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Geometry of R -spaces canonically embedded in Kähler C -spaces as Lagrangian submanifolds

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This article is dedicated to Professor Young Jin Suh on the occasion of his 65th birthday.

Abstract. An R -space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. It is known that any R -space has the canonical embedding into a Kähler C -space as a real form and thus it is a compact totally geodesic Lagrangian submanifold. In this article we provide an exposition on such nice properties of R -spaces as Lagrangian submanifolds and our recent work on minimal Maslov number of R -spaces canonically embedded in Einstein-Kähler C -spaces ([20]).

1 Introduction

A smooth immersion (resp. embedding) $\varphi : L \rightarrow M$ of a smooth manifold L into a symplectic manifold (M, ω) is called a *Lagrangian immersion* (resp. *Lagrangian embedding*) if $2 \dim L = \dim M$ and $\varphi^* \omega = 0$. For a Lagrangian immersion $\varphi : L \rightarrow M$, we have the vector bundle isomorphism $\varphi^{-1}TM/\varphi_*TL \ni v \leftrightarrow \alpha_v := \omega(v, \cdot) \in T^*L$. A smooth family of Lagrangian immersions $\varphi_t : L \rightarrow M$ with $\varphi_0 = \varphi$ is called a *Lagrangian deformation* of φ , which is characterized by the closedness of the 1-form $\alpha_{V_t} \in \Omega^1(L)$ corresponding to the variational vector field $V_t := \frac{\partial \varphi_t}{\partial t} \in \varphi^{-1}TM$ for each t . A Lagrangian deformation $\varphi_t : L \rightarrow M$ with

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$\varphi_0 = \varphi$ is called a *Hamiltonian deformation* of φ if $\alpha_{V_t} \in \Omega^1(L)$ is exact for each t . Suppose that $[\omega] \in H^2(M, \mathbb{R})$ is an integral class, that is, there is a complex line bundle E over M and a $U(1)$ -connection ∇ of E whose curvature form is equal to $2\pi\sqrt{-1}\omega$. It is known that a Lagrangian deformation $\varphi_t : L \rightarrow M$ with $\varphi_0 = \varphi$ is a Hamiltonian deformation if and only if a family of flat connections $\{\varphi_t^{-1}\nabla\}$ has the same holonomy homomorphism $\rho : \pi_1(L) \rightarrow U(1)$.

Two group homomorphisms $I_{\mu,L} : \pi_2(M, L) \rightarrow \mathbb{Z}$ and $I_{\omega,L} : \pi_2(M, L) \rightarrow \mathbb{R}$ are defined for any Lagrangian submanifold of a symplectic manifold in general (see Section 3) so that $I_{\mu,L}$ is a symplectic invariant and $I_{\omega,L}$ is not a symplectic invariant but a Hamiltonian invariant. The minimal Maslov number of a Lagrangian submanifold in a symplectic manifold is defined as the positive generator of $\text{Im} I_{\mu,L} \subset \mathbb{Z}$ as additive groups. It is very fundamental to the study of the Floer homology for intersections of Lagrangian submanifolds. The *monotonicity* for a Lagrangian submanifold of a symplectic manifold is defined by the condition that $I_{\mu,L} = \lambda I_{\omega,L}$ ($\exists \lambda > 0$). The Floer homology theory for the intersection of monotone Lagrangian submanifolds was initiated and well-developed by Y.-G. Oh ([15], [16], [17], [18] and so on). It is known that any compact minimal Lagrangian submanifold of an Einstein-Kähler manifold with positive Einstein constant is monotone (Cieliebak-Goldstein [2], Hajime Ono [21]). Moreover he ([21]) gave a nice formula of the minimal Maslov number for a compact monotone Lagrangian submanifold in a simply connected Einstein-Kähler manifold with positive Einstein constant (see the formula (3.1) in Section 3).

An R -space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. Note that an R -space is not a symmetric space in general and it is called a *symmetric R -space* when it is a symmetric space. It is known that each R -space has the canonical embedding into a Kähler C -space as a real form. A Kähler C -space is a simply connected compact homogeneous complex manifold which admits invariant Kähler metrics, and it is also called a generalized flag manifold. A real form means the fixed point subset by an anti-holomorphic isometry of a Kähler C -space and thus it is a compact embedded totally geodesic Lagrangian submanifold. So R -spaces canonically embedded in Kähler C -spaces constitute a nice class of Lagrangian submanifolds. As explained in Section 2 any R -space can be canonically embedded in an Einstein-Kähler C -space. In this case it is a compact monotone Lagrangian submanifold and so we can use H. Ono's formula in order to study the minimal Maslov number for R -spaces canonically embedded in Einstein-Kähler C -spaces. In [20] we showed a Lie theoretic formula for the minimal Maslov number of such an R -space and some examples of the calculation by that formula.

In this article we provide an exposition on such nice properties of R -spaces as Lagrangian submanifolds and our related recent work ([20]). This article is organized as follows: In Section 2 we review the definitions and elementary properties of R -spaces and their canonical embeddings into Kähler C -spaces and the description of the invariant symplectic structures, invariant complex structures, invariant Kähler metrics and invariant Einstein-Kähler metrics on a Kähler C -space. We also discuss several properties from the viewpoint of geometry of Lagrangian sub-

manifolds such as an anti-symplectic involutive diffeomorphism, the moment maps, Morse theory and related intersection problem. In Section 3 we recall the definitions of two Hamiltonian invariants $I_{\mu,L}$ and $I_{\omega,L}$ and the monotonicity for Lagrangian submanifolds of general symplectic manifolds. Moreover we refer the monotonicity theorem and minimal Maslov number formula by Cieliebak-Goldstein and H. Ono for Lagrangian submanifolds in Einstein-Kähler manifolds. and mention our applications to the case of the Gauss images of isoparametric hypersurfaces. In Section 4 we describe the construction of the Lie theoretic formula for minimal Maslov number for R -spaces canonically embedded in Einstein-Kähler C -spaces.

Throughout this article any manifold is smooth and connected.

2 The canonical embeddings of an R -space into a Kähler C -space

In this section we review the definitions and elementary properties of R -spaces and their canonical embeddings into Kähler C -spaces from the viewpoint of geometry of *Lagrangian submanifolds* (cf. [1], [22], [27], [23], [24], [25], [20]).

Let (G, K, θ) be a Riemannian symmetric pair with an involutive automorphism θ . Suppose that G is a connected compact semi-simple Lie group with Lie algebra \mathfrak{g} and K is a connected compact Lie subgroup of G with Lie algebra \mathfrak{k} . We choose an Ad_G - and θ -invariant inner product $\langle \cdot, \cdot \rangle$ of \mathfrak{g} and extend it to the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} by the complex bi-linearity. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of \mathfrak{g} with respect to (G, K, θ) . Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Choose a maximal abelian subalgebra \mathfrak{t} of \mathfrak{g} containing \mathfrak{a} . Then we have $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}$, $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ and \mathfrak{t} is invariant by θ . Let (\cdot, \cdot) denote an inner product of \mathfrak{t} defined by a restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{t} . The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to \mathfrak{t} is given as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} := \{X \in \mathfrak{g}^{\mathbb{C}} \mid (\text{ad} \xi)(X) = \sqrt{-1}(\alpha, \xi)X \ (\forall \xi \in \mathfrak{t})\}$$

and $\Sigma(\mathfrak{g}) \subset \mathfrak{t}$ denotes the set of all roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to \mathfrak{t} . Set $\Sigma_0(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \mathfrak{b}$. Define the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\gamma = \bar{\alpha} \mid \alpha \in \Sigma(\mathfrak{g})\}$, where $\bar{\alpha}$ denotes the \mathfrak{a} -component of $\alpha \in \Sigma(\mathfrak{g}) \subset \mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}$. We define an involutive orthogonal transformation $\sigma \in O(\mathfrak{t})$ by $\sigma := -\theta|_{\mathfrak{t}}$. We choose a σ -order $>$ on \mathfrak{t} so that if $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$ and $\alpha > 0$, then $\sigma\alpha > 0$ and thus $\theta\alpha = -\sigma\alpha < 0$ ([22]). Set $\Sigma^+(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0\}$, $\Sigma_0^+(\mathfrak{g}) := \Sigma_0(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \gamma > 0\} = \{\bar{\alpha} \mid \alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})\}$.

We choose $E_{\alpha} \in \mathfrak{g}^{\alpha}$ for $\alpha \in \Sigma(\mathfrak{g})$ such that $[E_{\alpha}, E_{-\alpha}] = \sqrt{-1}\alpha$, $\langle E_{\alpha}, E_{-\alpha} \rangle = 1$, $\overline{E_{\alpha}} = E_{-\alpha}$ for each $\alpha \in \Sigma(\mathfrak{g})$ and let $\{\omega^{\alpha} \mid \alpha \in \Sigma(\mathfrak{g})\}$ be the linear forms on $\mathfrak{g}^{\mathbb{C}}$ dual to $\{E_{\alpha} \mid \alpha \in \Sigma(\mathfrak{g})\}$ so that $\omega^{\alpha}(\mathfrak{t}^{\mathbb{C}}) = \{0\}$, $\omega^{\alpha}(E_{\beta}) = \delta_{\alpha\beta}$ for each $\alpha, \beta \in \Sigma(\mathfrak{g})$. The restricted root space decompositions of \mathfrak{k} and \mathfrak{p} with respect to \mathfrak{a} are given as

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{k}_{\gamma}, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{p}_{\gamma},$$

where $\mathfrak{k}_0 := \{X \in \mathfrak{k} \mid (\text{ad}H)X = 0 \ (\forall H \in \mathfrak{a})\}$ and for each $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ set

$$\begin{aligned}\mathfrak{k}_\gamma &:= \{X \in \mathfrak{k} \mid (\text{ad}H)^2X = -(\gamma, H)^2X \ (\forall H \in \mathfrak{a})\}, \\ \mathfrak{p}_\gamma &:= \{X \in \mathfrak{p} \mid (\text{ad}H)^2Y = -(\gamma, H)^2Y \ (\forall H \in \mathfrak{a})\}.\end{aligned}$$

For $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, there are an orthonormal basis $\{S_{\gamma,i} \mid i = 1, \dots, m(\gamma)\}$ of \mathfrak{k}_γ and an orthonormal basis $\{T_{\gamma,i} \mid i = 1, \dots, m(\gamma)\}$ of \mathfrak{p}_γ , where $m(\gamma) := \dim \mathfrak{k}_\gamma = \dim \mathfrak{p}_\gamma^\sharp$, such that $[H, S_{\gamma,i}] = (\gamma, H)T_{\gamma,i}$, $[H, T_{\gamma,i}] = -(\gamma, H)S_{\gamma,i}$ for each $H \in \mathfrak{a}$.

Now we fix an arbitrary non-zero element Z of \mathfrak{a} . Set

$$\Sigma_Z(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid (\alpha, Z) = 0\} \quad \text{and} \quad \Sigma_Z^+(\mathfrak{g}) := \Sigma_Z(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g}).$$

The element Z is called *regular* if $\Sigma_Z(\mathfrak{g}) = \Sigma_0(\mathfrak{g})$. Define closed subgroups G_Z and K_Z of G by

$$G_Z := \{a \in G \mid \text{Ad}(a)Z = Z\}$$

and

$$K_Z := \{a \in K \mid \text{Ad}(a)Z = Z\} = K \cap G_Z.$$

It is well-known that G_Z is always connected. Denote by \mathfrak{g}_Z and \mathfrak{k}_Z Lie algebras of G_Z and K_Z , respectively. Note that $\theta(G_Z) = G_Z$, $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ and thus $(G_Z, K_Z, \theta|_{G_Z})$ is also a compact symmetric pair.

Definition. The compact homogeneous space $L := K/K_Z$ is called an *R-space*, and it has the *standard imbedding* into the vector space \mathfrak{p} defined by

$$(2.1) \quad \varphi_Z : L = K/K_Z \ni aK_Z \longmapsto \text{Ad}(a)Z \in \text{Ad}(K)Z \subset \mathfrak{p}.$$

If Z is a regular element of \mathfrak{a} , then $L = K/K_Z$ is called a *regular R-space*. Another compact homogeneous space $M := G/G_Z$ is called a *generalized flag manifold* or a *Kähler C-space*, and it also has the *standard imbedding* into the Lie algebra \mathfrak{g}

$$(2.2) \quad \Phi_Z : M = G/G_Z \ni aG_Z \longmapsto \text{Ad}(a)Z \in \text{Ad}(G)Z \subset \mathfrak{g}.$$

The *canonical embedding* of K/K_Z into G/G_Z is a map defined by

$$(2.3) \quad \iota_Z : L = K/K_Z \ni aK_Z \longmapsto aG_Z \in G/G_Z = M.$$

We take the orthogonal direct sum decompositions of \mathfrak{g} and \mathfrak{k} as $\mathfrak{g} = \mathfrak{g}_Z + \mathfrak{m}$, $\mathfrak{m} \cong T_{eG_Z}M$ and $\mathfrak{k} = \mathfrak{k}_Z + \mathfrak{l}$, $\mathfrak{l} \cong T_{eK_Z}L$. Note that $\mathfrak{k}_Z = \mathfrak{k} \cap \mathfrak{g}_Z$. By using the property $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ one can show that ι_Z is an embedding and $2 \dim L = \dim M$.

The author has heard from Professor Masaru Takeuchi that the “R-space” was named first by Jacques Tits ([31]). Here we should notice that an R-space is not a symmetric space in general, and however the R-space can be considered as a class

of the most important compact homogeneous spaces related to symmetric spaces. An R -space K/K_Z is called a *symmetric R -space* if K/K_Z is a symmetric space. It is known that an R -space is a symmetric R -space if and only if one of the following conditions is satisfied:

- (1) (K, K_Z) is a symmetric pair.
- (2) There is an element $Z \in \mathfrak{p}$ satisfying the equation $(\text{ad}Z)^3 + (\text{ad}Z) = 0$ such that $L = K/K_Z$ and $G = G/G_Z$.
- (3) (G, G_Z) is a Hermitian symmetric pair.
- (4) The standard imbedding φ_Z has the parallel second fundamental form (Dirk Ferus [3]).
- (5) $\varphi_Z(L)$ is an (extrinsic) symmetric submanifold in Euclidean space \mathfrak{p} (Dirk Ferus [4]).

For such Z , we can define a G -invariant symplectic form ω_Z on $M = G/G_Z$ by

$$\omega_Z(X, Y) := \langle [Z, X], Y \rangle \text{ for each } X, Y \in \mathfrak{g}.$$

and ω_Z can be also expressed as

$$\omega_Z = -\sqrt{-1} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} (Z, \alpha) \omega^{-\alpha} \wedge \overline{\omega^{-\alpha}}.$$

Then the canonical embedding $\iota_Z : L = K/K_Z \rightarrow G/G_Z = M$ is a Lagrangian embedding with respect to ω_Z .

The involutive automorphism θ of G induces an involutive diffeomorphism

$$(2.4) \quad \hat{\theta}_Z : M = G/G_Z \ni aG_Z \mapsto \theta(a)G_Z \in G/G_Z = M$$

which is equivariant with respect to the Lie group automorphism $\theta : G \rightarrow G$. Then $\hat{\theta}_Z : G/G_Z \rightarrow G/G_Z$ is anti-symplectic with respect to ω_Z , that is, $\hat{\theta}_Z^* \omega_Z = -\omega_Z$.

Define the fixed point subset of M by $\hat{\theta}_Z$ as

$$(2.5) \quad \text{Fix}(M, \hat{\theta}_Z) := \{p \in M \mid \hat{\theta}_Z(p) = p\}.$$

Then $\iota_Z(K/K_Z) \subset \text{Fix}(M, \hat{\theta}_Z) \subset G/G_Z$.

The natural left action of G on a symplectic manifold $(M = G/G_Z, \omega_Z)$ is a Hamiltonian group action with the moment map

$$(2.6) \quad \mu_G := \Phi_Z : G/G_Z \longrightarrow \mathfrak{g} \cong \mathfrak{g}^*.$$

Moreover the natural left action of $K \subset G$ on $(M = G/G_Z, \omega_Z)$ is also a Hamiltonian group action with the moment map

$$(2.7) \quad \mu_K := \pi_{\mathfrak{k}} \circ \mu_G = \pi_{\mathfrak{k}} \circ \Phi_Z : G/G_Z \longrightarrow \mathfrak{k} \cong \mathfrak{k}^*.$$

Here $\pi_{\mathfrak{k}} : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \rightarrow \mathfrak{k}$ denotes the orthogonal projection of \mathfrak{g} onto \mathfrak{k} . Then $\mu_G \circ \hat{\theta}_Z = -\theta \circ \mu_G$ and $\mu_K \circ \hat{\theta}_Z = -\mu_K$. It follows from these formulas that

$$(2.8) \quad \text{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0).$$

Since K and M are compact, by a result of Kirwan ([12, p.549, (3.1)]) $\mu_K^{-1}(0)$ is connected and thus $\text{Fix}(M, \hat{\theta}_Z)$ is also connected. Therefore we obtain

$$(2.9) \quad \iota_Z(K/K_Z) = \text{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0).$$

The Weyl group of (G, K) is defined by $W(G, K) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. By the action of the Weyl group $W(G, K)$, we may assume that $Z \in \mathfrak{a} \subset \mathfrak{k}$ satisfies $(\alpha, Z) \geq 0$ for $\forall \alpha \in \Sigma^+(\mathfrak{g})$.

Now we describe an invariant complex structure on $M = G/G_Z$ corresponding to Z . Note that $Z \in \mathfrak{c}(\mathfrak{g}_Z) \subset \mathfrak{t} \subset \mathfrak{g}_Z$. Then

$$\begin{aligned} \mathfrak{g}_Z^{\mathbb{C}} &= \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha}, \\ T_{eG_Z}(G/G_Z)^{\mathbb{C}} &\cong \mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha} \end{aligned}$$

Note that $\mathfrak{g}^{\alpha} = \overline{\mathfrak{g}^{-\alpha}}$. Thus we can define a G -invariant complex structure J_Z on G/G_Z such that

$$T_{eG_Z}(G/G_Z)^{1,0} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha}, \quad T_{eG_Z}(G/G_Z)^{0,1} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

Since

$$\theta \left(\sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

the involutive diffeomorphism $\hat{\theta}_Z : G/G_Z \rightarrow G/G_Z$ is anti-holomorphic with respect to J_Z , that is, $J_Z \circ d\hat{\theta}_Z = -d\hat{\theta}_Z \circ J_Z$.

Moreover the corresponding G -invariant Kähler metric g_Z on $M = G/G_Z$ is defined by

$$(2.10) \quad \omega_Z(X, Y) = (-1)g_Z(J_Z X, Y) \quad \text{for each } X, Y \in \mathfrak{m}$$

or

$$g_Z = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z^+(\mathfrak{g})} (Z, \alpha) \omega^{-\alpha} \cdot \overline{\omega^{-\alpha}}.$$

Hence the diffeomorphism $\hat{\theta}_Z : M \rightarrow M$ is an isometry of M with respect to g_Z .

Let $\Pi := \Pi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_\ell\}$ be the fundamental root system of \mathfrak{g} with respect to the σ -order $<$ of \mathfrak{t} . Set $\Pi(\mathfrak{g})_0 := \Pi(\mathfrak{g}) \cap \mathfrak{b}$. For the above Z , set $\Pi_Z := \Pi_Z(\mathfrak{g}) := \{\alpha_i \in \Pi(\mathfrak{g}) \mid (\alpha_i, Z) = 0\}$. Note that $\Pi_0(\mathfrak{g}) \subset \Pi_Z(\mathfrak{g})$ and

thus $\Pi(\mathfrak{g}) \setminus \Pi_Z(\mathfrak{g}) \subset \Pi(\mathfrak{g}) \setminus \Pi_0(\mathfrak{g})$. Let $\{\Lambda_1, \dots, \Lambda_\ell\} \subset \mathfrak{t}$ be the fundamental weight system of \mathfrak{g} corresponding to $\Pi(\mathfrak{g})$ defined by

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (i, j = 1, \dots, \ell).$$

Now we set

$$\mathfrak{c}_Z := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}\Lambda_i \subset \mathfrak{t}, \quad \mathbb{Z}_{\mathfrak{c}_Z} := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}\Lambda_i \subset \mathfrak{c}_Z.$$

Note that \mathfrak{c}_Z coincides with the center $\mathfrak{c}(\mathfrak{g}_Z)$ of \mathfrak{g}_Z . Define

$$\mathfrak{c}_Z^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}^+ \Lambda_i \subset \mathfrak{c}_Z, \quad \mathbb{Z}_{\mathfrak{c}_Z}^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}^+ \Lambda_i \subset \mathfrak{c}_Z^+$$

where \mathbb{R}^+ and \mathbb{Z}^+ denote all positive real numbers and all positive integers, respectively. Obviously we have $Z \in \mathfrak{c}_Z^+$.

For each $\lambda \in \mathfrak{c}_Z^+$, since $\Pi_\lambda = \Pi_Z$, $\Sigma_\lambda(\mathfrak{g}) = \Sigma_Z(\mathfrak{g})$, we have $\mathfrak{g}_\lambda^{\mathbb{C}} = \mathfrak{g}_Z^{\mathbb{C}}$ and thus $\mathfrak{g}_\lambda = \mathfrak{g}_Z$. By the connectedness of G_ξ and G_Z , we obtain $G_\lambda = G_Z$ and $G/G_\lambda = G/G_Z = M$. In particular ω_λ is a G -invariant symplectic form on $M = G/G_Z = G/G_\lambda$. However λ and H define the same G -invariant complex structure $J_\lambda = J_H$ on $M = G/G_H = G/G_\lambda$.

Since $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ and thus $\theta(\mathfrak{c}(\mathfrak{g}_Z)) = \mathfrak{c}(\mathfrak{g}_Z)$, there is a direct sum decomposition

$$\mathfrak{c}(\mathfrak{g}_Z) = \mathfrak{c}_Z = (\mathfrak{c}_Z \cap \mathfrak{b}) + (\mathfrak{c}_Z \cap \mathfrak{a}).$$

For each $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$, since $G_H = G_Z$ and $G/G_H = G/G_Z$, we have $K_H = K \cap G_H = K \cap G_Z = K_Z$ and thus $K/K_H = K/K_Z = L$. Hence all $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$ correspond to the same R -space $L = K/K_Z$ and the convex set $\mathfrak{c}_Z^+ \cap \mathfrak{a}$ parametrizes orbits of the same type K_Z .

Let $\mathcal{J}_G^2(M)$ denote the real vector space of all G -invariant closed 2-forms on $M = G/G_Z$. Then the natural linear map $\mathfrak{w} : \mathcal{J}_G^2(M) \ni \omega \mapsto [\omega] \in H^2(M, \mathbb{R})$ is a linear isomorphism and there is a linear isomorphism $\omega : \frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_Z \rightarrow \mathcal{J}_G^2(M)$ defined by

$$\omega \left(\frac{1}{2\pi\sqrt{-1}}\lambda \right) (X, Y) := -\frac{1}{2\pi} \langle [\lambda, X], Y \rangle \quad (X, Y \in \mathfrak{m})$$

for each $\lambda \in \mathfrak{c}_Z$. Moreover the linear isomorphism $\tau = \mathfrak{w} \circ \omega : \frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_Z \rightarrow \mathcal{J}_G^2(M) \rightarrow H^2(M, \mathbb{R})$ given by the transgression operator is restricted to a \mathbb{Z} -module isomorphism $\tau = \mathfrak{w} \circ \omega : \frac{1}{2\pi\sqrt{-1}}\mathbb{Z}_{\mathfrak{c}_Z} \rightarrow H^2(M, \mathbb{Z})$. The 2nd cohomology and homology groups of G/G_Z are described as follows:

$$\begin{array}{ccc} \mathfrak{c}_Z = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}\Lambda_{\alpha_i} & \ni \lambda \longleftrightarrow & \left[\frac{1}{2\pi\sqrt{-1}}\omega(\lambda) \right] \in H^2(G/G_Z, \mathbb{R}) \\ \cup & & \cup \\ \mathbb{Z}_{\mathfrak{c}_Z} = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}\Lambda_{\alpha_i} & \longleftrightarrow & H^2(G/G_Z, \mathbb{Z}) \end{array}$$

and if we set $\alpha_i^* := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$,

$$\begin{aligned} \mathfrak{c}_Z^* &:= \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}\alpha_i^* && \ni \sum_i x_i \alpha_i^* \longleftrightarrow \sum_i x_i [S^2(\alpha_i^*)] \in H_2(G/G_Z, \mathbb{R}) \\ \cup &&& \cup \\ \mathbb{Z}_{\mathfrak{c}_Z^*} &:= \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}\alpha_i^* && \longleftrightarrow H_2(G/G_Z, \mathbb{Z}) \cong \pi_2(G/G_Z). \end{aligned}$$

For each $\lambda \in \mathfrak{c}_Z^+$, define a G -invariant Kähler metric on a complex manifold $(M = G/G_Z, J_Z)$ by

$$g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right) := \frac{1}{2\pi} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} (\lambda, \alpha) \omega^{-\alpha} \cdot \bar{\omega}^{-\alpha}$$

whose Kähler form coincides with $\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$ as

$$\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(X, Y) = g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(J_Z X, Y).$$

Note that $Z \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$ and $\omega\left(\frac{1}{2\pi\sqrt{-1}}2\pi Z\right) = -\omega_Z$. Namely, the convex open set \mathfrak{c}_Z^+ of the vector space \mathfrak{c}_Z parametrizes all G -invariant Kähler metrics on $M = G/G_Z$ relative to the complex structure J_Z . So the parameter spaces of all G -inv. Kähler metrics on G/G_Z are given as

$$\begin{aligned} \mathfrak{c}_Z^+ &= \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}^+ \Lambda_{\alpha_i} && \ni \lambda \longleftrightarrow \omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right) \in \{G\text{-inv. Kähler met. on } G/G_Z\} \\ \cup &&& \cup \\ \mathbb{Z}_{\mathfrak{c}_Z^+} &= \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}^+ \Lambda_{\alpha_i} && \longleftrightarrow \{G\text{-inv. Hodge met. on } G/G_Z\}. \end{aligned}$$

For each $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$, the diffeomorphism $\hat{\theta}_Z : M = G/G_Z \rightarrow M = G/G_Z$ preserves a G -invariant Kähler metric $g\left(\frac{1}{2\pi\sqrt{-1}}H\right)$ on M , that is, $\hat{\theta}_Z : M = G/G_Z \rightarrow M = G/G_Z$ is an isometry with respect to $g\left(\frac{1}{2\pi\sqrt{-1}}H\right)$. Hence the canonically embedded R -space $\iota_Z(K/K_Z)$ is a *real form*, that is, the fixed point subset of a Kähler C -space $M = G/G_Z$ by the anti-holomorphic isometry $\hat{\theta}_Z$ with respect to J_Z and a Kähler metric $g\left(\frac{1}{2\pi\sqrt{-1}}H\right)$ for any $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$.

Next we mention the characterization of a G -invariant Einstein-Kähler metric on $M = G/G_Z$. Set

$$\delta_{\mathfrak{m}} := \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \alpha \in \mathfrak{t}.$$

Lemma 2.1 ([1]).

$$(2.11) \quad 2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \alpha \in \mathbb{Z}_{\mathfrak{c}_H^+} = \bigoplus_{\alpha \in \Pi \setminus \Pi_H} \mathbb{Z}^+ \Lambda_{\alpha}.$$

and it corresponds to the first Chern class of the complex manifold (M, J_H) :

$$c_1(M) = \left[\omega \left(\frac{1}{2\pi\sqrt{-1}} 2\delta_m \right) \right] = \tau \left(\frac{1}{2\pi\sqrt{-1}} 2\delta_m \right).$$

Proposition 2.2 ([24]). *The G -invariant Kähler metric $g = g \left(\frac{1}{2\pi\sqrt{-1}} \lambda \right)$ on M is Einstein if and only if $\lambda = b\delta_m$ for some $b > 0$.*

Then we can show that $2\delta_m \in \mathfrak{a}$ ([20]). Therefore we obtain

Proposition 2.3. *The element $Z^{ein} := 2\delta_m \in Z_{\mathfrak{c}_Z^+} \cap \mathfrak{a} \subset \mathfrak{c}_Z^+ \cap \mathfrak{a}$ corresponds to the canonical embedding $\iota_{Z^{ein}}$ of the same R -space $L = K/K_Z$ into an Einstein-Kähler C -space $\left(M = G/G_Z, \omega_{Z^{ein}}, J_Z, g \left(\frac{1}{2\pi\sqrt{-1}} Z^{ein} \right) \right)$. Moreover, the element Z^{ein} is such a unique element of $\mathfrak{c}_Z^+ \cap \mathfrak{a}$ up to the multiplication by a positive constant.*

Here we shall mention about geometry of R -spaces as homogeneous spaces of noncompact real semisimple Lie groups. Set $\mathfrak{p}^\sharp := \sqrt{-1}\mathfrak{p}$. Then $\mathfrak{g}^\sharp := \mathfrak{k} + \mathfrak{p}^\sharp$ is the Cartan decomposition of a noncompact real semisimple Lie algebra \mathfrak{g}^\sharp with Cartan involution τ . Let $G^\mathbb{C}$ be a connected complex Lie group without center with Lie algebra $\mathfrak{g}^\mathbb{C}$ and then G can be regarded as an analytic subgroup of $G^\mathbb{C}$. Let G^\sharp be a connected real semisimple Lie subgroup of $G^\mathbb{C}$ with Lie algebra \mathfrak{g}^\sharp . The root space decomposition of \mathfrak{g}^\sharp with respect to $\sqrt{-1}\mathfrak{a}$ is given as

$$\mathfrak{g}^\sharp = \mathfrak{g}_0^\sharp + \bigoplus_{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\gamma^\sharp,$$

where $\mathfrak{g}_0^\sharp := \{X \in \mathfrak{g}^\sharp \mid [\sqrt{-1}H, X] = 0 \ (\forall H \in \mathfrak{a})\}$ and for each $\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})$

$$\mathfrak{g}_\gamma^\sharp := \{X \in \mathfrak{g}^\sharp \mid [\sqrt{-1}H, X] = (\gamma, H)X \ (\forall H \in \mathfrak{a})\}.$$

Then

$$\mathfrak{u} := \mathfrak{g}_0^\sharp + \bigoplus_{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}), \gamma(Z) \geq 0} \mathfrak{g}_\gamma^\sharp.$$

is a parabolic subalgebra of \mathfrak{g}^\sharp . Let U be a parabolic subgroup of G^\sharp with Lie algebra \mathfrak{u} , which is always connected. The complexification of \mathfrak{u}

$$\mathfrak{u}^\mathbb{C} = (\mathfrak{g}_0^\sharp)^\mathbb{C} + \bigoplus_{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}), \gamma(Z) \geq 0} (\mathfrak{g}_\gamma^\sharp)^\mathbb{C} = \mathfrak{t}^\mathbb{C} + \bigoplus_{\alpha \in \Sigma(\mathfrak{g}), \alpha(Z) \geq 0} \mathfrak{g}_\alpha^\mathbb{C}.$$

is a complex parabolic subalgebra of $\mathfrak{g}^\mathbb{C}$. Let $U^\mathbb{C}$ be a complex parabolic subgroup of $G^\mathbb{C}$ with Lie algebra $\tilde{\mathfrak{u}}$, which is always connected. Then we know ([23]) that

$$(2.12) \quad KU = G^\sharp, \quad K \cap U = K_Z, \quad \text{and thus } L = K/K_Z \cong G^\sharp/U,$$

$$(2.13) \quad GU^\mathbb{C} = G^\mathbb{C}, \quad G \cap U^\mathbb{C} = G_Z, \quad \text{and thus } G/G_Z \cong M = G^\mathbb{C}/U^\mathbb{C}.$$

The induced complex structure of M under the identification of $M = G/G_Z$ with the complex homogeneous space $G^{\mathbb{C}}/U^{\mathbb{C}}$ coincides with the G -invariant complex structure J_Z of M .

Define two subgroups of K and K_Z as

$$N_K(\mathfrak{a}) := \{k \in K \mid \text{Ad}(k)\mathfrak{a} = \mathfrak{a}\} \subset K, \quad N_{K_Z}(\mathfrak{a}) := N_K(\mathfrak{a}) \cap K_Z \subset K_Z.$$

Note that $N_{K_Z}(\mathfrak{a})$ is not a normal subgroup of $N_K(\mathfrak{a})$, $C_K(\mathfrak{a}) \subset N_{K_Z}(\mathfrak{a})$, and if $Z \in \mathfrak{a}$ is regular, then $C_K(\mathfrak{a}) = N_{K_Z}(\mathfrak{a})$:

$$1 \longrightarrow N_{K_Z}(\mathfrak{a})/C_K(\mathfrak{a}) \longrightarrow W(G, K) = N_K(\mathfrak{a})/C_K(\mathfrak{a}) \longrightarrow N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}) \longrightarrow 1$$

Theorem 2.4 (Masaru Takeuchi [23]). *Let $k_1, \dots, k_b \in N_K(\mathfrak{a})$ be complete representatives of $N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}) = \{[k_1], \dots, [k_b]\}$. Then the orbits Nk_1o, \dots, Nk_bo of N through the origin $o = eU \in G^{\sharp}/U = L$ provide a cellular decomposition of L as $L = G^{\sharp}/U = Nk_1o \cup \dots \cup Nk_bo$ and these cells are all cycles mod 2 of L . In particular, $\dim H_*(L; \mathbb{Z}_2) = \sum_{i=0}^{\dim L} \dim H_i(L; \mathbb{Z}_2) = b$.*

We briefly discuss related results of Masaru Takeuchi and Shoshichi Kobayashi ([27]) on perfect Morse functions on R -spaces. For each $X \in \mathfrak{g}$, we define a linear function $u_X : \mathfrak{g} \rightarrow \mathbb{R}$ defined by $u_X(\xi) := \langle \xi, X \rangle$ for each $\xi \in \mathfrak{g}$. A smooth function \tilde{f}_X on $M = G/G_Z$ is defined by

$$\tilde{f}_X := u_X \circ \Phi_Z = u_X \circ \mu_G = \langle \mu_G, X \rangle : M = G/G_Z \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R}.$$

Then by the moment map equation and (2.10) we have

$$(2.14) \quad d\tilde{f}_X = d\langle \mu_G, X \rangle = \omega_Z(\tilde{X}, \cdot) = -g_Z(J_Z \tilde{X}, \cdot),$$

where \tilde{X} denotes a vector field on $M = G/G_Z$ induced by a one-parameter subgroup $\{\exp(tX) \mid t \in \mathbb{R}\}$ of G , which is a Killing vector field on M with respect to a Kähler metric g_Z . Hence the gradient vector field $\text{grad}(\tilde{f}_X)$ of the function \tilde{f}_X on $M = G/G_Z$ with respect to the invariant Kähler metric g_Z is equal to $-J_Z \tilde{X}$:

$$(2.15) \quad -\text{grad}(\tilde{f}_X) = J_Z \tilde{X} = \widetilde{(\sqrt{-1}X)}.$$

Here $J_Z \tilde{X} = \widetilde{(\sqrt{-1}X)}$ is a holomorphic vector field on $M = G^{\mathbb{C}}/U^{\mathbb{C}}$ induced by a one-parameter subgroup $\{\exp(t\sqrt{-1}X) \mid t \in \mathbb{R}\}$ of $G^{\mathbb{C}}$.

Now assume that $X \in \mathfrak{p}$. A smooth function f_X on $L = K/K_Z$ is defined by

$$f_X = \tilde{f}_X \circ \iota_Z = u_X \circ \varphi_Z = \langle \mu_G \circ \iota_Z, X \rangle : L = K/K_Z \longrightarrow \mathfrak{p} \longrightarrow \mathbb{R}$$

By pulling back the equation (2.14) by the canonical embedding ι_Z , we have

$$(2.16) \quad df_X = \iota_Z^* d\tilde{f}_X = \iota_Z^* \omega_Z(\tilde{X}, \cdot) = -g_Z((\widetilde{(\sqrt{-1}X)}) \circ \iota_Z, (\iota_Z)_*(\cdot)).$$

Since $\sqrt{-1}X \in \sqrt{-1}\mathfrak{p} = \mathfrak{p}^\sharp \subset \mathfrak{g}^\sharp \subset \mathfrak{g}^\mathbb{C}$, $\exp(t\sqrt{-1}X) \mid t \in \mathbb{R}$ is a one-parameter subgroup of G^\sharp and it induces a vector field $(\widehat{\sqrt{-1}X})$ on $L = G^\sharp/U$. Since $(\iota_Z)_*((\widehat{\sqrt{-1}X})) = (\widehat{\sqrt{-1}X}) \circ \iota_Z = J_Z \tilde{X} \circ \iota_Z$, the equation (2.16) becomes

$$(2.17) \quad df_X = -(\iota_Z^* g_Z)((\widehat{\sqrt{-1}X}), \cdot)$$

Hence the gradient vector field $\text{grad}(f_X)$ on $L = K/K_Z$ with respect to the induced Riemannian metric $\iota_Z^* g_Z$ is equal to a vector field $-(\widehat{\sqrt{-1}X})$ on $L = K/K_Z = G^\sharp/U$ induced by $-\sqrt{-1}X \in \sqrt{-1}\mathfrak{p} \subset \mathfrak{g}^\sharp$. In particular, the critical point set of f_X on L coincides with the zero set $\text{Zero}(\widehat{\sqrt{-1}X}) = \text{Zero}(\tilde{X} \circ \iota_Z)$ of vector fields $(\widehat{\sqrt{-1}X})$ and $\tilde{X} \circ \iota_Z$ on L . In [27] they showed that for each $X = \text{Ad}(k)H \in \mathfrak{p}$ ($k \in K, H \in \mathfrak{a}$), it holds $\ell_k(N_K(\mathfrak{a})eK_Z) \subset \text{Zero}(\widehat{\sqrt{-1}X})$ and if X is regular, then $\ell_k(N_K(\mathfrak{a})eK_Z) = \text{Zero}(\widehat{\sqrt{-1}X})$. Here $\ell_k : K/K_Z \rightarrow K/K_Z$ denotes the left natural action by $k \in K$ on K/K_Z . Therefore, for each regular $X \in \mathfrak{p}$, the number of critical points of f_X is equal to $\sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}))$ and thus $b = \dim H_*(K/K_Z, \mathbb{Z}_2)$. In particular f_X is a perfect Morse function on the R -space L for each regular $X \in \mathfrak{p}$ ([27]). We can also observe that for each regular $X \in \mathfrak{p}$ the equality $\sharp(\exp(tX)\iota_Z(L) \cap \iota_Z(L)) = \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}))$ holds for any sufficiently small $t \neq 0$.

Here we recall some fundamental results from the structure theory of a compact symmetric space G/K (cf. [6], [26]). They are necessary to discuss the geometry of R -spaces canonically embedded in Kähler C -spaces. Set $A := \exp \mathfrak{a} \subset G$ and $\hat{A} := A(eK) \subset G/K$. Then the K -equivariant map

$$(2.18) \quad \psi : K/Z_K(\mathfrak{a}) \times \hat{A} \ni (kZ_K(\mathfrak{a}), \hat{a}) \mapsto k\hat{a} \in G/K$$

is a surjective smooth map. Define the diagram of a compact symmetric pair (G, K) by

$$\mathbf{D}(\mathfrak{g}, \mathfrak{a}) := \{H \in \mathfrak{a} \mid (\gamma, H) \in \pi\mathbb{Z} \ (\exists \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}))\}$$

and thus we have $\mathfrak{a} \setminus \mathbf{D}(\mathfrak{g}, \mathfrak{a}) = \{H \in \mathfrak{a} \mid (\gamma, H) \notin \pi\mathbb{Z} \ (\forall \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}))\}$. Set $\hat{A}_s := (\exp \mathbf{D}(\mathfrak{g}, \mathfrak{a}))eK \subset \hat{A}$ and $\hat{A}_r := \hat{A} \setminus \hat{A}_s = (\exp(\mathfrak{a} \setminus \mathbf{D}(\mathfrak{g}, \mathfrak{a})))eK$. Each element of \hat{A}_r (resp. \hat{A}_s) is called a *regular* (resp. *singular*) element of \hat{A} . Then $G/K = (G/K)_r \cup (G/K)_s$ (disjoint union), where $(G/K)_s := \psi(K/Z_K(\mathfrak{a}) \times \hat{A}_s)$ is a closed set of codimension at least 2 and

$$(2.19) \quad (G/K)_r := \psi(K/Z_K(\mathfrak{a}) \times \hat{A}_r)$$

is a connected open dense subset of G/K . Each element of $(G/K)_r$ is called a *regular* element of G/K . The surjective smooth map

$$(2.20) \quad \psi : K/Z_K(\mathfrak{a}) \times \hat{A}_r \longrightarrow (G/K)_r$$

is a covering map whose covering transformation group is the right natural action of $W(G, K)$ on $K/Z_K(\mathfrak{a}) \times \hat{A}_r$, and thus it induces a diffeomorphism $\bar{\psi} : (K/Z_K(\mathfrak{a}) \times \hat{A}_r)/W(G, K) \longrightarrow (G/K)_r$ which is equivariant with the actions of K . Here note that $K/Z_K(\mathfrak{a}) \times \hat{A}_r$ is not connected in general.

Using the geometry of a compact symmetric space G/K , we discuss the intersection property of $a\iota_Z(L)$ and $\iota_Z(L)$ under the left group action of $a \in G$ on $M = G/G_Z$.

For any $a \in G$, by the surjectivity of ψ we have $aK = \psi(kZ_K(\mathfrak{a}), \exp(H)eK)$ for some $k \in K$ and some $H \in \mathfrak{a}$ and thus $ak_1 = kk_0 \exp(H)$ for some $k_0 \in Z_K(\mathfrak{a})$ and some $k_1 \in K$. Thus $ak_1G_Z = kk_0 \exp(H)G_Z = kG_Z \in G/G_Z$ and hence $a\iota_Z(k_1K_Z) = \iota_Z(kK_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ gives an intersection point of $a\iota_Z(K/K_Z)$ and $\iota_Z(K/K_Z)$. Moreover, for any $k' \in N_K(\mathfrak{a})$, we have $ak_1k' = kk'(k'^{-1}k_0k') \exp(\text{Ad}(k'^{-1})H)$ where note that $k'^{-1}k_0k' \in Z_K(A)$ and $\text{Ad}(k'^{-1})H \in \mathfrak{a}$. Thus $ak_1k'G_Z = kk'(k'^{-1}k_0k') \exp(\text{Ad}(k'^{-1})H)G_Z = kk'G_Z \in G/G_Z$ and hence $a\iota_Z(k_1k'K_Z) = \iota_Z(kk'K_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ also gives an intersection point of $a\iota_Z(K/K_Z)$ and $\iota_Z(K/K_Z)$. Note that if $kk'K_Z = kk''K_Z$ for $k', k'' \in N_K(\mathfrak{a})$, then $k'N_{K_Z}(\mathfrak{a}) = k''N_{K_Z}(\mathfrak{a})$. Therefore, combining the above argument with Theorem 2.4, we obtain

Proposition 2.5. *For any $a \in G$, it holds*

$$\sharp(a\iota_Z(L) \cap \iota_Z(L)) \geq \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a})) = \dim H_*(L, \mathbb{Z}_2).$$

Next we mention about the transversality condition of the intersection $a\iota_Z(L) \cap \iota_Z(L)$. Suppose that $p \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$. Then $p = a\iota_Z(k_2K_Z) = \iota_Z(k_1K_Z)$ ($\exists k_1, k_2 \in K$). Since $k_1^{-1}ak_2 \in G_Z$, using the symmetric Lie algebra $\mathfrak{g}_Z = \mathfrak{k}_Z + \mathfrak{p}_Z$ of a compact symmetric pair (G_Z, K_Z) so that $\mathfrak{a} \subset \mathfrak{p}_Z$, there are $k_Z, k'_Z \in K_Z$ and $H_p \in \mathfrak{a}$ such that $k_1^{-1}ak_2 = k_Z(\exp H_p)k_Z^{-1}k'_Z$. Thus $ak_2 = k_1k_Z(\exp H_p)k_Z^{-1}k'_Z$. Since

$$\begin{aligned} (\Phi_Z)_*(T_p a\iota_Z(K/K_Z)) &= \text{Ad}(ak_2)[\mathfrak{k}, Z] \\ &= \text{Ad}(k_1k_Z(\exp H_p)k_Z^{-1}k'_Z)[\mathfrak{k}, Z] \\ &= \text{Ad}(k_1)\text{Ad}(k_Z)\text{Ad}(\exp H_p)[\mathfrak{k}, Z], \\ (\Phi_Z)_*(T_p \iota_Z(K/K_Z)) &= \text{Ad}(k_1)[\mathfrak{k}, Z] = \text{Ad}(k_1)\text{Ad}(k_Z)[\mathfrak{k}, Z], \end{aligned}$$

the transversality of $a\iota_Z(L) \cap \iota_Z(L)$ at p is equivalent to the transversality of $\text{Ad}(\exp H_p)[\mathfrak{k}, Z]$ and $[\mathfrak{k}, Z]$: $\text{Ad}(\exp H_p)[\mathfrak{k}, Z] \cap [\mathfrak{k}, Z] = \{0\}$. Then by a simple computation using the basis $\{S_{\gamma,i}, T_{\gamma,i}\}$ we can show

Lemma 2.6. *$a\iota_Z(L)$ intersects transversally with $\iota_Z(L)$ in $M = G/G_Z$ if and only if at each intersection point p such an $H_p \in \mathfrak{a}$ satisfies $(\gamma, H) \notin \pi\mathbb{Z}$ for each $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ with $(\gamma, Z) \neq 0$.*

First we suppose that $a \in G$ satisfies $aK \in (G/K)_r$. Then $a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is transversal at each intersection point. Fix an intersection point $\iota_Z(k_1K_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$. Then

$$\begin{aligned} aK &= k_1k_{Z,1} \exp(H_1)K \quad (\exists H_1 \in \mathfrak{a} \setminus \mathbb{D}(\mathfrak{g}, \mathfrak{a}), \exists k_Z \in K_Z) \\ &= \psi(k_1k_{Z,1}Z_K(\mathfrak{a}), \exp(H_1)K) \end{aligned}$$

For an arbitrary intersection point $\iota_Z(k_2K_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$,

$$\begin{aligned} aK &= k_2k_{Z,2} \exp(H_2)K \quad (\exists H_2 \in \mathfrak{a} \setminus \mathbb{D}(G, K), k_{Z,2} \in K_Z) \\ &= \psi(k_2k_{Z,2}Z_K(\mathfrak{a}), \exp(H_2)K) \end{aligned}$$

Then there is $s = [k'] \in W(G, K) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ such that

$$k_2k_{Z,2}Z_K(\mathfrak{a}) = k_1k_{Z,1}Z_K(\mathfrak{a})s = k_1k_{Z,1}k'Z_K(\mathfrak{a})$$

and

$$(\exp H_2)K = s^{-1}(\exp H_1)K = (\exp s^{-1}H_1)K = (\exp \text{Ad}(k')^{-1}H_1)K.$$

Thus $k_2K_Z = k_1k_{Z,1}k'K_Z$ and hence $\iota_Z(k_2K_Z) = \iota_Z(k_1k_{Z,1}k'K_Z)$. Therefore we obtain

$$a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) = \{\iota_Z(k_1k_{Z,1}k'K_Z) \mid [k'] \in N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a})\}.$$

In particular $\sharp(a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}))$.

In general suppose that $a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is transversal at each intersection point. Particularly $a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is a finite set. We may assume that $aK \in (G/K)_s$. Since $(G/K)_r$ is open and dense in G/K , we choose a smooth perturbation $a_t \in G$ of $a_0 = a$ such that $a_tK \in (G/K)_r$ ($0 < \forall t < \varepsilon$). Then $a_t\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is also transversal at each intersection point and for sufficiently small $t > 0$ we have

$$\sharp(a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(a_t\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(\mathfrak{a})/N_{K_0}(\mathfrak{a})).$$

Therefore we obtain

Proposition 2.7. *For any $a \in G$ with transversal $a\iota_Z(L) \cap \iota_Z(L)$, it holds*

$$\sharp(a\iota_Z(L) \cap \iota_Z(L)) = \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a})).$$

Combining it with Theorem 2.4, we see

Corollary 2.8. *For any $a \in G$ with transversal $a\iota_Z(L) \cap \iota_Z(L)$, it holds*

$$\sharp(a\iota_Z(L) \cap \iota_Z(L)) = \dim H_*(L, \mathbb{Z}_2).$$

Such a property is called the *global tightness* for a Lagrangian submanifolds in a Kähler C -space ([14], [9], [5]). It was proved by [28] in the case when L is a symmetric R -space. It is still an open problem to classify compact globally tight or simply tight Lagrangian submanifolds of Kähler C -spaces. More generally the intersection theory and Floer homology for two real forms in Kähler C -spaces are discussed in [7], [11].

3 Minimal Maslov number and monotonicity of Lagrangian submanifolds in symplectic manifolds and Einstein-Kähler manifolds

Let L be a Lagrangian submanifold of a symplectic manifold (M, ω) . Define two kinds of group homomorphisms $I_{\mu, L} : \pi_2(M, L) \rightarrow \mathbb{Z}$ and $I_{\omega, L} : \pi_2(M, L) \rightarrow \mathbb{R}$. For a smooth map $u : (D^2, \partial D^2) \rightarrow (M, L)$ with $A = [u] \in \pi_2(M, L)$, choose a trivialization of the pull-back bundle as a symplectic vector bundle (which is unique up to the homotopy). $u^{-1}TM \cong D^2 \times \mathbb{C}^n$. This gives a smooth map $\tilde{u} : S^1 = \partial D^2 \rightarrow \Lambda(\mathbb{C}^n)$. Here $\Lambda(\mathbb{C}^n)$ denotes the Grassmann manifold of Lagrangian vector subspaces of \mathbb{C}^n . Using the Moslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$, we define $I_{\mu, L}(A) := \mu(\tilde{u})$. Another homomorphism $I_{\omega, L} : \pi_2(M, L) \rightarrow \mathbb{R}$ is defined by $I_{\omega, L}(A) := \int_{D^2} u^* \omega$. Note that $I_{\mu, L}$ is invariant under symplectic diffeomorphisms and $I_{\omega, L}$ is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.

A Lagrangian submanifold L of (M, ω) is called *monotone* ([15]) if $I_{\mu, L} = \lambda I_{\omega, L}$ ($\exists \lambda > 0$). If $I_{\mu, L} = 0$, we define $\Sigma_L = 0$. We assume that $I_{\mu, L} \neq 0$. Denote by $\Sigma_L \in \mathbb{Z}^+$ the positive generator of $\text{Im}(I_{\mu, L})$ as an additive subgroup of \mathbb{Z} . We call Σ_L is called the *minimal Maslov number* of LD .

Theorem 3.1 ([2], [21]). *Suppose that (M, ω, J, g) is an Einstein-Kähler manifold of positive Einstein constant. If L is a compact minimal Lagrangian submanifold of M , then L is monotone.*

Suppose that (M, ω, J, g) is a *simply connected* Einstein-Kähler manifold with positive Einstein constant and L is a compact monotone Lagrangian submanifold of M . Then Hajime Ono ([21]) showed the formula for Σ_L :

$$(3.1) \quad n_L \Sigma_L = 2\gamma_{c_1},$$

where we set

$$\begin{aligned} \gamma_{c_1} &:= \min\{c_1(M)(A) \mid A \in H_2(M; \mathbb{Z}), c_1(M)(A) > 0\}, \\ n_L &:= \min\{k \in \mathbb{Z}^+ \mid \otimes^k E \text{ trivial}\}. \end{aligned}$$

$$\begin{array}{ccc} E|_L & \longrightarrow & E \text{ cplx. line bdle.} \\ \pi_L \downarrow \text{flat} & & \pi_E \downarrow U(1)\text{-connection } \nabla \\ L & \xrightarrow{\text{Lag.}} & M \text{ Einstein-Kähler mfd.} \end{array}$$

Here $\frac{1}{\gamma} \omega = c_1(E, \nabla)$ for some constant $\gamma > 0$.

As an application of that formula (3.1), we mention the minimal Maslov number formula for the Gauss images of isoparametric hypersurfaces in the standard sphere.

Let $N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an oriented hypersurface of $S^{n+1}(1)$ and let $\hat{N}^n := \{(\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$ be the Legendrian lift of N^n to $T^1 S^{n+1}$. Then we have the following diagram:

$$\begin{array}{ccc}
N \subset S^{n+1}(1) & & \text{unit sphere tangent bundle of } S^{n+1} \\
& \parallel & = \text{Stiefel mfd. of o.n. 2-frames of } \mathbb{R}^{n+2} \\
\hat{L} = \hat{N} & \xrightarrow{\text{Leg.}} & P = T^1 S^{n+1} = V_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(n)} \\
\text{Gauss map } \mathcal{G} & \downarrow \rho(\pi_1(L)) & \downarrow \pi \quad SO(2) \cong U(1) \\
L = \mathcal{G}(N) & \xrightarrow{\text{Lag.}} & M = Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(2) \times SO(n)} \\
\text{Gauss image} & & \text{complex hyperquadric} \\
& & = \text{real ori. 2-plane Grassmann mfd.}
\end{array}$$

The *Gauss Map* is defined by

$$\begin{array}{ccccc}
\mathcal{G}: N & \longrightarrow & \hat{N} & \longrightarrow & \mathcal{G}(N) \subset Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \\
p & \longmapsto & (\mathbf{x}(p), \mathbf{n}(p)) & \longmapsto & [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p)
\end{array}$$

Suppose that $N^n \subset S^{n+1}(1)$ is an isoparametric hypersurface with g distinct principal curvatures. Let $\hat{N}^n := \{(\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$ be the Legendrian lift of N^n to $T^1 S^{n+1}$. The following diagram becomes as follows:

$$\begin{array}{ccc}
N \subset S^{n+1}(1) & & \text{simply conn.} \\
& \parallel & \text{Einstein Sasakian homog. sp.} \\
\hat{L} = \hat{N} & \xrightarrow{\text{min. Leg. embed.}} & P = V_2(\mathbb{R}^{n+2}) = \frac{SO(n+2)}{SO(n)} = T^1 S^{n+1} \\
\text{Gauss map } \mathcal{G} & \downarrow \rho(\pi_1(L)) & \downarrow \pi \quad SO(2) \\
& \cong \mathbb{Z}_g & \cong U(1) \\
\text{Gauss image } L = \mathcal{G}(N) & \xrightarrow{\text{min. Lag. embed.}} & M = Q_n(\mathbb{C}) = \frac{SO(n+2)}{SO(2) \times SO(n)} \\
& & \text{Einstein-Kähler Herm. sym. sp.} \\
& & \text{complex hyperquadric}
\end{array}$$

Then it holds $\Sigma_L = \frac{2n}{g}$ ([19], [13]). This formula was crucial to the Hamiltonian non-displaceability theorem for Gauss images of isoparametric hypersurfaces ([8]).

4 Minimal Maslov number of R -spaces canonically embedded in Einstein-Kähler C -spaces

We take the universal cover $\tilde{G} \rightarrow G$ of G . Let $(\tilde{G}, \tilde{K}, \theta)$ be a Riemannian symmetric pair of compact type with simply connected \tilde{G} and connected \tilde{K} .

By Proposition 2.3 we can choose $Z = Z^{ein} = 2\delta_m$. Then $\iota_Z : L = K/K_Z \rightarrow M = G/G_Z$ is the canonical embedding of an R -space into an Einstein-Kähler C -space. As in [24] we use expression

$$2\delta_m = \sum_{\alpha_i \in \Pi \setminus \Pi_Z} k_i \Lambda_i = \kappa(M) \sum_{\alpha_i \in \Pi \setminus \Pi_Z} \kappa_i \Lambda_i \quad (k_i \in \mathbb{Z}^+),$$

where $\kappa(M)$ denotes the greatest common divisor of $\{k_i \mid \alpha_i \in \Pi \setminus \Pi_Z\}$ and set $\kappa_i := \frac{k_i}{\kappa(M)}$ for each $\alpha_i \in \Pi \setminus \Pi_Z$. Then the invariant γ_{c_1} in (3.1) is given as $\gamma_{c_1} = \kappa(M)$ (cf. [20]).

$$\begin{array}{ccc}
\hat{L} = \tilde{K}/\tilde{K}'_Z & \xrightarrow{\text{tot.geod.Leg.}} & \hat{P} = \tilde{G}/\tilde{G}'_Z \\
\hat{\pi} \downarrow \rho(\pi_1(L)) & & \pi \downarrow U(1) \cong S^1 \\
L = \tilde{K}/\tilde{K}'_Z & \xrightarrow{\text{canon. embed.}} & M = \tilde{G}/\tilde{G}'_Z \\
R\text{-sp.} & & \text{Einstein-Kähler } C\text{-sp.} \\
& & \text{tot.geod.Lag.}
\end{array}$$

simply conn.
Einstein Sasakian homog. sp.

Here we take the orthogonal direct sum decomposition $\mathfrak{g}_Z = \mathbb{R} \cdot 2\delta_m \oplus \mathfrak{g}'_Z$. Denote by \tilde{G}'_Z a connected Lie subgroup of \tilde{G}_Z with Lie algebra \mathfrak{g}'_Z and set $\tilde{K}'_Z := \tilde{K} \cap \tilde{G}'_Z$. Then $n_L = \sharp(\tilde{K}_Z/\tilde{K}'_Z)$ ([20]). Therefore by the formula (3.1) we obtain

Theorem 4.1 ([20]). *The minimal Maslov number Σ_L of an R -space L canonically embedded in an Einstein-Kähler C -space M is given by the formula*

$$(4.1) \quad \Sigma_L = \frac{2\kappa(M)}{\sharp(\tilde{K}_H/\tilde{K}'_H)}.$$

Some concrete examples of computations by this formula are given in [20] in the case when (1) $(G, K) = (SU(n+1), SO(n+1))$ and L is $\mathbb{R}P^n$ or a regular R -space, (2) L is a maximal flag manifold of a compact semisimple Lie group K , (3) L is an irreducible symmetric R -space. It is an interesting problem to study the minimal Maslov number for all other R -spaces by this formula.

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