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Asymptotic properties of critical points for subcritical Trudinger-Moser functional

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Abstract On a smooth bounded domain we study the Trudinger-Moser functional

$$E_\alpha(u) := \int_\Omega \left(e^{\alpha u^2} - 1\right) dx, \quad u \in H^1(\Omega)$$

for $\alpha \in (0, 2\pi)$ and its restriction $E_\alpha|_{\Sigma_\lambda}$, where $\Sigma_\lambda := \left\{u \in H^1(\Omega) \;\middle|\; \int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) dx = 1\right\}$ for $\lambda > 0$. By applying the asymptotic analysis and the variational method, we obtain asymptotic behavior of critical points of $E_\alpha|_{\Sigma_\lambda}$ both as $\lambda \to 0$ and as $\lambda \to +\infty$. In particular, we prove that when α is sufficiently small, maximizers for $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ tend to 0 in $C(\overline{\Omega})$ as $\lambda \to +\infty$.

Keywords asymptotic behavior · Neumann problem · subcritical · Trudinger-Moser inequality · two dimension

Mathematics Subject Classification (2000) 35A09 · 35B38 · 35B40 · 35J15 · 35J61

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. It is well-known that there is a Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{2p/(2-p)}(\Omega)$ for $p \in [1, 2)$. If we look at the limiting Sobolev case $p = 2$, then $H_0^1(\Omega) := W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \geq 1$, but $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. To fill in this gap, it is natural to look for the maximal growth function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \leq 1}} \int_\Omega g(u) dx < +\infty,$$

where $\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2 dx$ denotes the Dirichlet norm of u. Pohozaev [12] and Trudinger [15] proved independently that the maximal growth is of exponential type and more pre-

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cisely that there exists a constant α such that

$$\sup_{\substack{u\in H_0^1(\Omega)\\ \|\nabla u\|_2\le 1}} \int_\Omega e^{\alpha u^2}\,dx < +\infty.$$

Later, this inequality was sharpened by Moser [8] as follows:

$$\sup_{\substack{u\in H_0^1(\Omega)\\ \|\nabla u\|_2\le 1}} \int_\Omega e^{\alpha u^2}\,dx \begin{cases} < C|\Omega| & \text{if} \quad \alpha \le 4\pi \\ = +\infty & \text{if} \quad \alpha > 4\pi. \end{cases} \tag{1}$$

Lions [7] showed that for (1) there is a loss of compactness at the limiting exponent $\alpha = 4\pi$. But, despite the loss of compactness, the existence of a function which attains the supremum in (1) for $\alpha = 4\pi$ is shown by Carleson and Chang [1] if Ω is a unit ball. This result was extended to arbitrary bounded domains in $\mathbb{R}^2$ by Flucher [3].

In the case of the whole space $\Omega = \mathbb{R}^2$, Ruf [13] and Li and Ruf [5] showed that for $\alpha \le 4\pi$

$$d_\alpha := \sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)dx\le 1}} \int_{\mathbb{R}^2} \left(e^{\alpha u^2}-1\right)dx < +\infty \tag{2}$$

and that d_α is attained if $\alpha = 4\pi$. It is proved by Ishiwata [4] that there exists an explicit constant $C_{\mathbb{R}^2}$ such that d_α is attained for $C_{\mathbb{R}^2} < \alpha < 4\pi$, while d_α is not attained for α small enough, by vanishing loss of compactness.

In this paper, we consider positive critical points of

$$E_\alpha(u) := \int_\Omega \left(e^{\alpha u^2}-1\right)dx, \quad \alpha \in (0,2\pi)$$

constrained to the manifold

$$\Sigma_\lambda := \left\{ u \in H^1(\Omega) \;\middle|\; \int_\Omega \left(|\nabla u|^2+\lambda u^2\right)dx = 1 \right\},$$

where $\lambda > 0$ is a parameter. By the compactness of $E_\alpha|_{\Sigma_\lambda}$, i.e. by the continuity of E_α with respect to weak convergence sequence in Σ_λ, there is a maximizer for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$, which is a critical point of $E_\alpha|_{\Sigma_\lambda}$. Critical points of $E_\alpha|_{\Sigma_\lambda}$ correspond to solutions of the nonlocal problem

$$\begin{cases} -\Delta u + \lambda u = \dfrac{ue^{\alpha u^2}}{\int_\Omega u^2 e^{\alpha u^2}dx} & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where ν is the unit outer normal to $\partial\Omega$. In addition to maximizers for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$ the constant $(\lambda|\Omega|)^{-1/2}$ is also a solution of (3), where $|\Omega|$ denotes the Lebesgue measure of Ω. Obviously, u is a solution of (3) if and only if $u_\lambda(x) = u((x-p)/\sqrt{\lambda})$ is a solution of

$$\begin{cases} -\Delta u + u = \dfrac{ue^{\alpha u^2}}{\int_{\Omega_\lambda} u^2 e^{\alpha u^2}dx} & \text{in} \quad \Omega_\lambda, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_\lambda, \end{cases}$$

for $p \in \mathbb{R}^2$ and $\Omega_\lambda := \left\{ \sqrt{\lambda}x + p \;\middle|\; x \in \Omega \right\}$. So the parameter λ means the scaling of the domain. Our aim of this paper is to study asymptotic behavior of critical points of $E_\alpha|_{\Sigma_\lambda}$ both as $\lambda \to 0$ and as $\lambda \to +\infty$.

In [10,6,11,9], they considered the following Neumann problem for power type nonlinearity:

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

where ε is a parameter and f satisfies some conditions with $f(t) = O(t^p)$ as $t \to \infty$ for $p > 1$. In [10], it is shown that the constant solution is the only positive solution for (4) provided that ε is sufficiently large. In the case of small ε, it is proved by [6,11,9] that a solution at this least energy level for the Neumann problem possesses just one local maximum point, which lies on the boundary, and concentrates (up to subsequences) around a point where mean curvature maximizes. The method employed consists of a combination of the variational characterization of the solutions and exact estimates of the value of the energy functional based on a precise asymptotic analysis of the solutions.

To state our results, let us define the constant $I(\alpha,\lambda)$ by

$$I(\alpha,\lambda) := \sup_{u\in\Sigma_\lambda} E_\alpha(u)$$

for $\alpha \in (0,2\pi)$ and $\lambda > 0$. We make a remark that all maximizers for $I(\alpha,\lambda)$ are belong to $C^{2,\beta}(\overline{\Omega})$ and strictly positive in $\overline{\Omega}$. We also define I_α by

$$I_\alpha := \sup_{\substack{u\in H^1(\mathbb{R}^2_+) \\ \int_{\mathbb{R}^2_+}\left(|\nabla u|^2+u^2\right)dx\le 1}} \int_{\mathbb{R}^2_+} \left(e^{\alpha u^2} - 1\right) dx,$$

where $\mathbb{R}^2_+ := \left\{x \in \mathbb{R}^2 \;\middle|\; x_2 > 0\right\}$ is the half space. Then the constant α_* is defined by

$$\alpha_* := \inf\{\alpha \in (0,2\pi) \mid I_\alpha > \alpha\}.$$

Note that $\alpha_* \in (0,2\pi)$ holds. Indeed, by the radially symmetric rearrangement $I_\alpha = d_{2\alpha}/2$ holds, where $d_{2\alpha}$ is defined in (2) for 2α. Moreover, due to Ishiwata [4], $d_{2\alpha} > 2\alpha$ if α is close to 2π and $d_{2\alpha} = 2\alpha$ if α is small enough. Thus, $I_\alpha > \alpha$ holds if α is close to 2π and $I_\alpha = \alpha$ holds if α is small, which imply that $\alpha_* \in (0,2\pi)$.

In this setting, we obtain the following results:

Theorem 1 *Assume that $\alpha \in (\alpha_*, 2\pi)$. Let u_λ be a maximizer of $I(\alpha,\lambda)$ for $\lambda > 0$. Then the following statements hold:*

(I) There exist positive constants Λ_1, M_1 and M_2 such that for any $\lambda > \Lambda_1$ we have

$$M_1 \le \sup_{x\in\Omega} u_\lambda(x) \le M_2.$$

(II) For λ sufficiently large, u_λ has a unique maximum and the maximum point lies on the boundary of Ω.

(III) For any $\varepsilon > 0$, there exist positive constants R and Λ_2 such that for any $\lambda > \Lambda_2$ we have

$$u_\lambda(x) \le M_3 \varepsilon e^{-\mu_1 \delta(x)\sqrt{\lambda}} \quad \text{for} \quad x \in \overline{\Omega} \setminus B_{R/\sqrt{\lambda}}(x_\lambda),$$

where $x_\lambda \in \partial\Omega$ is the unique maximum point of u_λ, $\delta(x) = \min\left\{\operatorname{dist}(x, \partial B_{R/\sqrt{\lambda}}(x_\lambda)), \mu_2\right\}$ and M_3, μ_1, μ_2 are positive constants depending only on Ω.

Theorem 2 *Assume that $\alpha \in (0, \alpha_*)$. Let u_λ be a maximizer of $I(\alpha, \lambda)$ for $\lambda > 0$. Then we have*

$$u_\lambda \to 0 \quad \text{in} \quad C^0(\overline{\Omega})$$

and

$$\int_\Omega |\nabla u_\lambda|^2 dx \to 0, \quad \lambda \int_\Omega u_\lambda^2 dx \to 1$$

as $\lambda \to +\infty$.

In the case of $\alpha \in (\alpha_*, 2\pi)$, there is a maximizer for I_α. So the situation is similar to the case of power type nonlinearity (4). For large λ, a maximizer u_λ has a unique maximum which located on the boundary of the domain and u_λ can be made arbitrarily small in the outer region $\overline{\Omega} \setminus B_{R/\sqrt{\lambda}}(x_\lambda)$. In addition to Theorem 1, we derive that u_λ converges to some maximizer of I_α in some sense as $\lambda \to +\infty$, and it turns out that $\lim_{\lambda \to \infty} I(\alpha, \lambda) = I_\alpha$. In the case of $\alpha \in (0, \alpha_*)$, I_α is not attained by vanishing loss of compactness on maximizing sequences. The situation is completely different from the case of (4). Theorem 2 asserts that the vanishing phenomena occur for sequences of maximizers. Also in the case of $\alpha \in (0, \alpha_*)$, it follows from Theorem 2 that $\lim_{\lambda \to \infty} I(\alpha, \lambda) = I_\alpha$.

In the proofs of Theorems 1 and 2, we use a diffeomorphism straighting a boundary portion around a point on $\partial\Omega$ which was introduced in [6, 11, 9] and some results of the solution of the following equation:

$$-\Delta w + w = Lwe^{4\pi w^2} \quad \text{in} \quad \mathbb{R}^2, \quad L \in (0,1), \quad w \in H^1(\mathbb{R}^2).$$

Concerning the equation, it is known that all positive solutions are in $C^2(\mathbb{R}^2)$ and radially symmetric for any $L \in (0,1)$. Moreover, they and their first derivatives decay exponentially at infinity. By Ruf and Sani [14], it is proved that for each $L \in (0,1)$ there exists a solution which attains the ground state level. We use these result to reject the possibility that maximizer u_λ has infinitely many peak in $\overline{\Omega}$.

The following result is asymptotic behavior of positive critical points of $E_\alpha|_{\Sigma_\lambda}$ as $\lambda \to 0$.

Theorem 3 *Assume that $\alpha \in (0, 2\pi)$ and that v_λ is a positive critical point of $E_\alpha|_{\Sigma_\lambda}$ for $\lambda > 0$. Then we have*

$$(\lambda|\Omega|)^{\frac{1}{2}} v_\lambda \to 1 \quad \text{in} \quad C^2(\overline{\Omega})$$

as $\lambda \to 0$.

Theorem 3 means that $v_\lambda / \|v_\lambda\|_{L^\infty(\Omega)} \to 1$ in $C^2(\overline{\Omega})$ and $(\lambda|\Omega|)^{1/2} \|v_\lambda\|_{L^\infty(\Omega)} \to 1$ as $\lambda \to 0$. In order to prove the theorem, we show that $\|v_\lambda\|_{L^\infty(\Omega)} \to \infty$ as $\lambda \to 0$ and use a blow-up analysis. For small λ, the situation is more delicate than the case of (4) considered in [10], and then the uniqueness of the positive critical point of $E_\alpha|_{\Sigma_\lambda}$ is still open.

This paper is organized as follows. In Section 2, we will prove Theorems 1 and 2. By using asymptotic analysis, we will show that either "concentration at one point" or "vanishing" occurs on sequence of maximizers. In order to prove the claim, we will investigate the asymptotic behavior of maximizers in the region around concentration point as well as in the outer region. In Section 3, we will prove Theorem 3. In Section 4, the relationship between d_α and I_α will be discussed. In particular, we will show that α_* is the threshold dividing existence and non-existence of a maximizer for I_α.

2 Proofs of Theorems 1 and 2

In this section we prove Theorems 1 and 2. In order to derive the asymptotic behavior of u_λ, we study a nonlocal elliptic equation and estimate $I(\alpha,\lambda)$.

Before proving Theorems 1 and 2, we recall some facts about a diffeomorphism straightening a boundary portion around a point P on $\partial\Omega$, which was introduced in [6,11,9]. Fix $P\in\partial\Omega$. We may assume that P is the origin and the inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_2-axis, here $x=(x_1,x_2)\in\mathbb{R}^2$. In a neighborhood N of P, $\partial\Omega\cap N$ can be represented by

$$x_2=\psi(x_1)=\frac{1}{2}\gamma x_1^2+o(x_1^2),$$

where γ is the curvature of $\partial\Omega$ at P. Define a map $x=\Phi(y)=(\Phi_1(y),\Phi_2(y))$ by

$$\Phi_1(y)=y_1-y_2\frac{\partial\psi}{\partial x_1}(y_1),\quad \Phi_2(y)=y_2+\psi(y_1). \tag{5}$$

Since $\psi'(0)=0$, the differential map $D\Phi$ of Φ satisfies $D\Phi(0)=I$, the identity map. Thus Φ has the inverse mapping $y=\Phi^{-1}(x)$ for small $|x|$. We write as $\Psi(x)=(\Psi_1(x),\Psi_2(x))$ instead of $\Phi^{-1}(x)$.

For fixed $\alpha\in(0,2\pi)$ and a sequence λ_n such that $\lambda_n\to\infty$ as $n\to\infty$ a maximizer of $I(\alpha,\lambda_n)$ is denoted by u_n. The maximizer $u_n\in\Sigma_{\lambda_n}$ satisfies

$$\begin{cases}-\Delta u_n+\lambda_n u_n=\frac{u_n e^{\alpha u_n^2}}{\int_\Omega u_n^2 e^{\alpha u_n^2}dx} & \text{in} \quad \Omega,\\ \frac{\partial u_n}{\partial \nu}=0 & \text{on} \quad \partial\Omega.\end{cases} \tag{6}$$

2.1 Concentration profile

Proposition 1 *There exists a positive constant C_1 such that $\|u_n\|_{L^\infty(\Omega)}\le C_1$ for sufficiently large n.*

Proof Set $c_n:=\|u_n\|_{L^\infty(\Omega)}$ and assume that $x_n\in\overline{\Omega}$ satisfies $u_n(x_n)=c_n$. We assume that $c_n\to\infty$ as $n\to\infty$ and derive a contradiction. We define r_n such that

$$r_n^2=\frac{\int_\Omega u_n^2 e^{\alpha u_n^2}dx}{c_n^2 e^{\alpha c_n^2}},$$

and then, it follows that

$$r_n^2\le\frac{\int_\Omega u_n^2 dx}{c_n^2}\le\frac{1}{\lambda_n c_n^2}. \tag{7}$$

If $\mathrm{dist}(x_n,\partial\Omega)/r_n\to\infty$, we define $\Omega_n:=\{(x-x_n)/r_n\mid x\in\Omega\}$ and

$$\begin{cases}\phi_n(y):=c_n^{-1}u_n(r_n y+x_n) & y\in\Omega_n,\\ \eta_n(y):=c_n(u_n(r_n y+x_n)-c_n) & y\in\Omega_n.\end{cases}$$

Then, ϕ_n and η_n satisfy

$$-\Delta_y\phi_n+\lambda_n r_n^2\phi_n=c_n^{-2}\phi_n e^{\alpha c_n^2(\phi_n^2-1)},$$

$$-\Delta_y \eta_n + \lambda_n r_n^2 c_n^2 \phi_n = \phi_n e^{\alpha(1+\phi_n)\eta_n}. \tag{8}$$

Since $\mathrm{dist}(x_n, \partial\Omega)/r_n \to \infty$, for any $R > 0$ there exists K such that $B_R(x_n) \subset \Omega_n$ for any $n \geq K$. Thus, by (7), the elliptic regularity theory and the maximum principle we see that

$$\phi_n \to \phi_0 \equiv 1 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\phi_0 = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Using the behavior of ϕ_n, we estimate $\lambda_n r_n^2 c_n^2$ in (8). Since $u_n \in \Sigma_{\lambda_n}$, we have

$$\begin{aligned} 1 \geq \lambda_n \int_\Omega u_n^2 dx &\geq \lambda_n c_n^2 \int_{B_{Rr_n}(x_n)} \left(\frac{u_n}{c_n}\right)^2 dx = \lambda_n c_n^2 r_n^2 \int_{B_R} \phi_n^2 dy \\ &= \lambda_n c_n^2 r_n^2 \int_{B_R} (1+o(1))^2 dy = \lambda_n c_n^2 r_n^2 |B_R|(1+o(1)) \end{aligned}$$

for any $R > 0$, and thus $\lambda_n c_n^2 r_n^2 \to 0$ as $n \to \infty$. Applying the elliptic regularity theory to (8), we have

$$\eta_n \to \eta_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\eta_0 = e^{2\alpha\eta_0} \quad \text{in} \quad \mathbb{R}^2.$$

Moreover, it follows that

$$\int_{\mathbb{R}^2} e^{2\alpha\eta_0} dy = \lim_{R\to\infty}\lim_{n\to\infty} \int_{B_R} \phi_n^2 e^{\alpha(1+\phi_n)\eta_n} dy \leq \lim_{R\to\infty}\lim_{n\to\infty} \frac{\int_{B_{Rr_n}(x_n)} u_n^2 e^{\alpha u_n^2} dx}{\int_\Omega u_n^2 e^{\alpha u_n^2} dx} \leq 1, \tag{9}$$

and then

$$\eta_0 = -\frac{1}{\alpha}\log\left(1 + \frac{\alpha}{4}|y|^2\right).$$

Since $\alpha < 2\pi$, by a direct computation, we have

$$\int_{\mathbb{R}^2} e^{2\alpha\eta_0} dy = \frac{4\pi}{\alpha} > 2.$$

But this contradicts (9). Hence c_n is bounded if $\mathrm{dist}(x_n, \partial\Omega)/r_n \to \infty$.

In the case of $\mathrm{dist}(x_n, \partial\Omega) = O(r_n)$, we follow [11]. One may assume that $x_n \to x_0 \in \partial\Omega$ by passing to a subsequence if necessary. Take the diffeomorphism $y = \Psi(x)$ which straightens a boundary portion near x_0 as in (5). We may assume that $\Phi = \Psi^{-1}$ is defined in an open set containing the closed ball $\overline{B_{2\kappa}}$, $\kappa > 0$, and that $P_n := \Psi(x_n) \in B^+_\kappa$ for all n. Put

$$v_n(y) := u_n(\Phi(y)) \quad \text{for} \quad y \in \overline{B^+_{2\kappa}}$$

and extend it to $\overline{B_{2\kappa}}$ by reflection:

$$\tilde{v}_n(y) := \begin{cases} v_n(y) & \text{if} \quad y \in \overline{B^+_{2\kappa}}, \\ v_n((y_1, -y_2)) & \text{if} \quad y \in B^-_{2\kappa}, \end{cases}$$

where $B^-_{2\kappa} := \left\{ y \in \overline{B_{2\kappa}} \;\middle|\; y_2 < 0 \right\}$. Moreover, we define a scaled function $w_n(z)$ by

$$w_n(z) := \tilde{v}_n(r_n z + P_n) \quad \text{for} \quad z \in \overline{B_{\kappa/r_n}},$$

and then ϕ_n and η_n are defined by

$$\phi_n(z) := c_n^{-1} w_n(z),$$

$$\eta_n(z) := c_n(w_n(z) - c_n).$$

By (6), ϕ_n and η_n satisfy the following elliptic equations

$$\sum_{i,j=1}^{2} a_{ij}^n(z)\frac{\partial^2\phi_n}{\partial z_i\partial z_j}+r_n\sum_{j=1}^{2} b_j^n(z)\frac{\partial\phi_n}{\partial z_j}+\lambda_n r_n^2\phi_n=c_n^{-2}\phi_n e^{\alpha c_n^2(\phi_n^2-1)},$$

$$\sum_{i,j=1}^{2} a_{ij}^n(z)\frac{\partial^2\eta_n}{\partial z_i\partial z_j}+r_n\sum_{j=1}^{2} b_j^n(z)\frac{\partial\eta_n}{\partial z_j}+\lambda_n r_n^2 c_n^2\phi_n=\phi_n e^{\alpha(1+\phi_n)\eta_n},$$

where a_{ij}^n, b_j^n are defined as follows: First, put

$$a_{ij}(y)=\sum_{\ell=1}^{2}\frac{\partial\Psi_i}{\partial x_\ell}(\Phi(y))\frac{\partial\Psi_j}{\partial x_\ell}(\Phi(y))\quad 1\le i,j\le 2 \tag{10}$$

$$b_j(y)=(\Delta\Psi_j)(\Phi(y))\quad 1\le j\le 2. \tag{11}$$

Then set $P_n:=(p_n,q_nr_n)$ and

$$a_{ij}^n(z)=\begin{cases}a_{ij}(P_n+r_nz) & z_2\ge -q_n,\\ (-1)^{\delta_{i2}+\delta_{j2}}a_{ij}((p_n+r_nz_1,-(q_n+z_2)r_n) & z_2<q_n,\end{cases} \tag{12}$$

$$b_j^n(z)=\begin{cases}b_j(P_n+r_nz) & z_2\ge -q_n,\\ (-1)^{\delta_{j2}}b_j((p_n+r_nz_1),-(q_n+z_2)r_n) & z_2<-q_n,\end{cases} \tag{13}$$

where δ_{ij} is the Kronecker symbol. Using the elliptic regularity theory, we have

$$\phi_n\to\phi_0\equiv 1\quad\text{in}\quad C^2_{loc}(\mathbb{R}^2),\quad -\Delta\phi_0=0\quad\text{in}\quad\mathbb{R}^2,$$

$$\eta_n\to\eta_0\quad\text{in}\quad C^2_{loc}(\mathbb{R}^2),\quad -\Delta\eta_0=e^{2\alpha\eta_0}\quad\text{in}\quad\mathbb{R}^2.$$

We compute $\int_{\mathbb{R}^2}e^{2\alpha\eta_0}dz$ in the same way as in (9). It follows that

$$\int_{\mathbb{R}^2}e^{2\alpha\eta_0}dz=\lim_{R\to\infty}\lim_{n\to\infty}2\int_{B_R^+}\phi_n^2e^{\alpha(1+\phi_n)\eta_n}dz\le\lim_{R\to\infty}\lim_{n\to\infty}\frac{2\int_{\Omega\cap\Phi\left(B_{Rr_n}(P_n)\right)}u_n^2e^{\alpha u_n^2}dx}{\int_\Omega u_n^2e^{\alpha u_n^2}dx}\le 2. \tag{14}$$

Hence, we see that

$$\eta_0=-\frac{1}{\alpha}\log\left(1+\frac{\alpha}{4}|z|^2\right),$$

and

$$\int_{\mathbb{R}^2}e^{2\alpha\eta_0}dz=\frac{4\pi}{\alpha}.$$

But this equality and (14) contradict the hypothesis $\alpha<2\pi$. Thus, c_n is bounded if $\mathrm{dist}(x_n,\partial\Omega)=O(r_n)$. Consequently, in both cases, there exists a constant C_1 such that $c_n\le C_1$ for sufficiently large n.

Lemma 1 *There exist a positive constant C_2 such that*

$$\lambda_n\int_\Omega u_n^2e^{\alpha u_n^2}dx\in(1,C_2).$$

Proof Since u_n satisfies the equation (6) and $u_n > 0$ in $\overline{\Omega}$, we have

$$\lambda_n \int_\Omega u_n dx = \frac{\int_\Omega u_n e^{\alpha u_n^2} dx}{\int_\Omega u_n^2 e^{\alpha u_n^2} dx} > \frac{\int_\Omega u_n dx}{\int_\Omega u_n^2 e^{\alpha u_n^2} dx}.$$

Thus, we have $\lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx > 1$. Upper bound follows from Proposition 1. Indeed, assuming that C_1 is the constant obtained in Proposition 1 and setting $C_2 := e^{\alpha C_1^2}$, we have

$$\lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx \le C_2 \lambda_n \int_\Omega u_n^2 dx \le C_2.$$

Therefore, we conclude that the lemma holds.

The next proposition follows from Theorem 2.1 in [14].

Proposition 2 *For $L > 1$ and $\alpha > 0$ there exists a positive constant $\delta_{L,\alpha}$ such that for any $w \in H^1(\mathbb{R}^2)$ which is a solution of*

$$-\Delta w + w = \frac{w e^{\alpha w^2}}{L} \quad \text{in} \quad \mathbb{R}^2 \tag{15}$$

it holds that

$$\int_{\mathbb{R}^2} |\nabla w|^2 dx \ge \delta_{L,\alpha}.$$

Proof Assume that $L > 1$, $\alpha > 0$ and $w \in H^1(\mathbb{R}^2)$ is a solution of (15). Note that $w \in C^2_{loc}(\mathbb{R}^2)$ holds by the elliptic regularity theory. Set $\hat{w} = (\alpha/4\pi)^{1/2} w$. Then, $\hat{w}$ is a solution of

$$-\Delta w + w = \frac{w e^{4\pi w^2}}{L}, \tag{16}$$

and it follows from the Pohozaev identity that

$$\frac{1}{2} \int_{\mathbb{R}^2} \hat{w}^2 dx - \frac{1}{8\pi L} \int_{\mathbb{R}^2} \left(e^{4\pi \hat{w}^2} - 1 \right) dx = 0. \tag{17}$$

By Theorem 2.1 in [14], there exists a ground state solution w_* of (16), that is, w_* is a solution of (16) such that $I(w_*) = c_{*,L}$, where

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 \right) dx - \frac{1}{8\pi L} \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1 \right) dx, \quad u \in H^1(\mathbb{R}^2),$$

$$c_{*,L} := \inf \left\{ I(u) \;\middle|\; u \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of } (16) \right\}.$$

Combining the result and (17), we have

$$0 < c_{*,L} \le I(\hat{w}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \hat{w}|^2 dx = \frac{\alpha}{8\pi} \int_{\mathbb{R}^2} |\nabla w|^2 dx.$$

Taking $\delta_{L,\alpha} = 8\pi c_{*,L}/\alpha$, we obtain the desired lower bound.

Lemma 2 *Assume that there exist positive constant ε and a point $\tilde{x}_n \in \overline{\Omega}$ such that* $\lim_{n\to\infty} u_n(\tilde{x}_n) \ge \varepsilon$ *holds. Then, there exists $\tilde{w} \in H^1(\mathbb{R}^2)$ such that*

(i) $\tilde{w}$ is radially symmetric for some point $\tilde{p} \in \mathbb{R}^2$,
(ii) $\partial \tilde{w}/\partial r < 0$ for $r = |y - \tilde{p}| > 0$, $y \in \mathbb{R}^2$,

(iii) $\tilde{w}$ *is a solution of*

$$-\Delta w + w = \frac{we^{\alpha w^2}}{L} \quad \text{in} \quad \mathbb{R}^2$$

for some $L > 1$,

(iv) *if* $\sqrt{\lambda_n}\operatorname{dist}(\tilde{x}_n, \partial\Omega) \to \infty$ *as* $n \to \infty$, *then we have*

$$u_n\left(\frac{y}{\sqrt{\lambda_n}} + \tilde{x}_n\right) \to \tilde{w} \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2),$$

and if $\operatorname{dist}(\tilde{x}_n, \partial\Omega) = O(\sqrt{\lambda_n}^{-1})$, *then we have*

$$u_n\left(\Phi\left(\frac{z}{\sqrt{\lambda_n}} + \Psi(\tilde{x}_n)\right)\right) \to \tilde{w} \quad \text{in} \quad C^2_{loc}(\overline{\mathbb{R}^2_+}).$$

Proof In the case of $\sqrt{\lambda_n}\operatorname{dist}(\tilde{x}_n, \partial\Omega) \to \infty$, we set

$$w_n := u_n\left(\frac{y}{\sqrt{\lambda_n}} + \tilde{x}_n\right) \quad \text{for} \quad y \in \Omega_{\lambda_n} := \left\{ \sqrt{\lambda_n}(x - \tilde{x}_n) \;\middle|\; x \in \Omega \right\}.$$

Then, w_n is a solution of

$$-\Delta w + w = \frac{we^{\alpha w^2}}{\lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx}.$$

Since $\sqrt{\lambda_n}\operatorname{dist}(\tilde{x}_n, \partial\Omega) \to \infty$, for any $R > 0$ there exists K such that $B_R(\tilde{x}_n) \subset \Omega_{\lambda_n}$ for any $n \geq K$. By Lemma 1 and the elliptic regularity theory, there exists $\tilde{w}$ such that

$$w_n \to \tilde{w} \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2)$$

and $\tilde{w}$ is a solution of

$$-\Delta w + w = \frac{we^{\alpha w^2}}{L} \quad \text{in} \quad \mathbb{R}^2, \quad L \in [1, C_2].$$

Moreover,

$$\int_{\mathbb{R}^2} \left(|\nabla \tilde{w}|^2 + \tilde{w}^2\right) dx = \lim_{R\to\infty} \lim_{n\to\infty} \int_{B_R} \left(|\nabla w_n|^2 + w_n^2\right) dx \leq \lim_{n\to\infty} \int_\Omega \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) dx = 1,$$

and then

$$\tilde{w} \in H^1(\mathbb{R}^2).$$

Since $\tilde{w} \in C^2_{loc}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, using the Pohozaev identity, we have

$$\frac{1}{2}\int_{\mathbb{R}^2} \left[\tilde{w}^2 - \frac{1}{\alpha L}\left(e^{\alpha \tilde{w}^2} - 1\right)\right] dx = 0,$$

which implies $L > 1$. Since u_n is positive in $\overline{\Omega}$, we see that $\tilde{w}$ is positive in $\mathbb{R}^2$. Hence, $\tilde{w}$ is radially symmetric for some point in $\mathbb{R}^2$ and $\partial\tilde{w}/\partial r < 0$ for $r > 0$.

In the case of $\operatorname{dist}(\tilde{x}_n, \partial\Omega) = O(\sqrt{\lambda_n}^{-1})$, we may assume that $\tilde{x}_n \to \tilde{x}_0 \in \partial\Omega$ as $n \to \infty$ after passing to a subsequence. We use the diffeomorphism $y = \Psi(x)$ which straightens a boundary portion near $\tilde{x}_0 \in \partial\Omega$. For $\kappa > 0$, put

$$v_n(y) := u_n(\Phi(y)) \quad \text{for} \quad y \in \overline{B^+_{2\kappa}},$$

$$\tilde{v}_n(y) := \begin{cases} v_n(y) & \text{if} \quad y \in \overline{B^+_{2\kappa}}, \\ v_n((y_1, -y_2)) & \text{if} \quad y \in B^-_{2\kappa}, \end{cases}$$

$$w_n(z) := \tilde{v}_n\left(\frac{z}{\sqrt{\lambda_n}} + \tilde{P}_n\right) \quad \text{for} \quad z \in \overline{B_{\kappa\sqrt{\lambda_n}}},$$

where $\tilde{P}_n := \Psi(\tilde{x}_n) \in B^+_\kappa$. Set a_{ij}, b_j as in (10), (11), and then a^n_{ij}, b^n_j are defined as (12), (13) with replacing r_n and P_n by $\sqrt{\lambda_n}^{-1}$ and $\tilde{P}_n = (\tilde{p}_n, \tilde{q}_n/\sqrt{\lambda_n})$, respectively. In the setting, w_n satisfies

$$\sum_{i,j=1}^{2} a^n_{ij}(z)\frac{\partial^2 w_n}{\partial z_i \partial z_j} + \sqrt{\lambda_n}^{-1}\sum_{j=1}^{2} b^n_j(z)\frac{\partial w_n}{\partial z_j} + w_n = \frac{w_n e^{\alpha w_n^2}}{\lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx}.$$

Thus, by Lemma 1 and the elliptic regularity theory, we have

$$w_n \to \tilde{w} \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\tilde{w} + \tilde{w} = \frac{\tilde{w}e^{\alpha\tilde{w}^2}}{L} \quad \text{in} \quad \mathbb{R}^2, \quad L \in [1, C_2].$$

Computing in the same way as in the case of $\sqrt{\lambda_n}\mathrm{dist}(\tilde{x}_n, \partial\Omega) \to \infty$, we derive that $\tilde{w} \in H^1(\mathbb{R}^2)$, $L > 1$, $\tilde{w}$ is radially symmetric and $\partial\tilde{w}/\partial r < 0$ for $r > 0$.

Lemma 3 *The followings are equivalent.*

(i) There exists a positive constant C_3 such that $\lim_{n\to\infty} \|u_n\|_{L^\infty(\Omega)} \geq C_3$.
(ii) $\lim_{n\to\infty} \lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx > 1$.
(iii) There exists positive constant δ such that $\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \geq \delta$.

Proof First, we prove the equivalence of (i) and (ii). Set $L = \lim_{n\to\infty} \lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx$. Assume that (i) holds. Then applying Lemma 2 to a maximum point of u_n, we derive $L > 1$ by Lemma 2 (iii).

Suppose that (ii) holds. Assuming the contrary that $c_n := \|u_n\|_{L^\infty(\Omega)} \to 0$, we derive a contradiction. Under the assumption, it follows that

$$L = \lim_{n\to\infty} \lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx \leq \lim_{n\to\infty} \left(1 + Cc_n^2\right) \lambda_n \int_\Omega u_n^2 dx \leq 1 \tag{18}$$

for some positive constant C, which is a contradiction. Hence we have $c_n \geq C_3$ for some positive constant C_3.

Next, we show (iii) under the assumption (i). We apply Lemma 2 to a maximum point $x_n \in \overline{\Omega}$. If $\sqrt{\lambda_n}\mathrm{dist}(x_n, \partial\Omega) \to \infty$, by Lemma 2, there exists $w_0 \in H^1(\mathbb{R}^2)$ such that the conditions of Lemma 2 hold. Then, we have

$$\int_{\mathbb{R}^2} |\nabla w_0|^2 dx = \lim_{R\to\infty}\lim_{n\to\infty} \int_{B_{R/\sqrt{\lambda_n}}(x_n)} |\nabla u_n|^2 dx \leq \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx.$$

Moreover, it follows from Proposition 2 that $\int_{\mathbb{R}^2} |\nabla w_0|^2 dx \geq \delta_{L,\alpha}$. Hence $\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \geq \delta_{L,\alpha}$ holds.

In the case of $\mathrm{dist}(x_n, \partial\Omega) = O(\sqrt{\lambda_n}^{-1})$, by Lemma 2, there exists $w_0 \in H^1(\mathbb{R}^2)$ such that the conditions of Lemma 2 hold and

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla w_0|^2 dx \leq \int_\Omega |\nabla u_n|^2 dx + o(1).$$

This and Proposition 2 yield that $\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2dx \geq \delta_{L,\alpha}/2$ holds. Consequently, in both cases, we obtain $\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2dx \geq \delta$ with $\delta = \delta_{L,\alpha}/2$.

Finally, we prove (i) under the assumption (iii). Assuming the contrary that that $c_n := \|u_n\|_{L^\infty(\Omega)} \to 0$ as $n\to\infty$, we derive a contradiction. Combining Lemma 1 and (18), we have

$$1 \leq L \leq \lim_{n\to\infty}\left(1+Cc_n^2\right)\lambda_n\int_\Omega u_n^2dx \leq 1$$

for some positive constant C, and thus

$$\lim_{n\to\infty}\lambda_n\int_\Omega u_n^2dx = 1.$$

Since $u_n \in \Sigma_{\lambda_n}$ we have

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2dx = 0,$$

which is a contradiction. Therefore, we conclude that c_n is bounded from below.

Lemma 4 *Assume that there exists a positive constant* C_3 *such that* $\lim_{n\to\infty}\|u_n\|_{L^\infty(\Omega)} \geq C_3$. *Set* $k_0 := [2\delta_{L,\alpha}^{-1}]$ *which is the largest integer less than or equal to* $2\delta_{L,\alpha}^{-1}$, *where* $\delta_{L,\alpha}$ *is obtained in Proposition 2. Then there exist at most* k_0 *sequences* $\{x_n^i\}\subset\overline{\Omega}$, $i=1,\cdots,k_0$ *such that*

(i) for each i there exists a positive constant ε_i *such that*

$$\lim_{n\to\infty}u_n(x_n^i) \geq \varepsilon_i,$$

(ii) $\lim_{n\to\infty}\sqrt{\lambda_n}|x_n^i - x_n^j| = \infty$ *if* $i\neq j$.

Proof Assume that $\lim_{n\to\infty}\|u_n\|_{L^\infty(\Omega)} \geq C_3$ for some positive constant C_3. By Lemma 3, it holds that $\lim_{n\to\infty}\lambda_n\int_\Omega u_n^2e^{\alpha u_n^2}dx > 1$. Set $L := \lim_{n\to\infty}\lambda_n\int_\Omega u_n^2e^{\alpha u_n^2}dx$ and $k_0 := [2\delta_{L,\alpha}^{-1}]$. We assume the contrary that there exist (k_0+1) sequences $\{x_n^i\}\subset\overline{\Omega}$, $i=1,\cdots,k_0+1$ such that (i) and (ii) hold and derive a contradiction. Since $\{x_n^i\}$ satisfies (i) we can apply Lemma 2 to x_n^i. By Proposition 2 and Lemma 2, for each i it follows that

$$\frac{\delta_{L,\alpha}}{2} \leq \int_{A_{R,n}^i}|\nabla u_n|^2dx + o_n(1) + o_R(1),$$

where $o_n(1)\to 0$ as $n\to\infty$, $o_R(1)\to 0$ as $R\to\infty$ which is independent of n and

$$A_{R,n}^i = \begin{cases} B_{R/\sqrt{\lambda_n}}(x_n^i) & \text{if} \quad \sqrt{\lambda_n}\mathrm{dist}(x_n^i,\partial\Omega)\to\infty, \\ \Omega\cap\Phi\left(B_{R/\sqrt{\lambda_n}}(\Psi(x_n^i))\right) & \text{if} \quad \mathrm{dist}(x_n^i,\partial\Omega) = O(\sqrt{\lambda_n}^{-1}). \end{cases} \tag{19}$$

It follows from (ii) and the condition $u_n\in\Sigma_{\lambda_n}$ that

$$\begin{aligned} \frac{(k_0+1)\delta_{L,\alpha}}{2} &\leq \sum_{i=1}^{k_0+1}\int_{A_{R,n}^i}|\nabla u_n|^2dx + o_n(1) + o_R(1) \\ &= \int_{\cup_{i=1}^{k_0+1}A_{R,n}^i}|\nabla u_n|^2dx + o_n(1) + o_R(1) \\ &\leq \int_\Omega |\nabla u_n|^2dx + o_n(1) + o_R(1) \\ &\leq 1 + o_n(1) + o_R(1). \end{aligned}$$

But, this inequality contradicts the definition of k_0. Hence, we conclude that the lemma holds.

Lemma 5 *Assume that there exists a positive constant C_3 such that $\lim_{n\to\infty}\|u_n\|_{L^\infty(\Omega)} \geq C_3$. Fix $k < +\infty$ as the largest integer m such that m sequences $\{x_n^i\} \subset \overline{\Omega}$, $i = 1, \cdots, m$ satisfy the followings:*

(i) for each i there exists a positive constant ε_i such that $\lim_{n\to\infty} u_n(x_n^i) \geq \varepsilon_i$,
(ii) if $m \geq 2$, $\lim_{n\to\infty}\sqrt{\lambda_n}|x_n^i - x_n^j| = \infty$ for $i \neq j$,

such a k exists thanks to Lemma 4. In addition to the assumptions, for each i take $w_i \in H^1(\mathbb{R}^2)$ satisfying the conditions of Lemma 2 with replacing $\tilde{x}_n$ by x_n^i, such w_i is also exists by the condition (i). Then, we have

$$\tau_i := \int_{X_i} \left(|\nabla w_i|^2 + w_i^2\right) dx \leq 1, \quad \sum_{i=1}^{k} \tau_i \leq 1, \tag{20}$$

$$\lim_{n\to\infty} \lambda_n I(\alpha, \lambda_n) \leq \sum_{i=1}^{k} \int_{X_i} \left(e^{\alpha w_i^2} - 1\right) dx + \alpha \left(1 - \sum_{i=1}^{k} \tau_i\right), \tag{21}$$

where

$$X_i := \begin{cases} \mathbb{R}^2 & \text{if} \quad \sqrt{\lambda_n}\operatorname{dist}(x_n^i, \partial\Omega) = \infty, \\ \mathbb{R}^2_+ & \text{if} \quad \operatorname{dist}(x_n^i, \partial\Omega) = O(\sqrt{\lambda_n}^{-1}). \end{cases}$$

Proof It follows that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \int_\Omega \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) dx \\ &= \lim_{R\to\infty}\lim_{n\to\infty} \left[\sum_{i=1}^{k} \int_{A_{R,n}^i} \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) dx + \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) dx\right] \\ &= \sum_{i=1}^{k} \int_{X_i} \left(|\nabla w_i|^2 + w_i^2\right) dx + \lim_{R\to\infty}\lim_{n\to\infty} \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) dx, \end{aligned} \tag{22}$$

where $A_{R,n}^i$ is defined in (19). Thus, we obtain (20). Similarly, we observe that

$$\begin{aligned} \lim_{n\to\infty} \lambda_n I(\alpha, \lambda_n) &= \lim_{R\to\infty}\lim_{n\to\infty} \lambda_n \left[\sum_{i=1}^{k} \int_{A_{R,n}^i} \left(e^{\alpha u_n^2} - 1\right) dx + \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \left(e^{\alpha u_n^2} - 1\right) dx\right] \\ &= \sum_{i=1}^{k} \int_{X_i} \left(e^{\alpha w_i^2} - 1\right) dx + \lim_{R\to\infty}\lim_{n\to\infty} \lambda_n \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \left(e^{\alpha u_n^2} - 1\right) dx. \end{aligned} \tag{23}$$

Here, in order to obtain (21), we prove the following estimate:

$$\lim_{n\to\infty} \sup_{x\in\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} u_n(x) = o_R(1), \tag{24}$$

where $o_R(1) \to 0$ as $R \to \infty$.

Take any sequence $\{P_n^R\} \subset \overline{\Omega \setminus \left(\cup_{i=1}^k A_{R,n}^i\right)}$. If P_n^R satisfies $\lim_{n\to\infty}\sqrt{\lambda_n}|P_n^R - x_n^i| = \infty$ for all $i = 1, \cdots, k$, then it holds that $u_n(P_n^R) \to 0$ as $n \to \infty$ by the definition of k. Thus, we may assume that $|P_n^R - x_n^i| = O(\sqrt{\lambda_n}^{-1})$ for some i. In addition to this, since $\{P_n^R\} \subset$

$\overline{\Omega \setminus \left(\cup_{i=1}^k A_{R,n}^i\right)}$, we see that $|P_n^R - x_n^i| \geq \kappa R/\sqrt{\lambda_n}$ for $\kappa > 0$. Hence, after passing to a subsequence, there exists P_0^R such that

$$\lim_{n\to\infty} \sqrt{\lambda_n}\left(P_n^R - x_n^i\right) = P_0^R, \quad \lim_{R\to\infty} |P_0^R| = \infty \qquad \text{if} \quad \sqrt{\lambda_n}\text{dist}(x_n^i, \partial\Omega) = \infty,$$

$$\lim_{n\to\infty} \sqrt{\lambda_n}\left(\Psi\left(P_n^R\right) - \Psi\left(x_n^i\right)\right) = P_0^R, \quad \lim_{R\to\infty} |P_0^R| = \infty \qquad \text{if} \quad \text{dist}(x_n^i, \partial\Omega) = O(\sqrt{\lambda_n}^{-1}).$$

Recall that by Lemma 2,

$$u_n\left(\frac{y}{\sqrt{\lambda_n}} + x_n^i\right) \to w_i \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2) \quad \text{if} \quad \sqrt{\lambda_n}\text{dist}(x_n^i, \partial\Omega) = \infty,$$

$$u_n\left(\Phi\left(\frac{z}{\sqrt{\lambda_n}} + \Psi(x_n^i)\right)\right) \to w_i \quad \text{in} \quad C^2_{loc}(\overline{\mathbb{R}^2_+}) \quad \text{if} \quad \text{dist}(x_n^i, \partial\Omega) = O(\sqrt{\lambda_n}^{-1}),$$

and then we have

$$u_n(P_n^R) \to w_i(P_0^R)$$

as $n \to \infty$. We observe that $w_i(x) \to 0$ as $|x| \to \infty$ since $w_i \in H^1(\mathbb{R}^2)$. Thus it holds that $\lim_{n\to\infty} u_n(P_n^R) = o_R(1)$. Consequently, we obtain (24).

Set $\tau_i = \int_{X_i} \left(|\nabla w_i|^2 + w_i^2\right) dx$ for each i. It follows from (22) and (24) that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \left(e^{\alpha u_n^2} - 1\right) dx \\
&= (1 + o_R(1)) \lim_{n\to\infty} \lambda_n \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \alpha u_n^2 dx \\
&\leq \alpha\,(1 + o_R(1)) \lim_{n\to\infty} \int_{\Omega\setminus\left(\cup_{i=1}^k A_{R,n}^i\right)} \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) dx \\
&= \alpha\,(1 + o_R(1)) \left(1 - \sum_{i=1}^k \tau_i + o_R(1)\right).
\end{aligned}$$

Combining the estimate and (23), we derive (21). Consequently, we obtain the desired estimates.

Proposition 3 *It holds that*

$$\lim_{n\to\infty} \lambda_n I(\alpha, \lambda_n) \geq I_\alpha,$$

where I_α is defined by

$$I_\alpha := \sup_{\substack{u\in H^1(\mathbb{R}^2_+) \\ \int_{\mathbb{R}^2_+}\left(|\nabla u|^2 + u^2\right) dx \leq 1}} \int_{\mathbb{R}^2_+} \left(e^{\alpha u^2} - 1\right) dx.$$

Proof Without loss of generality, we may assume that $0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}^2_+$. Let $\{w_\ell\} \subset H^1(\mathbb{R}^2_+)$ be a maximizing sequence of I_α and set

$$W_\ell(x) := w_\ell\left(\sqrt{\lambda_n} x\right).$$

Since $\int_{\mathbb{R}^2_+}\left(|\nabla w_\ell|^2+w_\ell^2\right)dx=1$, we have

$$\int_\Omega \left(|\nabla W_\ell|^2+\lambda_n W_\ell^2\right)dx \le \int_{\mathbb{R}^2_+}\left(|\nabla W_\ell|^2+\lambda_n W_\ell^2\right)dx=\int_{\mathbb{R}^2_+}\left(|\nabla w_\ell|^2+w_\ell^2\right)dx=1.$$

Then, it follows that

$$I(\alpha,\lambda_n)\ge \int_\Omega\left(e^{\alpha W_\ell^2}-1\right)dx \ge \int_{\Omega\cap B_{R/\sqrt{\lambda_n}}}\left(e^{\alpha W_\ell^2}-1\right)dx=\lambda_n^{-1}\int_{\Omega_{\lambda_n}\cap B_R}\left(e^{\alpha w_\ell^2}-1\right)dx,$$

where $\Omega_{\lambda_n}:=\left\{\sqrt{\lambda_n}x \;\middle|\; x\in\Omega\right\}$. The smoothness of the boundary of Ω gives

$$\lim_{n\to\infty}\lambda_n I(\alpha,\lambda_n)\ge \int_{B_R^+}\left(e^{\alpha w_\ell^2}-1\right)dx.$$

Letting $R\to\infty$ and $\ell\to\infty$, we conclude that

$$\lim_{n\to\infty}\lambda_n I(\alpha,\lambda_n)\ge I_\alpha.$$

2.2 Proof of Theorem 1 completed

Now, we are in position to prove Theorem 1. In the case of $\alpha>\alpha_*$ it holds that $I_\alpha>\alpha$ and I_α is attained. First, we prove that (I). Assuming that $\|u_n\|_{L^\infty(\Omega)}\to 0$ as $n\to\infty$, we derive a contradiction. In this case, it follows that

$$\lim_{n\to\infty}\lambda_n I(\alpha,\lambda_n)=\lim_{n\to\infty}\lambda_n\int_\Omega\left(e^{\alpha u_n^2}-1\right)dx\le \lim_{n\to\infty}\left(\alpha+C\|u_n\|^2_{L^\infty(\Omega)}\right)\lambda_n\int_\Omega u_n^2dx\le\alpha<I_\alpha$$

for some positive constant C. But, this contradicts Proposition 3. Hence, there exists a positive constant M_1 such that $\|u_n\|_{L^\infty(\Omega)}\ge M_1$. This fact and Proposition 1 yield (I).

Next, we prove (II). Since Theorem 1 (I) holds, we can use Lemma 5. By Lemma 5 and Proposition 3, we have

$$I_\alpha\le\sum_{i=1}^k\int_{X_i}\left(e^{\alpha w_i^2}-1\right)dx+\alpha\left(1-\sum_{i=1}^k\tau_i\right), \tag{25}$$

where X_i,w_i,τ_i are defined in Lemma 5. For each i, since the function e^s-1 is convex, we have

$$\int_{X_i}\left(e^{\alpha w_i^2}-1\right)dx\le\tau_i\int_{X_i}\left(e^{\alpha\frac{w_i^2}{\tau_i}}-1\right)dx\le\tau_i\sup_{\substack{w\in H^1(X_i)\\ \int_{X_i}\left(|\nabla w|^2+w^2\right)dx=1}}\int_{X_i}\left(e^{\alpha w^2}-1\right)dx. \tag{26}$$

If $X_i=\mathbb{R}^2$, by the convexity of e^s-1 we have

$$\sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)dx=1}}\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)dx=2I_{\alpha/2}\le I_\alpha. \tag{27}$$

Thus, (25), (26), (27) and the inequality $I_\alpha>\alpha$ yield

$$I_\alpha\le\sum_{i=1}^k\int_{X_i}\left(e^{\alpha w_i^2}-1\right)dx+\alpha\left(1-\sum_{i=1}^k\tau_i\right)\le\sum_{i=1}^k\tau_iI_\alpha+\left(1-\sum_{i=1}^k\tau_i\right)I_\alpha=I_\alpha. \tag{28}$$

Hence, all inequalities in (28) become equalities. Since I_α is attained, the inequality in (27) becomes strict inequality. Thus $X_i \neq \mathbb{R}^2$, and $X_i = \mathbb{R}^2_+$. Moreover, equality of (26) holds if and only if $\tau_i = 1$ and w_i is a maximizer of I_α for some i. These conditions give the equality in (28). Consequently, $k = 1$, $X_1 = \mathbb{R}^2_+$ and w_1 is a maximizer of I_α.

In order to prove that u_n has a unique maximum, we use the following lemma which is introduced in [11].

Lemma 6 *Let $\xi_* \in C^2(\overline{B_a})$ be a radial function satisfying $\xi_*'(0) = 0$ and $\xi_*''(r) < 0$ for $0 \le r \le a$. Then there exists a $\delta > 0$ such that if $\xi \in C^2(\overline{B_a})$ satisfies (i) $\nabla\xi(0) = 0$ and (ii) $\|\xi - \xi_*\|_{C^2(\overline{B_a})} \le \delta$, then $\nabla\xi \neq 0$ for $x \neq 0$.*

Let x_n be a maximum point of u_n with $x_n \to x_0$ as $n \to \infty$. Computing in the same way as the proof of (24), we have

$$\lim_{n\to\infty} \sup_{x\in\Omega\setminus\Phi\left(B_{R/\sqrt{\lambda_n}}(\Psi(x_n))\right)} u_n(x) = o_R(1), \tag{29}$$

where $o_R(1) \to 0$ as $R \to \infty$. Thus, all maximum points are located in $\Omega \cap \Phi\left(B_{R/\sqrt{\lambda_n}}(\Psi(x_n))\right)$ for large $R > 0$ and n. Take the diffeomorphism $y = \Psi(x)$ which straightens a boundary portion near x_0 and define $P_n = \Psi(x_n) = \left(p_n, q_n/\sqrt{\lambda_n}\right)$. Then, set

$$w_n^1 = \begin{cases} u_n\left(\Phi\left(\frac{z}{\sqrt{\lambda_n}} + P_n\right)\right) & \text{if} \quad z_2 \ge -q_n, \\ u_n\left(\Phi\left(\frac{z_1}{\sqrt{\lambda_n}} + p_n, -\frac{z_2+q_n}{\sqrt{\lambda_n}}\right)\right) & \text{if} \quad z_2 < -q_n \end{cases}$$

by the reflection. Since $z = 0$ is a maximum point of w_n^1, $z = (0, -2q_n)$ is also maximum point of w_n^1. Computing in the same way as in the proof of Lemma 2, we have $w_n^1 \to w_1$ in $C^2_{loc}(\mathbb{R}^2)$. Applying Lemma 6 in the Ball $\overline{B_R}$ for large $R > 0$, we deduce that $q_n = 0$ for large n. Similarly, if $z = (p, 0)$ is also a maximum point, then we have $p = 0$ by Lemma 6. Consequently, u_n has a unique maximum point and the maximum point is located on the boundary for large n.

To end the proof of Theorem 1, we estimate u_n on the outside of $B_{R/\sqrt{\lambda_n}}(x_n)$. For fixed R, there exist positive constants R_1, R_2 such that

$$\Omega \cap B_{R_1/\sqrt{\lambda_n}}(x_n) \subset \Omega \cap \Phi\left(B_{R/\sqrt{\lambda_n}}(\Psi(x_n))\right) \subset \Omega \cap B_{R_2/\sqrt{\lambda_n}}(x_n).$$

Thus, by (29), u_n satisfies

$$\sup_{x\in\Omega\setminus B_{R_2/\sqrt{\lambda_n}}(x_n)} u_n(x) \to o_R(1)$$

as $n \to \infty$. Since u_n satisfies (6) and $\lim_{n\to\infty} \lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx > 1$, we have

$$\frac{1}{\lambda_n}\Delta u_n - \left(1 - \frac{e^{\alpha u_n^2}}{\lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx}\right) u_n = 0, \quad 1 - \frac{e^{\alpha u_n^2}}{\lambda_n \int_\Omega u_n^2 e^{\alpha u_n^2} dx} > 0 \quad \text{in} \quad \Omega \setminus B_{R_2/\sqrt{\lambda_n}}(x_n)$$

for large n. To prove (III), we use the following proposition which is introduced in [2].

Proposition 4 (Lemma 4.2 in [2]) *Assume that $\varepsilon > 0$ and $\mathscr{A}$ is a domain. Let ϕ be a C^2 function satisfying $L\phi := \varepsilon^2 \partial_i(a_{ik}\partial_k\phi) + q(x,\varepsilon)\phi = 0$ in $\mathscr{A}$, with $q(x,\varepsilon) < -a < 0$ in $\mathscr{A}$. Then there exists a positive constant $\mu = \mu(a_{ik}, a, \mathscr{A})$ such that*

$$|\phi(x)| \le 2\left(\sup|\phi(x)|\right) e^{-\frac{\mu\delta}{\varepsilon}}$$

where $\delta(x) = \mathrm{dist}(x, \partial\mathscr{A})$.

In the interior of $\Omega \setminus B_{R_2/\sqrt{\lambda_n}}(x_n)$, we can apply Proposition 4 to u_n directly. In the neighborhood around $\partial\Omega \setminus B_{R_2/\sqrt{\lambda_n}}(x_n)$, defining $\hat{w}_n$ as the extension of u_n by taking the diffeomorphism straightening a boundary portion at each point of $\partial\Omega$ and the reflection, we apply Proposition 4 to $\hat{w}_n$. Hence we obtain (III). Consequently, the proof of Theorem 1 is completed.

2.3 Proof of Theorem 2

Assuming the contrary that $\|u_n\|_{L^\infty(\Omega)} \geq \varepsilon > 0$ for large n, we derive a contradiction. Under the assumption, we can use Lemma 5, and the inequality (25) holds. In the case of $\alpha \in (0,\alpha_*)$, $I_\alpha = \alpha$ and I_α is not attained. Moreover, we see that $d_\alpha = \alpha$ and d_α is not attained. Thus, in (26), the second inequality becomes strict inequality for any i. The strict inequality and (25) yield

$$I_\alpha < \sum_{i=1}^{k} \tau_i I_\alpha + \alpha \left(1 - \sum_{i=1}^{k} \tau_i\right) = I_\alpha,$$

which is a contradiction. Hence, we obtain $\|u_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$. By Lemma 3, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx = 0,$$

and thus

$$\lim_{n\to\infty} \lambda_n \int_\Omega u_n^2 dx = 1 - \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx = 1.$$

Consequently, we conclude that Theorem 2 holds.

3 Proof of Theorem 3

In this section, we fix $\alpha \in (0,2\pi)$ and v_λ denotes a positive critical point of $E_\alpha|_{\Sigma_\lambda}$ for $\lambda > 0$. Then v_λ is a solution of

$$\begin{cases} -\Delta v + \lambda v = \frac{v e^{\alpha v^2}}{\int_\Omega v^2 e^{\alpha v^2} dx} & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{30}$$

We first prove the following proposition:

Proposition 5 *For any positive solution v of (30) it holds that*

$$\inf_{x\in\Omega} v(x) \leq (\lambda_n|\Omega|)^{-\frac{1}{2}} \leq \sup_{x\in\Omega} v(x).$$

Moreover, one of the inequalities becomes equality if and only if $v \equiv \lambda|\Omega|^{-1/2}$, which is equivalent to that all equalities hold.

Proof Since $v > 0$, multiplying (30) by v^{-1} and integrating over Ω, we have

$$-\int_\Omega \frac{|\nabla v|^2}{v^2} dx + \lambda|\Omega| = \frac{\int_\Omega e^{\alpha v^2} dx}{\int_\Omega v^2 e^{\alpha v^2} dx}.$$

We see that

$$\int_\Omega \frac{|\nabla v|^2}{v^2} dx \geq 0, \quad \frac{\int_\Omega e^{\alpha v^2} dx}{\int_\Omega v^2 e^{\alpha v^2} dx} \geq \left(\sup_{x\in\Omega} v(x)\right)^{-2}, \tag{31}$$

and then we have

$$(\lambda|\Omega|)^{-\frac{1}{2}} \leq \sup_{x\in\Omega} v(x).$$

The equalities hold on the estimates (31) if and only if v is a constant, and hence $v \equiv (\lambda|\Omega|)^{-1/2}$.

Multiplying (30) by v and integrating over Ω, we see that

$$\int_\Omega \left(|\nabla v|^2 + \lambda v^2\right) dx = 1.$$

Thus,

$$1 = \int_\Omega \left(|\nabla v|^2 + \lambda v^2\right) dx \geq \lambda \int_\Omega v^2 dx \geq \lambda|\Omega| \left(\inf_{x\in\Omega} v(x)\right)^2. \tag{32}$$

Hence the estimate

$$\inf_{x\in\Omega} v(x) \leq (\lambda|\Omega|)^{-\frac{1}{2}}$$

follows immediately. In (32), all equalities hold if and only if $v \equiv (\lambda|\Omega|)^{-1/2}$. Consequently, we conclude that the proposition holds.

In the following, let λ_n be a sequence such that $\lambda_n \to 0$ as $n \to \infty$ and let $v_n := v_{\lambda_n}$. In addition to the setting, assume that $x_n \in \overline{\Omega}$ is a maximum point of v_n and set

$$c_n = \sup_{x\in\Omega} v_n(x), \quad \underline{c_n} = \inf_{x\in\Omega} v_n(x).$$

Lemma 7 *We have*

$$\frac{e^{\alpha c_n^2}}{\int_\Omega v_n^2 e^{\alpha v_n^2} dx} \to 0$$

as $n \to \infty$.

Proof Assuming the contrary that there exists a positive constant ε such that

$$\frac{e^{\alpha c_n^2}}{\int_\Omega v_n^2 e^{\alpha v_n^2} dx} \geq \varepsilon$$

holds, we derive a contradiction. Define r_n such that

$$r_n^2 = \frac{\int_\Omega v_n^2 e^{\alpha v_n^2} dx}{c_n^2 e^{\alpha c_n^2}},$$

and by the assumption, we have

$$r_n^2 \leq \frac{1}{\varepsilon c_n^2} = O(c_n^{-2}). \tag{33}$$

We follow the proof of Proposition 1.

If $\mathrm{dist}(x_n, \partial\Omega)/r_n \to \infty$, we define $\Omega_n := \{(x - x_n)/r_n \mid x \in \Omega\}$ and

$$\begin{cases} \phi_n(y) := c_n^{-1} v_n(r_n y + x_n) & y \in \Omega_n, \\ \eta_n(y) := c_n(v_n(r_n y + x_n) - c_n) & y \in \Omega_n. \end{cases}$$

Then, ϕ_n and η_n satisfy

$$-\Delta_y \phi_n + \lambda_n r_n^2 \phi_n = c_n^{-2} \phi_n e^{\alpha c_n^2 (\phi_n^2 - 1)},$$

$$-\Delta_y \eta_n + \lambda_n r_n^2 c_n^2 \phi_n = \phi_n e^{\alpha(1+\phi_n)\eta_n}.$$

By (33), the elliptic regularity theory and the maximum principle we see that

$$\phi_n \to \phi_0 \equiv 1 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta \phi_0 = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Then, since $\lambda_n \to 0$, we have

$$\eta_n \to \eta_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta \eta_0 = e^{2\alpha\eta_0} \quad \text{in} \quad \mathbb{R}^2.$$

Moreover, computing in the same way as in (9), we derive that

$$\int_{\mathbb{R}^2} e^{2\alpha\eta_0} dy \le 1, \tag{34}$$

and then

$$\eta_0 = -\frac{1}{\alpha} \log\left(1 + \frac{\alpha}{4}|y|^2\right).$$

Since $\alpha < 2\pi$, by a direct computation, we have

$$\int_{\mathbb{R}^2} e^{2\alpha\eta_0} dy = \frac{4\pi}{\alpha} > 2.$$

But this contradicts (34).

In the case of $\mathrm{dist}(x_n, \partial\Omega) = O(r_n)$, we may assume that $x_n \to x_0 \in \partial\Omega$ by passing to a subsequence if necessary. Put

$$\tilde{v}_n(y) := v_n(\Phi(y)) \quad \text{for} \quad y \in \overline{B^+_{2\kappa}}$$

for $\kappa > 0$ and

$$\hat{v}_n(y) := \begin{cases} \tilde{v}_n(y) & \text{if} \quad y \in \overline{B^+_{2\kappa}}, \\ \tilde{v}_n((y_1, -y_2)) & \text{if} \quad y \in B^-_{2\kappa}. \end{cases}$$

Moreover, set $P_n := \Psi(x_n) = (p_n, q_n r_n)$, and define $w_n(z)$ by

$$w_n(z) := \hat{v}_n(r_n z + P_n) \quad \text{for} \quad z \in \overline{B_{\kappa/r_n}}.$$

Then, ϕ_n and η_n are defined by

$$\phi_n(z) := c_n^{-1} w_n(z),$$

$$\eta_n(z) := c_n(w_n(z) - c_n).$$

Set a_{ij}, b_j as in (10), (11), and then a^n_{ij}, b^n_j are defined by (12), (13). Since v_n is a solution of (30) for λ_n, ϕ_n and η_n satisfy the elliptic equations

$$\sum_{i,j=1}^{2} a^n_{ij}(z) \frac{\partial^2 \phi_n}{\partial z_i \partial z_j} + r_n \sum_{j=1}^{2} b^n_j(z) \frac{\partial \phi_n}{\partial z_j} + \lambda_n r_n^2 \phi_n = c_n^{-2} \phi_n e^{\alpha c_n^2 (\phi_n^2 - 1)},$$

$$\sum_{i,j=1}^{2} a_{ij}^n(z)\frac{\partial^2 \eta_n}{\partial z_i \partial z_j} + r_n \sum_{j=1}^{2} b_j^n(z)\frac{\partial \eta_n}{\partial z_j} + \lambda_n r_n^2 c_n^2 \phi_n = \phi_n e^{\alpha(1+\phi_n)\eta_n}.$$

Using the elliptic regularity theory, we have

$$\phi_n \to \phi_0 \equiv 1 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\phi_0 = 0 \quad \text{in} \quad \mathbb{R}^2,$$

$$\eta_n \to \eta_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\eta_0 = e^{2\alpha\eta_0} \quad \text{in} \quad \mathbb{R}^2.$$

We compute $\int_{\mathbb{R}^2} e^{2\alpha\eta_0}dz$ in the same way as in (14). It follows that

$$\int_{\mathbb{R}^2} e^{2\alpha\eta_0}dz \leq 2. \tag{35}$$

Hence, we see that

$$\eta_0 = -\frac{1}{\alpha}\log\left(1+\frac{\alpha}{4}|z|^2\right),$$

and then by direct computation

$$\int_{\mathbb{R}^2} e^{2\alpha\eta_0}dz = \frac{4\pi}{\alpha}.$$

But this equality and (35) contradict the hypothesis $\alpha < 2\pi$. Consequently, it holds that

$$\lim_{n\to\infty}\frac{e^{\alpha c_n^2}}{\int_\Omega v_n^2 e^{\alpha v_n^2}dx} = 0.$$

Proof (Proof of Theorem 3 completed) Set $\xi_n = v_n/c_n$. Since v_n is a solution of (30) for λ_n, ξ_n satisfies

$$\begin{cases} -\Delta\xi_n + \lambda_n\xi_n = \frac{\xi_n e^{\alpha v_n^2}}{\int_\Omega v_n^2 e^{\alpha v_n^2}dx} & \text{in} \quad \Omega, \\ \frac{\partial\xi_n}{\partial\nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By Lemma 7 and the elliptic regularity theory, we have

$$\xi_n \to \xi_0 \quad \text{in} \quad C^2(\overline{\Omega}) \tag{36}$$

and ξ_0 satisfies

$$\begin{cases} -\Delta\xi_0 = 0 & \text{in} \quad \Omega, \\ \frac{\partial\xi_0}{\partial\nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Thus, ξ_0 is a constant. Since $\|\xi_0\|_{L^\infty(\Omega)} = 1$, we deduce that $\xi_0 \equiv 1$.

To end the proof of Theorem 3, we prove

$$c_n(\lambda_n|\Omega|)^{\frac{1}{2}} \to 1. \tag{37}$$

By Proposition 5, (36) and $\xi_0 \equiv 1$ we have

$$1+o(1) \leq \underline{c_n}c_n^{-1} \leq (\lambda_n|\Omega|)^{-\frac{1}{2}}c_n^{-1} \leq 1,$$

which implies (37). Consequently, employing (36), (37) and the fact that $\xi_0 \equiv 1$ again, we conclude that Theorem 3 holds.

4 Appendix

Define

$$I_\alpha := \sup_{\substack{u\in H^1(\mathbb{R}^2_+)\\ \int_{\mathbb{R}^2_+}(|\nabla u|^2+u^2)dx\le 1}} \int_{\mathbb{R}^2_+}\left(e^{\alpha u^2}-1\right)dx, \quad d_\beta := \sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2}(|\nabla u|^2+u^2)dx\le 1}} \int_{\mathbb{R}^2}\left(e^{\beta u^2}-1\right)dx.$$

In this section, we summarize the properties of I_α and d_β. By Ishiwata [4], it is proved that $d_\beta \ge \beta$ for all $\beta \in (0,4\pi)$. Moreover, it is proved that if β is close to 4π, then $d_\beta > \beta$ and d_β is attained, while if β is sufficiently small, then $d_\beta = \beta$ and d_β is not attained.

The following relationship between I_α and d_β holds.

Proposition 6 *For $\alpha \in (0,2\pi)$, we have $I_\alpha = d_{2\alpha}/2$. Moreover, attainability of I_α is equivalent to that of $d_{2\alpha}$.*

Proof Let $u_n \in H^1(\mathbb{R}^2_+)$ be a maximizing sequence of I_α and let $\tilde{u}_n \in H^1(\mathbb{R}^2_+)$ denote the extension of u_n by the reflection. It holds that

$$\int_{\mathbb{R}^2}\left(|\nabla \tilde{u}_n|^2+\tilde{u}_n^2\right)dx = 2\int_{\mathbb{R}^2_+}\left(|\nabla u_n|^2+u_n^2\right)dx \le 2.$$

Then, we have

$$\begin{aligned}
I_\alpha &= \lim_{n\to\infty}\int_{\mathbb{R}^2_+}\left(e^{\alpha u_n^2}-1\right)dx \\
&\le \sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2}(|\nabla u|^2+u^2)dx\le 2}} \frac{1}{2}\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)dx \\
&= \frac{1}{2}\sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2}(|\nabla u|^2+u^2)dx\le 1}} \int_{\mathbb{R}^2}\left(e^{2\alpha u^2}-1\right)dx \\
&= \frac{1}{2}d_{2\alpha}.
\end{aligned}$$

By virtue of the radially symmetric rearrangement, we can assume that maximizing sequence of $d_{2\alpha}$ is a radially symmetric, nonnegative function. Thus,

$$\begin{aligned}
d_{2\alpha} &= \sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2}(|\nabla u|^2+u^2)dx\le 1}} \int_{\mathbb{R}^2}\left(e^{2\alpha u^2}-1\right)dx \\
&\le \sup_{\substack{u\in H^1(\mathbb{R}^2)\\ \int_{\mathbb{R}^2_+}(|\nabla u|^2+u^2)dx\le \frac{1}{2}}} 2\int_{\mathbb{R}^2_+}\left(e^{2\alpha u^2}-1\right)dx \\
&\le 2\sup_{\substack{u\in H^1(\mathbb{R}^2_+)\\ \int_{\mathbb{R}^2_+}(|\nabla u|^2+u^2)dx\le 1}} \int_{\mathbb{R}^2_+}\left(e^{\alpha u^2}-1\right)dx \\
&= 2I_\alpha
\end{aligned}$$

Hence, we have $I_\alpha = d_{2\alpha}/2$.

If u_* is a maximizer of I_α, then the extension of u_* by the reflection is a maximizer of $d_{2\alpha}$. Conversely, if v_* is a maximizer of d_β, then $v_*|_{\mathbb{R}^2_+}$ is a maximizer of $I_{\beta/2}$. Thus, the existence of a maximizer for I_α is equivalent to that for $d_{2\alpha}$.

Proposition 7 *Assume that*

$$\alpha_* = \inf\{\alpha \in (0,2\pi) \mid I_\alpha > \alpha\}.$$

Then, we have $\alpha_* \in (0,2\pi)$, and

(i) for $\alpha \in (\alpha_*, 2\pi)$ it holds that $I_\alpha > \alpha$ and I_α is attained,
(ii) for $\alpha \in (0,\alpha_*)$, it holds that $I_\alpha = \alpha$ and I_α is not attained.

Proof Define

$$\beta_* := \inf\left\{\beta \in (0,4\pi) \;\middle|\; d_\beta > \beta\right\}. \tag{38}$$

By the results of Ishiwata [4], we see that $\beta_* \in (0,4\pi)$. In order to prove the proposition it suffices to show that (i)' if $\beta \in (\beta_*, 4\pi)$, then $d_\beta > \beta$ and d_β is attained and (ii)' if $\beta \in (0,\beta_*)$, then $d_\beta = \beta$ and d_β is not attained. Indeed, for such β_*, $\alpha_* = \beta_*/2$ and α_* satisfies (i) and (ii) of the proposition by Proposition 6.

First, we prove that if $d_{\tilde\beta} > \tilde\beta$ for some $\tilde\beta$, then $d_\beta > \beta$ and d_β is attained for any $\beta \in [\tilde\beta, 4\pi)$. Since $d_{\tilde\beta} > \tilde\beta$, we can show the existence of a maximizer $\tilde u$ for $d_{\tilde\beta}$ by applying Section 2.3 in [4]. Hence, since the function $e^s - 1$ is convex, we have

$$d_\beta \ge \int_{\mathbb{R}^2} \left(e^{\beta \tilde u^2} - 1\right) dx \ge \frac{\beta}{\tilde\beta} \int_{\mathbb{R}^2} \left(e^{\tilde\beta \tilde u^2} - 1\right) dx = \frac{\beta}{\tilde\beta} d_{\tilde\beta} > \beta.$$

Applying Section 2.3 in [4] again, we obtain the existence of a maximizer for d_β. Thus, $d_\beta > \beta$ and d_β is attained for any $\beta \in [\tilde\beta, 4\pi)$.

Next, we prove that if $d_{\hat\beta} = \hat\beta$ for some $\hat\beta$, then $d_\beta = \beta$ and d_β is not attained for all $\beta \in (0,\hat\beta)$. Assume the contrary that d_β is attained by u for some $\beta \in (0,\hat\beta)$. Then, we have

$$d_{\hat\beta} \ge \int_{\mathbb{R}^2} \left(e^{\hat\beta u^2} - 1\right) dx > \frac{\hat\beta}{\beta} \int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1\right) dx = \frac{\hat\beta}{\beta} d_\beta \ge \hat\beta,$$

which is a contradiction. Hence, $d_\beta = \beta$ and d_β is not attained for all $\beta \in (0,\hat\beta)$.

Finally, we set β_* as in (38). Then, by the definition of β_*, $d_{\beta_*} = \beta_*$ and $d_\beta > \beta$ for any $\beta \in (\beta_*, 4\pi)$, and hence β_* satisfies (i)' and (ii)'. Consequently, by Proposition 6, $\alpha_* = \beta_*/2 \in (0,2\pi)$ holds and α_* satisfies (i) and (ii).

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