Boundedness of solutions to a parabolic attraction-repulsion chemotaxis system in $\mathbb{R}^2$: the attractive dominant case

Toshitaka NAGAI Yukihiro SEKI Tetsuya YAMADA

Boundedness of solutions to a parabolic attraction-repulsion chemotaxis system in $\mathbb{R}^2$: the attractive dominant case

Toshitaka NAGAI* Yukihiro SEKI† Tetsuya YAMADA ‡

Abstract

We discuss the Cauchy problem for a parabolic attraction-repulsion chemotaxis system:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (\beta_1 u \nabla v_1) + \nabla \cdot (\beta_2 u \nabla v_2), & t > 0,\ x \in \mathbb{R}^2, \\ \partial_t v_j = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(0, x) = u_0(x),\ v_{j0}(0, x) = v_{j0}(x), & x \in \mathbb{R}^2 \quad (j = 1, 2) \end{cases}$$

with positive constants $\beta_j, \lambda_j > 0$ $(j = 1, 2)$ satisfying $\beta_1 > \beta_2$. In our companion paper, the authors proved the existence of global-in-time solutions for any initial data with $(\beta_1 - \beta_2) \int_{\mathbb{R}^2} u_0 \, dx < 8\pi$. In this paper, we prove that every solution stays bounded as $t \to \infty$ provided that $(\beta_1 - \beta_2) \int_{\mathbb{R}^2} u_0 \, dx < 4\pi$.

Key words: Global existence; A priori estimate; Boundedness
2020 Mathematics subject classification: 35A01; 35B45; 35K45; 35Q92

1 Introduction

In this paper, we consider the Cauchy problem:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (\beta_1 u \nabla v_1) + \nabla \cdot (\beta_2 u \nabla v_2), & t > 0,\ x \in \mathbb{R}^2, \\ \partial_t v_j = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(0, x) = u_0(x),\ v_j(0, x) = v_{j0}(x), & x \in \mathbb{R}^2 \quad (j = 1, 2), \end{cases} \tag{CP}$$

where β_j, λ_j $(j = 1, 2)$ are positive constants and u_0, v_{j0} are nonnegative functions satisfying

$$u_0 \geq 0,\ u_0 \not\equiv 0,\ u_0 \in L^1 \cap L^\infty(\mathbb{R}^2), \quad v_{j0} \geq 0, \quad v_{j0},\ |\nabla v_{j0}| \in L^1 \cap L^\infty(\mathbb{R}^2).$$

This system was proposed in [6] to describe the aggregation process of *Microglia*, in which functions $u(t, x)$, $v_1(t, x)$, and $v_2(t, x)$ represent the density of *Microglia*, the chemical concentration of attractive, and repulsive signals, respectively.

*Department of Mathematics, Hiroshima University, Higashihiroshima, 739-8526, Japan. E-mail address:tnagai@hiroshima-u.ac.jp
†Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka 558-8585, Japan. E-mail address:seki@sci.osaka-cu.ac.jp
‡Course of General Education, National Institute of Technology, Fukui College, Sabae, Fukui 916-8507, Japan. E-mail address:yamada@fukui-nct.ac.jp (Corresponding author)

The Cauchy–Neumann problem (CP) on bounded domains have been studied by many researchers (cf. [1, 3, 4, 5] and references cited therein), whereas only a few results were obtained for the Cauchy problem (CP) in $\mathbb{R}^2$. In what follows, the symbols for the integral over the whole space, Lebesgue spaces, and their norms are abbreviated as $\int dx := \int_{\mathbb{R}^2} dx$, $L^p := L^p(\mathbb{R}^2)$, and $\|\cdot\|_p := \|\cdot\|_{L^p}$ $(1 \le p \le \infty)$, respectively. Jin–Liu [2] proved that every solution (u, v_1, v_2) to the Cauchy problem (CP) is globally bounded provided that $\beta_1 = \beta_2$. They also proved that for all $1 < p \le \infty$,

$$\sup_{t>0} (1+t)^{1-1/p} \|u(t)\|_p < \infty, \tag{1.1a}$$

$$\lim_{t\to\infty} t^{1-1/p} \|u(t) - \|u_0\|_1 G(t)\|_p = 0 \tag{1.1b}$$

as well as the same asymptotic profiles for v_1 and v_2, where $G(t) = G(x,t)$ denotes the usual heat kernel in $\mathbb{R}^2$. For the repulsion-dominant case $\beta_1 < \beta_2$, the third author [10] has recently proven that every solution is bounded globally in time. The most delicate situation is the attraction-dominant case $\beta_1 > \beta_2$ since it is expected that the attraction can dominate over repulsive and diffusive effects, so that finite or infinite time blow-up can occur. In this case, the authors [7] have proven that every nonnegative solution with $(\beta_1 - \beta_2)\|u_0\|_1 < 8\pi$ exists globally in time. However, there is no result as to whether or not it remains bounded as $t \to \infty$. The goal of this paper is to solve this last problem under an additional condition on initial data. We are now in a position to state our main result.

Theorem 1.1. *Assume $\beta_1 > \beta_2$ and*

$$\int u_0 \, dx < \frac{4\pi}{\beta_1 - \beta_2}.$$

Then the nonnegative solution of (CP) *exists globally in time and satisfies*

$$\sup_{t>0} (\|u(t)\|_\infty + \|v_1(t)\|_\infty + \|v_2(t)\|_\infty) < \infty. \tag{1.2}$$

Remark 1.2. Once the boundedness is established, the same analysis as in [2] (which goes back to [8]) on asymptotic profile works without any change (even for $\beta_1 \ne \beta_2$), and therefore (1.1) holds as well.

2 Proof of Theorem 1.1

We first recall the following inequality, which is a crucial key to show Theorem 1.1:

Lemma 2.1 ([9, Lemma 2.3])**.** *For $0 < \varepsilon < 1$ and nonnegative functions $g \in L^1 \cap W^{1,2}(\mathbb{R}^2)$,*

$$\int g^2 \, dx \le \frac{1+\varepsilon}{4\pi} \left(\int g \, dx \right) \left(\int \frac{|\nabla g|^2}{1+g} \, dx \right) + \frac{2}{\varepsilon} \int g \, dx.$$

In what follows, let $0 < T < \infty$ and (u, v_1, v_2) be the nonnegative solution to (CP) on $[0,T] \times \mathbb{R}^2$.

Proof of Theorem 1.1. By (CP), the fact of $\int \partial_t u dx = 0$, and an integration by parts, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int (1+u)\log(1+u)\,dx + \int \frac{|\nabla u|^2}{1+u}\,dx \\
&= \int \nabla\cdot(\nabla u - u\nabla(\beta_1 v_1 - \beta_2 v_2))\log(1+u)\,dx + \int \frac{|\nabla u|^2}{1+u}\,dx \\
&= \int \nabla u\cdot\nabla(\beta_1 v_1 - \beta_2 v_2)\,dx - \int \nabla\log(1+u)\cdot\nabla(\beta_1 v_1 - \beta_2 v_2)\,dx. \qquad (2.1)
\end{aligned}
$$

Set

$$
\psi = \beta_1 v_1 - \beta_2 v_2, \qquad h = \lambda_2\beta_2 v_2 - \lambda_1\beta_1 v_1, \tag{2.2a}
$$

$$
\beta = \beta_1 - \beta_2, \tag{2.2b}
$$

where β is positive by assumption. Due to (CP), we have

$$
\partial_t\psi = \Delta\psi + h + \beta u. \tag{2.3}
$$

Integrating by parts in (2.1) and using (2.3), we obtain, after re-grouping of terms,

$$
\begin{aligned}
&\frac{d}{dt}\int (1+u)\log(1+u)\,dx + \int \frac{|\nabla u|^2}{1+u}\,dx \\
&= -\beta\int u\log(1+u)dx + \int (u-\log(1+u))h dx + \beta\int u^2 dx - \int u\partial_t\psi dx + \int \partial_t\psi\log(1+u)dx. \qquad (2.4)
\end{aligned}
$$

Let us write

$$
-\beta\int u\log(1+u)dx = -\beta\int (1+u)\log(1+u)dx + \beta\int \log(1+u)dx. \tag{2.5}
$$

By use of $x \geq \log(1+x)$, Hölder's and Young's inequalities as well as mass conservation, we obtain

$$
\beta\int \log(1+u)\,dx \leq \beta\int u\,dx = \beta\|u_0\|_1, \tag{2.6}
$$

$$
\int (u-\log(1+u))|h|\,dx \leq 2\left(\int u^2\,dx\right)^{1/2}\left(\int h^2\,dx\right)^{1/2} \leq \varepsilon\int u^2\,dx + \frac{1}{\varepsilon}\int h^2\,dx, \tag{2.7}
$$

where the constant $\varepsilon > 0$ is arbitrary. Multiply $-\partial_t\psi/\beta$ for the both sides of (2.3) and integrate the resulted identity over $\mathbb{R}^2$. An integration by parts then shows

$$
-\frac{1}{\beta}\int (\partial_t\psi)^2\,dx = \frac{1}{\beta}\int \nabla\psi\cdot\nabla\partial_t\psi\,dx - \frac{1}{\beta}\int h\partial_t\psi\,dx - \int u\partial_t\psi\,dx.
$$

Since $\partial_t\left(|\nabla\psi|^2\right) = 2\nabla\psi\cdot\nabla(\partial_t\psi)$, a similar argument to the one used to derive (2.7) shows

$$
\begin{aligned}
-\int u\partial_t\psi\,dx &= -\frac{1}{\beta}\int (\partial_t\psi)^2\,dx - \frac{1}{2\beta}\frac{d}{dt}\left(\int |\nabla\psi|^2\,dx\right) + \frac{1}{\beta}\int h\partial_t\psi\,dx \\
&\leq -\frac{3}{4\beta}\int (\partial_t\psi)^2\,dx - \frac{1}{2\beta}\frac{d}{dt}\left(\int |\nabla\psi|^2\,dx\right) + \frac{1}{\beta}\int h^2\,dx. \qquad (2.8)
\end{aligned}
$$

Since $\sqrt{x} \geq \log(1+x)$, it follows by Hölder's and Young's inequalities as well as mass conservation that

$$\begin{aligned}\int \partial_t \psi \log(1+u)\,dx \leq& \int u^{1/2} |\partial_t \psi|\,dx \\ \leq& \|u_0\|_1^{1/2} \left(\int (\partial_t \psi)^2\,dx \right)^{1/2} \\ \leq& \frac{1}{4\beta} \int (\partial_t \psi)^2\,dx + \beta \|u_0\|_1. \end{aligned} \tag{2.9}$$

Putting (2.4)–(2.9) together, we have

$$\begin{aligned}&\frac{d}{dt}\left(\int (1+\ u) \log(1+u)\,dx + \frac{1}{2\beta} \int |\nabla \psi|^2\,dx \right) + \int \frac{|\nabla u|^2}{1+u}\,dx + \frac{1}{2\beta} \int (\partial_t \psi)^2\,dx \\ &\qquad \leq -\beta \int (1+u) \log(1+u)\,dx + (\beta+\varepsilon) \int u^2\,dx + 2\beta \|u_0\|_1 + C_1(\varepsilon) \int h^2\,dx. \end{aligned} \tag{2.10}$$

Due to (2.3) and an integration by parts, we readily obtain

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\left(\int \psi^2\,dx \right) + \frac{1}{2} \int |\nabla \psi|^2\,dx =& \frac{1}{2} \int h\psi\,dx + \frac{\beta}{2} \int u\psi\,dx \\ \leq& \frac{1}{4} \int \left(h^2 + \psi^2 \right)\,dx + \varepsilon \int u^2\,dx + \frac{\beta^2}{16\varepsilon} \int \psi^2\,dx, \end{aligned} \tag{2.11}$$

where Young's inequality has been used as well. Adding the inequality (2.11) to (2.10) and $(\beta/4) \int \psi^2\,dx$ to the both sides of the resulted inequality yields that

$$\begin{aligned}&\frac{d}{dt}\left(\int (1+\ u) \log(1+u)\,dx + \frac{1}{2\beta} \int |\nabla \psi|^2\,dx + \frac{1}{4} \int \psi^2\,dx \right) \\ &+ \beta \int (1+u)\log(1+u)\,dx + \frac{1}{2} \int |\nabla \psi|^2\,dx + \int \frac{|\nabla u|^2}{1+u}\,dx + \frac{1}{2\beta} \int (\partial_t \psi)^2\,dx + \frac{\beta}{4} \int \psi^2\,dx \\ &\leq (\beta + 2\varepsilon) \int u^2\,dx + 2\beta\|u_0\|_1 + C_1(\varepsilon) \int h^2\,dx + C_2(\varepsilon) \int \psi^2\,dx. \end{aligned}$$

This is rewritten as

$$\frac{d}{dt}\mathcal{F} + \beta \mathcal{F} + \frac{1}{2\beta} \int (\partial_t \psi)^2\,dx + \int \frac{|\nabla u|^2}{1+u}\,dx \leq (\beta + 2\varepsilon) \int u^2\,dx + \mathcal{G}(\varepsilon) \tag{2.12}$$

with

$$\begin{aligned}\mathcal{F} :=& \int (1+u)\log(1+u)\,dx + \frac{1}{2\beta} \int |\nabla \psi|^2\,dx + \frac{1}{4} \int \psi^2\,dx, \\ \mathcal{G}(\varepsilon) :=& 2\beta\|u_0\|_1 + C_1(\varepsilon) \int h^2\,dx + C_2(\varepsilon) \int \psi^2\,dx. \end{aligned} \tag{2.13}$$

Applying Lemma 2.1 with $g = u$, we obtain

$$\int u^2\,dx \leq \frac{1+\varepsilon}{4\pi} \|u_0\|_1 \int \frac{|\nabla u|^2}{1+u}\,dx + C_3(\varepsilon)\|u_0\|_1. \tag{2.14}$$

Due to our assumption $\|u_0\|_1 < 4\pi/\beta$, there exists a small constant $\varepsilon = \varepsilon_0 > 0$ such that

$$(\beta + 2\varepsilon_0)\frac{1+\varepsilon_0}{4\pi}\|u_0\|_1 < 1. \tag{2.15}$$

We deduce from (2.12), (2.14), and (2.15) that

$$\frac{d}{dt}\mathcal{F} + \beta\mathcal{F} \le \mathcal{G}(\varepsilon_0) + C_3(\varepsilon_0)\|u_0\|_1. \tag{2.16}$$

We now estimate each term that constitutes $\mathcal{G}$ (cf. (2.13)). By standard computations, one may rewrite equations $\partial_t v_j = \Delta v_j - \lambda_j v_j + u$ $(j = 1, 2)$ to equivalent integral equations. Applying the L^p-L^q estimates ($q = 1$ or $q = p$) for the heat semigroup to the resulted equations, we then obtain

$$\|v_j(t)\|_p \le e^{-\lambda_j t}\|e^{t\Delta}v_{j0}\|_p + \int_0^t e^{-\lambda_j(t-s)}\|e^{(t-s)\Delta}u(s)\|_p\,ds \le C(\|v_{j0}\|_p, \lambda_j, \|u_0\|_1)$$

for any $1 \le p < \infty$, $j = 1, 2$, $0 < t < T$. Therefore quantities $\|\psi(t)\|_2$ and $\|h(t)\|_2$ (cf. (2.2a)) are bounded by a positive constant depending only on β_1, β_2, λ_1, λ_2, $\|u_0\|_1$, $\|v_{10}\|_2$, and $\|v_{20}\|_2$. Hence the application of Gronwall's inequality to (2.16) shows that

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{-\beta t} + C(\beta_1, \beta_2, \lambda_1, \lambda_2, \varepsilon_0, \|u_0\|_1, \|v_{10}\|_2, \|v_{20}\|_2) \tag{2.17}$$

for $0 < t < T$. Since the uniform bound (2.17) with respect to T is in hand, a standard iteration argument (cf. [5, Section 3]) yields a uniform bound on $\sup_{0<t<T}\|u(t)\|_\infty$. Consequently, the nonnegative solution to (CP) may be extended globally in time and

$$\sup_{t>0}\|u(t)\|_\infty < \infty. \tag{2.18}$$

The combination of (2.18) and the L^∞-L^∞ estimate for the heat semigroup readily yields uniform bounds for $\|v_1(t)\|_\infty$ and $\|v_2(t)\|_\infty$, whence (1.2). The proof is now complete. $\square$

Acknowledgement. The second author was partly supported by Grant-in-Aid for scientific research (18K03373) and by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849).

References

[1] T. Cieślak, K. Fujie, Some remarks on well–posedness of the higher–dimensional chemorepulsion system, Bull. Pol. Acad. Sci. Math. **67** (2019) 165–178.

[2] H. Y. Jin, Z. Liu, Large time behavior of the full attraction–repulsion Keller–Segel system in the whole space, Appl. Math. Lett. **47** (2015) 13–20.

[3] H. Y. Jin, Z.A. Wang, Boundedness, blowup and critical mass phenomenon in competing chemotaxis, J. Differential Equations **260** (2016) 162–196.

[4] K. Lin, C. Mu, Global existence and convergence to steady states for an attraction–repulsion chemotaxis system, Nonlinear Anal. Real World Appl. **31** (2016) 630–642.

[5] D. Liu, Y. Tao, Global boundedness in a fully parabolic attraction–repulsion chemotaxis model, Math. Methods Appl. Sci. **38** (2015) 2537–2546.

[6] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, A. Mogilner, Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: Is there a connection? Bull. Math. Biol. **65** (2003) 693–730.

[7] T. Nagai, Y. Seki, T. Yamada, Global existence of solutions to a parabolic attraction-repulsion chemotaxis system in $\mathbb{R}^2$: the attractive dominant case, submitted.

[8] T. Nagai, R. Syukuinn, M. Umesako, Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in $\mathbb{R}^n$, Funkcial. Ekvac. **46** (2003), 383–407.

[9] T. Nagai, T. Yamada, Boundedness of solutions to the Cauchy problem for an attraction-repulsion chemotaxis system in two-dimensional space, Rend. Istit. Mat. Univ. Trieste, **52**(2020), 1–19 (electronic preview).

[10] T. Yamada, Global existence and boundedness of solutions to a parabolic attraction–repulsion chemotaxis system in $\mathbb{R}^2$: the repulsive dominant case, preprint.