Z}^3)$. Therefore it suffices to check $\mu(Q_I) = 1$ when $|I| = 1$. At vertices $Q_1 \cap Q_4 \cap Q_+$, $Q_2 \cap Q_4 \cap Q_+$ and $Q_1 \cap Q_5 \cap Q_-$, we have

$$|\det(v_1, v_4, v_+)| = |\det(v_2, v_4, v_+)| = |\det(v_1, v_5, v_-)| = 1$$

and hence $\mu(Q_I) = 1$ for every I with $|I| = 1$ except $I = \{3\}$ again by the remark in Section 5. In order to see $\mu(Q_3) = 1$, we note that $\{v_3, v_4, v_+\}$ is a base of N and

$$v_1 = -v_3 - dv_4, \quad v_2 = v_3 + 2dv_4 - dv_+.$$

Therefore, the images of v_1 and v_2 by the quotient map $\pi_{\{3\}}\colon N \to N(\{3\}) = N/\langle v_3\rangle$ are $(-d, 0)$ and $(2d, -d)$ with respect to the base $\{\pi_{\{3\}}(v_4), \pi_{\{3\}}(v_+)\}$. Thus the corresponding primitive vectors are $(-1, 0)$ and $(2, -1)$ which form a base of $N(\{3\})$. Hence $\mu(Q_3) = 1$.

We shall compute $H^3(X)$. Take a plane in $\mathbb{R}^3$ which meets the facets Q_1, Q_2, Q_3 transversally and does not meet the other facets of Q. Cutting Q along the plane, we divide Q into two polytopes, denoted P_+ and P_- containing Q_+ and Q_- respectively. Let $\pi\colon X \to Q$ be the quotient map and set

$$Y_\epsilon := \pi^{-1}(P_\epsilon) \text{ for } \epsilon = \pm, \quad Y := Y_+ \cap Y_-, \quad P := P_+ \cap P_-.$$

The quotient space P_ϵ/P can be regarded as a prism. The characteristic function v on Q induces a characteristic function on P_ϵ/P, denoted w_ϵ, and $X/Y_+ = Y_-/Y$ (resp. $X/Y_- = Y_+/Y$) is homeomorphic to $X(P_-/P, w_-)$ (resp. $X(P_+/P, w_+)$). The same argument as above shows that μ takes 1 on all faces of the prism P_ϵ/P, so

$$H^*(X, Y_\epsilon) \text{ and } H^*(Y_\epsilon, Y) \text{ are torsion free and vanish in odd degrees} \tag{8.1}$$

by Proposition 7.4.

Let $\tilde{Q}$ be a nice manifold with corners obtained from Q by collapsing $Q_4 \cup Q_+$ and $Q_5 \cup Q_-$ to a point respectively. The $\tilde{Q}$ has three facets coming from Q_1, Q_2, Q_3 and the characteristic function v on Q induces a characteristic function $\tilde{v}$ on $\tilde{Q}$. Since

$$v_1 = (1, 0, 0), \quad v_2 = (-1, d, -d), \quad v_3 = (-1, -d, 0),$$

one can see that $H^4(X(\tilde{Q}, \tilde{v})) \cong \mathbb{Z}/d$ by Corollary 4.2, and since $X(\tilde{Q}, \tilde{v})$ is homeomorphic to the suspension of Y, we obtain

$$H^3(Y) \cong \mathbb{Z}/d. \tag{8.2}$$

Now, consider the exact sequence in cohomology for the pair (Y_+, Y):

$$\to H^3(Y_+, Y) \to H^3(Y_+) \to H^3(Y) \to H^4(Y_+, Y) \to . \tag{8.3}$$

Since $H^3(Y_+, Y) = 0$ and $H^4(Y_+, Y)$ is torsion free by (8.1) and $H^3(Y)$ is a torsion group by (8.2), it follows from the exact sequence (8.3) that

$$H^3(Y_+) \cong H^3(Y) \cong \mathbb{Z}/d. \tag{8.4}$$

Next, consider the exact sequence in cohomology for the pair (X, Y_+):

$$\to H^3(X, Y_+) \to H^3(X) \to H^3(Y_+) \to H^4(X, Y_+) \to . \tag{8.5}$$

Similarly to the above argument, $H^3(X, Y_+) = 0$ and $H^4(X, Y_+)$ is torsion free by (8.1) and $H^3(Y_+)$ is a torsion group by (8.4), so it follows from the exact sequence (8.5) that

$$H^3(X) \cong H^3(Y_+) \cong \mathbb{Z}/d.$$

Thus $X = X(Q, v)$ is the desired example when $d \geq 2$.

Appendix

In this appendix, we observe that when X is a compact simplicial toric variety of complex dimension n, a result of Fischli [5] or Jordan [9] implies that $H^{2n-1}(X) \cong N/\hat{N}$ and $\operatorname{Tor} H^{2n-2}(X) \cong \wedge^2 N/(\hat{N} \wedge N)$, where $\operatorname{Tor} H^{2n-2}(X)$ denotes the torsion part of $H^{2n-2}(X)$. This result agrees with Proposition 2.2 since Q is contractible in this case.

Let Δ be a simplicial complete fan of dimension n and let X be the associated compact simplicial toric variety. Let M is the free abelian group dual to N. According to [5, Theorem 2.3] or [9, Theorem 2.5.5],

$$H^{2n-1}(X) \cong \operatorname{coker} \delta_1, \quad \operatorname{Tor} H^{2n-2}(X) \cong \operatorname{coker} \delta_2,$$

where

$$\delta_r \colon \bigoplus_{\tau \in \Delta^{(1)}} \wedge^{n-r}(\tau^\perp \cap M) \to \wedge^{n-r} M \qquad (r = 1, 2) \tag{8.6}$$

is the sum of inclusion maps, $\Delta^{(1)}$ denotes the set of one-dimensional cones in Δ and $\tau^\perp$ denotes the subspace of $M \otimes \mathbb{R}$ which vanish on τ.

We shall interpret the above in terms of N. Let σ be a cone of dimension $n-k$ in Δ. Then we have

$$\begin{aligned} \wedge^{\ell}(\sigma^{\perp}\cap M) &\cong \mathrm{Hom}(\wedge^{k-\ell}(\sigma^{\perp}\cap M),\mathbb{Z}) \quad (\because \operatorname{rank}\sigma^{\perp}\cap M=k)\\ &\cong \wedge^{k-\ell}(N/N_\sigma) \qquad (\because N/N_\sigma \text{ is dual to } \sigma^{\perp}\cap M)\\ &\cong (\wedge^{n-k}N_\sigma)\wedge(\wedge^{k-\ell}N) \end{aligned} \tag{8.7}$$

where N_σ is the intersection of N with the subspace of $N\otimes\mathbb{R}$ spanned by σ. The last isomorphism above is given as follows. Choose a base $\rho_1,\dots,\rho_{n-k}$ of N_σ. Since N_σ is of rank $n-k$, $\wedge^{n-k}N_\sigma$ is a free abelian group of rank one and $\rho_1\wedge\cdots\wedge\rho_{n-k}$ is its generator. For $w\in N$, we denote by $[w]$ the element of N/N_σ determined by w. Then the following correspondence

$$[w_1]\wedge\cdots\wedge[w_{k-\ell}]\to\rho_1\wedge\cdots\wedge\rho_{n-k}\wedge w_1\wedge\cdots\wedge w_{k-\ell}$$

is well defined and gives the desired isomorphism from $\wedge^{k-\ell}(N/N_\sigma)$ to $(\wedge^{n-k}N_\sigma)\wedge(\wedge^{k-\ell}N)$. This isomorphism is independent of the choice of the base $\rho_1,\dots,\rho_{n-k}$ up to sign. Namely, the isomorphism (8.7) depends only on the choice of orientations on M (or N) and σ.

Applying (8.7) to $\sigma=\tau\in\Delta^{(1)}$ and $\sigma=0$, we obtain

$$\begin{aligned} \wedge^{n-1}(\tau^{\perp}\cap M)&\cong N_\tau, & \wedge^{n-1}M&\cong N,\\ \wedge^{n-2}(\tau^{\perp}\cap M)&\cong N_\tau\wedge N, & \wedge^{n-2}M&\cong \wedge^2 N. \end{aligned}$$

Since δ_r is the sum of inclusion maps, the image of δ_1 (resp. δ_2) in (8.6) can be identified with $\hat{N}$ (resp. $\hat{N}\wedge N$) and hence

$$H^{2n-1}(X)\cong E_2^{n,n-1}\cong N/\hat{N},\quad \operatorname{Tor}H^{2n-2}(X)\cong E_2^{n,n-2}\cong\wedge^2N/(\hat{N}\wedge N).$$

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