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Abstract. We introduce the concept of a homotopy motion of a subset in a manifold, and give a systematic study of homotopy motions of surfaces in closed oriented 3-manifold. This notion arises from various natural problems in 3-manifold theory such as domination of manifold pairs, homotopical behavior of simple loops on a Heegaard surface, and monodromies of virtual branched covering surface bundles associated to a Heegaard splitting.

Introduction

For a manifold M and a compact subspace Σ, a *motion* of Σ in M is an ambient isotopy of M of compact support that ends up with a homeomorphism preserving the subset Σ. The *motion group* $\mathcal{M}(M,\Sigma)$ of Σ in M is the group made up of the equivalence classes of such motions where the product is defined by concatenation of ambient isotopies. The concept of a motion has its origin in the braid group, which can be regarded as the motion group of a finite set in the plane. In his 1962 PhD thesis [27] supervised by Fox, Dahm developed a general theory of motions and calculated the motion group of a trivial link in the Euclidean space. In [35], Goldsmith published an exposition of Dahm's thesis, and in the succeeding paper [36], she obtained generators and relations of the motion groups of torus links in S^3. Since then (variations of) motion groups have been studied by many researchers in various settings. (See [14, 28, 32] and references therein.)

In the case where M is a closed, orientable 3-manifold and Σ is a Heegaard surface, Johnson-Rubinstein [52] and Johnson-McCullough [51] studied the (smooth) motion group $\mathcal{M}(M,\Sigma)$ and its quotient group $G(M,\Sigma)$ defined by

$$\begin{aligned} G(M,\Sigma) =& \{[f] \in \mathrm{MCG}(\Sigma) \mid \text{There exists a motion } \{f_t\}_{t\in I} \text{ with } j\circ f = f_1|_\Sigma.\} \\ =& \{[f] \in \mathrm{MCG}(\Sigma) \mid j\circ f : \Sigma \to M \text{ is ambient isotopic to } j.\}, \end{aligned}$$

where $\mathrm{MCG}(\Sigma)$ is the mapping class group of Σ and $j : \Sigma \to M$ is the inclusion map. These groups are also intimately related to the pairwise mapping class group

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MCG(M,Σ) first studied by Goeritz [34] in 1933, which has been attracting attention of various researchers. See Section 4 for a brief summary.

Motivated by Minsky's question [37] on the homotopical behavior of simple loops on a Heegaard surface in the ambient 3-manifold (see Question 0.1 below) and the second authors' joint work with Donghi Lee [62] on the corresponding problem for 2-bridge spheres of 2-bridge links (see the paragraph after Question 0.1), we are naturally lead to a homotopy version of the motion group $\mathcal{M}(M,\Sigma)$ and that of the group $G(M,\Sigma)$.

A *homotopy motion* of a closed surface Σ in a compact 3-manifold M is a homotopy $F=\{f_t\}_{t\in I}:\Sigma\times I\to M$, such that the initial end f_0 is the inclusion map $j:\Sigma\to M$ and the terminal end f_1 is an embedding with image Σ, where $f_t:\Sigma\to M$ ($t\in I=[0,1]$) is the continuous map from Σ to M defined by $f_t(x)=F(x,t)$. The *homotopy motion group* $\Pi(M,\Sigma)$ is the group of equivalence classes of homotopy motions of Σ in M, where the product is defined by concatenation of homotopies (see Section 2 for the precise definition). There is a natural homomorphism $\partial_+:\Pi(M,\Sigma)\to \mathrm{MCG}(\Sigma)$ which assigns (the equivalence class of) a homotopy motion with (the mapping class represented by) its terminal end. We denote the image of ∂_+ by $\Gamma(M,\Sigma)$. Then we have

$$\Gamma(M,\Sigma)=\{[f]\in \mathrm{MCG}(\Sigma)\mid j\circ f:\Sigma\to M \text{ is homotopic to the inclusion map } j.\}.$$

By denoting the kernel of ∂_+ by $\mathcal{K}(M,\Sigma)$, we have the following exact sequence.

$$1\longrightarrow \mathcal{K}(M,\Sigma)\longrightarrow \Pi(M,\Sigma)\xrightarrow{\partial_+}\Gamma(M,\Sigma)\longrightarrow 1. \tag{1}$$

In the case where M is a closed, orientable 3-manifold and Σ is a Heegaard surface, the above exact sequence is a homotopy version of the following exact sequence studied by Johnson-McCullough [51].

$$1\longrightarrow \pi_1(\mathrm{Diff}(M))\longrightarrow \mathcal{M}(M,\Sigma)\longrightarrow G(M,\Sigma)\longrightarrow 1, \tag{2}$$

where $\mathrm{Diff}(M)$ is the space of diffeomorphisms of M. (The smooth motion group $\mathcal{M}(M,\Sigma)$ corresponds to $\mathcal{H}_1(M,\Sigma)$ in [51], the fundamental group of the space $\mathcal{H}(M,\Sigma)$ of Heegaard surfaces equivalent to (M,Σ).)

The purpose of this paper is to give a systematic study of the homotopy motion group $\Pi(M,\Sigma)$ and the related groups in the exact sequence (1) for a closed, orientable surface Σ in a closed, orientable 3-manifold M.

Before stating the main results, we explain our motivation. Let Σ be a Heegaard surface of a closed, orientable 3-manifold M, and let V_1 and V_2 be the handlebodies obtained by cutting M along Σ. Let $\Gamma(V_i)$ be the kernel of the homomorphism $\mathrm{MCG}(V_i)\to \mathrm{Out}(\pi_1(V_i)))$ $(i=1,2)$. Now, let $\mathcal{S}(\Sigma)$ be the set of the isotopy classes of essential loops on Σ. Let $\Delta_i\subset\mathcal{S}(\Sigma)$ be the set of (isotopy classes of) meridians, i.e., the essential loops on Σ that bound disks in V_i. Set $\Delta:=\Delta_1\cup\Delta_2$. Let $Z\subset\mathcal{S}(\Sigma)$

be the set of (isotopy classes of) essential loops on Σ that are null-homotopic in M. In [37, Question 5.4], Minsky raised the following question.

Question 0.1. When is Z equal to the orbit $\langle \Gamma(V_1), \Gamma(V_2)\rangle\Delta$?

This question also makes sense for bridge spheres of links in S^3, and the question for 2-bridge spheres for 2-bridge links was solved affirmatively by Lee-Sakuma [62], which in turn led to the complete characterization of epimorphisms among 2-bridge knot groups by Aimi-Lee-Sakai-Sakuma [3, Theorem 8.1]), showing that any such epimorphism essentially arises from the construction by Ohtsuki-Riley-Sakuma [78]. This paper is motivated by the natural question to what extent these results (and the related results explained later) hold in a general setting.

To formulate our question explicitly, note that the group $\Gamma(V_i)$ is identified with the group $\Gamma(V_i, \Sigma) = \partial_+(\Pi(V_i, \Sigma)) < \mathrm{MCG}(\Sigma)$. So the subgroup $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ is contained in the group $\Gamma(M, \Sigma) = \partial_+(\Pi(M, \Sigma))$. In particular, we have

$$\langle \Gamma(V_1), \Gamma(V_2)\rangle\Delta \ \subset \ \Gamma(M, \Sigma)\Delta \ \subset \ Z.$$

This means that it would be more natural to work with the group $\Gamma(M, \Sigma)$ than to work with its subgroup $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ for Question 0.1.

Question 0.2. Let Σ be a Heegaard surface of a closed, orientable 3-manifold M.

(1) When is Z equal to the orbit $\Gamma(M, \Sigma)\Delta$?
(2) Let $\kappa : \mathcal{S}(\Sigma)/\Gamma(M, \Sigma) \to \mathcal{S}(\Sigma)/\simeq_M$ be the projection, where $\simeq_M$ is the equivalence relation on $\mathcal{S}(\Sigma)$ induced by homotopy in M, namely two essential simple loops of Σ are equivalent with respect to $\simeq_M$ if they are homotopic in M. Then how far is the map κ from being injective? In particular, when is the restriction of κ to $(\mathcal{S}(\Sigma) - Z)/\Gamma(M, \Sigma)$ injective?

The corresponding question for 2-bridge spheres for 2-bridge links were completely solved by Lee-Sakuma [62, 64]. (Below, we use the same symbol (M, Σ) for a 2-bridge decomposition by abusing notation.) The first question was solved affirmatively in [62] as already noted, and the second question was solved as follows: the restriction of κ to $(\mathcal{S}(\Sigma) - Z)/\Gamma(M, \Sigma)$ is injective except for the Whitehead link (for which, κ is injective with precisely two exceptional pairs). These results were applied to establish a variation of McShane's identity for hyperbolic 2-bridge links [63]. (See [60] for summary and see [61] for further extension.) A partial answer to Question 0.2 in a general setting was given by Ohshika-Sakuma [77, Theorems C and E]. In the companion paper [58], we give an answer for the special case where Σ is a genus-1 Heegaard surface of a lens space, and show that the results of [64] imply the following results for 2-bridge decompositions.

- For a hyperbolic 2-bridge link, equivalently, for the case where the Hempel distance of the 2-bridge sphere is ≥ 3, we have

$$\Gamma(M, \Sigma) = \langle \Gamma(V_1), \Gamma(V_2)\rangle.$$

For a 2-bridge torus link (except for the trivial knot and the two-component trivial link), equivalently for the case where the Hempel distance of the 2-bridge sphere is 2, we have

$$\Gamma(M,\Sigma) \gneqq \langle \Gamma(V_1), \Gamma(V_2)\rangle.$$

To be precise, the index $[\Gamma(M,\Sigma) : \langle \Gamma(V_1), \Gamma(V_2)\rangle]$ is 2, and the gap arises from the open book structure of the link complement whose binding is the axis of the 2-strand braid representing the 2-bridge torus link (see [64, p.5] and Section 5).

Moreover, in both cases, the image of $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ in the automorphism group of the curve complex of the 4-times punctured sphere is isomorphic to the free product of those of $\Gamma(V_1)$ and $\Gamma(V_2)$.

Thus the following question naturally arises.

Question 0.3. (1) When is the group $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ equal to $\Gamma(M,\Sigma)$?

(2) When is the group $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ equal to the free product $\langle \Gamma(V_1)\rangle * \langle \Gamma(V_2)\rangle$?

A partial answer to the second question was given Bowditch-Ohshika-Sakuma in [77, Theorem B] (cf. Bestvina-Fujiwara [9, Section 3]), which says that if the Hempel distance is large enough, then the orientation-preserving subgroup $\langle \Gamma^+(V_1), \Gamma^+(V_2)\rangle$ is equal to the free product $\langle \Gamma^+(V_1)\rangle * \langle \Gamma^+(V_2)\rangle$.

A main purpose of this paper is to give the following partial answer to Question 0.3(1).

Theorem 8.1. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M induced from an open book decomposition. If M has an aspherical prime summand, then we have $\langle \Gamma(V_1), \Gamma(V_2)\rangle \lneqq \Gamma(M,\Sigma)$.*

To prove this theorem we construct a $\mathbb{Z}^2$-valued invariant of $\Gamma(M,\Sigma)$, i.e., a map $\mathrm{Deg} : \Gamma(M,\Sigma) \to \mathbb{Z}^2$, such that its mod 2 reduction vanishes on the subgroup $\langle \Gamma(V_1), \Gamma(V_2)\rangle$. This actually comes from a natural invariant $\widehat{\mathrm{Deg}} : \Pi(M,\Sigma) \to \mathbb{Z}^2$, where the well-definedness of Deg is equivalent to the vanishing of $\widehat{\mathrm{Deg}}$ on the subgroup $\mathcal{K}(M,\Sigma)$.

An element α of $\mathcal{K}(M,\Sigma)$ is represented by a homotopy motion $F = \{f_t\}_{t\in I} : \Sigma \times I \to M$, such that both f_0 and f_1 are equal to the inclusion map $j : \Sigma \to M$. Thus F determines a continuous map $\hat{F} : \Sigma \times S^1 \to M$. Though the homotopy class of $\hat{F}$ is not always uniquely determined by $\alpha \in \mathcal{K}(M,\Sigma)$, its degree is uniquely determined by α, and so we have a homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ (Lemma 1.6). The map $\mathrm{Deg} : \Gamma(M,\Sigma) \to \mathbb{Z}^2$ is well-defined if and only if the homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ vanishes (see the paragraph just before Proposition 8.11). The problem of whether this condition holds can be regarded as a special case of the problem of dominations among 3-manifolds, which has been a subject of extensive literatures (see e.g. [89, 59, 76] and references therein).

Definition 0.4. We say that a closed, orientable surface Σ in a closed, orientable 3-manifold M (or a pair (M, Σ)) is *dominated by* $\Sigma \times S^1$ if there exists a map $\phi : \Sigma \times S^1 \to M$ such that $\phi|_{\Sigma \times \{0\}}$ is an embedding with image $\Sigma \subset M$ and that the degree of ϕ is non-zero.

Clearly, the homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ vanishes if and only if (M, Σ) is not dominated by $\Sigma \times S^1$. In this regard, we obtain the following theorem.

Theorem 6.1. *Let M be a closed, orientable 3-manifold, and suppose that Σ is a Heegaard surface for M.*

(1) *If M has an aspherical prime summand, then (M, Σ) is not dominated by $\Sigma \times S^1$.*
(2) *If $M = \#_g(S^2 \times S^1)$ for some non-negative integer g, or M admits the geometry of S^3 or $S^2 \times \mathbb{R}$, then (M, Σ) is dominated by $\Sigma \times S^1$.*

By Kneser-Milnor prime decomposition theorem [56, 70] (cf. [41, 46]), every closed, orientable 3-manifold M admits a unique prime decomposition, and by the geometrization theorem established by Perelman [80, 81, 82] (see [10, 19, 55, 73, 74] for exposition), each prime factor admits a unique decomposition by tori into geometric manifolds, i.e., those which have one of Thurston's 8 geometries. This together with the sphere theorem implies that for a closed, orientable 3-manifold M, the following three conditions are equivalent: (i) M is aspherical, (ii) M is irreducible and $\pi_1(M)$ is not finite, (iii) the universal covering space of M is homeomorphic to $\mathbb{R}^3$. Thus, for example, a 3-manifold having positive Grmov norm satisfies the condition of Theorem 6.1(1), for the Gromov norm is additive under connected sum. Conversely, if M is non-aspherical, then either M is admits the geometry of S^3 or $S^2 \times \mathbb{R}$, or M is non-prime (cf. [10, Chapter 1]). Here $\mathbb{RP}^3 \# \mathbb{RP}^3$ is the unique geometric 3-manifold which is non-prime. Thus, Theorem 6.1 especially implies that, for a prime 3-manifold M and its Heegaard surface Σ, (M, Σ) is dominated by $\Sigma \times S^1$ if and only if M is non-aspherical.

Theorem 6.1 guarantees the existence of the map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ when M has an aspherical prime summand, and Theorem 8.1 is proved by using this fact. Theorem 6.1 is in fact a consequence of the following two theorems.

Theorem 6.2. *Suppose that $M = \#_g(S^2 \times S^1)$ for some non-negative integer g, or M admits the geometry of S^3 or $S^2 \times \mathbb{R}$. Let Σ be a Heegaard surface for M. Then (M, Σ) is dominated by $\Sigma \times S^1$. Moreover, the following hold for the image of the homomorphism* $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$.

(1) *If $M = \#_g(S^2 \times S^1)$, then* $\deg(\mathcal{K}(M, \Sigma)) = \mathbb{Z}$.
(2) *If M admits the geometry of S^3, then* $\deg(\mathcal{K}(M, \Sigma)) \supset |\pi_1(M)| \cdot \mathbb{Z}$
(3) *If M admits the geometry of $S^2 \times \mathbb{R}$, then* $\deg(\mathcal{K}(M, \Sigma)) = \mathbb{Z}$ *or* $2\mathbb{Z}$ *according to whether $M = S^2 \times \mathbb{R}$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.*

Theorem 7.1. *Let M be a closed, orientable, aspherical 3-manifold, and Σ be a Heegaard surface for M. Then (M, Σ) is not dominated by $\Sigma \times S^1$. To be precise, the following hold.*

(1) *The group $\mathcal{K}(M, \Sigma)$ is isomorphic to the center $Z(\pi_1(M))$ of $\pi_1(M)$. Thus if M is a Seifert fibered space with orientable base orbifold, then $\mathcal{K}(M, \Sigma)$ is isomorphic to $\mathbb{Z}^3$ or $\mathbb{Z}$ according to whether M is the 3-torus T^3 or not. Otherwise, $\mathcal{K}(M, \Sigma)$ is the trivial group.*
(2) *The homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ vanishes.*

We remark here that Kotschick-Neofytidis [59, Theorem 1] proved that a closed, orientable 3-manifold M is dominated by a product $\Sigma \times S^1$ for some closed, orientable surface Σ if and only if M is finitely covered by either a product $F \times S^1$, for some aspherical surface F, or a connected sum $\#_g(S^2 \times S^1)$ for some non-negative integer g. (In [59] and the present paper, we employ the usual convention that the empty connected sum $\#_0(S^2 \times S^1)$ represents S^3.) Thus Theorem 7.1(2) follows from their results. The proof of Theorem 6.2, however, require more subtle arguments, for we impose that the product $\Sigma \times S^1$ dominates not only the manifold M itself but also the pair (M, Σ). Thus our constructions of dominating maps are quite different from those in [59].

In this paper, we also study incompressible surfaces in Haken manifolds. In Theorem 2.3 and Corollary 2.6, we completely describe the structures of their homotopy motion groups and related groups. The proof of that theorem is inspired by the work of Jaco-Shalen [47] (see also [46, Chapter VII]), where they introduced the concept of a *spatial deformation* of a subset Σ in the boundary of a manifold. The concept of a homotopy motion is also regarded as a variation of that of a spatial deformation. As in [47] and [46, Chapter 5], the proof of Theorem 2.3 uses the covering spaces of compact 3-manifolds corresponding to the surface fundamental groups, and it is based on the positive solution of Simon's conjecture [86] (see [18, Theorem 9.2]) concerning manifold compactifications of such covering spaces.

The opposite case where Σ is *homotopically trivial*, in the sense that the inclusion map $j : \Sigma \to M$ is homotopic to the contant map, is studied as well (see Theorem 3.2). In that case, we prove that if M is aspherical then $\Pi(M, \Sigma) \cong \pi_1(M) \times \mathrm{MCG}(\Sigma)$: the factors $\pi_1(M)$ and $\mathrm{MCG}(\Sigma)$ correspond to $\mathcal{K}(M, \Sigma)$ and $\Gamma(M, \Sigma)$, respectively. Conversely, if $\Gamma(M, \Sigma) = \mathrm{MCG}(\Sigma)$ then Σ is homotopically trivial provided that M is irreducible (Corollary 3.4).

Our interest in the group $\Gamma(M, \Sigma)$ has also its origin in the virtual branched fibration theorem, which says that, for every Heegaard surface Σ of a closed, orientable 3-manifold M, there exists a double branched covering of M such that the inverse image of Σ is the union of two fiber surfaces ([83, Addendum 1]). We show that this theorem is intimately related to the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ of $\Gamma(M, \Sigma)$. Let $\mathcal{I}(V_i)$ ($\subset \mathrm{MCG}(\Sigma)$) be the set of torsion elements of $\Gamma(V_i)$. By slightly refining the arguments of Zimmermann [90, Proof of Corollary 1.3], we can see that this is nothing but the set of vertical I-bundle involutions of V_i (Lemma 9.3). Here, a *vertical I-bundle involution* of a handlebody V is an involution h for which there exists an I-bundle structure of V such that h preserves each fiber setwise and acts on it as a reflection. We then prove the following refinement of [83, Addendum 1].

Theorem 9.1. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M. Then there exists a double branched covering $p : \tilde{M} \to M$ that satisfies the following conditions.*

(i) $\tilde{M}$ is a surface bundle over S^1 whose fiber is homeomorphic to Σ.
(ii) The preimage $p^{-1}(\Sigma)$ of the Heegaard surface Σ is a union of two (disjoint) fiber surfaces.

Moreover, the set $D(M, \Sigma)$ of monodromies of such bundles is equal to the set $\{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\}$, up to conjugation and inversion.

This paper is organized as follows. In Section 1, we give formal definitions of the homotopy motion group $\Pi(M, \Sigma)$, its subgroup $\mathcal{K}(M, \Sigma)$ and its quotient group $\Gamma(M, \Sigma)$, and present basic properties of these groups. Section 2 is devoted to the case where Σ is an incompressible surface in a Haken manifold M. Section 3 treats the opposite case where Σ is homotopically trivial. The remaining sections are devoted to the case where Σ is a Heegaard surface. In Section 4, we recall various natural subgroups of $\mathrm{MCG}(\Sigma)$ associated with a Heegaard surface, and describe their relationships with the group $\Gamma(M, \Sigma)$. In Section 5, we consider the Heegaard splitting obtained from an open book decomposition, and introduce two homotopy motions, the half book rotation ρ and the unilateral book rotation σ, which play key roles in the proofs of the main theorems given in the succeeding three sections. In Sections 6 and 7, we study the group $\mathcal{K}(M, \Sigma)$ of a Heegaard surface Σ of a closed, orientable 3-manifold M. In Section 8, we discuss gaps between $\Gamma(M, \Sigma)$ and the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$, and prove Theorem 8.1, which provides a partial answer to Question 0.3(1). In Section 9, we prove the branched fibration theorem (Theorem 9.1), which gives another motivation for defining and studying the group $\Gamma(M, \Sigma)$.

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1. The homotopy motion groups

Let X and Y be topological spaces. We denote by $C(X,Y)$ the space of continuous maps from X to Y, endowed with the compact-open topology. For a subspace A of X, we denote by $J(A,X)$ the subspace of $C(A,X)$ consisting of embeddings of A into X with image $A = j(A)$, where $j : A \to X$ is the inclusion map. For subspaces $A_1, \ldots, A_n$ of X, let $\mathrm{Homeo}(X, A_1, \ldots, A_n)$ denote the topological group of self-homeomorphisms of X that preserves each A_i ($1 \le i \le n$). By $\mathrm{MCG}(X, A_1, \ldots, A_n)$ we mean the *mapping class group* of $(X, A_1, \ldots, A_n)$, i.e., the group of connected components of $\mathrm{Homeo}(X, A_1, \ldots, A_n)$. We usually do not distinguish notationally between $f \in \mathrm{Homeo}(X, Y_1, \ldots, Y_n)$ and the element $[f] \in \mathrm{MCG}(X, A_1, \ldots, A_n)$ represented by f. Note that we allow orientation-reversing homeomorphisms when X is an orientable manifold, so our $\mathrm{MCG}(X, A_1, \ldots, A_n)$ is what is often called the extended mapping class group. A "plus" symbol, as in $\mathrm{MCG}^+(X, A_1, \ldots, A_n)$, indicates the subgroup, of index 1 or 2, consisting of the elements represented by orientation-preserving homeomorphisms of X.

Throughout the paper, we identify S^1 with $\mathbb{R}/\mathbb{Z}$. In our notation, we will not distinguish between an element of S^1 and its representative in $\mathbb{R}$.

Let Σ be a subspace of a manifold M, and $j : \Sigma \to M$ the inclusion map. In this section, we first introduce formal definitions of the homotopy motion group $\Pi(M, \Sigma)$, its subgroup $\mathcal{K}(M, \Sigma)$, and the quotient group $\Gamma(M, \Sigma)$. We then describe their basic properties especially when M is a 3-manifold and Σ is a closed, orientable surface.

Let $C(\Sigma, M)$ be the space of continuous maps from Σ to M, and $J(\Sigma, M)$ the subspace of $C(\Sigma, M)$ consisting of embeddings of Σ into M with image $j(\Sigma)$. We call a path

$$\alpha : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$$

a *homotopy motion* of Σ. We call the maps $\alpha(0)$ and $\alpha(1)$ from Σ to M the *initial end* and the *terminal end*, respectively, of the homotopy motion. Two homotopy motions $(I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$ are said to be *equivalent* if they are homotopic via a homotopy through maps of the same form. We usually do not distinguish notationally between a homotopy motion and its equivalence class. We define

$$\Pi(M, \Sigma) := \pi_1(C(\Sigma, M), J(\Sigma, M), j)$$

to be the set of equivalence classes of homotopy motions, as usual in the definition of relative homotopy groups $\pi_n(X, A, x_0)$ for $x_0 \in A \subset X$, where X is a topological space. In general, the relative fundamental group $\pi_1(X, A, x_0)$ is defined only when X is a topological group and A is a subgroup (cf. [35, Remark 2.7]). Inspired by the definition of the relative fundamental group, however, we equip $\Pi(M,\Sigma)$ with a group structure as in the following way.

Let α and β be homotopy motions. Then the *concatenation*

$$\alpha\cdot\beta : (I,\{1\},\{0\}) \to (C(\Sigma,M), J(\Sigma,M), \{j\})$$

of them is defined by

$$\alpha\cdot\beta(t) = \begin{cases} \alpha(2t) & (0\le t\le 1/2) \\ \beta(2t-1)\circ\alpha(1) & (1/2\le t\le 1). \end{cases}$$

We can easily check that the concatenation naturally induces a product of elements of $\pi_1(C(\Sigma,M), J(\Sigma,M))$. The *identity deformation* $e : (I,\{1\},\{0\}) \to (C(\Sigma,M), J(\Sigma,M), \{j\})$ defined by $e(t) = j$ $(t\in I)$ represents the identity element of $\Pi(M,\Sigma)$. The *inverse* $\bar{\alpha}$ of a homotopy motion α is defined by

$$\bar{\alpha}(t) = \alpha(1-t)\circ\alpha(1)^{-1},$$

where we regard $\alpha(1)$ as a self-homeomorphism of Σ, and $\alpha(1)^{-1}$ denotes its inverse. Note that the inverse of $[\alpha]$ in the group $\pi_1(C(\Sigma,M), J(\Sigma,M))$ is given by $[\bar{\alpha}]$.

Definition 1.1. We call the group $\Pi(M,\Sigma)$ the *homotopy motion group* of Σ in M.

Remark 1.2. When Σ is a single point x_0, $\Pi(M,\Sigma)$ is nothing but the fundamental group $\pi_1(M,x_0)$ of M. Thus, the group $\Pi(M,\Sigma)$ is a sort of generalization of $\pi_1(M,x_0)$. See also Theorem 3.2 below.

Notation 1.3. For a homotopy motion

$$\alpha : (I,\{1\},\{0\}) \to (C(\Sigma,M), J(\Sigma,M), \{j\})$$

we employ the following notation.

(1) We occasionally regard α as a continuous map $\Sigma\times I\to M$ defined by $\alpha(x,t) = \alpha(x)(t)$.
(2) When we regard α as a continuous family of maps, we occasionally write $\alpha = \{f_t\}_{t\in I}$ where $f_t = \alpha(t) : \Sigma\to M$.
(3) When α is a closed path, i.e., $\alpha(1) = \alpha(0) = j$, α induces a continuous map $\Sigma\times S^1\to M$, which we denote by $\hat{\alpha}$, that sends $(x,t)\in\Sigma\times S^1$ to $\alpha(t)(x) = \alpha(x,t)$. The homotopy class of this map relative to $\Sigma\times\{0\}$ is uniquely determined by the element $[\alpha]\in\pi_1(C(\Sigma,M), j)$.

Since the inclusion map j is nothing but the identity if we think of the codomain of j as Σ, $J(\Sigma,M)$ can be canonically identified with $\mathrm{Homeo}(\Sigma)$. Thus, the terminal end $\alpha(1) = f_1$ of a homotopy motion $\alpha = \{f_t\}_{t\in I}$ can be regarded as an element of $\mathrm{Homeo}(\Sigma)$. Therefore, we obtain a map

$$\partial_+ : \Pi(M,\Sigma)\to \mathrm{MCG}(\Sigma)$$

by taking the equivalence class of a homotopy motion $\alpha = \{f_t\}_{t\in I}$ to the mapping class of $\alpha(1) = f_1\in\mathrm{Homeo}(\Sigma)$. Clearly, this map is a homomorphism. (To be precise, this holds when we think of $\mathrm{Homeo}(\Sigma)$ as acting on X from the right: under the usual convention where $\mathrm{Homeo}(\Sigma)$ acts on X from the left, which we employ in this paper, the map ∂_+ is actually an anti-homomorphism.)

Definition 1.4. We denote the image of ∂_+ by $\Gamma(M,\Sigma)$. Namely, $\Gamma(M,\Sigma)$ is the subgroup of the extended mapping class group $\mathrm{MCG}(\Sigma)$ defined by

$$\begin{aligned}\Gamma(M,\Sigma) &= \{[f]\in \mathrm{MCG}(\Sigma) \mid \text{There exists a homotopy motion } \{f_t\}_{t\in I} \text{ with } f=f_1.\}\\ &= \{[f]\in \mathrm{MCG}(\Sigma) \mid j\circ f:\Sigma\to M \text{ is homotopic to the inclusion map } j.\}.\end{aligned}$$

The kernel of ∂_+ is denoted by $\mathcal{K}(M,\Sigma)$: thus we have the exact sequence (1) in the introduction.

Throughout the remainder of this paper, Σ denotes a connected, closed, orientable surface embedded in a connected, orientable 3-manifold M, and $j:\Sigma\to M$ denotes the inclusion.

We now provide a few basic properties concerning the groups defined in the above, by using elementary arguments in homotopy theory.

Note that we have the following long exact sequence.

$$\begin{aligned}\cdots\to \pi_1(J(\Sigma,M),j) &\xrightarrow{\mathscr{I}} \pi_1(C(\Sigma,M),j)\to \pi_1(C(\Sigma,M),J(\Sigma,M),j)\\ &\to \pi_0(J(\Sigma,M))\to \pi_0(C(\Sigma,M)).\end{aligned}$$

The boundary map $\pi_1(C(\Sigma,M),J(\Sigma,M),j)\to\pi_0(J(\Sigma,M))$ respects the group structures of $\Pi(M,\Sigma)=\pi_1(C(\Sigma,M),J(\Sigma,M),j)$ and $\mathrm{MCG}(\Sigma)=\pi_0(J(\Sigma,M))$, and it is identical with the (anti-) homomorphism ∂_+. Thus we have the following description of the kernel $\mathcal{K}(M,\Sigma)$.

Lemma 1.5. *We have the isomorphism*

$$\mathcal{K}(M,\Sigma)\cong \pi_1(C(\Sigma,M),j)/\mathscr{I}(\pi_1(J(\Sigma,M),j)).$$

Moreover, if the genus of Σ is at least 2, then we have

$$\mathcal{K}(M,\Sigma)\cong \pi_1(C(\Sigma,M),j).$$

Proof. The first assertion is a direct consequence of the exact sequence. The second assertion follows from the fact that $J(\Sigma,M)$ can be canonically identified with $\mathrm{Homeo}(\Sigma)$ as discussed before, and the result of Hamstrom [39] that $\pi_1(J(\Sigma,M),j)$ is the trivial group when the genus of Σ is at least 2. □

Lemma 1.6. *Suppose that M is an oriented, closed 3-manifold and Σ is a closed, oriented surface in M. Then there is a homomorphism* $\deg:\mathcal{K}(M,\Sigma)\to\mathbb{Z}$ *such that* $\deg(\alpha)$ *is the degree of the map $\hat{\alpha}:\Sigma\times S^1\to M$ defined in Notation* 1.3.

Proof. As noted in Notation 1.3(3), each element α in $\pi_1(C(\Sigma,M),j)$ determines a continuous map $\hat{\alpha}:\Sigma\times S^1\to M$ whose homotopy class is uniquely determined by the element $\alpha\in\pi_1(C(\Sigma,M),j)$. Thus we obtain a homomorphism from

$\pi_1(C(\Sigma, M), j)$ to $\mathbb{Z}$ which sends α to the degree of $\hat{\alpha}$. If α belongs to the subgroup $\mathscr{I}(\pi_1(J(\Sigma, M), j))$, then image$(\hat{\alpha}) = \Sigma \times \{0\}$ and therefore α belongs to the kernel of the homomorphism. Hence it descends to the desired homomorphism

$$\deg : \mathcal{K}(M, \Sigma) \cong \pi_1(C(\Sigma, M), j)/\mathscr{I}(\pi_1(J(\Sigma, M), j)) \to \mathbb{Z}.$$

□

For an element $\alpha \in \mathcal{K}(M, \Sigma)$, we call the value $\deg(\alpha) \in \mathbb{Z}$ the *degree* of α.

The following lemma gives a characterization of the group $\Gamma(M, \Sigma)$ in terms of the induced homomorphisms between the fundamental groups. This lemma will be used in the companion paper [58] to determine the group $\Gamma(M, \Sigma)$ when M is a lens space and Σ is its genus-1 Heegaard surface.

Lemma 1.7. *Let Σ be a closed, orientable surface embedded in a 3-manifold M. Then for every mapping class $[f] \in \Gamma(M, \Sigma) < \mathrm{MCG}(\Sigma)$, the homomorphism $(j \circ f)_* : \pi_1(\Sigma) \to \pi_1(M)$ is equal to the homomorphism j_* modulo post composition of an inner-automorphism of $\pi_1(M)$. Moreover, if M is irreducible then the above condition is also a sufficient condition for a mapping class $[f] \in \mathrm{MCG}(\Sigma)$ to be contained in $\Gamma(M, \Sigma)$.*

Proof. The first assertion is obvious from the definition of $\Gamma(M, \Sigma)$. To prove the second assertion, assume that M is irreducible and let $[f] \in \mathrm{MCG}(\Sigma)$ be a mapping class satisfying the condition. We may choose the representative f so that it fixes a base point $x_0 \in \Sigma \subset M$. Then the condition means that there is a closed path $\delta : (I, \partial I) \to (M, x_0)$ such that $(j \circ f)_* : \pi_1(\Sigma, x_0) \to \pi_1(M, x_0)$ is equal to the composition of j_* and the inner-automorphism of $\pi_1(M, x_0)$ determined by $[\delta] \in \pi_1(M, x_0)$, namely, the following holds.

- For any closed path $\gamma : (I, \partial I) \to (M, x_0)$, the path $f \circ \gamma$ is homotopic rel. ∂I to the product path $\delta^{-1} \cdot \gamma \cdot \delta$.

We want to construct a map $F = \{f_t\}_{t \in I} : \Sigma \times I \to M$ such that $f_0 = j$ and $f_1 = f$. The condition for F determines the restriction, F', of the desired map to the subspace $Y := \Sigma \times \{0, 1\}$. Our task is to extend F' to the whole space $\Sigma \times I$. This can be done through an elementary argument in homotopy theory as follows. Consider a (standard) cellular decomposition of Σ, consisting of the single 0-cell $e^0 = x_0$, 1-cells $e^1_1, \ldots, e^1_{2g}$ (g is the genus of Σ), and a single 2-cell e^2. Let $\mathcal{C}^3$ be the product cellular decomposition of $\Sigma \times I$, where the I-factor is endowed with the cellular decomposition consisting of two 0-cells $\{i\}$ $(i = 0, 1)$ and a single 1-cell $[0, 1]$. Then $\mathcal{C}^3$ consists of 0-cells $e^0 \times \{i\}$, 1-cells $e^1_k \times \{i\}$, $e^0 \times [0, 1]$, 2-cells $e^2 \times \{i\}$, $e^1_k \times [0, 1]$, and the unique 3-cell $e^2 \times [0, 1]$, where $i, j \in \{0, 1\}$ and $k \in \{1, \ldots, 2g\}$. Note that all 1-cells, except for $e^0 \times [0, 1]$, and all 2-cells, except for $e^1_k \times [0, 1]$, are contained in the domain of F'. We first extend F' to the map $F^{(1)}$ from the 1-skeleton of $\mathcal{C}^3$, by defining its restriction to the unique remaining 1-cell $e^0 \times [0, 1]$ by using the path δ, namely $F^{(1)}(e^0, t) = \delta(t)$. The condition for f stated in the above implies that the restriction of $F^{(1)}$ to the boundary of each of the remaining 2-cell $e^1_k \times [0,1]$ is homotopic to a constant map. Hence we can extend $F^{(1)}$ to the 2-skeleton of $\mathcal{C}^3$. The resulting map can be extended to the whole $\mathcal{C}^3$, because the assumption that M is irreducible together with the sphere theorem implies that $\pi_2(M) = 0$. □

The next lemma plays important roles in the proofs of Theorems 2.3, 3.2 and 7.1.

Lemma 1.8. *Let Σ be a closed, orientable surface embedded in a 3-manifold M, and $x_0 \in \Sigma$. Then there exists a homomorphism*

$$\Phi : \pi_1(C(\Sigma, M), j) \to Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0)),$$

where the target is the centralizer of $j_*(\pi_1(\Sigma, x_0))$ in $\pi_1(M, x_0)$, which maps $[\alpha] \in \pi_1(C(\Sigma, M), j)$ to the element of $\pi_1(M, x_0)$ represented by the loop

$$(I, \partial I) \to (M, \{x_0\}),\ t \mapsto \alpha(t)(x_0).$$

Moreover, if M is aspherical, then Φ is injective.

Proof. For an element $[\alpha] \in \pi_1(C(\Sigma, M), j)$, let $\hat{\alpha} : \Sigma \times S^1 \to M$ be the map defined in Notation 1.3(3). Let w be the element of $\pi_1(\Sigma \times S^1, (x_0, 0)) = \pi_1(\Sigma, x_0) \times \pi_1(S^1, 0)$ representing the generator of $\pi_1(S^1, 0)$. Then $\hat{\alpha}_*(w)$ belongs to $Z(j_*(\pi_1(\Sigma)), \pi_1(M, x_0))$, and it is represented by the based loop $t \mapsto \alpha(t)(x_0)$. Since the homotopy class of $\hat{\alpha}$ relative to $(x_0, 0)$ is uniquely determined by $[\alpha] \in \pi_1(C(\Sigma, M), j)$, we obtain the desired homomorphism Φ.

To prove the second assertion, assume that M is aspherical, and let $[\alpha]$ be an element of $\pi_1(C(\Sigma, M), j)$ which is contained in $\ker \Phi$. We want to construct a homotopy between the path α and the constant path e in $C(\Sigma, M)$ relative to the endpoints. By regarding α and e as maps from $\Sigma \times I$ to M (cf. Notation 1.3(1)), this is equivalent to constructing a continuous map $H = \{h_s\}_{s \in I} : (\Sigma \times I) \times I \to M$ such that $h_0 = \alpha$, $h_1 = e$, and $h_s|_{\Sigma \times \{0\}} = h_s|_{\Sigma \times \{1\}} = j$ for every $s \in I$ where we identify $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ with Σ; namely, $H(x, t, 0) = \alpha(x, t)$, $H(x, t, 1) = x$, and $H(x, 0, s) = H(x, 1, s) = x$ for every $x \in \Sigma$ and $t, s \in I$. The condition for H determines the restriction, H', of the desired map H to the subspace $Y := (\Sigma \times I \times \{0,1\}) \cup (\Sigma \times \{0,1\} \times I)$. Our task is to extend the map H' to the whole space $(\Sigma \times I) \times I$. To this end, consider a (standard) cellular decomposition of Σ in the proof of Lemma 1.7, and let $\mathcal{C}^4$ be the product cellular decomposition of $(\Sigma \times I) \times I$. Then $\mathcal{C}^4$ consists of 0-cells

$$e^0 \times \{i\} \times \{j\},$$

1-cells

$$e^1_k \times \{i\} \times \{j\},\ e^0 \times [0,1] \times \{j\},\ e^0 \times \{i\} \times [0,1],$$

2-cells

$$e^2 \times \{i\} \times \{j\},\ e^1_k \times [0,1] \times \{j\},\ e^1_k \times \{i\} \times [0,1],\ e^0 \times [0,1] \times [0,1],$$

and several 3 and 4-cells, where $i, j \in \{0, 1\}$ and $k \in \{1, \ldots, 2g\}$. Note that all 0-cells, 1-cells, and 2-cells, except for $e^0 \times [0,1] \times [0,1]$ are contained in the domain Y of the map H'. By the assumption that $[\alpha]$ belongs $\ker \Phi$, the restriction of H' to the boundary of the 2-cell $e^0 \times [0,1] \times [0,1]$ is homotopic to a constant map. So, we can extend the map H' to the 2-cell. Since M is aspherical, there is no obstruction to extending the map over 3 and 4-cells. Consequently, we have $[\alpha] = [e] \in \pi_1(C(\Sigma, M), j)$. □

2. The homotopy motion groups of incompressible surfaces in Haken manifolds

In this section, we consider the groups $\Pi(M, \Sigma)$ and $\Gamma(M, \Sigma)$ in the case where Σ is a closed, orientable, incompressible surface in a closed, orientable Haken manifold M. Let us begin with two examples of non-trivial elements of $\Pi(M, \Sigma)$. We will see soon in Theorem 2.3 that they are in fact the only elements necessary to generate $\Pi(M, \Sigma)$.

Example 2.1. Let φ be an element of $\mathrm{MCG}(\Sigma)$. Consider the 3-manifold $M := \Sigma \times \mathbb{R}/(x,t) \sim (\varphi(x), t+1)$, which is the Σ-bundle over S^1 with monodromy φ. We denote the image of $\Sigma \times \{0\}$ in M by the same symbol Σ and call it a *fiber surface*. Then we have a natural homotopy motion $\lambda = \{f_t\}$ of Σ in M defined by $f_t(x) = [x, t]$, where $[x, t]$ is the element of M represented by (x, t). Its terminal end is equal to φ^{-1}, because $f_1(x) = [x, 1] = [\varphi^{-1}(x), 0] = \varphi^{-1}(x)$. Thus φ belongs to $\Gamma(M, \Sigma)$.

Example 2.2. Let h be an orientation-reversing free involution of a closed, orientable surface Σ. Consider the 3-manifold $N := \Sigma \times [0,1]/(x,t) \sim (h(x), 1-t)$, which is the orientable twisted I-bundle over the closed, non-orientable surface Σ/h. The boundary ∂N is identified with Σ by the homeomorphism $\Sigma \to \partial N$ mapping x to $[x, 0]$, where $[x, t]$ denotes the element of N represented by (x, t). Then we have a natural homotopy motion $\mu = \{f_t\}_{t \in I}$ of $\Sigma = \partial N$ in N, defined by $f_t(x) = [x, t]$. Its terminal end is equal to h, because $f_1(x) = [x, 1] = [h(x), 0] = h(x)$ for every $x \in \Sigma = \partial N$. Let N' be any compact, orientable 3-manifold whose boundary is identified with Σ, i.e., a homeomorphism $\partial N' \cong \Sigma$ is fixed, and let $M = N \cup N'$ be the closed, orientable 3-manifold obtained by gluing N and N' along the common boundary Σ. Then the homotopy motion $\mu = \{f_t\}_{t \in I}$ of Σ in N defined as above can be regarded as that of Σ in M, and thus h is an element of $\Gamma(M, \Sigma)$. If N' is also a twisted I-bundle associated with an orientation-reversing involution h' of Σ, then we have another homotopy motion μ' of Σ in N' with terminal end $h' \in \Gamma(M, \Sigma)$.

The following theorem is proved by using the positive solution of Simon's conjecture [86] concerning manifold compactifications of covering spaces, with finitely generated fundamental groups, of compact 3-manifolds, which in turn is proved by using the geometrization theorem established by Perelman [80, 81, 82] and the tameness theorem of hyperbolic manifolds established by Agol [2] and Calegari-Gabai [17] (see also Soma [87] and Bowditch [13]). A proof of Simon's conjecture can be found in Canary's expository article [18, Theorem 9.2], where he attributes it to Long and Reid.

Theorem 2.3. *Let M be a closed, orientable Haken manifold, and suppose that Σ is a closed, orientable, incompressible surface in M. Then the following hold.*

(1) *If M is a Σ-bundle over S^1 with monodromy φ and Σ is a fiber surface, then $\Pi(M, \Sigma)$ is the infinite cyclic group generated by the homotopy motion λ described in Example* 2.1.
(2) *If Σ separates M into two submanifolds, M_1 and M_2, precisely one of which is a twisted I-bundle, then $\Pi(M, \Sigma)$ is the order-2 cyclic group generated by the homotopy motion μ described in Example* 2.2.
(3) *If Σ separates M into two submanifolds, M_1 and M_2, both of which are twisted I-bundles, then $\Pi(M, \Sigma)$ is the infinite dihedral group generated by the homotopy motions μ and μ' described in Example* 2.2.
(4) *Otherwise, $\Pi(M, \Sigma)$ is the trivial group.*

To show the above theorem, we require the following two lemmas.

Lemma 2.4. *Let Σ be a closed, orientable surface of genus at least* 1. *Then*

$$\Pi(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1.$$

Proof. Consider the projection $q : \Sigma \times \mathbb{R} \to \Sigma \times \{0\}$. Then for any homotopy motion $\alpha = \{f_t\}_{t \in I}$ of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$, the composition $\{q \circ f_t\}_{t \in I}$ is a homotopy of maps from $\Sigma \times \{0\}$ to itself with initial end $\mathrm{id}_{\Sigma \times \{0\}}$ and terminal end f_1. It follows form Baer [5] (cf. [29, Theorem 1.12]) that f_1 is isotopic to $\mathrm{id}_{\Sigma \times \{0\}}$. Hence $\Gamma(\Sigma \times \mathbb{R}, \Sigma \times \{0\})$ is trivial, and so $\Pi(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = \mathcal{K}(\Sigma \times \mathbb{R}, \Sigma \times \{0\})$.

Suppose first that the genus of Σ is at least 2. Then $\mathcal{K}(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) \cong \pi_1(C(\Sigma \times \{0\}, \Sigma \times \mathbb{R}), j)$ by Lemma 1.5, and this group admits an embedding into $Z(j_*(\pi_1(\Sigma \times \{0\})), \pi_1(\Sigma \times \mathbb{R})) \cong Z(\pi_1(\Sigma))$ by Lemma 1.8. Since $Z(\pi_1(\Sigma)) = 1$, we have $\mathcal{K}(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1$.

Suppose next that Σ is the torus. Then, by Lemmas 1.5 and 1.8, $\mathcal{K}(\Sigma \times \mathbb{R}, \Sigma \times \{0\})$ admits an embedding into the quotient of the centralizer $Z(j_*(\pi_1(\Sigma \times \{0\})), \pi_1(\Sigma \times \mathbb{R})) \cong Z(\pi_1(\Sigma))$ by $\Phi(\mathscr{I}(\pi_1(J(\Sigma \times \{0\}, \Sigma \times \mathbb{R}))))$, where $\mathscr{I}$ and Φ are homomorphisms in Lemmas 1.5 and 1.8, respectively. Now, identify Σ with $\mathbb{R}^2/\mathbb{Z}^2$, and denote by $[x, y]$ the point of Σ represented by $(x, y) \in \mathbb{R}^2$. For an element $(m, n) \in \mathbb{Z}^2 = \pi_1(\Sigma)$, let $\xi_{m,n} = \{g_t\}_{t \in I}$ be the ambient isotopy of Σ, defined by $g_t([x, y]) = ([x + mt, y + nt)])$, and regard it as an element of $\pi_1(J(\Sigma \times \{0\}, \Sigma \times \mathbb{R}))$. Then we can easily check that $\Phi(\mathscr{I}(\xi_{m,n})) = (m, n)$. Thus $\Phi \circ \mathscr{I}$ is surjective, and therefore we again have $\mathcal{K}(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1$. Hence $\Pi(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1$ as desired. □

Lemma 2.5. *Let Σ be a closed, orientable surface of genus at least* 1. *For* $t \in [0,1]$, *let* $j_t : \Sigma \to \mathbb{R}$ *be the embedding defined by* $j_t(x) = (x,t)$, *and set* $\kappa := \{j_t\}_{t\in I} : I \to C(\Sigma, \Sigma \times \mathbb{R})$. *Then any continuous map* $\alpha = \{f_t\}_{t\in I} : (I, \{1\}, \{0\}) \to (C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$ *is homotopic to* κ *via a homotopy through maps of the same form.*

Proof. Let $\alpha = \{f_t\}_{t\in I}$ be a continuous map from $(I, \{1\}, \{0\})$ to $(C(\Sigma, \Sigma\times\mathbb{R}), J(\Sigma\times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$. Let $q : \Sigma \times \mathbb{R} \to \Sigma$ be the projection. Then, by Baer [5], $q \circ \alpha(1) = q \circ f_1$ is isotopic to the identity map as a self-homeomorphism of Σ. Thus deforming α by a homotopy through maps $(I, \{1\}, \{0\}) \to (C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$ if necessary, we may assume that $\alpha(1) = f_1 = j_1$.

Consider the path

$$a : (I, \{1\}, \{0\}) \to (\Sigma \times \mathbb{R}, \{(x_0, 0)\}, \{(x_0, 1)\}),\ t \mapsto f_t(x_0).$$

We see that the closed loop $q \circ a$ represents an element of the center $Z(\pi_1(\Sigma))$, through an argument as in the first part of the proof of Lemma 1.8. (Here, we use the map $\bar{\alpha} : \Sigma \times S^1 \to \Sigma$ defined by $\bar{\alpha}(x,t) = q(f_t(x))$ instead of α itself.) If Σ has genus at least 2, then $q \circ a$ represents the trivial element of $\pi_1(\Sigma)$. If Σ is a torus, using the ambient isotopy $\xi_{m,n}$ for $(m,n) \in \mathbb{Z}^2$ defined in the proof of Lemma 2.4, we can deform α by a homotopy through maps $(I, \{1\}, \{0\}) \to (C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$ so that $q \circ a$ represents the trivial element of $\pi_1(\Sigma)$. Hence the path a is homotopic to the path

$$b : (I, \{1\}, \{0\}) \to (\Sigma \times \mathbb{R}, \{(x_0, 0)\}, \{(x_0, 1)\}),\ t \mapsto j_t(x_0)$$

relative to the end points.

The remaining arguments, which we briefly describe here, actually run in the same way as those of Lemma 1.8. Our aim is to construct a continuous map $H = \{\varphi_s\}_{s\in I} : (\Sigma \times I) \times I \to M$ such that $\varphi_0(x,t) = f_t(x)$, $\varphi_1(x,t) = j_t(x) = (x,t)$ for any $x \in \Sigma$, $t \in I$, and $\varphi_s(x,0) = (x,0)$, $\varphi_s(x,1) = (x,1)$ for any $x \in \Sigma$. Consider the product cellular decomposition $\mathcal{C}^4$ of $(\Sigma \times I) \times I$ as in the proof of Lemma 1.8. The condition for H determines the restriction, H', of the desired map H to the subspace consisting of all 0-cells, 1-cells, and 2-cells, except for $e^0 \times [0,1] \times [0,1]$. Since a and b are homotopic relative to the endpoints, the restriction of H' to the boundary of the 2-cell $e^0 \times [0,1] \times [0,1]$ is homotopic to a constant map. So, we can extend H' to the 2-cell. Since $\Sigma \times \mathbb{R}$ is aspherical, there is no obstruction to extending the resulting map over 3 and 4-cells. This completes the proof. $\square$

Proof of Theorem 2.3. Let $p : \tilde{M} \to M$ be the covering corresponding to $\pi_1(\Sigma) < \pi_1(M)$. Then, by the positive solution of Simon's conjecture (see [18, Theorem 9.2]), $\tilde{M}$ admits a manifold compactification, that is, there exists a compact 3-manifold $\hat{M}$ with boundary, such that $\tilde{M}$ is homeomorphic to $\hat{M} - \hat{C}$, where $\hat{C}$ is a closed subset of $\partial\hat{M}$. We actually have $\hat{C} = \partial\hat{M}$ and $\tilde{M} \cong \operatorname{Int}\hat{M}$, because M is closed. Brown's theorem [16, Theorem 3.4] implies that $\hat{M} \cong \Sigma \times [-\infty, \infty]$, where $[-\infty, \infty]$

is the closed interval that is obtained by compactifying $\mathbb{R} = (-\infty, \infty)$. Thus $\tilde{M}$ is identified with $\Sigma \times \mathbb{R}$. We assume that the restriction of the covering projection p to $\Sigma \times \{0\}$ is given by $p(x, 0) = x$. In other words, the inclusion map $j : \Sigma \to M$ has a lift $\tilde{j} : \Sigma \to \tilde{M} = \Sigma \times \mathbb{R}$ such that $\tilde{j}(x) = (x, 0)$.

Suppose that $\Pi(M, \Sigma)$ has a nontrivial element $\alpha = \{f_t\}_{t \in I}$ with $f_1 = f$. Let $\tilde{\alpha} = \{\tilde{f}_t\}_{t \in I}$ be the lift of α with $\tilde{f}_0 = \tilde{j}$, and let $\tilde{f}$ be the lift of f defined by $\tilde{f} = \tilde{f}_1$. Since $f(\Sigma) = \Sigma$, the image $\tilde{f}(\Sigma)$ is a component of $p^{-1}(\Sigma)$ to which the restriction of p is a homeomorphism onto Σ. In particular, we have either $\tilde{f}(\Sigma) = \Sigma \times \{0\}$ or $\tilde{f}(\Sigma) \cap (\Sigma \times \{0\}) = \emptyset$. If $\tilde{f}(\Sigma) = \Sigma \times \{0\}$, the map $\tilde{\alpha}$ is homotopic to the constant map from I to $\tilde{j} \in C(\Sigma, \tilde{M})$ via a homotopy through maps $(I, \{1\}, \{0\}) \to (C(\Sigma, \tilde{M}), J(\Sigma \times \{0\}, \tilde{M}) \circ \tilde{j}, \{\tilde{j}\})$ by Lemma 2.4. This homotopy projects to a homotopy from α to the trivial homotopy motion of $\Sigma \subset M$. This contradicts the assumption that α is a nontrivial element of $\Pi(M, \Sigma)$. Therefore we have $\tilde{f}(\Sigma) \cap (\Sigma \times \{0\}) = \emptyset$. Then, by [16], $\tilde{f}(\Sigma)$ is parallel to $\Sigma \times \{0\}$ in $\tilde{M} = \Sigma \times \mathbb{R}$. We may choose the product structure so that $\tilde{f}(\Sigma) = \Sigma \times \{1\}$. Since p is a covering and since Σ is incompressible, we see that $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1))$ is a finite disjoint union of compact surfaces that are incompressible in $\tilde{M}$ and so in $\Sigma \times \mathbb{R}$. The result of [16] implies that all components of $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1))$ are parallel to $\Sigma \times \{0\}$ in $\tilde{M}$. Hence there exists a component that is closest to $\Sigma \times \{0\}$. We choose α so that $\tilde{f}(\Sigma) = \Sigma \times \{1\}$ is the closest component, namely $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1)) = \emptyset$.

Fix an orientation of the surface $\Sigma \subset M$, and orient the surfaces $\Sigma \times \{t\} \subset \tilde{M} = \Sigma \times \mathbb{R}$ ($t \in \mathbb{R}$) via the canonical identification with the oriented Σ. Consider the homeomorphism $\psi : \Sigma \times \{0\} \to \Sigma \times \{1\}$ defined by $\psi = (p|_{\Sigma \times \{1\}})^{-1} \circ p|_{\Sigma \times \{0\}}$. It should be noted that ψ is a "local covering transformation", in the sense that ψ extends to a homeomorphism between a neighborhoods of $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ that commutes with the covering projection p. Let $q : \Sigma \times \mathbb{R} \to \Sigma$ be the projection to the first factor, which is identified with the surface Σ in M.

Case 1. Suppose that ψ is orientation-preserving. Consider the 3-manifold $M' := (\Sigma \times [0, 1])/(x, 0) \sim (q(\psi(x)), 1)$. Then the restriction of the covering projection p to $\Sigma \times [0, 1]$ descends to a continuous map $p' : M' \to M$, which is a local-homeomorphism at the image of $\Sigma \times (0, 1)$ in M'. The condition that ψ is orientation-preserving implies that p' is also a local homeomorphism at an open neighborhood of the image of $\Sigma \times \{0\}$ (which is equal to that of $\Sigma \times \{1\}$) in M'. (Here, we use the fact that ψ is a local covering transformation.) Thus $p' : M' \to M$ is a local homeomorphism. Since M is a compact, connected manifold, it follows that p' has the path-lifting property, and hence p' is a covering (see e.g. Forster [30, Theorem 4.19]). Since $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1)) = \emptyset$, the preimage of a point in $\Sigma \subset M$ by p' is a singleton. Hence p' is a homeomorphism and so M is identified with the Σ-bundle $(\Sigma \times \mathbb{R})/(x, t) \sim (\varphi(x), t + 1)$ over S^1, where the monodromy φ is defined by $\varphi = q \circ \psi$.

Case 2. Suppose that ψ is orientation-reversing. Consider the submanifold $M' := \Sigma \times [0,1]$ of $\tilde{M}$ and its image $M_1 := p(M')$ in M. Since $p^{-1}(\Sigma) \cap (\Sigma \times (0,1)) = \emptyset$, $p(\operatorname{int} M')$ is disjoint from Σ. This together with the assumption that ψ is orientation-reversing implies that M_1 is a submanifold of M with boundary Σ. Moreover, as in Case 1, we can see that the restriction $p' : M' \to M_1$ of p to M' is a covering and that it has geometric degree 2. (Note that the preimage of a point $x \in \Sigma = \partial M_1$ by p' consists of the two points $(x, 0)$ and $(\psi(x), 1)$ of $M' = \Sigma \times [0, 1]$.) By [41, Theorem 10.3], this implies that M_1 is a twisted I-bundle, $\Sigma \times [0, 1]/(x, t) \sim (h(x), 1 - t)$, associated with some orientation-reversing free involution h of Σ.

The above arguments show that if $\Pi(M, \Sigma)$ is nontrivial, then either (i) M is a Σ-bundle over S^1 or (ii) Σ separates M into two submanifolds, at least one of which is a twisted I-bundle. In particular, we obtain the assertion (4) of the theorem.

Suppose that the conclusion (i) holds, namely $M \cong (\Sigma \times \mathbb{R})/(x, t) \sim (\varphi(x), t + 1)$ for some $\varphi \in \mathrm{MCG}_+(\Sigma)$. Consider the map $\zeta : \Pi(M, \Sigma) \to \mathbb{Z}$ that sends the homotopy motion $\alpha = \{f_t\}_{t \in I}$ to $n \in \mathbb{Z}$ given by $\tilde{f}_1(\Sigma) = \Sigma \times \{n\}$, where $\tilde{\alpha} = \{\tilde{f}_t\}_{t \in I}$ is the lift of α with $\tilde{f}_0 = \tilde{j}$. Then ζ is injective, because if two homotopy motions α and α' are mapped to the same element $n \in \mathbb{Z}$, then the homotopy between $\tilde{\alpha}$ and $\tilde{\alpha}'$, given by Lemmas 2.4 and 2.5 according to whether $n = 0$ or not, projects to a homotopy which gives the equivalence of α and α' as elements of $\Pi(M, \Sigma)$. It is obvious that ζ is a group homomorphism and maps λ to 1, where λ is the homotopy motion described in Example 2.1. Hence $\Pi(M, \Sigma)$ is the infinite cyclic group generated by λ, proving the assertion (1).

Suppose that the conclusion (ii) holds, namely $M = M_1 \cup_\Sigma M_2$ and $M_1 = \Sigma \times [0, 1]/(x, t) \sim (h(x), 1 - t)$, where h is an orientation-reversing involution of Σ. By the preceding argument, we may assume that $\tilde{M} = \Sigma \times \mathbb{R}$ and the restriction of the covering projection p to $\Sigma \times [0, 1]$ is the double covering of M_1 that maps (x, t) to the point $[x, t] \in M_1$ it represents. Note that the homotopy motion $\mu = \{f_t\}_{t \in I}$ defined in Example 2.2 lifts to the map $\tilde{\mu} = \{\tilde{f}_t\}_{t \in I} : \Sigma \to \Sigma \times \mathbb{R}$ given by $\tilde{f}_t(x) = (x, t)$. Moreover, Lemma 2.5 implies if a homotopy motion $\alpha = \{f'_t\}$ has a lift $\tilde{\alpha} = \{\tilde{f}'_t\}_{t \in I} : \Sigma \to \Sigma \times \mathbb{R}$ such that $\tilde{f}'_0(x) = (x, 0)$ and $\mathrm{image}(\tilde{f}'_1) = \Sigma \times \{1\}$, then it is equivalent to μ. We can easily see from the definition of the concatenation that $\mu \cdot \mu$ is equivalent to the identity deformation, thus, the order of μ in $\Pi(M, \Sigma)$ is 2.

Suppose that there exists a nontrivial element $\beta = \{g_t\}_{t \in I}$ of $\Pi(\Sigma, M)$ that is not equivalent to α. Then the previous arguments imply that $\tilde{g}_1(\Sigma)$ is equal to neither $\Sigma \times \{0\}$ nor $\Sigma \times \{1\}$, and so $\tilde{g}_1(\Sigma) \cap (\Sigma \times [0, 1]) = \emptyset$. (Here, $\tilde{\beta} = \{\tilde{g}_t\}_{t \in I}$ is the lift of β with $\tilde{g}_0 = \tilde{j}$.) By choosing $\beta = \{g_t\}_{t \in I}$ suitably, we may assume that $\tilde{g}_1(\Sigma)$ is the lift of Σ closest to $\Sigma \times \{0\}$ in $\Sigma \times (-\infty, 0)$. Then the argument in Case 2 implies that M_2 is a twisted I-bundle and that the terminal end g_1 is (represented by) the involution h' corresponding to the twisted I-bundle structure of M_2. This, in particular, proves the assertion (2). In order to prove the assertion (3), observe that $p : \tilde{M} \to M$ is a regular covering and that the covering transformation group is the infinite dihedral group generated by the two involutions γ and γ' of $\tilde{M} = \Sigma \times \mathbb{R}$ defined by

$$\gamma(x,t) = (h(x), 1-t), \quad \gamma'(x,t) = (h'(x), -1-t).$$

In this case, $p^{-1}(\Sigma) = \Sigma \times \mathbb{Z}$, and the argument for the case (i) implies that $\Pi(\Sigma, M)$ is the infinite dihedral group generated by the two elements μ and μ' of order 2. This completes the proof of the assertion (3). $\square$

Corollary 2.6. *Let M be a closed, orientable Haken manifold M, and suppose that Σ is an incompressible surface in M. Then the following hold.*

(1) *If M is a Σ-bundle over S^1 with monodromy φ and Σ is a fiber surface, then $\Gamma(M,\Sigma)$ is the cyclic group $\langle\varphi\rangle$, and $\mathcal{K}(M,\Sigma)$ is the (possibly trivial) subgroup generated by λ^n of the infinite cyclic group $\Pi(M,\Sigma) = \langle\lambda\rangle$, where n is the order of φ. Moreover, the homomorphism $\deg : \mathcal{K}(M,\Sigma) = \langle\lambda^n\rangle \to \mathbb{Z}$ is given by $\deg(\lambda^n) = n$ under a suitable orientation convention.*

(2) *If Σ separates M into two submanifolds, M_1 and M_2, precisely one of which is a twisted I-bundle, then $\Gamma(M,\Sigma)$ is the order-2 cyclic group generated by the orientation-reversing involution of Σ associated with the twisted I-bundle structure, and $\mathcal{K}(M,\Sigma)$ is the trivial group.*

(3) *If Σ separates M into two submanifolds, M_1 and M_2, both of which are twisted I-bundles, then $\Gamma(M,\Sigma)$ is the (finite or infinite, and possibly cyclic) dihedral group generated by the two orientation-reversing involutions h_1 and h_2 of Σ associated with the twisted I-bundle structures, and $\mathcal{K}(M,\Sigma)$ is the subgroup of the inifinite dihedral group $\Pi(M,\Sigma) = \langle\mu,\mu' \mid \mu^2, \mu'^2\rangle$ generated by $(\mu\mu')^n$, where n is the order of hh'. Moreover, the homomorphism $\deg :$ $\mathcal{K}(M,\Sigma) = \langle(\mu\mu')^n\rangle \to \mathbb{Z}$ is given by $\deg((\mu\mu')^n) = 2n$ under a suitable orientation convention.*

(4) *Otherwise, both $\Gamma(M,\Sigma)$ and $\mathcal{K}(M,\Sigma)$ are the trivial group.*

Proof. The assertions except for those concerning the homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ follow immediately from Theorem 2.3 and the exact sequence (1). It is also easy to see that $\deg(\lambda^n) = n$. The identity $\deg((\mu\mu')^n) = 2n$ can be verified by considering the double covering of M, which is the Σ-bundle over S^1 with monodromy h_1h_2. $\square$

3. Homotopy motion groups of homotopically trivial surfaces

In this section, we study the case contrastive to that treated in the previous section. We say that a closed, orientable surface Σ embedded in a closed, orientable 3-manifold M is *homotopically trivial* if the inclusion map $j : \Sigma \to M$ is homotopic to a constant map.

Lemma 3.1. *A closed, orientable surface Σ embedded in a closed, orientable, irreducible 3-manifold M is homotopically trivial if and only if $j_* : \pi_1(\Sigma) \to \pi_1(M)$ is the trivial map.*

Proof. The "only if" part is obvious. The "if" part can be proved by an argument similar to the proof of Lemma 1.7. □

We have the following theorem.

Theorem 3.2. *Let Σ be a closed, orientable surface embedded in a closed, orientable 3-manifold M. Then the following hold.*

(1) *If Σ is homotopically trivial and if M is aspherical, then $\Pi(M,\Sigma) \cong \pi_1(M) \times \mathrm{MCG}(\Sigma)$. To be more precise, $\Gamma(M,\Sigma) = \mathrm{MCG}(\Sigma)$, and $\mathcal{K}(M,\Sigma)$ is identified with the factor $\pi_1(M)$. Moreover, the homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ vanishes.*

(2) *Conversely, if $\Gamma(M,\Sigma) = \mathrm{MCG}(\Sigma)$ and if M is irreducible, then Σ is homotopically trivial.*

Proof. (1) Suppose that Σ is a homotopically trivial and M is aspherical. Pick a base point $x_0 \in \Sigma \subset M$, and define a homomorphism $\Psi : \Pi(M,\Sigma) \to \pi_1(M,x_0)$ as follows. For an element of $\Pi(M,\Sigma)$, choose a representative homotopy motion α such that $\alpha(1)(x_0) = x_0$. Then the element of $\pi_1(M,x_0)$ represented by the closed path

$$(I,\partial I) \to (M,x_0),\ t \mapsto \alpha(t)(x_0)$$

does not depend on the choice of a representative α, by the following reason. Two such closed paths are related, up to homotopy relative to ∂I, by concatenation of a closed path on Σ based at x_0. However, since Σ is homotopically trivial in M, any closed path on Σ is null-homotopic in M. Thus two such closed paths represent the same element of $\pi_1(M,x_0)$. We define $\Psi([\alpha]) \in \pi_1(M,x_0)$ to be that element.

Suppose first that the genus of Σ is at least 2. By Lemma 1.5, $\mathcal{K}(M,\Sigma)$ can be canonically identified with $\pi_1(C(\Sigma,M),j)$, and the restriction of Ψ to $\pi_1(C(\Sigma,M),j)$ is nothing but the map Φ defined in Lemma 1.8. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(C(\Sigma,M),j) & \longrightarrow & \Pi(M,\Sigma) & \xrightarrow{\partial_+} & \Gamma(M,\Sigma) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Psi\times\partial_+} & & \Big\downarrow{\scriptstyle \iota} & & \\
1 & \longrightarrow & \pi_1(M,x_0) & \longrightarrow & \pi_1(M,x_0)\times \mathrm{MCG}(\Sigma) & \longrightarrow & \mathrm{MCG}(\Sigma) & \longrightarrow & 1,
\end{array}$$

where the two rows are exact, and $\iota : \Gamma(M,\Sigma) \to \mathrm{MCG}(\Sigma)$ is the inclusion map. It suffices to show that both Φ and ι are isomorphisms. Φ is injective by Lemma 1.8, and ι is obviously injective.

To prove the surjectivity of Φ and ι, let $\{h_t : \Sigma \to M\}_{t\in I}$ be the homotopy such that $h_0 = j$ and $h_1(\Sigma) = x_0$, which exists by the assumption that Σ is homotopically trivial. For a given loop $a : (I,\partial I) \to (M,x_0)$, let $\alpha = \{f_t\}_{t\in I}$ be the element of

$\pi_1(C(\Sigma, M), j)$ defined by

$$f_t(x) = \begin{cases} h_{3t}(x) & (0 \leq t \leq 1/3) \\ a(3t-1) & (1/3 \leq t \leq 2/3) \\ h_{3-3t}(x) & (2/3 \leq t \leq 1). \end{cases}$$

Then $\Phi([\alpha]) = [a] \in \pi_1(M, x_0)$, and so Φ is surjective.

The sujectivity of ι is proved as follows. Given $g \in \mathrm{MCG}(\Sigma)$, consider a homotopy motion $\beta = \{g_t\}_{t \in I}$ defined by

$$g_t = \begin{cases} h_{2t} & (0 \leq t \leq 1/2) \\ h_{2-2t} \circ g & (1/2 \leq t \leq 1). \end{cases}$$

Then we have $\partial_+(\beta) = g$, which implies that ι is surjective. Consequently, both Φ and ι are isomorphisms, so does $\Psi \times \partial_+$.

Suppose that the genus of Σ is less than 2. In that case, by replacing $\pi_1(C(\Sigma, M), j)$ with $\pi_1(C(\Sigma, M), j)/\mathscr{I}(\pi_1(J(\Sigma, M), j))$, the same argument as above still works because the map Φ vanishes on $\mathscr{I}(\pi_1(J(\Sigma, M), j))$ due to the assumption that Σ is homotopically trivial.

The vanishing of $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ can be seen as follows. Suppose that $\deg(\alpha) \neq 0$ for some $\alpha \in \mathcal{K}(M, \Sigma)$. Then the image of $\hat{\alpha}_* : \pi_1(\Sigma \times S^1) \to \pi_1(M)$ has finite index in $\pi_1(M)$ (cf. [41, Lemma 15.12]). Since M is an aspherical, closed, orientable 3-manifold, this implies that the cohomological dimension of image$(\hat{\alpha}_*)$ is 3. On the other hand, since Σ is homotopically trivial, image$(\hat{\alpha}_*)$ is cyclic. This is a contradiction, because the cohomological dimension of a cyclic group is 0, ∞, or 1, according as it is trivial, nontrivial finite cyclic, or infinite cyclic. This completes the proof of (1).

(2) Note that the assertion is trivial when $\Sigma = S^2$. We assume that the genus of Σ is at least 1, and we show the assertion by induction on the genus g of Σ. Suppose that $g = 1$ and $\Gamma(M, \Sigma) = \mathrm{MCG}(\Sigma)$. By Corollary 2.6, Σ cannot be incompressible in M. Thus, there exists an essential simple closed curve on Σ that is null-homotopic in M. Since $\mathrm{MCG}(\Sigma)$ acts on the set $\mathcal{S}(\Sigma)$ of (isotopy classes of) essential simple closed curves on Σ transitively, every simple closed curve on Σ is null-homotopic in M. Thus, the map $j_* : \pi_1(\Sigma) \to \pi_1(M)$ vanishes, which implies by Lemma 3.1 that Σ is homotopically trivial in M, as M is irreducible. For the inductive step, suppose that the assertion holds for any surface Σ with genus at most g. Let Σ be a closed, orientable surface of genus $g+1$ embedded in M so that $\Gamma(M, \Sigma) = \mathrm{MCG}(\Sigma)$. Again, by Corollary 2.6, Σ is compressible. Let D be a compression disk for Σ. If ∂D is non-separating in Σ, the proof runs as in the case of $g = 1$. Suppose that ∂D is separating. Let Σ_1 and Σ_2 be the surface obtained by compressing Σ along D. Then we can see that $\Gamma(M, \Sigma_i) = \mathrm{MCG}(\Sigma_i)$ $(i = 1, 2)$ as follows. Let Σ'_1 and Σ'_2 $(i = 1, 2)$ be the closures of components of $\Sigma - \partial D$. We regard Σ_i as $\Sigma'_i \cup D$ $(i = 1, 2)$, so $\Sigma_1 \cap \Sigma_2 = D$. Let f_1 be an arbitrary element of $\mathrm{MCG}(\Sigma_1)$. We show that $f_1 \in \Gamma(M, \Sigma_1)$. We can assume that $f_1(D) = D$. Then there exists an element

$f_2 \in \mathrm{MCG}(\Sigma_2)$ such that $f_2(D) = D$ and $f_1|_D = f_2|_D$. Let f be the element of $\mathrm{MCG}(\Sigma)$ defined by

$$f(x) = \begin{cases} f_1(x) & (x \in \Sigma_1') \\ f_2(x) & (x \in \Sigma_2'). \end{cases}$$

By the assumption, there exists a homotopy motion $\alpha : \Sigma \times I \to M$ with terminal end f. Now let $\bar{f} : \Sigma \cup D \to M$ be an arbitrary extension of f. Then we can extend α to a homotopy motion $\bar{\alpha} : (\Sigma \cup D) \times I \to M$ with terminal end $\bar{f}$ because M is irreducible and hence $\pi_2(M) = 0$. By restriction, $\bar{\alpha}$ determines a homotopy motion of $\Sigma_1 (\subset \Sigma \cup D)$ with terminal end f_1, which implies that $\Gamma(M, \Sigma_1) = \mathrm{MCG}(\Sigma_1)$. Clearly, the same consequence holds for Σ_2. By the assumption of induction, both Σ_1 and Σ_2 are homotopically trivial in M. Hence the image of $\pi_1(\Sigma \cup D)) \cong \pi_1(\Sigma_1) * \pi_1(\Sigma_2)$ in $\pi_1(M)$ is trivial. Thus $j_* : \pi_1(\Sigma) \to \pi_1(M)$ is the trivial homomorphism, and so Σ is homotopically trivial in M, by Lemma 3.1. □

Remark 3.3. The assumption that M is aspherical in Theorem 3.2(1) is essential. In fact, in Example 6.6 we will see that when Σ is a Heegaard surface of S^3, which is homotopically trivial, the kernel of the map $\Psi \times \partial_+ : \Pi(S^3, \Sigma) \to \pi_1(S^3) \times \mathrm{MCG}(\Sigma) = \mathrm{MCG}(\Sigma)$ defined in the above proof consists of infinitely many elements.

Corollary 3.4. *Let Σ be a closed, orientable surface embedded in a closed, orientable, irreducible* 3*-manifold M. Then Σ is homotopically trivial and if and only if $\Gamma(M, \Sigma) = \mathrm{MCG}(\Sigma)$.*

Proof. The "if" part is nothing other than Theorem 3.2(2), and the "only if" part follows from Lemma 1.7. □

4. The group $\Gamma(M, \Sigma)$ for a Heegaard surface and its friends

From this section, we are going to study the homotopy motion group $\Pi(M, \Sigma)$ and related groups $\Gamma(M, \Sigma)$ and $\mathcal{K}(M, \Sigma)$ for a Heegaard surface Σ of a closed, orientable 3-manifold M. Recall that a closed surface Σ in a closed orientable 3-manifold M is called a *Heegaard surface* if Σ separates M into two handlebodies V_1 and V_2. Such a decomposition $M = V_1 \cup_\Sigma V_2$ is then called a *Heegaard splitting* of M, and the *genus* of the splitting is defined to be the genus of Σ. In this section, we mainly consider the group $\Gamma(M, \Sigma)$ rather than $\Pi(M, \Sigma)$. We recall various natural subgroups of $\mathrm{MCG}(\Sigma)$ associated with a Heegaard surface, and describe their relationships with the group $\Gamma(M, \Sigma)$. We also describe the group $\Gamma(M, \Sigma)$ and give answers to Questions 0.2 and 0.3 for the very special cases where Σ is either an arbitrary Heegaard surface of $M = S^3$ and where Σ is a minimal genus Heegaard surface of $M = \#_g(S^2 \times S^1)$.

By definition, the group $\Gamma(M, \Sigma)$ is a subgroup of the extended mapping class group $\mathrm{MCG}(\Sigma)$ of the Heegaard surface Σ. For a Heegaard splitting $M = V_1 \cup_\Sigma V_2$, many other (and similar) subgroups of $\mathrm{MCG}(\Sigma)$ associated with the Heegaard splitting $M = V_1 \cup_\Sigma V_2$ has been studied as follows.

(1) The *handlebody group* $\mathrm{MCG}(V_i)$ of the handlebody V_i, which is identified with a subgroups of $\mathrm{MCG}(\Sigma)$, by restricting a self-homeomorphism of V_i to its boundary $\partial V_i = \Sigma$. This has been a target of various works (see a survey by Hensel [43] and references therein).
(2) The intersection $\mathrm{MCG}(V_1) \cap \mathrm{MCG}(V_2)$, which is identified with $\mathrm{MCG}(M, V_1, V_2)$. This group or its orientation-subgroup $\mathrm{MCG}^+(M, V_1, V_2)$ is called the *Goeritz group* of the Heegaard splitting $M = V_1 \cup_\Sigma V_2$ and it has been extensively studied. In particular, the problem of when this group is finite, finitely generated, or finitely presented attracts attention of various researchers (cf. Minsky [37, Question 5.1]). The work on this problem goes back to Goeritz [34], which gave a finite generating set of the Goeritz group of the genus-2 Heegaard splitting of S^3. In these two decades, great progress was achieved by many authors [84, 75, 4, 21, 48, 49, 22, 23, 24, 25, 31, 26, 45], however, it still remains open whether the Goeritz group a Heegaard splitting of S^3 is finitely generated when the genus is at least 4.
(3) The group $\langle \mathrm{MCG}(V_1), \mathrm{MCG}(V_2) \rangle$ generated by $\mathrm{MCG}(V_1)$ and $\mathrm{MCG}(V_2)$. Minsky [37, Question 5.2] asked when this subgroup is the free product with amalgamated subgroup $\mathrm{MCG}(V_1) \cap \mathrm{MCG}(V_2)$. A partial answer to this question was given by Bestvina-Fujiwara [9].
(4) The mapping class group $\mathrm{MCG}(M, \Sigma)$ of the pair (M, Σ). This contains $\mathrm{MCG}(M, V_1, V_2)$ as a subgroup of index 1 or 2. The result of Scharlemann-Tomova [85] says that the natural map $\mathrm{MCG}(M, \Sigma)$ to $\mathrm{MCG}(M)$ is surjective if the Hempel distance $d(\Sigma)$ (see [42]) is greater than $2g(\Sigma)$. On the other hand, it is proved by Johnson [48], improving the result of Namazi [75], that the natural map $\mathrm{MCG}(M, \Sigma)$ to $\mathrm{MCG}(M)$ is injective if the Hempel distance $d(\Sigma)$ is greater than 3. Hence, the natural map gives an isomorphism $\mathrm{MCG}(M, \Sigma) \cong \mathrm{MCG}(M)$ if $g(\Sigma) \geq 2$ and $d(\Sigma) > 2g(\Sigma)$. Building on the work of McCullough-Miller-Zimmermann [69] on finite group actions on handlebodies, finite group actions on the pair (M, Σ) are extensively studied (see Zimmermann [91, 92] and references therein).
(5) The subgroup $G(M, \Sigma) := \ker(\mathrm{MCG}(M, \Sigma) \to \mathrm{MCG}(M))$, which forms a subgroup of the group $\Gamma(M, \Sigma)$. We can write this group as

$$G(M, \Sigma) = \{[f] \in \mathrm{MCG}(\Sigma) \mid j \circ f \text{ is ambient isotopic to } j.\},$$

where $j : \Sigma \to M$ is the inclusion map, and thus we can think of $\Gamma(M, \Sigma)$ as a "homotopy version" of $G(M, \Sigma)$. Johnson-Rubinstein [52] gave systematic constructions of periodic, reducible, pseudo-Anosov elements in this group. Johnson-McCullough [51] called this group the Goeritz group instead of the one we described in (2), and they used this group to study the homotopy type of the space of Heegaard surfaces. In particular, they prove that if Σ is a Heegaard surface of a closed, orientable, aspherical 3-manifold M, then, except the case where M is a non-Haken infranilmanifold, the exact sequence (2) in the introduction is refined to the following exact sequence

$$1 \to Z(\pi_1(M)) \to \mathcal{M}(M, \Sigma) \to G(M, \Sigma) \to 1,$$

where $\mathcal{M}(M, \Sigma)$ is the smooth motion group of Σ in M [51, Corollary 1].

(6) The group $\Gamma(V_i) := \ker(\mathrm{MCG}(V_i) \to \mathrm{Out}(\pi_1(V_i)))$. As noted in the introduction, the group $\Gamma(V_i)$ is identified with the group $\Gamma(V_i, \Sigma) < \mathrm{MCG}(\Sigma)$. It was shown by Luft [66] that its index-2 subgroup $\Gamma^+(V_i) := \ker(\mathrm{MCG}^+(V_i) \to \mathrm{Out}(\pi_1(V_i)))$ is the *twist group*, that is, the subgroup of $\mathrm{MCG}^+(V_i)$ generated by the Dehn twists about meridian disks. McCullough [68] proved that $\Gamma(V_i)$ is not finitely generated by showing that it admits a surjection onto a free abelian group of infinite rank. A typical orientation-reversing element of $\Gamma(V_i)$ ($< \mathrm{MCG}(\Sigma)$) is the restriction to $\Sigma = \partial V_i$ of a vertical I-bundle involution of V_i.

(7) The group $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ generated by $\Gamma(V_1)$ and $\Gamma(V_2)$, which is contained in $\Gamma(M, \Sigma)$. It was proved by Bowditch-Ohshika-Sakuma in [77, Theorem B] (see also Bestvina-Fujiwara [9, Section 3]) that its orientation-preserving subgroup $\langle \Gamma^+(V_1), \Gamma^+(V_2) \rangle$ is the free product $\Gamma^+(V_1) * \Gamma^+(V_2)$ if the Hempel distance $d(\Sigma)$ is high enough. (The question of whether the same conclusion holds for $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ is still an open question.)

In summary, the subgroups of $\mathrm{MCG}(\Sigma)$ introduced above are related as follows:

$$G(M, \Sigma) < \Gamma(M, \Sigma) \cap \mathrm{MCG}(M, \Sigma) < \Gamma(M, \Sigma),$$
$$\langle \Gamma(V_1), \Gamma(V_2) \rangle < \Gamma(M, \Sigma) \cap \langle \mathrm{MCG}(V_1), \mathrm{MCG}(V_2) \rangle < \Gamma(M, \Sigma).$$

As noted in the introduction, our interest in $\Gamma(M, \Sigma)$ was motivated by Minsky's Question 0.1 and its refinement Question 0.2, and our main concern is Question 0.3(1) about the relationship between $\Gamma(M, \Sigma)$ and its subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$. We end this section by giving an answer to Questions 0.2 and 0.3 in two very special cases.

Example 4.1. Let $S^3 = V_1 \cup_\Sigma V_2$ be the genus-g Heegaard splitting of S^3. Recall that the (orientation-preserving) mapping class group $\mathrm{MCG}^+(\Sigma)$ is generated by the Dehn twists about certain $3g - 1$ simple closed curves on Σ by Lickorish [65], where g is the genus of Σ. Since we can find those simple closed curves in Δ, we have $\langle \Gamma^+(V_1), \Gamma^+(V_2) \rangle = \mathrm{MCG}^+(\Sigma)$. It is thus easy to see that

$$\langle \Gamma(V_1), \Gamma(V_2) \rangle = \Gamma(S^3, \Sigma) = \mathrm{MCG}(\Sigma)$$

and

$$\langle \Gamma(V_1), \Gamma(V_2) \rangle \Delta = \Gamma(S^3, \Sigma) \Delta = \mathcal{S}(\Sigma) = Z.$$

We note that the group $\Gamma(M, \Sigma)$ detects the 3-sphere as in the following meaning.

Proposition 4.2. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold. Then we have $\Gamma(M, \Sigma) = \mathrm{MCG}(\Sigma)$ if and only if $M = S^3$.*

Proof. This is straightforward from Corollary 3.4 and the Poincaré conjecture proved by Perelman [80, 81, 82]. □

Example 4.3. Let $M = \#_g(S^2 \times S^1)$, and $M = V_1 \cup_\Sigma V_2$ the genus-g Heegaard splitting. In this case, M is the double of the handlebody V_1, and thus $\Delta = \Delta_i = Z$ ($i = 1, 2$). Further, we can check easily that

$$\Gamma(V_i) = \langle \Gamma(V_1), \Gamma(V_2) \rangle = \Gamma(M, \Sigma)$$

and

$$\langle \Gamma(V_1), \Gamma(V_2) \rangle \Delta = \Gamma(M, \Sigma)\Delta = Z.$$

In the above easy examples, the group $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ coincides with the whole group $\Gamma(M, \Sigma)$ in an obvious way. However, this is not the case in general, as indicated in the introduction and proved in Theorem 8.1.

5. Open book rotations

In this section, we first recall the definition of an open book decomposition and the Heegaard splitting obtained from an open book decomposition. We then introduce two homotopy motions of the Heegaard surface, the "half book rotation" ρ and the "unilateral book rotation" σ, which play key roles in the subsequent three sections.

Let M be a closed, orientable 3-manifold. Recall that an *open book decomposition* of M is defined to be the pair (L, π), where

(1) L is a (fibered) link in M; and
(2) $\pi : M - L \to S^1$ is a fibration such that $\pi^{-1}(\theta)$ is the interior of a Seifert surface Σ_θ of L for each $\theta \in S^1$.

We call L the *binding* and Σ_θ a *page* of the open book decomposition (L, π). The monodromy of the fibration π is called the *monodromy* of (L, π). We think of the monodromy φ of (L, π) as an element of $\mathrm{MCG}(\Sigma_0, \mathrm{rel}\ \partial\Sigma_0)$, the mapping class group of Σ_0 relative to $\partial\Sigma_0$, i.e., the group of self-homeomorphisms of Σ_0 that fix $\partial\Sigma_0$, modulo isotopy fixing $\partial\Sigma_0$. The pair (M, L), as well as the projection π, is then recovered from Σ_0 and φ. Indeed, we can identify (M, L) with

$$(\Sigma_0 \times \mathbb{R}, \partial\Sigma_0 \times \mathbb{R})/\sim,$$

where $\sim$ is defined by $(x, s) \sim (\varphi(x), s + 1)$ for $x \in \Sigma_0$ and $s \in \mathbb{R}$, and $(y, 0) \sim (y, s)$ for $y \in \partial\Sigma_0$ and any $s \in \mathbb{R}$. So, we occasionally denote the open book decomposition (L, π) by (Σ_0, φ). Under this identification, the Seifert surface Σ_θ is identified with the image $\Sigma \times \{\theta\}$. We define an $\mathbb{R}$-action $\{r_t\}_{t \in \mathbb{R}}$ on M, called a *book rotation*, by $r_t([x, s]) = [x, s + t]$, where $[x, s]$ denotes the element of M represented by (x, s).

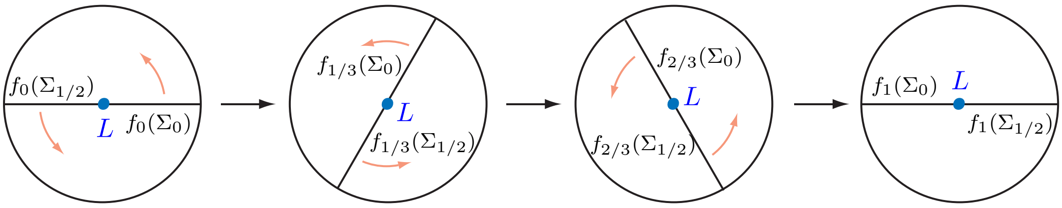

FIGURE 1. The homotopy motion $\rho = \{f_t\}_{t \in I}$.

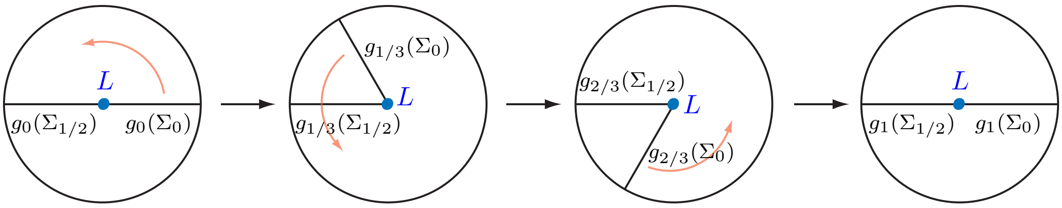

FIGURE 2. The homotopy motion $\sigma = \{g_t\}_{t \in I}$.

Given an open book decomposition (L, π) of M, we obtain a Heegaard splitting $M = V_1 \cup_\Sigma V_2$, where

$$\begin{aligned} V_1 &= \mathrm{cl}(\pi^{-1}([0,1/2])) = \pi^{-1}([0,1/2]) \cup L = \cup_{0 \le \theta \le 1/2} \Sigma_\theta, \\ V_2 &= \mathrm{cl}(\pi^{-1}([1/2,1])) = \pi^{-1}([1/2,1]) \cup L = \cup_{1/2 \le \theta \le 1} \Sigma_\theta, \\ \Sigma &= \Sigma_0 \cup \Sigma_{1/2}. \end{aligned}$$

We call this the Heegaard splitting of M *induced from* the open book decomposition (L, π). For the resulting Heegaard surface Σ, we define two particular homotopy motions in M. The first one, $\rho = \rho_{(L,\pi)} = \rho_{(\Sigma_0,\varphi)}$, is defined by restricting the book rotation, with time parameter rescaled by the factor $1/2$, to the Heegaard surface Σ, namely $\rho(t) = r_{t/2}|_\Sigma$, see Figure 1. The second one, $\sigma = \sigma_{(L,\pi)} = \sigma_{(\Sigma_0,\varphi)}$, is defined by

$$\sigma(t)(x) = \begin{cases} r_t(x) & (x \in \Sigma_0) \\ x & (x \in \Sigma_{1/2}), \end{cases}$$

see Figure 2. We call ρ and σ, respectively, the *half book rotation* and the *unilateral book rotation* associated with the open book decomposition (L, π) (or (Σ_0, φ)).

The elements of the group $\Gamma(M, \Sigma)$ obtained as the terminal ends $\rho(1) = \partial_+(\rho)$ and $\sigma(1) = \partial_+(\sigma)$ play a key role in the proof of the main Theorem 8.1. We note that $\rho(1)$ is orientation-reversing whereas $\sigma(1)$ is orientation-preserving, and they are related as follows.

Lemma 5.1. *$\sigma(1) = \rho(1) \circ h$, where h is the restriction to Σ of the vertical I-bundle involution on V_1 with respect to the natural I-bundle structure given by (L, π).*

Proof. Under the identification $(M, L) = (\Sigma_0 \times \mathbb{R}, \partial\Sigma_0 \times \mathbb{R})/\sim$, the following formulas hold for every $x \in \Sigma_0$.

$$\begin{aligned} h([x,0]) &= [x,1/2], & h([x,1/2]) &= [x,0], \\ \rho(1)([x,0]) &= [x,1/2], & \rho(1)([x,1/2]) &= [x,1] = [\varphi^{-1}(x),0], \\ \sigma(1)([x,0]) &= [x,1] = [\varphi^{-1}(x),0], & \sigma(1)([x,1/2]) &= [x,1/2]. \end{aligned}$$

By using these formulas, we see that the following hold for every $x \in \Sigma_0$, which in turn imply the desired identity.

$$\begin{aligned} \rho(1) \circ h([x,0]) &= \rho(1)([x,1/2]) = [\varphi^{-1}(x),0] = \sigma(1)([x,0]), \\ \rho(1) \circ h([x,1/2]) &= \rho(1)([x,0]) = [x,1/2] = \sigma(1)([x,1/2]). \end{aligned}$$

□

For open book decompositions with trivial monodromies, we have the following lemma. Though it should be well-known and the proof is straightforward, we provide a brief proof here, for this plays a key role in Section 6.

Lemma 5.2. *Let M be a closed, orientable 3-manifold that admits an open book decomposition $(\Sigma_0, \mathrm{id}_{\Sigma_0})$ with trivial monodromy id_{Σ_0}, where Σ_0 is a compact, connected surface embedded in M. Let Σ be the Heegaard surface of M associated with the open book decomposition $(\Sigma_0, \mathrm{id}_{\Sigma_0})$. Then the following hold.*

(1) *$M \cong \#_g(S^2 \times S^1)$, where g is the first Betti number of Σ_0, and Σ is the unique minimal genus Heegaard surface of M.*
(2) *The unilateral book rotation σ associated with $(\Sigma_0, \mathrm{id}_{\Sigma_0})$ determines a non-trivial element of $\mathcal{K}(M, \Sigma)$ of degree 1.*

Proof. Let $\{\delta_i\}_{1\le i\le g}$ be a complete non-separating arc system of Σ_0, namely a family of disjoint non-separating arcs which cuts Σ_0 into a disk. Then the image of $\{\delta_i \times \mathbb{R}\}_{1\le i\le g}$ in $M = (\Sigma_0 \times \mathbb{R}, \partial\Sigma_0 \times \mathbb{R})/\sim$ gives a family of disjoint non-separating spheres which cut M into a 3-ball. Hence $M \cong \#_g(S^2 \times S^1)$ and Σ is a genus g Heegaard surface of M. Since $\#_g(S^2 \times S^1)$ admits a unique Heegaard splitting of genus g by Waldhausen [88], Bonahon-Otal [12] and Haken [38], we obtain the assertion (1). Since the monodromy of the open book decomposition is the identity map, the terminal end $\sigma(1)$ of σ is the identity map. Thus σ determines an element of $\mathcal{K}(M, \Sigma)$. Obviously, $\deg(\sigma) = \deg(\hat{\sigma} : \Sigma \times S^1 \to M) = 1$, and so we obtain the assertion (2). □

6. The group $\mathcal{K}(M,\Sigma)$ for non-aspherical manifolds

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold, and $j : \Sigma \to M$ the inclusion map. Recall the homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ introduced in Lemma 1.6, and the fact that this homomorphism does not vanish if and only if the pair (M,Σ) is dominated by $\Sigma \times S^1$ (cf. Definition 0.4). In this section and the next, we discuss the problem of which pair (M,Σ) of a closed, orientable 3-manifold and its Heegaard surface Σ is dominated by $\Sigma \times S^1$, and prove the following result.

Theorem 6.1. *Let M be a closed, orientable 3-manifold, and suppose that Σ is a Heegaard surface for M.*

(1) *If M has an aspherical prime summand, then (M,Σ) is not dominated by $\Sigma \times S^1$.*
(2) *If $M = \#_g(S^2 \times S^1)$ for some non-negative integer g, or M admits the geometry of S^3 or $S^2 \times \mathbb{R}$, then (M,Σ) is dominated by $\Sigma \times S^1$.*

This theorem is obtained as a consequence of Theorem 6.2 for non-aspherical manifolds and Theorem 7.1 for aspherical manifolds. In this section, we discuss the case where M is non-aspherical, and prove the following theorem.

Theorem 6.2. *Suppose that $M = \#_g(S^2 \times S^1)$ for some non-negative integer g, or M admits the geometry of S^3 or $S^2 \times \mathbb{R}$. Let Σ be a Heegaard surface for M. Then (M,Σ) is dominated by $\Sigma \times S^1$. Moreover, the following hold for the image of the homomorphism* $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$.

(1) *If $M = \#_g(S^2 \times S^1)$, then* $\deg(\mathcal{K}(M,\Sigma)) = \mathbb{Z}$.
(2) *If M admits the geometry of S^3, then* $\deg(\mathcal{K}(M,\Sigma)) \supset |\pi_1(M)| \cdot \mathbb{Z}$.
(3) *If M admits the geometry of $S^2 \times \mathbb{R}$, then* $\deg(\mathcal{K}(M,\Sigma)) = \mathbb{Z}$ *or* $2\mathbb{Z}$ *according to whether $M = S^2 \times \mathbb{R}$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.*

The proof is given at the end of this section, after providing case by case construction, for Heegaard surfaces Σ of non-aspherical manifolds M, of non-zero degree maps $(\Sigma \times S^1, \Sigma \times \{0\}) \to (M,\Sigma)$ that realize dominations of (M,Σ) in the sense of Definition 0.4. We call such a map a Σ-*domination* of (M,Σ).

To begin with, recall that a *stabilization* of a Heegaard splitting $M = V_1 \cup_\Sigma V_2$ (or a Heegaard surface $\Sigma \subset M$) is an operation to obtain a Heegaard splitting $M = V_1' \cup_{\Sigma'} V_2'$ (or a Heegaard surface $\Sigma' \subset M$) of higher genus by adding V_1 a trivial 1-handle, that is, a 1-handle whose core is parallel to Σ in V_2, and removing that from V_2.

Lemma 6.3. *Let M be a closed, orientable 3-manifold, Σ a Heegaard surface for M, and Σ' a Heegaard surface obtained by a stabilization from Σ. If there exists a degree-d Σ-domination of (M,Σ), then there exists a degree-d Σ'-domination of (M,Σ') as well.*

Proof. Suppose that there exists a degree-d Σ-domination $\phi : (\Sigma \times S^1, \Sigma \times \{0\}) \to (M, \Sigma)$. Without loss of generality we can assume that $\phi(x, 0) = x$ for any $x \in \Sigma$. Further, we can assume that the stabilization is performed in a 3-ball B in M that intersects Σ in a disk, thus, $\Sigma - B = \Sigma' - B$. Then there exists a homotopy $F = \{f_t\}_{t \in I} : \Sigma' \times I \to M$ such that

(1) $f_0(x) = x$, for $x \in \Sigma'$;
(2) $f_t(x) = x$ for $x \in \Sigma' - B$, $t \in I$;
(3) $f_t(x) \in B$ for $x \in B \cap \Sigma$, $t \in I$; and
(4) $f_1(\Sigma') = \Sigma$.

Using this homotopy, we can define a Σ'-domination $\phi' : (\Sigma' \times S^1, \Sigma' \times \{0\}) \to (M, \Sigma')$ by

$$\phi'(x, \theta) = \begin{cases} f_{3\theta}(x) & (0 \leq \theta \leq 1/3) \\ \phi(f_1(x), 3\theta - 1) & (1/3 \leq \theta \leq 2/3) \\ f_{3-3\theta}(x) & (2/3 \leq \theta \leq 1). \end{cases}$$

Since the homotopy F moves Σ' only inside the local 3-ball B, the degree of ϕ' is d. $\square$

In each example given below, a particular choice of a Heegaard splitting does not matter.

Example 6.4. *Let $M = \#_g(S^2 \times S^1)$, and suppose that Σ is a Heegaard surface of M. Then there exists a degree-d Σ-domination of (M, Σ) for any integer d.*

Remark 6.5. It is proved in [59, Proposition 4] that there exists a double branched covering map from $\Sigma \times S^1$ to $\#_g(S^2 \times S^1)$ where Σ is a closed, orientable surface of genus g. (See [76, Lemma 2.3] for a related interesting result.) That map actually gives a domination of the minimal genus Heegaard surface of $\#_g(S^2 \times S^1)$ by $\Sigma \times S^1$. However, this does not imply the full statement of the example, because the map has degree 2 and it gives domination of only the minimal genus Heegaard surface.

Proof. Let $V_1 \cup_\Sigma V_2$ be the unique genus-g Heegaard splitting of $M = \#_g(S^2 \times S^1)$. Then by Lemma 5.2, there exists a degree-1 Σ-domination of (M, Σ). Since deg : $\mathcal{K}(M, \Sigma) \to \mathbb{Z}$ is a homomorphism, there exists a degree-d Σ-domination of (M, Σ) for any integer d. On the other hand, by Waldhausen [88], Bonahon-Otal [12] and Haken [38], any Heegaard splitting of M is $V_1 \cup_\Sigma V_2$ itself or its stabilization. Hence we obtain the desired result for any Heegaard splitting by Lemma 6.3. $\square$

In the following, we observe the existence of Σ-dominations of (M, Σ) when M admits the geometry of S^3 or $S^2 \times \mathbb{R}$.

For M with the geometry of S^3, we have the following.

Example 6.6. *Let M be a closed, orientable 3-manifold which admits the S^3 geometry with $|\pi_1(M)| = n$, and suppose that Σ is a Heegaard surface for M. Then there exists a degree-d Σ-domination of (M, Σ) for any integer d with $n|d$. Moreover, in* *the case where M is a lens space $L(p,q)$, there exists a degree-d Σ-domination of (M,Σ) if and only if $d \in p\mathbb{Z}$.*

Proof. The case of $M = S^3$ follows immediately from Example 6.4, however, we first describe a degree-1 Σ-domination $\phi_{st} : (\Sigma \times S^1, \Sigma \times \{0\}) \to (S^3, \Sigma)$ for any given Heegaard surface Σ of S^3 in a slightly different (but essentially the same) way for the later use. Let Σ be a Heegaard surface of S^3. Let D be a disk in Σ with the boundary K, which is the trivial knot. Then there exists an open book decomposition (K,π) of S^3, where each page $D_\theta = \pi^{-1}(\theta)$ is a disk. We can assume that $D = D_0$. Let $\{r_t\}_{t\in\mathbb{R}}$ be the book rotation with respect to (K,π). Since the monodromy of (K,π) is the identity, we can define, by modifying the construction using the unilateral rotation in Lemma 5.2(2), a degree-1 Σ-domination $\phi_{st} : (\Sigma\times S^1, \Sigma\times\{0\}) \to (S^3,\Sigma)$ as follows.

$$\phi_{st}(x,\theta) = \begin{cases} r_\theta(x) & (x \in D = D_0) \\ x & (x \in \Sigma - D). \end{cases}$$

Next we consider the general case. Let M be an arbitrary closed orientable 3-manifold that admits the S^3 geometry with $|\pi_1(M)| = n$, and suppose that Σ is a Heegaard surface for M. Let $p : S^3 \to M$ be the universal covering. It is easy to see that the preimage $\tilde{\Sigma} = p^{-1}(\Sigma)$ is a Heegaard surface for S^3, and every covering transformation of p restricts to an orientation-preserving self-homeomorphism of $\tilde{\Sigma}$. Let D be a disk in Σ. Then the preimage $p^{-1}(D)$ consists of n disjoint disks $\tilde{D}_1, \ldots, \tilde{D}_n$ in $\tilde{\Sigma}$. For each $i \in \{1,\ldots,n\}$, there exists a unique element τ_i in the covering transformation group of p satisfying $\tau_i(D_1) = D_i$. For the Heegaard surface $\tilde{\Sigma}$ for S^3 with a disk $\tilde{D}_1 \subset \tilde{\Sigma}$, we obtain a degree-1 $\tilde{\Sigma}$-domination $\phi_{st} : (\tilde{\Sigma} \times S^1, \tilde{\Sigma}\times\{0\}) \to (S^3,\tilde{\Sigma})$ as described in the previous paragraph. Now, define a map $\tilde{\phi} : (\tilde{\Sigma}\times S^1, \tilde{\Sigma}\times\{0\}) \to (S^3,\tilde{\Sigma})$ by

$$\tilde{\phi}(x,\theta) = \begin{cases} \tau_i \circ \phi_{st}(\tau_i^{-1}(x),\theta)) & (x \in D_i) \\ x & (x \in \Sigma - (D_1 \sqcup \cdots \sqcup D_n)). \end{cases}$$

By definition, this map is equivariant with respect to the action of the covering transformation group, and we have $\deg(\tilde{\phi}) = n$. Therefore, we can define a Σ-domination $\phi : (\Sigma\times S^1, \Sigma\times\{0\}) \to (M,\Sigma)$ by $\phi\circ p' = p\circ\tilde{\phi}$, where $p' := p|_{\tilde{\Sigma}} \times \mathrm{id}_{S^1}$. Since the degree of each of $\tilde{\phi}$, p and p' is n, we have $\deg(\phi) = n$. Since $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ is a homomorphism, this implies the first assertion of the lemma.

When M is a lens space $L(p,q)$, there exists a degree-d map from $\Sigma\times S^1$ to $L(p,q)$ only if p divides d by Hayat-Legrand-Wang-Zieschang [40, Theorem 2]. This completes the proof. □

There exist only two closed, orientable 3-manifolds that admits the geometry of $S^2\times\mathbb{R}$: $S^2\times S^1$ and $\mathbb{RP}^3\#\mathbb{RP}^3$, where we note that $\mathbb{RP}^3\#\mathbb{RP}^3$ is the only closed, non-prime 3-manifold that possesses a geometric structure. The former case has

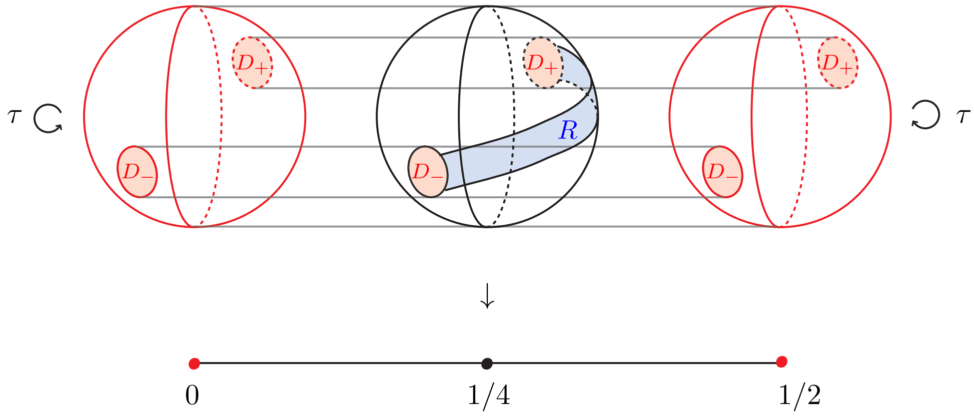

FIGURE 3. The manifold $\mathbb{RP}^3\#\mathbb{RP}^3$ as an S^2-bundle over the orbifold S^1/η

already observed in Example 6.4. The next example gives an answer for the latter case.

Example 6.7. *Let $M = \mathbb{RP}^3\#\mathbb{RP}^3$, and suppose that Σ is a Heegaard surface for M. Then there exists a degree-d Σ-domination of (M, Σ) if and only if $d \in 2\mathbb{Z}$.*

Proof. We first show the "if" part. By Montesinos-Safont [72] and Haken [38], any Heegaard splitting of M is a stabilization of the unique genus-2 Heegaard splitting $M = V_1 \cup_\Sigma V_2$. Therefore, by Lemma 6.3 it suffices to show that there exists a degree-d Σ-domination of (M, Σ) for any even integer d. For this, it is enough to find a degree-2 Σ-domination of (M, Σ).

Let τ be the antipodal map of S^2, and η the involution of S^1 defined by $\eta(\theta) = -\theta$. Identify $M = \mathbb{RP}^3\#\mathbb{RP}^3$ with $(S^2 \times S^1)/(\tau \times \eta)$, and let $p : S^2 \times S^1 \to M$ be the covering projection. Thus we can regard M as an S^2-bundle over the orbifold S^1/η with underlying space $[0, 1/2]$. Choose disjoint disks D_- and D_+ in S^2 with $\tau(D_-) = D_+$. Let $R = I \times I$ be a rectangle in S^2 such that $R \cap D_- = \{0\} \times I$ and $R \cap D_+ = \{1\} \times I$. Then $\tilde{V}_1 := ((D_- \cup D_+) \times S^1) \cup (R \times [1/6, 2/6]) \cup (\tau(R) \times [4/6, 5/6])$ is a $(\tau \times \eta)$-invariant handlebody of genus 3, and its exterior $\tilde{V}_2 := \mathrm{Cl}(M - \tilde{V}_1)$ is also a $(\tau \times \eta)$-invariant handlebody of genus 3. Thus the pair $(\tilde{V}_1, \tilde{V}_2)$ determines a $(\tau \times \eta)$-invariant Heegaard splitting of $S^2 \times S^1$, and it projects to the genus-2 Heegaard splitting (V_1, V_2) of M, where $V_i := p(\tilde{V}_i)$ $(i = 1, 2)$. See Figure 3. We are going to construct a domination of the Heegaard surface $\Sigma := V_1 \cap V_2$, by constructing an equivariant domination of the Heegaard surface $\tilde{\Sigma} = p^{-1}(\Sigma) = \tilde{V}_1 \cap \tilde{V}_2$, as in the proof of Example 6.6.

To this end, consider an annulus $A := \delta \times S^1 \subset \tilde{\Sigma}$, where δ is an arc in ∂D_- disjoint from the rectangle R. Then there is an open book decomposition (L, π) with $L = \partial A$, such that A is the page $\pi^{-1}(0) \cup L$. Let $\{r_t\}_{t \in \mathbb{R}}$ be the book rotation with respect to (L, π). Now consider the conjugate of the above open book decomposition by the covering involution $\tau \times \eta$, and let $\{r'_t\}_{t \in \mathbb{R}}$ be the associated book rotation obtained from $\{r_t\}_{t \in \mathbb{R}}$ through conjugation by $\tau \times \eta$. Observe that the page $A' := (\tau \times \eta)(A)$ is disjoint from A. Since the monodromy of the open book decomposition (L, π) is the identity, we can construct a $\mathbb{Z}/2\mathbb{Z}$-equivariant map $\tilde{\phi} : \tilde{\Sigma} \times S^1 \to S^2 \times S^1$ by

$$\tilde{\phi}(x, t) = \begin{cases} r_t(x) & (x \in A) \\ r'_t(x) & (x \in A') \\ x & (x \in \tilde{\Sigma} - (A \cup A')). \end{cases}$$

This map naturally induces a Σ-domination $\phi : (\Sigma \times S^1, \Sigma \times \{0\}) \to (M, \Sigma)$ whose restriction to $\Sigma \times \{0\}$ is a homeomorphism onto the Heegaard surface Σ, that is actually the identity map under a natural identification of the two surfaces. Since the degree of each of $\tilde{\phi}$, p and the map $\tilde{\Sigma} \times S^1 \to \Sigma \times S^1$ is 2, the degree of ϕ is 2.

To show the other direction, suppose that ϕ is a degree-d map from $\Sigma \times S^1$ to $\mathbb{RP}^3 \# \mathbb{RP}^3$. Here we do not need to require that $\phi(\Sigma \times \{0\})$ is a Heegaard surface of $\mathbb{RP}^3 \# \mathbb{RP}^3$. Let $p : \mathbb{RP}^3 \# \mathbb{RP}^3 \to \mathbb{RP}^3$ be a degree-1 map defined by pinching one summand $\mathbb{RP}^3$ to a 3-ball in the other summand $\mathbb{RP}^3$. Then the composition $p \circ \phi$ is a degree-d map from $\Sigma \times S^1$ to $\mathbb{RP}^3$. From Hayat-Legrand-Wang-Zieschang [40, Theorem 2] it follows that d should be an even number. This completes the proof. □

Proof of Theorem 6.2. The proof is straightforward from Examples 6.4, 6.6, and 6.7. In fact, if $M = \#_g(S^2 \times S^1)$, then the homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ is surjective by Example 6.4. If M admits the geometry of S^3, then $\deg(\mathcal{K}(M, \Sigma)) \supset |\pi_1(M)| \cdot \mathbb{Z}$ by Example 6.6. Finally, if M admits the geometry of $S^2 \times S^1$, then we have $M = S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$. The former case has already discussed above, and in the latter case, we have $\deg(\mathcal{K}(M, \Sigma)) = 2\mathbb{Z}$ by Example 6.7. □

7. The group $\mathcal{K}(M, \Sigma)$ for aspherical manifolds

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold, and $j : \Sigma \to M$ the inclusion map. In this section, we consider the group $\mathcal{K}(M, \Sigma)$ in the case where M is non-aspherical. We note that in this case, the genus of Σ is at least 2, and thus, we can canonically identify $\mathcal{K}(M, \Sigma)$ with $\pi_1(C(\Sigma, M), j)$ by Lemma 1.5. Thus, the homomorphism Φ in Lemma 1.8 determines an injective homomorphism

$$\Phi : \mathcal{K}(M, \Sigma) \to Z(\pi_1(M)),$$

because $j_*(\pi_1(\Sigma)) = \pi_1(M)$. Then we have the following theorem, which completely determine $\mathcal{K}(M, \Sigma)$ in the aspherical case. We note that this theorem may be regarded as an (easier) analogy of the result of Johnson-McCullough [51, Corollary

1] concerning the fundamental group of the space of Heegaard splittings, i.e., the (smooth) motion group of Heegaard surfaces, explained in (5) in Section 4.

Theorem 7.1. *Let M be a closed, orientable, aspherical 3-manifold, and Σ a Heegaard surface of M. Then (M,Σ) is not dominated by $\Sigma \times S^1$. To be precise, the following hold.*

1. *The map $\Phi : \mathcal{K}(M,\Sigma) \to Z(\pi_1(M))$ is an isomorphism. Thus if M is a Seifert fibered space with orientable base orbifold, then $\mathcal{K}(M,\Sigma)$ is isomorphic to $\mathbb{Z}^3$ or $\mathbb{Z}$ according to whether M is the 3-torus T^3 or not. Otherwise, $\mathcal{K}(M,\Sigma)$ is the trivial group.*
2. *The homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ vanishes.*

Proof. To prove (1), we have only to show that $\Phi : \mathcal{K}(M,\Sigma)) \to Z(\pi_1(M))$ is surjective. If $Z(\pi_1(M)) = 1$, there is nothing to prove. Suppose that $Z(\pi_1(M))$ is non-trivial. By the Seifert fiber space conjecture proved by Gabai [33] and Casson-Jungreis [20], M is then a Seifert fibered space with orientable base orbifold. If M is not the 3-torus T^3, then, M admits a unique Seifert fibration with orientable base orbifold, and the center $Z(\pi_1(M))$ is generated by an (infinite order) element corresponding to a regular fiber of the Seifert fibration of M, see Jaco [46, VI]. When $M = T^3$, we have $Z(\pi_1(M)) = \pi_1(M) = \mathbb{Z}^3$, and any primitive element of $Z(\pi_1(M))$ can be realized as a regular fiber of a Seifert fibration of M. In any case, let z be a primitive element of $Z(\pi_1(M))$. Equip M with a Seifert fibration where z is represented by its regular fiber. Fix a faithful action of $S^1 = \mathbb{R}/\mathbb{Z}$ on M that is compatible with the Seifert fibration. Let α_z be the homotopy motion of Σ defined by $\alpha_z(t)(x) = t \cdot x$ for $t \in I$ and $x \in \Sigma$, where $t \cdot x$ is the image of x by the action of $t \in S^1$. Then we see that α_z determines an element of $\mathcal{K}(M,\Sigma)$ and that $\Phi(\alpha_z) = z$. Hence $\Phi : \mathcal{K}(M,\Sigma) \to Z(\pi_1(M))$ is surjective, completing the proof of (1).

Next, we prove (2). Though this follows from Kotschick-Neofytidis [59, Theorem 1], we give a direct geometric proof here. We may suppose that M is a Seifert fibered space with orientable base orbifold. Let z be a primitive element of $Z(\pi_1(M))$. It suffices to show that the degree of the element $\alpha_z \in \mathcal{K}(M,\Sigma)$, defined as in the previous paragraph, is equal to 0, namely the degree of the map $\hat{\alpha}_z : \Sigma \times S^1 \to M$ is 0. To this end, let Y_1 be a spine of the handlebody V_1 in M bounded by the Heegaard surface Σ, and let $\{r_t\}_{t\in I}$ be a strong deformation retraction of V_1 onto Y_1, namely $r_0 = \mathrm{id}_{V_1}$, $r_t|_{Y_1} = \mathrm{id}_{Y_1}$ $(t \in I)$, and $r_1(V_1) = Y_1$. Define a map $H = \{h_s\}_{s\in I} : (\Sigma \times S^1) \times I \to M$ by $H(x,t,s) = t \cdot r_s(x)$. Then $h_0 = \hat{\alpha}_z$ and $h_1(\Sigma \times S^1) = S^1 \cdot Y_1$. Since Y_1 is 1-dimensional, the image $h_1(\Sigma \times S^1)$ is strictly smaller than M. Hence $\deg(\hat{\alpha}_z) = \deg(h_1) = 0$. This completes the proof. □

We now give a proof of Theorem 6.1.

Proof of Theorem 6.1. (1) Suppose that M has an aspherical prime summand, namely M is a connected sum $M_1 \# M_2$ of an aspherical prime manifold M_1 and another 3-manifold M_2, which is possibly S^3. Suppose on the contrary that there is a Heegaard

surface Σ of M, such that (M,Σ) admits a Σ-domination $\phi : (\Sigma \times S^1, \Sigma \times \{0\}) \to (M,\Sigma)$. By Haken's theorem on Heegaard surfaces of composite manifolds (see [46, Theorem II.7]), (M,Σ) is a pairwise connected sum $(M_1,\Sigma_1)\#(M_2,\Sigma_2)$ where Σ_i is a Heegaard surface of M_i $(i=1,2)$. By pinching (M_2,Σ_2) into a point as in the proof of Lemma 6.3, we obtain from ϕ a Σ_1-domination of (M_1,Σ_1). This contradicts Theorem 7.1.
(2) This is an immediate consequence of Theorem 6.2. □

As noted in the introduction, the problem of the existence of Σ-domination of (M,Σ) is completely solved by Theorem 6.1 especially when M is prime. We do not know, however, what happens when M is non-prime and M has no aspherical prime summand, in other words, each prime summand of M is $S^2 \times S^1$, or has the geometry of S^3, except when $M = \#_g(S^2 \times S^1)$ or $\mathbb{RP}^3\#\mathbb{RP}^3$.

Question 7.2. Let $M = \#_{i=1}^{n} M_i$ $(n \geq 2)$ be a closed, orientable non-prime 3-manifold such that each M_i is either $S^2 \times S^1$ or admits the geometry of S^3. When is a Heegaard surface Σ of M dominated by $\Sigma \times S^1$?

8. Gap between $\Gamma(M,\Sigma)$ and the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$

In this section, we show the following theorem, which gives a partial answer to Question 0.3(2).

Theorem 8.1. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M induced from an open book decomposition. If M has an aspherical prime summmand, then we have $\langle \Gamma(V_1), \Gamma(V_2) \rangle \lneq \Gamma(M,\Sigma)$.*

In fact, we will see that neither $\rho(1)$ nor $\sigma(1)$, defined in Section 5, is not contained in $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ under the assumption of Theorem 8.1. To show this, we will define a $\mathbb{Z}^2$-valued invariant $\mathrm{Deg}(f)$ for elements f of $\Gamma(M,\Sigma)$. To this end, we first define a $\mathbb{Z}^2$-valued invariant $\widehat{\mathrm{Deg}}(\alpha)$ for elements α of the homotopy motion group $\Gamma(M,\Sigma)$, and study its basic properties. We then show, by using Theorem 7.1, that it descends to an invariant for elements of $\Gamma(M,\Sigma)$ when M satisfies the assumption of Theorem 8.1.

Remark 8.2. Let Σ be a Heegaard surface of a closed, orientable 3-manifold M. The existence of a gap between $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ and $\Gamma(M,\Sigma)$ given in the above theorem implies, in particular, that the Seifert-van Kampen-like theorem for the homotopy motion group $\Pi(M,\Sigma)$ is no longer valid as in the following meaning, though $\Pi(M,\Sigma)$ is regarded as a generalization of the fundamental group (cf. Remark 1.2 and Theorem 3.2). Consider the homotopy motion groups $\Pi(V_1,\Sigma)$ and $\Pi(V_2,\Sigma)$. Recall that the group $\Gamma(V_i)$ $(i=1,2)$ is the image of the natural map $\partial_+ : \Pi(V_i,\Sigma) \to \mathrm{MCG}(\Sigma)$. Since a homotopy motion of Σ in V_i is that of Σ in M as well, we have a canonical map $I_i : \Pi(V_i,\Sigma) \to \Pi(M,\Sigma)$. Since the manifold M is obtained by gluing V_1 and V_2 along Σ, one might expect that

$$\langle I_1(\Pi(V_1,\Sigma)), I_2(\Pi(V_2,\Sigma))\rangle = \Pi(M,\Sigma),$$

that is, $\Pi(M,\Sigma)$ is generated by elements of $I_1(\Pi(V_1,\Sigma))$ and $I_2(\Pi(V_2,\Sigma))$ as we see in the Seifert-van Kampen theorem. The incoincidence $\langle \Gamma(V_1), \Gamma(V_2)\rangle \lneq \Gamma(M,\Sigma)$, however, implies that this is not true because

$$\partial_+(\langle I_1(\Pi_1(V_1,\Sigma)), I_2(\Pi_1(V_2,\Sigma))\rangle) = \langle \Gamma(V_1), \Gamma(V_2)\rangle$$

while

$$\partial_+(\Pi(M,\Sigma)) = \langle \Gamma(M,\Sigma)\rangle,$$

and they are different.

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M. We will adopt the following convention. Given an orientation of M, or equivalently, a fundamental class $[M] \in H_3(M)$, we always choose the fundamental classes $[V_i] \in H_3(V_i, \partial V_i)$ $(i = 1, 2)$ and $[\Sigma] \in H_2(\Sigma)$ so as to satisfy the following.

Convention 8.3. $[M] = [V_2] - [V_1]$ and $[\Sigma] = [\partial V_1] = [\partial V_2]$, where $[\partial V_i]$ is the one induced from $[V_i]$.

By $[I] \in H_1(I; \partial I)$ we always mean the fundamental class corresponding to the canonical orientation of I.

We define a map $\widehat{\mathrm{Deg}} : \Pi(M,\Sigma) \to \mathbb{Z}^2$ as follows. First, we fix an orientation of M. Let $\alpha = \{f_t\}_{t\in I} : \Sigma \times I \to M$ be a homotopy motion. Consider the homomorphism

$$\alpha_* : H_3(\Sigma \times I, \Sigma \times \partial I) \to H_3(M,\Sigma) \cong H_3(V_1, \partial V_1) \oplus H_3(V_2, \partial V_2),$$

and let (d_1, d_2) be the pair of integers such that $\alpha_*([\Sigma \times I]) = d_1[V_1] + d_2[V_2]$, where $[\Sigma \times I]$ is the cross product of $[\Sigma]$ and $[I]$. This pair is uniquely determined by the equivalence class of α. We then define $\widehat{\mathrm{Deg}}(\alpha) = (d_1, d_2)$. We note that this invariant does not depend on the orientation of M under the above convention. The following examples can be easily checked.

Example 8.4. Let f be a Dehn twist about a meridian of V_1. This is an element of $\Gamma^+(V_1)$. In fact, we can construct a homotopy motion α of $\Sigma = \partial V_1$ in V_1 with terminal end f as follows. As in the proof of Theorem 7.1(2), let Y_1 be a spine of V_1 and let $\{r_t\}_{t\in I}$ be a strong deformation retraction of V_1 onto Y_1. Define a homotopy motion α of Σ in $M = V_1 \cup V_2$ by

$$\alpha(t)(x) = \begin{cases} r_{2t}(x) & (0 \le t \le 1/2) \\ r_{2-2t}(f(x)) & (1/2 \le t \le 1). \end{cases}$$

Then we have $\widehat{\mathrm{Deg}}(\alpha) = (0, 0)$.

Example 8.5. Let h be a vertical I-bundle involution on V_1, which is an element of $\Gamma^-(V_1)$. Then we can construct a homotopy motion α of $\Sigma = \partial V_1$ in V_1 with terminal end h as follows. For a point $x \in \Sigma$, let $I_x \cong [-1,1]$ be the fiber containing x where x corresponds to $1 \in [-1,1]$. Then $\alpha(t)$ maps x to the point corresponding to $1-2t \in [-1,1]$ of I_x. Regarding α as a homotopy motion of Σ in $M = V_1 \cup V_2$, we have $\widehat{\mathrm{Deg}}(\alpha) = (-2,0)$. Similarly, if h is a vertical I-bundle involution on V_2, then we have $\widehat{\mathrm{Deg}}(\alpha) = (0,-2)$ for the corresponding homotopy motion α of Σ in M.

Example 8.6. Recall the homomorphism $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ introduced in Lemma 1.6. For each $\alpha \in \mathcal{K}(M,\Sigma)$, we have

$$\widehat{\mathrm{Deg}}(\alpha) = (\deg(\alpha), \deg(\alpha)).$$

Example 8.7. Suppose that $M = V_1 \cup_\Sigma V_2$ is the Heegaard spitting induced from an open book decomposition, and let ρ and σ, respectively, be the half rotation and the unilateral rotation of Σ associated with the open book decomposition. Then we have

$$\widehat{\mathrm{Deg}}(\rho) = (-1,-1), \quad \widehat{\mathrm{Deg}}(\sigma) = (-1,1).$$

Examples 8.4, 8.5, and 8.7 allow us to predict that $\rho(1)$ and $\sigma(1)$ should give a gap between $\Gamma(M,\Sigma)$ and the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ for a Heegaard spitting $M = V_1 \cup_\Sigma V_2$ induced from an open book decomposition. We are going to verify that when M has an aspherical prime summand.

Lemma 8.8. *The invariant $\widehat{\mathrm{Deg}} : \Pi(M,\Sigma) \to \mathbb{Z}^2$ has the following properties.*

(1) *Let α be an element of $\Pi(M,\Sigma)$, and let $\widehat{\mathrm{Deg}}(\alpha) = (d_1, d_2)$. Then we have*

$$d_1 + d_2 = -1 + \deg \partial_+(\alpha).$$

(2) *For any pair α, β of elements of $\Pi(M,\Sigma)$, we have*

$$\widehat{\mathrm{Deg}}(\alpha \cdot \beta) = \widehat{\mathrm{Deg}}(\alpha) + \deg \partial_+(\alpha) \cdot \widehat{\mathrm{Deg}}(\beta).$$

In the above lemma, $\deg \partial_+(\alpha) \in \{\pm 1\}$ is the degree of the terminal end $\partial_+(\alpha) = \alpha(1) \in \Gamma(M,\Sigma) < \Gamma(\Sigma)$ of α, as a mapping class of the closed, orientable surface Σ.

Proof. (1) Let α be a homotopy motion of Σ in M with terminal end $\alpha(1) = f$. Consider the following commutative diagram:

$$\begin{array}{ccc} H_3(\Sigma \times I, \Sigma \times \partial I) & \xrightarrow{\alpha_*} & H_3(M,\Sigma) \\ \downarrow & & \downarrow \\ H_2(\Sigma \times \partial I) & \xrightarrow{\alpha_*} & H_2(\Sigma), \end{array}$$

where the vertical allows are the connecting maps. By the connecting map $H_3(\Sigma \times I, \Sigma \times \partial I) \to H_2(\Sigma \times \partial I)$, the fundamental class $[\Sigma \times I]$ is mapped to $-[\Sigma \times \{0\}] + [\Sigma \times \{1\}]$. The induced map $\alpha_* : H_2(M, \Sigma) \to H_2(\Sigma)$ then takes this to

$$j_*(-[\Sigma]) + (j \circ f)_*([\Sigma]) = (-1 + \deg f)[\Sigma],$$

where $j : \Sigma \to M$ is the inclusion map. On the other hand, the induced map $\alpha_* : H_3(\Sigma \times I, \Sigma \times \partial I) \to H_3(M, \Sigma)$ takes $[\Sigma \times I]$ to $d_1[V_1] + d_2[V_2]$, and then, the connecting map $H_3(M, \Sigma) \to H_2(\Sigma)$ takes this to $(d_1 + d_2)[\Sigma]$. This implies $d_1 + d_2 = -1 + \deg f$.
(2) Let $f = \partial_+(\alpha) = \alpha(1)$ be the terminal end of α. Then the concatenation $\alpha \cdot \beta$ is given by

$$\alpha \cdot \beta(x, t) = \begin{cases} \alpha(x, 2t) & (0 \le t \le 1/2) \\ \beta(f(x), 2t - 1) & (1/2 \le t \le 1). \end{cases}$$

Let $\widehat{\mathrm{Deg}}(\alpha) = (d_1, d_2)$ and $\widehat{\mathrm{Deg}}(\beta) = (e_1, e_2)$. The assertion then follows from

$$\begin{aligned}(\alpha \cdot \beta)_*([\Sigma \times I]) &= \alpha_*([\Sigma \times I]) + (\beta_* \circ (f \times \mathrm{id}_I)_*)([\Sigma \times I]) \\ &= (d_1[V_1] + d_2[V_2]) + \deg(f \times \mathrm{id}_I)(e_1[V_1] + e_2[V_2]) \\ &= (d_1[V_1] + d_2[V_2]) + \deg f \cdot (e_1[V_1] + e_2[V_2]).\end{aligned}$$

□

The following corollary generalizes Examples 8.4 and 8.5.

Corollary 8.9. *Let α be a homotopy motion of Σ in V_i ($i = 1$ or 2) with terminal end f, and regard it as a homotopy motion of Σ in M. Then $\widehat{\mathrm{Deg}}(\alpha) = (-1 + \deg f, 0)$ or $(0, -1 + \deg f)$ according to whether $i = 1$ or 2.*

Proof. Put $(d_1, d_2) = \widehat{\mathrm{Deg}}(\alpha)$ and suppose that α comes from a homotopy motion in V_1. Then the image of α is equal to V_1, and so we have $d_2 = 0$. By Lemma 8.8(1), $d_1 = -1 + \deg f - d_2 = -1 + \deg f$, completing the proof for the case $i = 1$. The remaining case $i = 2$ is proved by the same argument. □

The following corollary is a consequence of Lemma 8.8(2) and the definition of a semi-direct product.

Corollary 8.10. *Let $C_2 = \{\pm 1\}$ be the order-2 cyclic group, and consider its action on $\mathbb{Z}^2$ defined by $(-1) \cdot (d_1, d_2) = (-d_1, -d_2)$. Let $\mathbb{Z}^2 \rtimes C_2$ be the semi-direct product determined by this action. Then the map $\Pi(M, \Sigma) \to \mathbb{Z}^2 \rtimes C_2$ defined by $\alpha \mapsto (\widehat{\mathrm{Deg}}(\alpha), \deg \partial_+(\alpha))$ is a group homomorphism.*

By the above corollary we can define a map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ so that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{K}(M,\Sigma) & \longrightarrow & \Pi(M,\Sigma) & \xrightarrow{\partial_+} & \Gamma(M,\Sigma) & \longrightarrow & 1 \\ & & & & {\scriptstyle \widehat{\mathrm{Deg}}}\downarrow & \swarrow{\scriptstyle \mathrm{Deg}} & & & \\ & & & & \mathbb{Z}^2 & & & & \end{array}$$

commutes if and only if $\widehat{\mathrm{Deg}}$ vanishes on $\mathcal{K}(M, \Sigma)$. By Example 8.6, the latter condition is satisfied if and only if the homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ vanishes, namely (M, Σ) is not dominated by $\Sigma \times S^1$. Hence, Theorem 6.1 implies the following proposition.

Proposition 8.11. *Let M be a closed, orientable 3-manifold, and suppose that Σ is a Heegaard surface for M. Then if M has an aspherical prime summand, then the map $\mathrm{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ is well-defined.*

From the properties of $\widehat{\mathrm{Deg}}$ we have the following.

Lemma 8.12. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M, and assume that the map $\widehat{\mathrm{Deg}}$ vanishes on $\mathcal{K}(M, \Sigma)$ and so the map $\mathrm{Deg} : \Gamma(M, \Gamma) \to \mathbb{Z}$ is defined. Then the following hold.*

1. *Let f be an element of $\Gamma(M, \Sigma)$, and let $\mathrm{Deg}(f) = (d_1, d_2)$. Then we have $d_1 + d_2 = -1 + \deg f$.*
2. *For any $f, g \in \Gamma(M, \Sigma)$, we have $\mathrm{Deg}(g \circ f) = \mathrm{Deg}(f) + \deg f \cdot \mathrm{Deg}(g)$*
3. *For any $f \in \Gamma(V_1)$, we have $\mathrm{Deg}(f) = (-1+\deg f, 0)$; and for any $f \in \Gamma(V_2)$, we have $\mathrm{Deg}(f) = (0, -1 + \deg f)$.*
4. *If $f \in \langle \Gamma(V_1), \Gamma(V_2) \rangle$, then $\mathrm{Deg}(f)$ is one of $(2k, -2k)$ and $(2k - 2, -2k)$ for some $k \in \mathbb{Z}$, according to whether f is orientation-preserving or reversing. In particular, the mod 2 reduction of $\mathrm{Deg}(f)$ is $(0, 0) \in (\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. The assertions (1) and (2) follow from Lemma 8.8, and the assertion (3) follows from Corollary 8.9. To prove the assertion (4), let f be an element of $\langle \Gamma(V_1), \Gamma(V_2) \rangle$. Then we can write

$$f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1,$$

where $f_i \in \Gamma(V_1)$ for odd i and $f_i \in \Gamma(V_2)$ for even i (or vice verca). We show the proposition by induction on n. If $n = 1$, the assertion follows from the assertion (3) of this lemma. Assume that $\mathrm{Deg}(f_{n-1} \circ \cdots \circ f_2 \circ f_1)$ is of the desired form. Suppose first that $f_{n-1} \circ \cdots \circ f_2 \circ f_1$ is orientation-preserving. Then by the assertion (2) of this lemma, we have

$$\begin{aligned}\mathrm{Deg}(f) &= \mathrm{Deg}(f_{n-1}\circ\cdots\circ f_2\circ f_1) + \deg f_n \\ &= (2k, -2k) + \begin{cases}(0,0) & (f_n \in \Gamma^+(V_1)\cup\Gamma^+(V_2)) \\ (-2,0) & (f_n\in\Gamma^-(V_1)) \\ (0,-2) & (f_n\in\Gamma^-(V_2)),\end{cases}\end{aligned}$$

for some $k \in \mathbb{Z}$. Since $\deg f = \deg f_n$, we see that $\mathrm{Deg}(f)$ is of desired form. Suppose next that $f_{n-1}\circ\cdots\circ f_2\circ f_1$ is orientation-reversing. Then by the assertion (2) of this lemma, we have

$$\begin{aligned}\mathrm{Deg}(f) &= \mathrm{Deg}(f_{n-1}\circ\cdots\circ f_2\circ f_1) - \deg f_n \\ &= (2k-2, -2k) - \begin{cases}(0,0) & (f_n \in \Gamma^+(V_1)\cup\Gamma^+(V_2)) \\ (-2,0) & (f_n\in\Gamma^-(V_1)) \\ (0,-2) & (f_n\in\Gamma^-(V_2)),\end{cases}\end{aligned}$$

for some $k \in \mathbb{Z}$. Since $\deg f = -\deg f_n$, we see that $\mathrm{Deg}(f)$ is of desired form. □

Now we are ready to prove Theorem 8.1.

Proof of Theorem 8.1. Let $M = V_1 \cup_\Sigma V_2$ be the Heegaard splitting of a closed, orientable 3-manifold M induced from an open book decomposition, and assume that M has an aspherical prime summand. Then by Proposition 8.11, the map $\mathrm{Deg} : \Gamma(M,\Sigma) \to \mathbb{Z}^2$ is well-defined. Let ρ and σ be the half rotation and the unilateral rotation of Σ, respectively. Then by Example 8.7, we have $\mathrm{Deg}(\rho(1)) = (-1,-1)$ and $\mathrm{Deg}(\sigma(1)) = (-1,1)$. Therefore, $\rho(1)$ and $\sigma(1)$ do not belong to $\langle\Gamma(V_1),\Gamma(V_2)\rangle$ by Lemma 8.12(4), as desired. □

In the above proof, we have shown that neither $\sigma(1)$ nor $\rho(1)$ is contained in $\langle\Gamma(V_1),\Gamma(V_2)\rangle$. Clearly, the same consequence holds for any odd power of $\sigma(1)$ and $\rho(1)$ by Lemma 8.12. We do not know, however, whether $\sigma(1)^2$ or $\rho(1)^2$ is contained in $\langle\Gamma(V_1),\Gamma(V_2)\rangle$.

Question 8.13. Under the assumption of Theorem 8.1, is $\sigma(1)^2$ or $\rho(1)^2$ contained in $\langle\Gamma(V_1),\Gamma(V_2)\rangle$?

We see from a result in the companion paper [58] that, for the genus-1 Heegaard surface Σ of a lens space $L(p,q)$, there is a gap between $\Gamma(L(p,q),\Sigma)$ and $\langle\Gamma(V_1),\Gamma(V_2)\rangle$ generically. This and Theorem 8.1 are the only examples of Heegaard splittings we know for which there are gaps between the two groups. On the other hand, the Hempel distance of a Heegaard splitting induced from an open book decomposition is at most 2.

Question 8.14. Let Σ be a Heegaard surface of genus at least 2 of a closed, orientable 3-manifold M. Is it true that $\Gamma(M,\Sigma) = \langle\Gamma(V_1),\Gamma(V_2)\rangle$ if Σ has high Hempel distance?

9. The virtual branched fibration theorem and the group $\langle \Gamma(V_1), \Gamma(V_2)\rangle$

In this section, we give yet another motivation for studying the group $\Gamma(M, \Sigma)$ and its subgroup $\langle \Gamma(V_1), \Gamma(V_2)\rangle$ associated with a Heegaard splitting $M = V_1 \cup_\Sigma V_2$. To describe this, let $\mathcal{I}(V_i)$ ($\subset \mathrm{MCG}(\Sigma)$) be the set of torsion elements of $\Gamma(V_i)$. (In fact, this set will turn out to be equal to the set of vertical I-bundle involutions of V_i as shown in Lemma 9.3.) Then we have the following theorem, which refines the observation [83, Addendum 1] that every closed, orientable 3-manifold M admits a surface bundle as a double branched covering space.

Theorem 9.1. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M. Then there exists a double branched covering $p : \tilde{M} \to M$ that satisfies the following conditions.*

- (*i*) *$\tilde{M}$ is a surface bundle over S^1 whose fiber is homeomorphic to Σ.*
- (*ii*) *The preimage $p^{-1}(\Sigma)$ of the Heegaard surface Σ is a union of two (disjoint) fiber surfaces.*

Moreover, the set $D(M, \Sigma)$ of monodromies of such bundles is equal to the set $\{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\}$, up to conjugation and inversion.

Example 9.2. Let $M = \#_g(S^2 \times S^1)$, and $V_1 \cup_\Sigma V_2$ the genus-g Heegaard splitting. Recall that $\Gamma(V_1) = \Gamma(V_2)$ (cf. Example 4.3). Pick an element $h_1 = h_2$ from $\mathcal{I}(V_1) = \mathcal{I}(V_2)$. Then $h_1 \circ h_2 = \mathrm{id}_\Sigma$ and hence the above theorem implies that $\Sigma \times S^1$ is a double branched covering space of $M = \#_g(S^2 \times S^1)$, and so $\Sigma \times S^1$ dominates $\#_g(S^2 \times S^1)$. This gives the construction by Kotschick-Neofytidis [59, Proposition 4].

We first prove the lemma below, following and correcting the arguments of Zimmermann [90, Proof of Corollary 1.3].

Lemma 9.3. *Let V be a handlebody with $\partial V = \Sigma$. Then an element of $\Gamma(V) < \mathrm{MCG}(\Sigma)$ is a nontrivial torsion element if and only if it is represented by (the restriction to $\Sigma = \partial V$ of) a vertical I-bundle involution of V.*

Proof. Since the "if" part is clear, we prove the "only if" part. If the genus of V is 0 or 1, then the assertion can be easily proved by using the facts that $\mathrm{MCG}(B^3) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{MCG}(S^1 \times D^2) \cong \mathrm{MCG}^+(S^1 \times D^2) \rtimes C_2 \cong D_\infty \rtimes C_2$, the semi-direct product of the infinite dihedral group D_∞ and the order-2 cyclic group C_2. Assume that the genus of V is greater than 1. Let h be a torsion element of $\Gamma(V) \subset \mathrm{MCG}(\Sigma)$. Then, by the solution of the Nielsen realization problem (see Kerckhoff [53]), there exists a conformal structure on $\Sigma = \partial V$ and a conformal (or anti-conformal) map h' of the Riemann surface Σ which is isotopic to h. By Bers [8, Theorem 3], the Riemann surface Σ admits a Schottky uniformization, i.e., there is a Schottky group G such that the Riemann surface Σ is conformally equivalent to the Riemann surface $\partial\Omega(G)/G$, where $\Omega(G)$ is the domain of discontinuity of G, and such that the identification of Σ with $\partial\Omega(G)/G$ extends to an identification of V with $V(G) := (\mathbb{H}^3 \cup \Omega(G))/G$. By Marden's isomorphism theorem [67, Theorem 8.1], h' extends to an isometry of $V(G)$, which we continue to denote by h'. Let $\tilde{h}$ be the lift of h' to $\mathbb{H}^3$. Then, by the assumption that $h \in \Gamma(V)$, the conjugation action of $\tilde{h}$ on G is an inner-automorphism of G, that is, there exists an element $k \in G$ such that $\tilde{h} \circ g \circ \tilde{h}^{-1} = k \circ g \circ k^{-1}$ for every $g \in G$. Thus $\tilde{h} \circ k^{-1}$ belongs to the centralizer, Z, of G in $\mathrm{Isom}\,\mathbb{H}^3$. Since G is a free group of rank ≥ 2, it follows that Z is trivial except when G preserves a hyperbolic plane, in which case Z is the order-2 cyclic group generated by the reflection in the hyperbolic plane. In the exceptional case, we may assume that $\tilde{h}$ is the reflection in the hyperbolic plane preserved by G. This implies that the isometry h' of $V(G)$ is a vertical I-bundle involution. This completes the proof of the lemma. □

Remark 9.4. The assertion in the proof that there exists a Schottky group G such that h is realized by an isometry of $V(G)$ is proved by Zimmermann [90, Theorem 1.1] under a more general setting. In fact, [90, Theorem 1.1] says that any finite subgroup of $\mathrm{MCG}(V)$ is realized as a subgroup of the isometry group of $V(G)$. His proof is based on Zieschang's partial solution of the Nielsen realization problem, which was available at that time, and some delicate consideration on the group structure, which guarantees that Zieschang's result is applicable to his setting. Since we only need to consider cyclic groups, we do not need the consideration of the group structure, or we may simply appeal to Kerckhoff's full solution of the Nielsen realization problem [53]. In our terminology, [90, Corollary 1.3] should be read as follows: the orientation-preserving subgroup of $\Gamma^+(V)$ of $\Gamma(V)$ is torsion-free. (A similar proof of this result was also given by Otal [79, Proposition 1.7], and an outline of a similar proof, suggested by Minsky, is included in [9, Introduction].) Thus Lemma 9.3 is a slight extension of [90, Corollary 1.3].

Lemma 9.5. *Let V be a handlebody with $\partial V = \Sigma$, and suppose that h is an orientation-reversing involution of Σ that extends to a vertical I-bundle involution of V. Then there exists a double branched covering projection $p : \Sigma \times [-1,1] \to V$ satisfying the following conditions.*

- *(i)* $p(x,1) = p(h(x),-1) = x \in \Sigma = \partial V$ *for every* $x \in \Sigma$.
- *(ii) The covering transformation is given by the involution* $\hat{h} := h \times (-1)$ *of* $\Sigma \times [-1,1]$ *defined by* $\hat{h}(x,t) = (h(x),-t)$. *In particular, the branch set of p is equal to the image of* $\mathrm{Fix}(h) \times \{0\} \subset \Sigma \times [-1,1]$ *in V.*

Proof. Let $\hat{V}$ be the quotient of $\Sigma \times [-1,1]$ by the orientation-preserving involution $\hat{h}$ defined by the formula in (ii), and let $\hat{p} : \Sigma \times [-1,1] \to \hat{V}$ be the projection. Then $\hat{p}$ is a double branched covering projection with branched set the image of $\mathrm{Fix}(h) \times \{0\} \subset \Sigma \times [-1,1]$ in $\hat{V}$, and the restriction of $\hat{p}$ to $\Sigma \times \{1\}$ is a homeomorphism onto $\partial\hat{V}$. We identify $\partial\hat{V}$ with Σ via this homeomorphism, i.e., identify each $x \in \Sigma$

with $\hat{p}(x,1) \in \partial\hat{V}$. We show that the identification of $\Sigma = \partial V$ with $\partial\hat{V}$ extends to a homeomorphism from V to $\hat{V}$. To this end, recall the assumption that h extends to a vertical I-bundle involution of V, that is, there exists an I-bundle structure of V such that h preserves each fiber setwise and acts on it as a reflection. Then the base space of the I-bundle structure is identified with the quotient surface $F := \Sigma/h$ and we can construct a complete meridian system of V as follows. Pick a complete arc system $\{\delta_i\}_{i=1}^g$. Then the preimages of these arcs by the I-bundle projection form a complete meridian disk system of V. Let $\{\alpha_i\}_{i=1}^g$ be the family of essential loops on $\Sigma = \partial V$ obtained as the boundaries of these meridian disks. Note that the involution h preserves each α_i and that $\alpha_i/h = \delta_i \subset \Sigma/h = F$. This implies that the quotient $(\alpha_i \times [-1,1])/\hat{h}$ is a meridian disk of the handlebody $\hat{V} = \Sigma \times [-1,1]/\hat{h}$ bounded by the loop $\alpha_i \subset \Sigma = \partial\hat{V}$. Since the meridian loop α_i of V remains to be a meridian loop of $\hat{V}$ under the identification of $\Sigma = \partial V$ with $\partial\hat{V}$, the identification homeomorphism extends to a homeomorphism from V to $\hat{V}$. Thus the composition of the branched covering projection $\hat{p} : \Sigma \times [-1,1] \to \hat{V}$ and the identification homeomorphism $\hat{V} \cong V$ determines the desired branched covering projection $p : \Sigma \times [-1,1] \to V$. □

By using the result of Kim-Tollefson [54, Theorem A] on involutions of product spaces, we can obtain the following converse to Lemma 9.5.

Lemma 9.6. *Let V be a handlebody with $\partial V = \Sigma$, and $p : \Sigma \times [-1,1] \to V$ a double branched covering projection such that the restriction $p|_{\Sigma\times\{1\}} : \Sigma \times \{1\} \to \partial V = \Sigma$ is the identity, i.e., $p(x,1) = x$ for every $x \in \Sigma$. Then there exists an orientation-reversing involution h of Σ that extends to a vertical I-bundle involution of V such that p is equivalent to the covering projection constructed from h as indicated in Lemma* 9.5. *To be precise, there exists a self-homeomorphism of $\Sigma \times [-1,1]$ that fixes $\Sigma \times \{1\}$ such that the composition of this homeomorphism and p is equal to the covering projection constructed in Lemma* 9.5.

Proof. Let g be the covering transformation of the double branched covering p. Since g interchanges the two components of $\Sigma \times \partial I$, the result [54, Theorem A] implies that there exists an orientation-reversing involution h of Σ such that g is equivalent to the involution $h \times (-1)$. To be more precise, we can see that g is conjugate to $h \times (-1)$ by a self-homeomorphism of $\Sigma \times [-1,1]$ that fixes $\Sigma \times \{1\}$. Thus we may assume that $g = h \times (-1)$. By the assumption, $\Sigma \times [-1,1]/g$ is identified with the handlebody V in such a way that the point $[x,1]$ of $\Sigma \times [-1,1]/g$ represented by $(x,1)$ is identified with the point $x \in \Sigma = \partial V$ for every $x \in \Sigma$. Now consider the involution $h \times 1$ of $\Sigma \times [-1,1]$. This map is commutative with the involution $g = h \times (-1)$ and so it descends to an involution $\bar{h}$ of $V = \Sigma \times [-1,1]/g$. The restriction of $\bar{h}$ to $\partial V = \Sigma$ is equal to h. Moreover, $\bar{h}$ is a vertical I-bundle involution of V, as shown below. Note that $V = \Sigma \times [-1,1]/g = \Sigma \times [0,1]/(x,0) \sim (h(x),0)$, and so there exists a deformation retraction of V onto the subspace $F := \Sigma \times \{0\}/(x,0) \sim (h(x),0)$. Thus

F is a compact surface with nonempty boundary, which is embedded in the interior of V and is a deformation retract of V. Note that $\mathrm{Fix}(\bar{h})$ is equal the union of the image of $\mathrm{Fix}(h) \times [0,1]$ and the image of $\Sigma \times \{0\}$. The former is a disjoint union of annuli and the latter is equal to F. Thus $\mathrm{Fix}(\bar{h})$ is a surface properly embedded in V and contains F as its deformation retract. This implies that $\bar{h}$ is an I-bundle involution, where $\mathrm{Fix}(\bar{h}) \cong F$ is the base space of the I-bundle structure of V. Thus we have proved that the involution h of $\Sigma = \partial V$ extends to the vertical I-bundle involution $\bar{h}$ of V. Since the covering involution g of the double branched covering projection $p : \Sigma \times [-1,1] \to V$ is given by $g = h \times (-1)$, we can say that p is obtained from h, satisfying the prescribed condition, as indicated in Lemma 9.5. □

Proof of Theorem 9.1. For $i = 1,2$, pick an element h_i of $\mathcal{I}(V_i) \subset \mathrm{MCG}(\Sigma)$. By Lemma 9.3, h_i is represented by an orientation-reversing involution of Σ that extends to a vertical I-bundle involution of V_i. We continue to denote the orientation-reversing involution of Σ by h_i. Let $p_i : \Sigma \times [-1,1] \to V_i$ the double branched covering projection given by Lemma 9.5. Take two copies $[-1,1]_i$ of $[-1,1]$, and regard p_i as a map $\Sigma \times [-1,1]_i \to V_i$. Let $\tilde{M}$ be the space obtained from the disjoint union $\sqcup_{i=1}^{2} \Sigma \times [-1,1]_i$ through the identification

$$(x,1)_1 \sim (x,1)_2, \quad (h_1(x),-1)_1 \sim (h_2(x),-1)_2 \quad (x \in \Sigma).$$

Here $(x,t)_i$ denotes the point in $\Sigma \times [-1,1]_i$ corresponding to $(x,t) \in \Sigma \times [-1,1]$. Then $\tilde{M}$ is a Σ-bundle over S^1 with monodromy $h_1^{-1} \circ h_2 = h_1 \circ h_2$. Moreover we can glue the branched covering projections $p_i : \Sigma \times [-1,1]_i \to V_i$ $(i = 1,2)$ together to obtain a continuous map $p : \tilde{M} \to M = V_1 \cup_\Sigma V_2$, because

$$p_1((h_1(x),-1)_1) = p_1((x,1)_1) = x = p_2((x,1)_2) = p_2((h_2(x),-1)_2).$$

Then p is a branched covering projection whose branch set is the union of those of p_1 and p_2. Hence the Σ-bundle over S^1 with monodromy $h_1 \circ h_2$ is a double branched covering space of M. Moreover the preimage $p^{-1}(\Sigma)$ of the Heegaard surface $\Sigma = \partial V_1 = \partial V_2$ is the image of $\Sigma \times \partial[-1,1]_1$ (and so is that of $\Sigma \times \partial[-1,1]_2$) in $\tilde{M}$. Thus, $p^{-1}(\Sigma)$ is a union of two fiber surfaces. This completes the proof of the first assertion of Theorem 9.1 and the assertion $\{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\} \subset D(M,\Sigma)$.

We prove $D(M,\Sigma) \subset \{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\}$. To this end, let $p : \tilde{M} \to M$ be a double branched covering satisfying the conditions (i) and (ii) of Theorem 9.1, and let τ be the covering involution. By the condition (ii), $p^{-1}(\Sigma)$ consists of two (distinct and so disjoint) fiber surfaces, Σ_0 and Σ_1, and τ interchanges these two components. Set $\tilde{V}_i = p^{-1}(V_i)$ $(i = 1,2)$. Then $\tilde{V}_1 \cap \tilde{V}_2 = \partial\tilde{V}_1 = \partial\tilde{V}_2 = \Sigma_0 \sqcup \Sigma_1$ and $\tilde{V}_1 \cong \tilde{V}_2 \cong \Sigma \times [-1,1]$. We identify the fiber surface Σ_0 with the Heegaard surface Σ via the restriction $p|_{\Sigma_0}$. Then there exists a homeomorphism $\psi_i : \tilde{V}_i \to \Sigma \times [-1,1]$ such that $\psi_i(x) = (p(x),1)$ for every $x \in \Sigma_0$. Let τ_i be the involution of $\Sigma \times [-1,1]$ defined by $\psi_i \circ \tau|_{\tilde{V}_i} \circ \psi_i^{-1}$. Then $p_i := p|_{\tilde{V}_i} \circ \psi_i^{-1} : \Sigma \times [-1,1] \to V_i$ is a double branched covering whose restriction to $\Sigma \times \{1\}$ is the identity map onto $\Sigma = \partial V_i$.

Hence, by Lemma 9.6, there exists an element $h_i \in \mathcal{I}(V_i)$ such that the covering p_i is equivalent to that constructed from h_i as indicated in Lemma 9.5. This implies that the monodromy of the Σ-bundle $\tilde{M}$ is equal to $h_1 \circ h_2$. □

The characterization of $D(M,\Sigma)$ in Theorem 9.1 reminds us of the result of A'Campo [1, Corollary 1] which says that the geometric monodromy of an isolated complex hypersurface singularity, which is defined by a real equation, is the composition of two orientation-reversing involutions of the fiber, one of which is the restriction of the complex conjugation. Brooks [15] and Montesinos [71] independently proved that $D(M,\Sigma)$ contains a pseudo-Anosov element whenever $g(\Sigma) \geq 2$. Hirose and Kin [44] studied the asymptotic behavior of the minimum of the dilatations of pseudo-Anosov elements contained in $D(S^3,\Sigma_g)$ as $g \to \infty$, where Σ_g is the genus-g Heegaard surface of S^3.

When $g(\Sigma) = 1$, we will see in the companion paper [58] that, for any element ϕ of $D(M,\Sigma)$, the minimum translation length $d(\phi)$ of the action of ϕ on the curve graph is comparable with $2d(\Sigma)$, where $d(\Sigma)$ is the Hempel distance of Σ. We expect that this toy example may be extended to a result for general Heegaard splittings.

Question 9.7. For a Heegaard splitting $M = V_1 \cup_\Sigma V_2$ of a closed, orientable 3-manifold M and for an element $\phi \in D(M,\Sigma)$, is there an estimate of $d(\phi)$, the translation length or the asymptotic translation length of the action of ϕ on the curve graph of Σ, in terms of the Hempel distance $d(\Sigma)$?

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