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Preprojective Algebras and Calabi-Yau Algebras

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Abstract

This workshop held March 1–4, 2022 to conduct international research exchanges on Preprojective Algebras and Calabi-Yau Algebras.

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Preprojective Algebras, Calabi-Yau Algebras, cluster algebras,
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Preface

This volume of OCAMI Reports summarizes the workshop *Preprojective Algebras and Calabi-Yau Algebras* held from March 1 to March 4 in 2022 online by Zoom (because of the COVID-19 pandemic). This workshop was supported by Osaka city University, Advanced Mathematical Institute MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics.

Preprojective algebras are algebras determined by a quiver, which are not only important in their representation theory but are also fundamental objects used in the Lusztig and Nakajima quiver varieties, and they have a deep connection between representation theory of Lie algebras and quantum groups. The purpose of this workshop is to bring together researchers focusing on the representation theory of preprojective algebras and Calabi-Yau algebras and the connection between related combinatorics, geometry and integrable systems.

We have three lecturers : Professor Anne Dranowski gave a series of lectures about *Crystals and Preprojective Algebra Modules.* Professor Matthew Pressland gave a series of lectures about *Dimer models: consistency, Calabi-Yau properties and categorification.* Professor Bernard Leclerc gave a series of lectures about *Generalized preprojective algebras.* Moreover, we have two Speakers : Professor Akishi Ikeda gave a talk about *Calabi-Yau algebras and canonical bundles.* Professor Kota Murakami gave a talk about *Deformed Cartan matrices and generalized preprojective algebras.*

All speakers gave talks on recent developments, which promoted the fruitful discussions for especially young researchers. The organizers are convinced that discussions will lead to future research projects and a discovery of a new relation.

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Contents

Dimer models: consistency, Calabi–Yau properties and categorification

Matthew Pressland

Abstract. A dimer model is a bipartite graph drawn in a surface. First introduced in the context of statistical mechanics, dimer models became a significant topic in string theory around fifteen years ago. In mathematics, a key development at this time was the study of consistency conditions, and the use of dimer models on the torus to construct non-commutative crepant resolutions of 3-dimensional Gorenstein singularities. A key property of a consistent dimer model is that its associated non-commutative dimer algebra is 3-Calabi–Yau. More recently, dimer models have reappeared, now on surfaces with boundary and sometimes called plabic graphs or Postnikov diagrams, in the context of categorifying cluster algebras with coefficients, notably the cluster structure on the Grassmannian and its positroid strata. In this lecture series, I will survey these ideas.

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DIMER MODELS: CONSISTENCY, CALABI–YAU PROPERTIES AND CATEGORIFICATION

MATTHEW PRESSLAND

1. Dimer models

A *dimer model* D consists of a finite bipartite graph drawn in the interior of an oriented topological surface Σ, together with a finite collection of *half-edges* which connect nodes of this graph to distinct points on the boundary $\partial\Sigma$. We require that $\Sigma^\circ \setminus D$ is a union of open discs, each of which is called a *face* of D. Dimer models have found many applications in mathematics and in physics, relating to statistical mechanics [22, 34], string theory [12, 18], non-commutative geometry and singularity theory [3, 6, 9], and most recently to the theory of cluster algebras and categories [1, 8, 14, 24, 26, 27, 32]. We are particularly interested in the case that either Σ is the torus (which is a closed surface, so that D is an honest bipartite graph, without half-edges) or that Σ is the disc.

To a dimer model D we may associate the dual quiver $Q = Q_D$, with vertex set Q_0 given by the faces of D, and arrow set Q_1 given by the edges (and half-edges) of D. An arrow in Q_1 connects the two faces incident along the corresponding edge, and is oriented so that the black node of this edge is on the left of the arrow (or, if the edge is a half-edge with only a white node, this white node is on the right of the arrow). Due to its embedding in Σ, the quiver Q also has a natural set Q_2 of *faces*, distinguished cycles in Q bounding components of $\Sigma \setminus Q$, which are in bijection with the nodes of D. The bipartite structure of D is reflected in the orientations of these distinguished cycles, which can be either positive or negative, with two faces sharing an arrow having opposite orientations. Positive quiver faces correspond to black nodes of D, while negative ones correspond to white nodes.

Definition 1. The *dimer algebra* A_D of a dimer model D is the complete path algebra of Q_D modulo the closed ideal generated by relations $p - q$, where $p, q \colon i \to i$ both bound a face, and $p_a^+ - p_a^-$ for $a \in Q_1 \setminus F_1$, where ap_a^+ and ap_a^- are, respectively, the boundaries of the positively and negatively oriented faces incident with a.

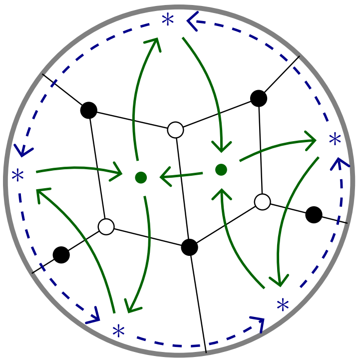

Figure 1. A dimer model in the disc, together with its dual quiver. Frozen vertices of this quiver are indicated by asterisks, and frozen arrows are dashed.

In the case that D is connected (which is automatic if Σ is closed), the relations of the form $p - q$ are redundant, and so A_D is an instance of a *frozen Jacobian algebra*. Here we treat Q as an *ice quiver*, in the sense of [1, 29, 30] (so there are frozen arrows as well as frozen vertices, and not

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every arrow between frozen vertices has to be frozen), by taking the frozen vertices to be faces of D incident with $\partial\Sigma$ and the frozen arrows to be the half-edges of D. To define the frozen Jacobian algebra we also put a potential on Q, which is the sum of positively oriented faces minus the sum of negatively oriented faces; thus each relation $p_a^+ - p_a^-$ is the derivative of this potential with respect to the unfrozen arrow a. Note that the frozen arrows, which correspond to half-edges, lie in the boundary of only one face, and so do not contribute relations.

2. Consistency and the Calabi–Yau condition

A dimer model D has a collection of *zig-zag paths*, walks along the edges which turn maximally to the right at black nodes and maximally to the left at white nodes. Thus each edge of the dimer model may be treated as the transverse intersection of two zig-zag paths, which traverse this edge in opposite directions. Each zig-zag path is either a closed loop in the interior of Σ, or a path with both endpoints on $\partial\Sigma$. In the following definition, the zig-zag paths starting and ending on a half-edge of D are taken to cross transversely at this half edge if they are distinct (cf. Figure 2), but not if they are the same: in this second case we consider the zig-zag path to be a closed loop at the boundary point on the half-edge.

Definition 2. Let D be a dimer model, and let $\widetilde{D}$ be its lift to the universal cover of Σ; note that the zig-zag paths of $\widetilde{D}$ lift those of D. We say that D is *consistent* if

(a) its zig-zag paths have no transverse self-intersections, and
(b) if two zig-zag paths on $\widetilde{D}$ intersect twice, they are oriented in opposite directions between these two intersection points.

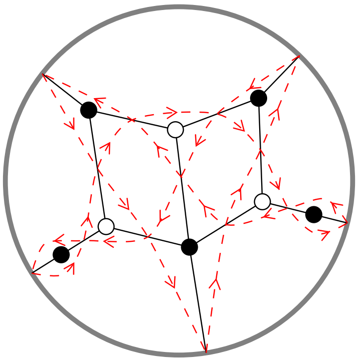

Figure 2. The zig-zag paths of the dimer model in Figure 1, lifted off of the edges for readability. We see from these paths that this dimer model is consistent; while some pairs of zig-zag paths on $\widetilde{D} = D$ intersect twice, they are oriented in opposite directions between the crossings.

Consistent dimer models will turn out to have Calabi–Yau dimer algebras, in the sense of the following definition.

Definition 3 ([29])**.** Let A be a Noetherian algebra with enveloping algebra $A^\varepsilon = A \otimes A^{\mathrm{op}}$, and let $e \in A$ be an idempotent element, i.e. $e^2 = e$. Write $\mathcal{D}^{\mathrm{b}}_e(A) = \{X \in \mathcal{D}^{\mathrm{b}}(A) : eX = 0\}$ and $\Omega_A = \mathbf{R}\mathrm{Hom}_{A^\varepsilon}(A, A^\varepsilon)$. Say that (A, e) is (bimodule) internally d-Calabi–Yau if

(a) $\operatorname{gl.dim} A \leqslant d$ and ${}_{A^\varepsilon}A$ has a finite resolution by finitely-generated projective A^ε-modules;
(b) there is a triangle

$$\Sigma^{-d}A \longrightarrow \Omega_A \longrightarrow X \longrightarrow \Sigma^{1-d}A$$

in $\mathcal{D}(A^\varepsilon)$ such that $\mathbf{R}\mathrm{Hom}_A(X, M) = 0 = \mathbf{R}\mathrm{Hom}_{A^{\mathrm{op}}}(X, N)$ for any $M \in \mathcal{D}^{\mathrm{b}}_e(A)$ and $N \in \mathcal{D}^{\mathrm{b}}_e(A^{\mathrm{op}})$.

We say A is (bimodule) d-Calabi–Yau if we may choose $X = 0$ in (b), so that $A \cong \Sigma^d\Omega_A$ in $\mathcal{D}(A^\varepsilon)$. In this case the condition on the global dimension in (a) is redundant, and $(A, 0)$ is internally d-Calabi–Yau.

When (A,e) is internally d-Calabi–Yau, we may show using a result of Keller [23, Lem. 4.1] that there is a bifunctorial isomorphism

$$\operatorname{Ext}^i_A(M,N)=\operatorname{Ext}^{d-i}_A(N,M)^*$$

for any $M\in\operatorname{Mod}A$ and any finite-dimensional $N\in\operatorname{mod}A/AeA$, where $(-)^*$ denotes duality over the ground field. Thus we recover the traditional Calabi–Yau duality formula, at least under certain conditions on the A-modules involved.

Theorem 4. *Let D be a consistent dimer model on a surface Σ, with dimer algebra A. Let $e=\sum_{i\in F_0}e_i$ be the sum of the vertex idempotents at frozen vertices (so $e=0$ if Σ is closed). Then (A,e) is internally 3-Calabi–Yau if*

(a) Σ is a closed surface with positive genus [9] (see also [6] for the case that Σ is a torus), or
(b) if Σ is a disc and D is connected [28].

We remark that if Σ is closed with genus at least 2, then the dimer algebra of a consistent dimer model on Σ is not Noetherian [4], and so the definition of internally Calabi–Yau algebra does not strictly apply: we use instead the more general definition (with more technical assumptions) as originally stated in [29].

3. Categorification

In general, internally Calabi–Yau algebras may be used to define stably Calabi–Yau Frobenius categories with cluster-tilting objects. Here a *Frobenius category* is an exact category $\mathcal{E}$ with enough projective and injective objects, which coincide; a prototypical example is the module category of a finite-dimensional selfinjective algebra. The *stable category* $\underline{\mathcal{E}}$ of $\mathcal{E}$ is the quotient by morphisms factoring over a projective object, and is canonically triangulated [19]. We say $\mathcal{E}$ is *stably d-Calabi–Yau* if $\underline{\mathcal{E}}$ is *d-Calabi–Yau*, i.e. if Σ^d is a Serre functor on this triangulated category. An object $T\in\mathcal{E}$ is *d-cluster-tilting* if

$$\begin{aligned}\operatorname{add}T&=\{X\in\mathcal{E}:\operatorname{Ext}^i_{\mathcal{E}}(T,X)=0\text{ for }i=1,\dots,d-1\}\\&=\{X\in\mathcal{E}:\operatorname{Ext}^i_{\mathcal{E}}(X,T)=0\text{ for }i=1,\dots,d-1\}.\end{aligned}$$

The second equality is automatic when $\mathcal{E}$ is stably d-Calabi–Yau. It is immediate from this definition that $\operatorname{add}T$ must contain all projective-injective objects in $\mathcal{E}$, and that a 1-cluster-tilting object is the same as an additive generator, i.e. an object T such that $\operatorname{add}T=\mathcal{E}$. Frobenius categories with these properties (for $d=3$) play an important role in the categorification of cluster algebras with frozen variables [10, 13, 15, 21, 28]. Further examples, with $d=2$, are given by categories of Cohen–Macaulay modules over Kleinian singularities, as appearing in the classical McKay correspondence; see [16, 25].

Theorem 5 ([29]). *Let A be a Noetherian algebra, and let $e\in A$ be an idempotent such that A/AeA is finite-dimensional and (A,e) is internally d-Calabi–Yau. Write $B=eAe$. Then*

(a) B is Iwanaga–Gorenstein (meaning that B is Noetherian and has finite injective dimension as both a left and a right B-module), so the category

$$\operatorname{GP}(B)=\{X\in\operatorname{mod}B:\operatorname{Ext}^i_B(X,B)=0\textit{ for all }i>0\}$$

of Gorenstein projective B-modules is Frobenius,
(b) the stable category $\underline{\operatorname{GP}}(B)$ is $(d-1)$-Calabi–Yau,
(c) $eA\in\operatorname{GP}(B)$ is $(d-1)$-cluster-tilting in this category, and
(d) the natural maps $A\to\operatorname{End}_B(eA)^{\mathrm{op}}$ and $A/AeA\to\underline{\operatorname{End}}_B(eA)^{\mathrm{op}}$ (the latter denoting the endomorphism algebra of eA in $\underline{\operatorname{GP}}(B)$) are isomorphisms.

This theorem may be applied in the case that A is a dimer algebra from a consistent connected dimer model on the torus or on the disc; the necessary Calabi–Yau property is the result of Theorem 4. In the case of the disc we take e to be the boundary idempotent, whereas in the case of the torus e must be non-zero so that A/AeA is finite-dimensional. While dimer algebras on higher genus closed surfaces are Calabi–Yau, they are not Noetherian. The categories and algebras appearing in Theorem 5 have useful applications, to (non-commutative) algebraic geometry in the case of dimers on the torus, and to cluster algebras [11] in the case of dimers on the disc.

3.1. **The torus.** For D a consistent dimer model on the torus, choose any vertex $0 \in Q_0$, and set $e = e_0$. Then $B = eAe$ coincides with the centre of A (in particular, it is commutative), and is a Gorenstein toric 3-fold singularity [6]. Moreover [17, 33], every such singularity appears in this way. In this case $\mathrm{GP}(B) = \mathrm{CM}(B)$ is the category of Cohen–Macaulay modules over this Gorenstein ring, and the stable category $\underline{\mathrm{GP}}(B)$ is the singularity category of B [7]. Furthermore, $A \cong \mathrm{End}_B(eA)^{\mathrm{op}}$ is a non-commutative crepant resolution of $\operatorname{Spec} B$, and hence [2, 5] $\mathcal{D}^{\mathrm{b}}(A) \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{coh} X)$ where $X \to \operatorname{Spec}(B)$ is a crepant resolution and $\operatorname{coh} X$ denotes the category of coherent sheaves on X. Modifying the dimer model by a local move known as Seiberg duality corresponds to modifying this crepant resolution via a flop, and modifying the cluster-tilting object eA via mutation, cf. [35].

This is a 3-dimensional analogue of the classical McKay correspondence in dimension 2. In that setting, one considers a Kleinian singularity B, i.e. the ring of functions on the quotient of $\mathbb{C}^2$ by a finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$ (localised at the origin). Such subgroups are classified by Dynkin diagrams, and if we take A to be the preprojective algebra of the corresponding extended (or affine) diagram, then A is 2-Calabi–Yau. In particular, (A, e) is internally Calabi–Yau when e is the vertex idempotent at the extending vertex. In this case $eAe = B$ (again isomorphic to the centre of A), and A/AeA is the preprojective algebra of the unextended Dynkin diagram, which is finite-dimensional. Again $\mathrm{GP}(B) = \mathrm{CM}(B)$ is the category of Cohen–Macaulay modules for the commutative ring B and $\mathcal{D}^{\mathrm{b}}(A) \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{coh} X)$ for X the (unique) crepant resolution of the Kleinian singularity. It follows from Theorem 5 that $\mathrm{GP}(B) = \mathrm{CM}(B)$ has a 1-cluster-tilting object; this is equivalent to the fact that $\mathrm{CM}(B)$ has finitely many indecomposable objects up to isomorphism, as originally proved by Herzog [20].

3.2. **The disc.** When D is a connected consistent dimer model on the disc, we take e to be the boundary idempotent of A. Since D typically has multiple boundary faces, this means that B is not commutative. However, it is shown in [28] that in this case the category $\mathrm{GP}(B)$ categorifies the cluster algebra structure on an open positroid variety in the Grassmannian, as defined by Galashin–Lam [14], and by Serhiyenko–Sherman-Bennett–Williams [32] in some special cases. This generalises work of Jensen–King–Su [21], who provide a categorification for the unique dense positroid variety, whose cluster structure is due to Scott [31], and of Baur–King–Marsh [1], relating this categorification to dimer models.

More precisely, a dimer model D in the disc with n half-edges determines a permutation π of this (cyclically ordered) set of half-edges, by the condition that the zig-zag path starting at half-edge e ends at half-edge $\pi(e)$. If D is connected, then this permutation has no fixed points; otherwise we should 'decorate' it by marking each fixed point as either positive or negative depending on the orientation of the zig-zag path starting and ending at this point. As explained by Postnikov [27], this permutation (with its decoration if it has fixed points) determines an open subvariety of the Grassmannian Gr_k^n of k-dimensional subspaces of $\mathbb{C}^n$, called an open positroid variety. Here n is simply the number of half-edges of D, while k is a little harder to describe, but is still determined combinatorially by D or π. It is shown in [14, 31, 32] how to use D to construct the initial seed of a cluster algebra structure on the coordinate ring of this variety, the quiver of this seed being Q_D. It is this structure which we categorify via $\mathrm{GP}(B)$ when D is connected. Modifying D via Seiberg duality corresponds to mutating this initial seed.

In the case that $\pi(e)$ is always exactly k clockwise steps from e around the boundary of the disc (as in Figure 2, where $k = 3$), the corresponding positroid variety is dense in Gr_k^n, and so the results of Scott [31], Jensen–King–Su [21] and Baur–King–Marsh [1] apply. One consequence of Baur–King–Marsh's work is that in this case the categorification $\mathrm{GP}(B)$ described via the dimer model coincides with Jensen–King–Su's (who defined B directly via a quiver with relations in this special case). The isomorphism $A \xrightarrow{\sim} \mathrm{End}_B(eA)^{\mathrm{op}}$ appearing in Theorem 5 also recovers a result of Baur–King–Marsh for the dense positroid variety, as well as extending it to positroid varieties for more general permutations.

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Crystals and Preprojective Algebra Modules

Anne Dranowski

Abstract. Underlying every irreducible representation of a semisimple simply-laced Lie algebra is a colored directed graph called its crystal. Crystals can be realized combinatorially, geometrically, and categorically. For example, components of Springer fibres yield a geometric realization, which is compatible with the combinatorial realization given by semistandard Young tableaux. In joint work with B. Elek, J. Kamnitzer and C. Morton-Ferguson, we generalize the latter well-known type A specific combinatorial realization. In place of tableaux we work with reverse plane partitions, and to establish the crystal structure, we relate these to a space of modules for the preprojective algebra. In these lectures we will begin by reviewing the classical results on tableaux and Springer fibres, as well as the notion of a g-crystal. Next we will cover the combinatorics necessary for working with reverse plane partition (Stembridge). We will then discuss the crystal structures on components of quiver Grassmannians and cores of quiver varieties (Nakajima, Savage and Tingley). Finally, we will discuss the crystal structure on reverse plane partitions, as well as future directions, including possible connections to recent work of Garver, Patrias, and Thomas, as well as Baumann, Knutson, and Kamnitzer.

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CRYSTALS AND PREPROJECTIVE ALGEBRA MODULES

Based on arXiv: 2202.02490
with Elek, Kamnitzer, and Morton-Ferguson

Plan

I) see patterns, mostly blackboxing the objects involved; state generalization; literature review

II) give precise definitions uncovering the key ingredients in our generalization.

$$\Pi(Q), \Lambda, Gr(M), \ldots$$

III) RPP combinatorics coming from

(a) tensor product quiver varieties via quiver grassmannians

(b) nilpotent filtrations of Π-modules.

IV) Open questions related to canonical bases and cluster algebras.

Patterns (Demonstrating cool connections!)

Consider the set $SSYT_4(1^2)$ of semistd Young tableaux in $\{1,2,3,4\}$ of shape $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$.

This set is in interesting bijection bijection with several other (po)sets.

① Irr $F_4(1^2)$ where $F_4(1^2)$ is the 4-step Springer fibre preserved by the Jordan type $(1^2)^t = (2)$ normal form: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\{(0 \overset{1}{\subseteq} V_1 \overset{0}{\subseteq} V_2 \overset{1}{\subseteq} V_3 \overset{0}{\subseteq} \mathbb{C}^2) : AV_i \subseteq V_{i-1}\}$$

Exer This set is size $\frac{4!}{4} := \frac{|S_4|}{|S_2 \times S_2|}$

$\boxed{\text{Eg}}$ $F_4(1^2)_{\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}} = \{V_\bullet \in F_4(12) : \dim V_i/V_{i-1} = \mu_i\}$

$\mu = wt\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\right)$

$wt\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\right) = (1010)$

$$AV_1 \subseteq 0 \Rightarrow \ker A \supseteq V_1 \Rightarrow V_1 = \langle e_1 \rangle$$

... This irreducible component is the point

$$\{0 \subseteq \langle e_1 \rangle \subseteq \langle e_1 \rangle \subseteq \mathbb{C}^2 \subseteq \mathbb{C}^2\}$$

② $GT_4(1^2)$ the set of Gelfand-Tsetlin patterns of shape $(1,1,0,0)$

i.e. arrays (g^{i}_{i})

$$\begin{array}{ccccccc}
 & & & g^1_1 & & & \\
 & & \geq & & \geq & & \\
 & & g^2_1 & & g^2_2 & & \\
 & \geq & \geq & & \geq & \geq & \\
 & g^3_1 & & g^3_2 & & g^3_3 & \\
\geq & & \geq & \geq & \geq & & \geq \\
g^4_1 = 1 & & g^4_2=1 & & g^4_3=0 & & g^4_4=0
\end{array} \quad \leftarrow \qquad \text{s.t.} \quad \begin{array}{ccc}
 & g^i_k & \\
\geq & & \geq \\
g^{i+1}_k & & g^{i+1}_{k+1}
\end{array}$$

Eg $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \rightsquigarrow g = \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 0 & & \\
 & 1 & & 1 & & 0 & \\
1 & & 1 & & 0 & & 0
\end{array}$

In general, a tableax τ defines

$$(g^{i}_{i}) = \text{shape of } \tau|_{\{1\dots i\}}$$

Spoiler: GT patterns will give reverse plane partitions!

③ $\boxed{J}(\boxed{H}(2132))$ $\boxed{\text{order ideals}}$ in the $\boxed{\text{heap}}$ of $s_2 s_1 s_3 s_2$

$\underline{w} = (2,1,3,2) = (i_1, i_2, i_3, i_4)$

Then $H(\underline{w})$ is the poset obtained by taking transitive closure the relation

$$a \prec b \quad \text{if} \quad a > b \text{ and } s_{i_a} s_{i_b} \neq s_{i_b} s_{i_a}$$

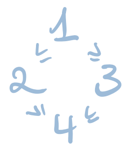
on $\{1, 2, 3, 4\}$ (more generally, $\{1, 2, .. l\}$ where $l = \mathrm{len}(w)$)

In this example $\underline{w} = (\underline{2}\ \underline{1}\ 3\ 2)$
$\underline{\underline{1}}\ \underline{\underline{2}}\ \underline{3}\ \underline{4}$

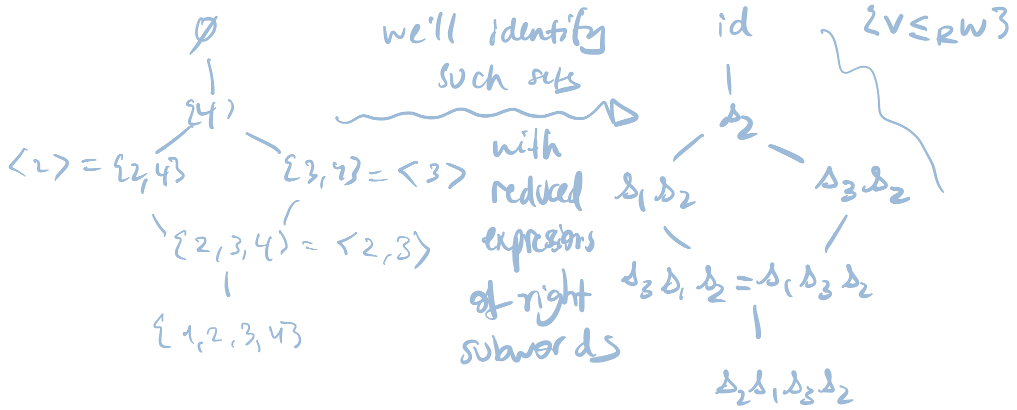
Can compare

1, 2; 1, 3; 3, 4; 2, 4

So

$H(\underline{w}) =$


This heap has order ideal poset:



we'll identify such sets with reduced expressions of right subwords

For example $\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$, of weight (1010), is associated with $s_2 \in J(H(\underline{w}))$

because $s_2(1100) = (1010)$

and s_2 is minimal for this requirement.

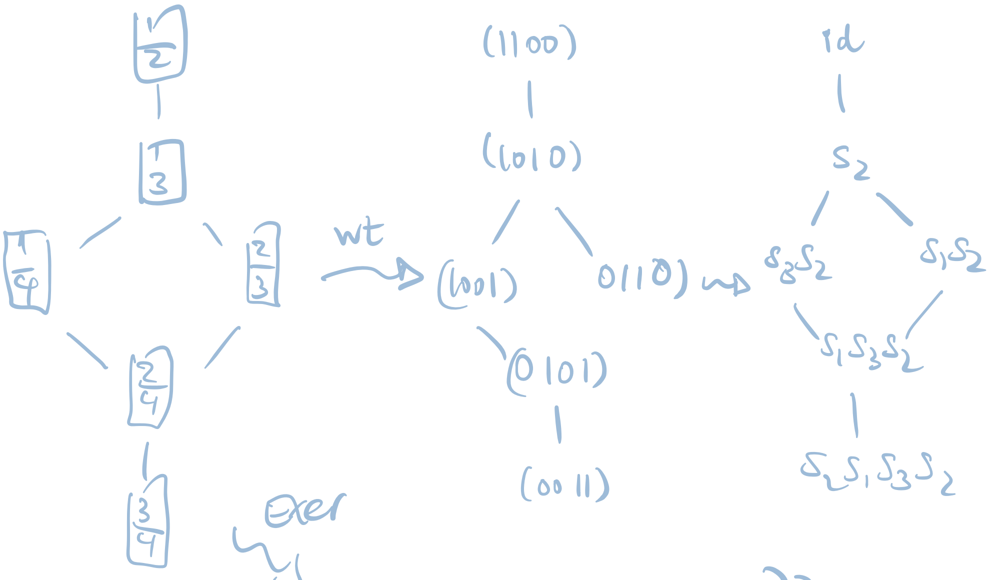
Remark $\underline{w} = (2\,1\,3\,2)$ is fully commutative, which means that every reduced word of $s_2 s_1 s_3 s_2$ can be obtained by swapping adjacent commuting pairs of generators.

This attribute allows us to write unambiguously $H(w)$ for $w \in S_4$ because w is fully commutative.

So any choice of reduced word will result in the same heap.

④ The orbit set $S_4 \cdot (1100)$

Summary



Exer ↝ poset of GT patterns??

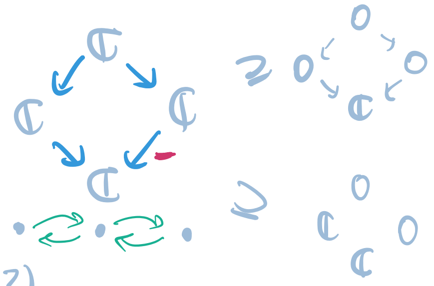
⑤ Irr $\underbrace{Gr(I(1^2))}$

quiver grassmannian of modules for $\Pi(A_3)$ which are submodules of $I(1^2)$ the injective hull of $S(2)$ the simple over vertex 2.

$I(2) = I(1^2) =$

$\uparrow$

ω_2



Exer...

$I(3) = I(\omega_3) \leftarrow S(3)$

These posets are actually in crystal bijection.

Eg $X < Y$ in one of these posets

$\Leftrightarrow f_i Y = X$ where f_i denotes a lowering operator in the crystal structure intrinsic to the poset

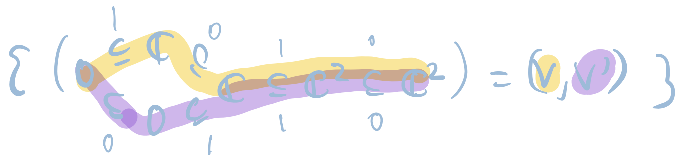
Compatibility $\tau = \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{f_1} \tau' = \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$

[Hong, Kang]

① $f_1 F_4(1^2)_\tau = F_4(1^2)_{\tau'}$

indeed in this context crystal operators f_i are defined using correspondences.

$$\begin{tikzcd}
 & \{(V, V', A) : V_k = V'_k \quad k \neq i \quad V'_i \subseteq V_i \quad \dim V_i = \dim V'_i + 1\} \arrow[dl] \arrow[dr] & \\
\{\cdot\} \overset{Ex}{\cong} F(\lambda)_\mu & & F(\lambda)_{\mu - \alpha_i} = \{\cdot\}
\end{tikzcd}$$

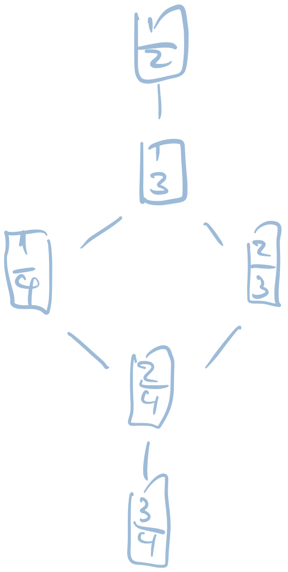


Indeed $SSYT_4\left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right)_{0110} = \left\{ \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \right\}$.

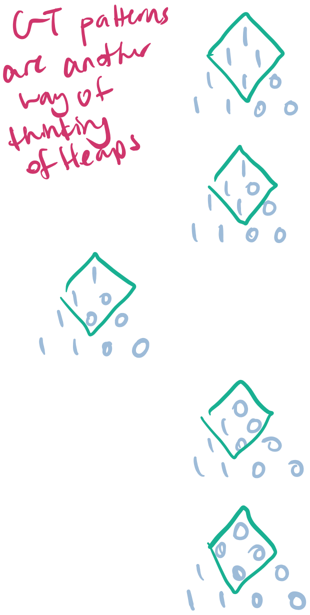
→ Similarly, in the heap model s_i will act on a right subword by premultiply by s_i

→ And, in $S_4 \cdot (1100)$, s_i will act by permuting μ by s_i

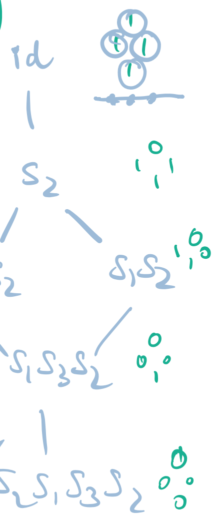
Generalization



GT patterns are another way of thinking of Heaps



$GT(\omega_n)=GT\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right)$ can always be truncated to rectangular subarray.



"Schutzenberger involution" rotate the

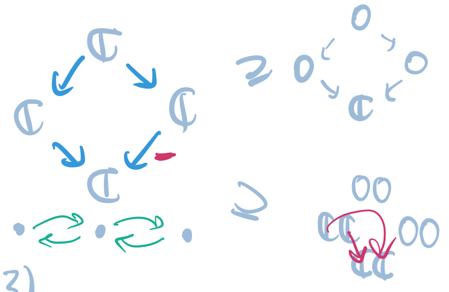
90° ↺ and flip in a vertical axis

⑤ $\mathrm{Irr}\ \underbrace{Gr(I(1^2))}$

quiver grassmannian of modules for $\Pi(A_3)$ which are submodules of $I(1^2)$ the injective hull of $S(2)$ the simple over vertex 2.

$I(2)=I(1^2)=$

$\uparrow$
ω_2



Exer...

$I(3)=I(\omega_3)\leftarrow S(3)$

False: $\forall\ M \subseteq I(i)^{\oplus n}$, $M = \bigoplus N$, $N \subseteq I(i)$.

We are witnessing an iso of crystals

$$\begin{tikzcd}
\mathrm{Irr}\, F_m(\lambda) \arrow[r] & SSYT_m(\lambda) & J(H(w)) \arrow[l] \\
\mathrm{Irr}\, Gr_{\Pi(A_{m-1})}(I(\lambda)) \arrow[ur] & GT_m(\lambda) \arrow[u] & W\cdot\lambda \arrow[ul]
\end{tikzcd}$$

where w is minimal for $w\lambda = w_0\lambda$

when λ is minuscule.

Our Goal Generalize

$$\mathrm{Irr}\, F_m(\lambda) \to SSYT_m(\lambda)$$

in a type-independent way, by using the geometry of quiver grassmannians

We replace $F_m(\lambda)$ by $Gr_{\Pi(A_{m-1})}(I(\lambda))$

for $\lambda = \sum \lambda_i \omega_i$

$$I(\lambda) := \bigoplus \underbrace{I(i)}_{\text{injective hull of simple } S(i).}{}^{\oplus \lambda_i}$$

We replace $SSYT_m(\lambda)$ by (disjoint unions of) reverse plane partitions of shape $H(w_0^i)$ where $\forall i$, w_0^i is minimal for $w_0^i\omega_i = w_0\omega_i$

Theorem A There is a crystal isomorphism (up to Schützenberger involution) of $SSYT(n\omega_i)$ and $RPP(w_0^i, n)$.

Theorem B There is a compatible isomorphism

$$Gr(I(i)^{\oplus n}) \longrightarrow F(n\omega_i)$$

of varieties!

Type ADE

Let λ be dominant minuscule weight. (W acts transitively on weights of $V(\lambda)$.) Let w be minimal $w\lambda = w_0\lambda$. Call w λ-minuscule. (Stembridge.)

To a λ-min. elt. w is associated a poset

$H(w)$ "heap of w".

If $\underline{w} = (i_1 \dots i_\ell)$ is a reduced word then $H(\underline{w}) = (\underset{[\ell]}{\underset{\shortparallel}{\{1, \dots, \ell\}}}, \bar{\prec})$ where

$\bar{\prec}$ is the trans. clos. of

$\forall a, b \in [\ell], a \prec b$ iff $a > b$ and $s_{i_a} s_{i_b} \neq s_{i_b} s_{i_a}$

Fact/Exer in type A heaps are Young diagrams.

An RPP of shape $H(w)$ and height n is an order reversing map

$$\Phi : H(w) \longrightarrow \{1, \dots, n\} \sqcup \{0\} = [n] \cup \{0\}.$$

$$a \preceq b \rightsquigarrow \Phi(a) \geq \Phi(b) \quad \text{(in the usual increasing order on } [n] \sqcup \{0\})$$

We denote the set of all such RPPs by $RPP(H(w), n)$ or $R(w,n)$.

Remark / Exer when $n=1$.

$$R(w, 1) = J(H(w))$$

Theorem There is a crystal bijection

$$\begin{tikzcd} R(w,n) \arrow[r] \arrow[d] & B(n\lambda) \arrow[d] \\ J(H(w))^{\times n} \arrow[r] & B(\lambda)^{\otimes n} \end{tikzcd}$$

and this diagram commute:

$$\begin{aligned} R(w,n) &\to J(H(w))^{\times n} \\ \Phi &\mapsto (\phi_1, \dots, \phi_n) \end{aligned}$$

where $\phi_k = \Phi^{-1}(\{n-k+1, \dots, n\})$.

Eg $\underline{w} = (1\,3\,2)$

$H(\underline{w}) =$

$$R(\underline{w}, 2) = \left\{ \begin{array}{llll} {}^{0}{}_{0}{}^{0}, & {}^{0}{}_{1}{}^{0}, & {}^{0}{}_{2}{}^{0} & {}^{0}{}_{2}{}^{2} \\ & {}^{1}{}_{1}{}^{0}, & {}^{1}{}_{2}{}^{0} & {}^{2}{}_{2}{}^{0} \\ & {}^{0}{}_{1}{}^{1}, & {}^{0}{}_{2}{}^{1} & {}^{2}{}_{2}{}^{2} \\ & {}^{1}{}_{1}{}^{1} & {}^{1}{}_{2}{}^{1} & \end{array} \right\}$$

Eg $\Phi = {}^{1}{}_{2}{}^{0} \longrightarrow ({}^{1}{}_{1}{}^{0}, {}^{0}{}_{1}{}^{0})$

The result will be ordered wrt the order on $J(H(w))$.

Crystals Let $\mathfrak{g}$ be ADE. simple. P - weight.
$\{\alpha_i\}$ simple roots

Def The data $(B, wt, \varepsilon_i, \varphi_i, e_i, f_i)$ defines a $\mathfrak{g}$-crystal which is upper semi-normal. or highest weight if (1) it's a crystal, i.e.

- $wt : B \rightarrow P$
- $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{N}$
- $e_i, f_i : B \rightarrow B \sqcup \{0\}$

Satisfying

- $\varphi_i(b) = \varepsilon_i(b) + wt(b)(\alpha_i^\vee)$
- $wt(e_i b) = wt(b) + \alpha_i$
  $wt(f_i b) = wt(b) - \alpha_i$ if $e_i b, f_i b \in B$
- $f_i(b_2) = b_1$ iff $e_i(b_1) = b_2$

(2) it is seminormal for the raising operator e_i

$$\varepsilon_i(b) = \max\{n \geq 0 : e_i^n(b) \neq 0\}$$

In other words, there is a distinguished highest weight elt. b_+ which we can reach to from every $b \in B$ by applying $e_{\underline{i}} = e_{i_1} \cdots e_{i_N}$

for some $\underline{i} \in I^N$ for some n. : $b_t = e_{\underline{i}} b$

$\uparrow$ vertex set of Γ - dynkin graph.

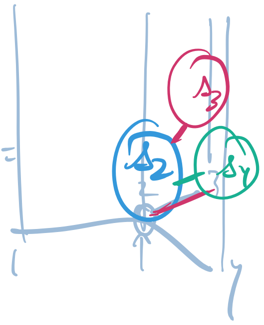
Example $W \cdot \omega_2$

There is a classification of

minuscule fundamentals in all types

$\omega_i \leftarrow$ vertex i of the Dynkin diagram can support a glass bead game.

$H(\underline{w}) =$

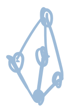


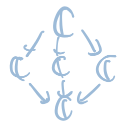
$\not\rightarrow$ Π-module.

$B_{\underline{w}}(\lambda)$

- non minuscule
- non-simply laced.

$\uparrow$ too many edges for nice module structure.





$\varepsilon : Q_1 \to \{\pm 1\}$.

$\underline{w} = 2 1 4 3 2$ when $\Gamma = D_4$ is not minuscule.

$\rightsquigarrow$

Next time we'll start with recalling the $\otimes$ rule on crystals.

$$R(\omega_n) \longrightarrow J(H(\omega))^{\times n}.$$

OSAKA 2022 Lecture III

HOUSEKEEPING

$<_L$ instead of $<_R$

the glass bead game

other influences: TODO

$\le_L$ instead of $\le_R$: notation for "is right subword"

s/b $\le_L$ as its std to define left weak order by $u \le_L w$ if $w = s_1 \cdots s_k u$ ...

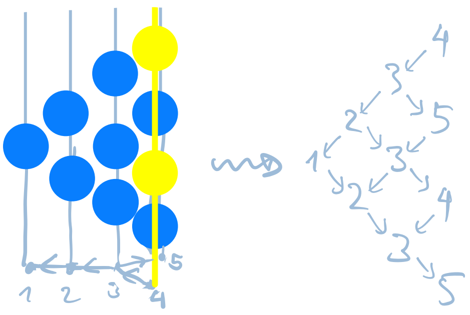
or Mikado

$H(w)$ as a bead game: we can visualizes heaps of words (après Stembridge, Viennot) as configurations of beads on a $\Gamma \times \mathbb{R}_{\geq 0}$ abacus where Γ is the relevant Dynkin diagram

Eg

$\Gamma = D_5$

$\underline{w} = (5\,3\,2\,4\,1\,3\,2\,5\,3\,4)$



Tensor product crystals

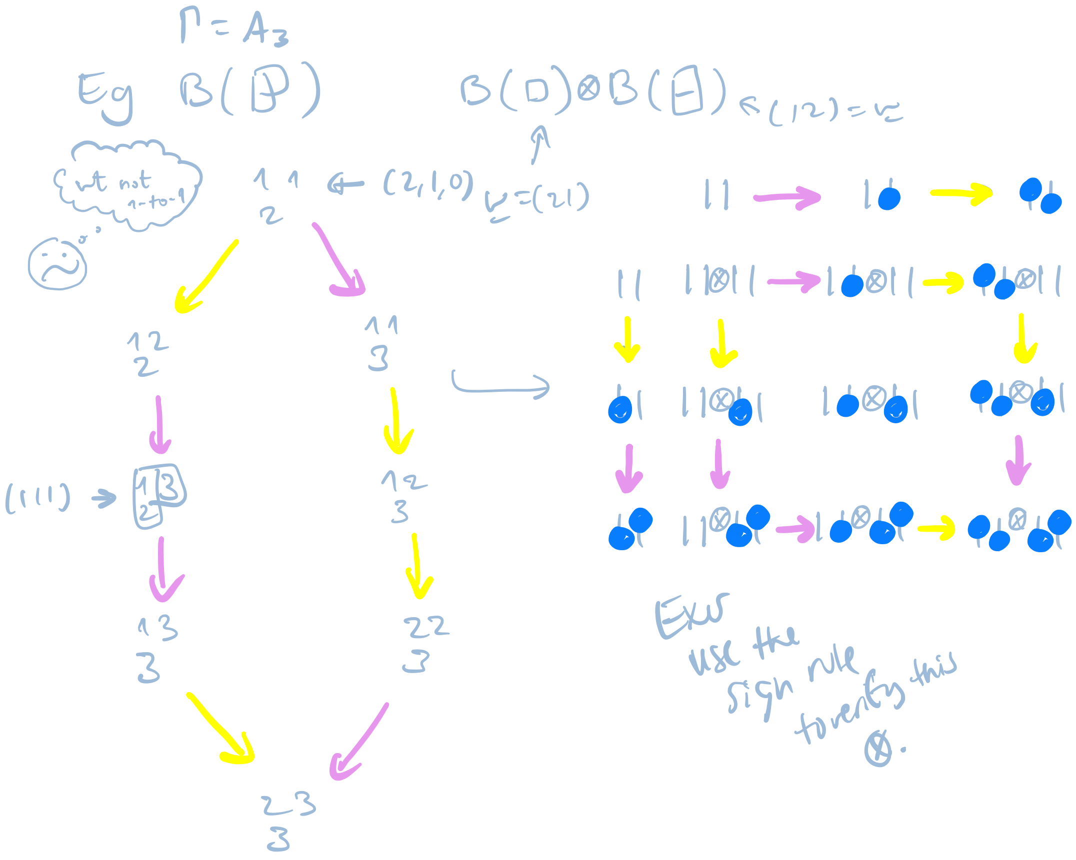
Given two $\mathfrak{g}$-crystals A, B, can endow $A \otimes B$ with crystal str with raising and lowering operators defined as:

$$e_i(a \otimes b) = \begin{cases} a \otimes e_i(b) & \text{if } \varepsilon_i(a) \le \varphi_i(b) \\ e_i(a) \otimes b & \text{else.} \end{cases}$$

$$f_i(a \otimes b) = \Big\{ \quad \text{Similar rule.}$$

In practice the sign pattern (signature rule) rewording of this rule to determine e_i or f_i of a $\otimes$ crictor.

Lem (Tingley) Given $b \in B$, denote by $S_i(b)$ the string made up of $\varphi_i(b)$ many $+$ signs followed by $\varepsilon_i(b)$ many $-$ signs. Call $S_i(b)$ the sign pattern of b. On tensor crictors define $S_i(a\otimes b) = S_i(a) S_i(b)$. The action of f_i is found by first cancelling in $S_i(a\otimes b)$ all $-+$ pairs. The result is a sequence of the form $+ - + - \cdots -$. The signature rule says that f_i will act on the elt contributing the rightmost $+$ if it exists, and by zero otherwise.



Exer use the sign rule to verify this ⊗.

Question can we avoid ⊗ combinatorics when $\lambda = \sum \lambda_i \omega_i$

Example $\Gamma = A_3$ $\quad \underline{w} = (2\,3\,1\,2)$ $\quad \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4$

$\underline{\Phi} =$

Use the signature rule to compute $f_2(\underline{\Phi})$, $e_2(\underline{\Phi})$

$$S_2(\phi_1 \otimes \cdots \otimes \phi_4) = - \;\square\; + - \rightsquigarrow -$$

so $f_2 \Phi = 0.$

Preprojective algebras

Let $Q=(Q_0, Q_1, h,t : Q_1 \to Q_0)$ be quiver with underlying graph $\Gamma = (Q_0, Q_1)$ or (I, E) among $(A_n, D_n, E_\bullet)$

Double Q by adding $j \xrightarrow{a^-} i$ for each $i \xrightarrow{a^+} j$ in Q_1 to get the associated doubled quiver $\bar{Q} = (Q_0, \bar{Q}_1, h,t : \bar{Q}_1 \to Q_0)$. Define $\bar{Q}_1 \xrightarrow{*} \bar{Q}_1$ by $a^{\pm} \mapsto a^{\mp}$ and **charge** $\bar{Q}$ by defining

$$c : \bar{Q}_1 \to \{-1, +1\} : a^{\pm} \mapsto \pm 1$$

Notice $c(a^*) = -c(a)$ for all $a \in \bar{Q}_1$

Def The preprojective algebra $\Pi(Q)$ over $\mathbb{C}$ is

$$\mathbb{C}\bar{Q} \Big/ \sum_{a \in \bar{Q}_1} c(a) a a^*$$

Preprojective relation.

Varieties of modules

Modules for $\Pi(Q)$

- Q_0-grading so that $\underset{\to}{\dim}\, M \in \mathbb{N}^{Q_0}$

- $M_a : \underset{i = t(a)}{M_i} \to \underset{j = h(a)}{M_j}$ whenever $i \xrightarrow{a} j \in \overline{Q}_1$
- $\sum_{\substack{a \in \overline{Q}_1 \\ h(a)=i}} M_a M_{a^*} c(a) = 0$ at each $i \in Q_0$

Given $\vec{v} = (v_i) \in \mathbb{N}^{Q_0}$, consider $\Lambda(\vec{v})$ the variety of $\Pi(Q)$-module structures on $\mathbb{C}^{v_1} \oplus \cdots \oplus \mathbb{C}^{v_n}$ $(|I|=n)$

$\Lambda = \bigsqcup \Lambda(\vec{v})$ is Lusztig's nilpotent variety.

The simple $\Pi(Q)$-mods are 1d. $S(i)$, $\underrightarrow{\dim} S(i) = e_i$

$(0..010\ 0)$

Def The socle of M is the maximal semisimple submodule of M.

Let $T(i)$ denote the injective hull of $S(i)$.

$\underrightarrow{\dim}\, T(i) = \vec{v} = (v_i)$ s.t. $\omega_i - w_0\omega_i = \sum v_i \alpha_i$

Soc $T(i) = S(i)$.

<u>Thm</u> (Baumann-Kamnitzer) $\forall w \in W$, $\exists!$ $T(i,w)$ with $\underrightarrow{\dim} = (v_i)$ s.t. $\omega_i - w\omega_i = \sum v_i \alpha_i$ and Soc $T(i,w) = S(i)$. **Maya modules.**

If $w\omega_i = \omega_i$, $T(i,w) = 0$. <u>Exercise</u> $T(i,w) \subseteq T(i)$.

More generally if $\lambda = \sum d_i \omega_i$

$$T(\lambda) := \bigoplus T(i)^{\oplus d_i} \qquad \lambda - w_0\lambda = \sum v_i \alpha_i$$

$$T(\lambda, w) := \bigoplus T(i, w)^{\oplus \lambda_i} \qquad \lambda - w\lambda_{.} = \sum v_i \alpha_i.$$

<u>Exr</u> $\operatorname{Soc} T(\lambda, w)$?

Quiver grassmannians

Define

$$\mathrm{Gr}(T(\lambda)) = \{M \overset{\pi}{\subseteq} T(\lambda) : M \in \Lambda\}$$

$\mathrm{Gr}(T(\lambda, w))$ defined analog.

The connected components are

$$\mathrm{Gr}(\vec{v}, \_) = \{M \subseteq \_ : \underset{\to}{\dim} M = \vec{v}\}.$$

Thm (Saito, Savage, Savage-Tingley)

① $\mathrm{Irr}\, \mathrm{Gr}(T(\lambda)) \cong B(\lambda)$

② $\mathrm{Irr}\, \mathrm{Gr}(T(\lambda, w)) \cong B_w(\lambda)$

From heaps to modules

Apparent in the glass bead visualization.
And can be derived from the def of a heap.

Heaps are equipped a map

$\pi : H(w) \to I$ with fibres $\pi^{-1}(i) =: H(w)_i$

Note $H(w)$ are totally ordered ⇝ filtration -
(later)

We can give I-graded vector space.

$$\mathbb{C}H_w := \operatorname{Span}_{\mathbb{C}}(H(w))$$

$$= \bigoplus_{i \in I} \underbrace{\mathbb{C}H(w)_i}_{\operatorname{Span} H(w)_i}$$

Promote this vectsp. to a $\Pi(Q)$-module.

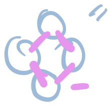
by defining $\forall x \in \mathbb{C}H_w$, $\forall a \in \bar{Q}_1$

$$a \cdot x = \pm y$$

if $x \geq y$ are adjacent in $H(w)$.

$\pi(x) = t(a)$

$\pi(y) = h(a)$.

Eg  $= H(2132)$ ⇝ $\mathbb{C}H_w$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] & \mathbb{C}^2 \arrow[l, bend left, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \arrow[r, bend left, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] & \mathbb{C} \arrow[l, bend left, "{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}"] \end{tikzcd}$$

Prop
- If w is minuscule, then $\mathbb{C}H(w)$ is a module for $\Pi(Q)$.
- If w is dominant λ-minuscule, $\mathbb{C}H(w) \cong T(\lambda, w)$.

In particular, when λ is minuscule,

$$\mathbb{C}H(w_0^J) \cong T(\lambda)$$

where w_0^J is the smallest rep of $w_0 W_J$

where $W_J = \langle s_j : j \in J \rangle$

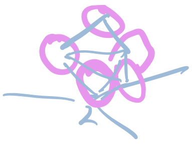
and $J = \{ j : s_j \lambda = \lambda \} \subseteq I$.

Eg (non) $\Gamma = D_4$

$H(2, 1, 3, 4, 2) \rightsquigarrow$

$$\begin{tikzcd} & \mathbb{C} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbb{C} \arrow[dr] & \mathbb{C} \arrow[d] & \mathbb{C} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

not a module for $\Pi(Q)$ for any choice of $\pm$ in def of $\mathbb{C}Q\,C^{\circ}H(w)$.



w is assumed minuscule.

Prop $^{\vee}$ $\forall \phi \in \mathcal{T}(H(w))$, $\mathbb{C}\phi \overset{\Pi}{\subseteq} \mathbb{C}Hw$

And $\phi \mapsto \mathbb{C}\phi$ yields a bij.

$$J(H(w)) \longrightarrow \operatorname{Irr} Gr(\mathbb{C}H(w))$$

~~Study Nakajima tensor product varieties.~~

Goal: $(X_1 \dots X_r) \in \operatorname{Irr} Gr(T(\lambda^1)) \times \cdots \times \operatorname{Irr} Gr(T(\lambda^r))$

$$\Rightarrow X_1 \otimes - \otimes X_r \in B(\lambda)$$

$$\big\downarrow$$

$$\otimes B(\lambda^i) \ni Z(X_1, \dots, X_r) := \overline{\{ M \in Gr(T(\lambda)) : M^k \in X_k \ \forall k = 1..r \}}$$

$\uparrow$

$\mathbb{C}\phi_k$

$\lambda = \lambda^1 + \cdots + \lambda^r$ is a composition.

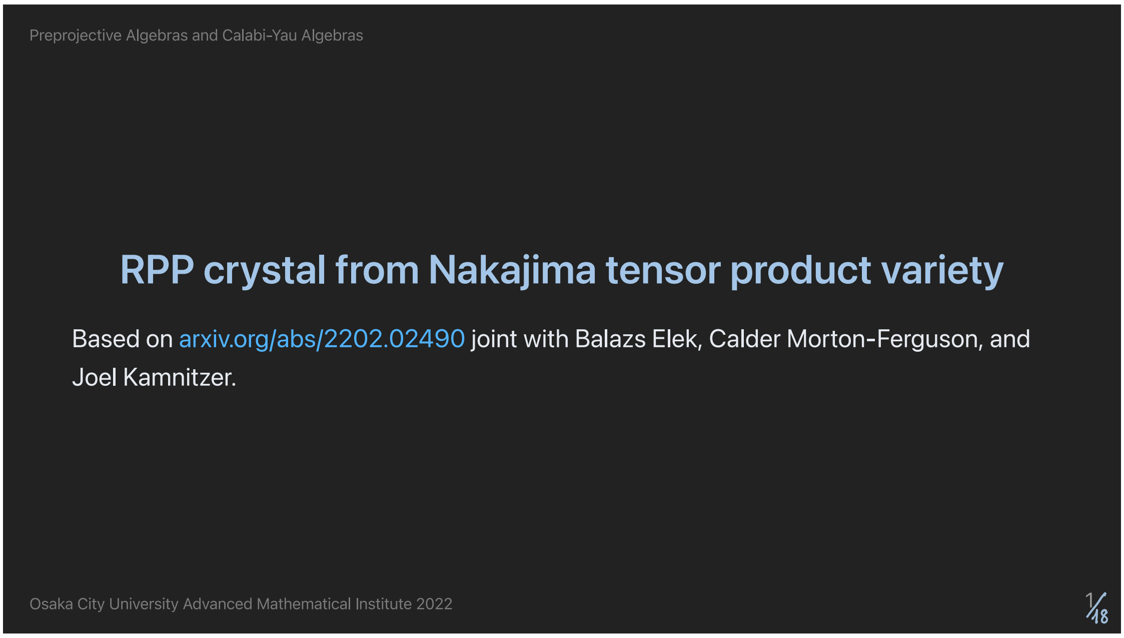
$\Phi_1 \otimes \cdots \otimes \Phi_r$

is a filtration of

an rpp Φ

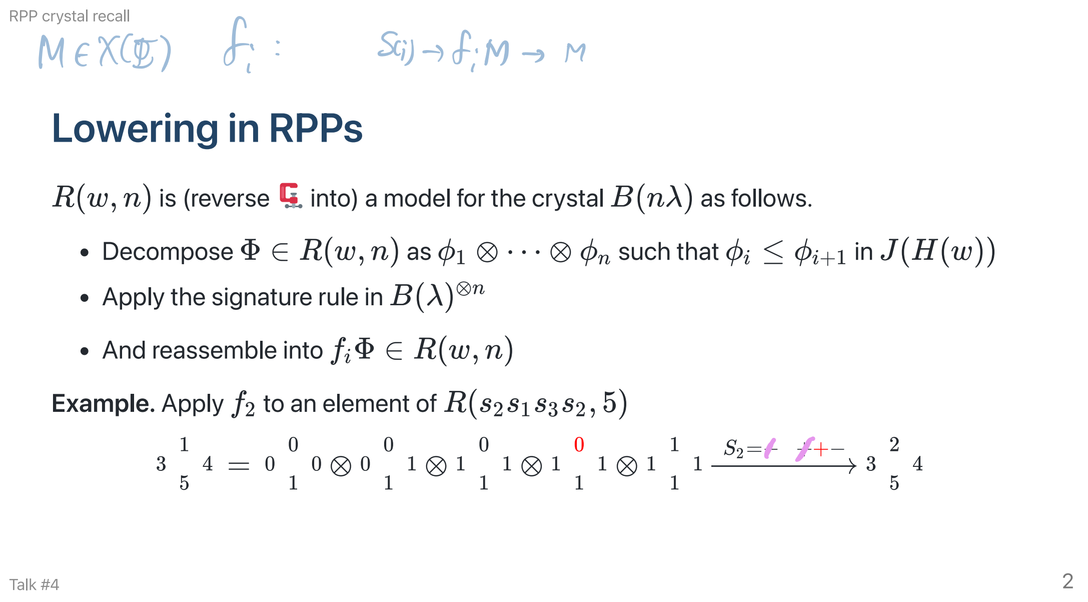
by order ideals.

$T(\lambda, w)$ are τ-rigid?



RPP crystal from Nakajima tensor product variety

Based on arxiv.org/abs/2202.02490 joint with Balazs Elek, Calder Morton-Ferguson, and Joel Kamnitzer.



$M \in X(\Phi)$ f_i : $S(i) \to f_i M \to M$

Lowering in RPPs

$R(w,n)$ is (reverse 🗜️ into) a model for the crystal $B(n\lambda)$ as follows.

- Decompose $\Phi \in R(w,n)$ as $\phi_1 \otimes \cdots \otimes \phi_n$ such that $\phi_i \leq \phi_{i+1}$ in $J(H(w))$
- Apply the signature rule in $B(\lambda)^{\otimes n}$
- And reassemble into $f_i\Phi \in R(w,n)$

Example. Apply f_2 to an element of $R(s_2 s_1 s_3 s_2, 5)$

$$3 \begin{smallmatrix}1\\ \\ 5\end{smallmatrix} 4 = 0 \begin{smallmatrix}0\\ \\ 1\end{smallmatrix} 0 \otimes 0 \begin{smallmatrix}0\\ \\ 1\end{smallmatrix} 1 \otimes 1 \begin{smallmatrix}0\\ \\ 1\end{smallmatrix} 1 \otimes 1 \begin{smallmatrix}{\color{red}0}\\ \\ 1\end{smallmatrix} 1 \otimes 1 \begin{smallmatrix}1\\ \\ 1\end{smallmatrix} 1 \xrightarrow{S_2=\cancel{+}-\ \cancel{-}{\color{red}+}-} 3 \begin{smallmatrix}2\\ \\ 5\end{smallmatrix} 4$$

Recall: Nakajima core quiver variety

Let $\lambda \in P_+$ and $\mu \in P$ be such that $\lambda = \sum w_i \omega_i$ and $\lambda - \mu = \sum v_i \alpha_i$ for some $\vec{w} = (w_i), \vec{v} = (v_i) \in \mathbb{N}^I$.

Let V, W be I-graded vector spaces of dimension $\vec{v}, \vec{w}$.

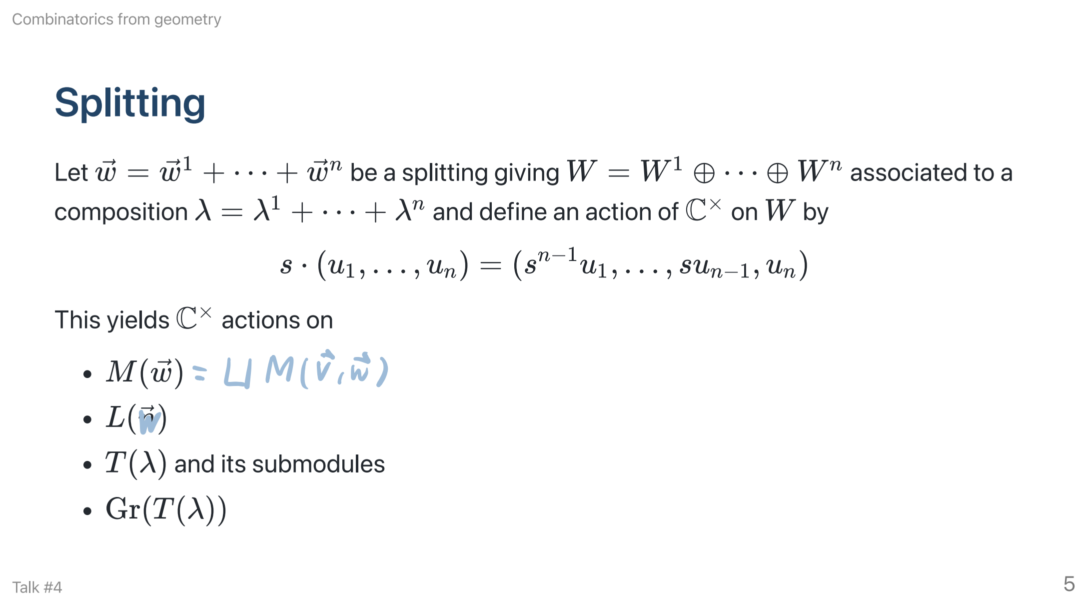
1. $\mathrm{Hom}(\vec{v}, \vec{w}) := \bigoplus \mathrm{Hom}(V_i, V_j) \oplus \bigoplus \mathrm{Hom}(V_i, W_i)$
2. $\prod GL(V_i) \circlearrowright T^*\mathrm{Hom}(\vec{v}, \vec{w}) \xrightarrow{\psi} \prod \mathfrak{gl}(V_i)$
3. $M_\chi(\vec{v}, \vec{w}) := \psi^{-1}(0) /\!\!/_\chi \prod GL(V_i) \to M_0(\vec{v}, \vec{w})$ $\chi =$ determinantal char
4. the preimage of zero under this projection is the so-called *core* quiver variety

$$L(\vec{w}) := \bigsqcup_{\vec{v}} L(\vec{v}, \vec{w})$$

Recall: Lusztig's isomorphism

Theorem. With $\lambda = \sum w_i \omega_i, \lambda - \mu = \sum v_i \alpha_i$ as above

$$\mathrm{Gr}(T(\lambda)) \cong L(\vec{w})$$



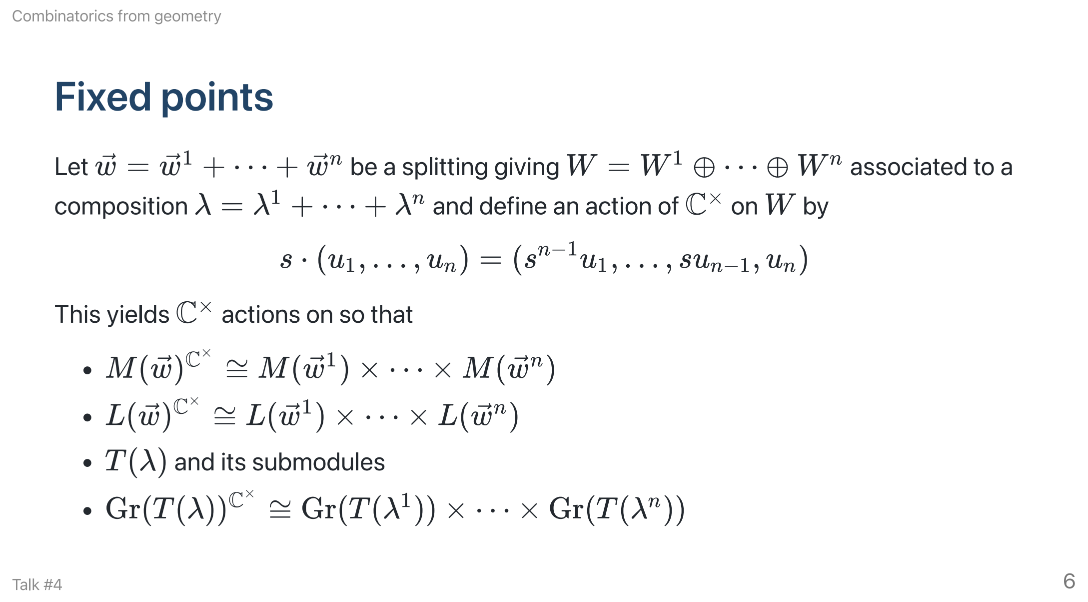
Splitting

Let $\vec{w} = \vec{w}^1 + \cdots + \vec{w}^n$ be a splitting giving $W = W^1 \oplus \cdots \oplus W^n$ associated to a composition $\lambda = \lambda^1 + \cdots + \lambda^n$ and define an action of $\mathbb{C}^\times$ on W by

$$s \cdot (u_1, \ldots, u_n) = (s^{n-1}u_1, \ldots, su_{n-1}, u_n)$$

This yields $\mathbb{C}^\times$ actions on

- $M(\vec{w}) = \sqcup M(\vec{v}, \vec{w})$
- $L(\vec{w})$
- $T(\lambda)$ and its submodules
- $\mathrm{Gr}(T(\lambda))$



Fixed points

Let $\vec{w} = \vec{w}^1 + \cdots + \vec{w}^n$ be a splitting giving $W = W^1 \oplus \cdots \oplus W^n$ associated to a composition $\lambda = \lambda^1 + \cdots + \lambda^n$ and define an action of $\mathbb{C}^\times$ on W by

$$s \cdot (u_1, \ldots, u_n) = (s^{n-1}u_1, \ldots, su_{n-1}, u_n)$$

This yields $\mathbb{C}^\times$ actions on so that

- $M(\vec{w})^{\mathbb{C}^\times} \cong M(\vec{w}^1) \times \cdots \times M(\vec{w}^n)$
- $L(\vec{w})^{\mathbb{C}^\times} \cong L(\vec{w}^1) \times \cdots \times L(\vec{w}^n)$
- $T(\lambda)$ and its submodules
- $\mathrm{Gr}(T(\lambda))^{\mathbb{C}^\times} \cong \mathrm{Gr}(T(\lambda^1)) \times \cdots \times \mathrm{Gr}(T(\lambda^n))$

Tensor product variety

Nakajima: Irreducible components of

$$Z(\vec{w}) = \left\{ x \in M(\vec{w}) : \lim_{s \to 0} s \cdot x \in L(\vec{w})^{\mathbb{C}^\times} \right\}$$

are in crystal bijection ⛵ with irreducible components of

$$L(\vec{w})^{\mathbb{C}^\times} \cong L(\vec{w}^1) \times \cdots \times L(\vec{w}^n)$$

The bijection ⛵ is

$$(X_1, \dots, X_n) \mapsto \overline{\left\{ x \in M(\vec{w}) : \lim_{s \to 0} s \cdot x \in X_1 \times \cdots \times X_n \right\}}$$

Tensor product variety

Fact: $L(\vec{w}) \subset Z(\vec{w})$ because it is a projective variety, so every point has a limit, and this inclusion yields an inclusion on irreducible components. Combined with ⛵ we have

Theorem. The crystal structure on $\mathrm{Irr} L(\vec{w})^{\mathbb{C}^\times}$ extends the crystal structure on $\mathrm{Irr} L(\vec{w})$. In other words, there is an inclusion of crystals.

$$\begin{aligned} \mathrm{Irr}\mathrm{Gr}(T(\lambda)) = B(\lambda) \to \prod \mathrm{Irr} L(\vec{w}^i) &= \bigotimes B(\lambda^i) \\ &\xrightarrow{\lambda^i = w_i \omega_i} \bigotimes B(\omega_i)^{\otimes w_i} \end{aligned}$$

Components and filtrations

$T(\lambda) = \oplus T(\lambda^i)$

$\lambda = \lambda^1 + \cdots + \lambda^n$

Submodules $M \subset T(\lambda)$ can be filtered by $M^{\leq i} := M \cap (T(\lambda^1) \oplus \cdots \oplus T(\lambda^i))$ and composition factors $M^i = M^{\leq i}/M^{\leq i-1}$ regarded as $M^i \subset T(\lambda^i)$.

This filtration is compatible with respect to the $\mathbb{C}^\times$ action:

$$\lim_{s \to 0} s \cdot M = M^1 \oplus \cdots \oplus M^n \in \mathrm{Gr}(T(\lambda))^{\mathbb{C}^\times} \longleftrightarrow (M^1, \ldots, M^n)$$

Theorem. Thus we get another version of the ⛵ crystal isomorphism

$$B(\lambda) \to B(\lambda^1) \otimes \cdots \otimes B(\lambda^n)$$
$$X_1 \otimes \cdots \otimes X_n \mapsto \overline{\{M \in \mathrm{Gr}\, T(\lambda) : M^i \in X_i \text{ for all } i\}}$$

Heaps and composition factors

Reducing again to the special case $\lambda = \overset{n}{\cancel{w}}\omega$ recall that $T(\lambda, w) = \mathbb{C}H(w)$ and irreducible components of the quiver grassmannian of $T(\lambda, w)$ are in bijection with $J(H(w))$.

Theorem. Given $\Phi =$ $(\phi^1, \ldots, \phi^n)$

$$Z(\Phi) = \overline{\{M \in \mathbb{C}H(w)^{\oplus \overset{n}{\cancel{w}}} : M^i = \mathbb{C}\phi^i \text{ for all } i\}}$$

is an irreducible component of $\mathrm{Gr}T(\lambda, w)$ and supplies the crystal isomorphism

$$R(w, n) = B_w(n\lambda)$$

Another filtration

Recall that $H(w)_i = \{x_1^i < \cdots < x_q^i\}$ so define $A_i : x_j^i \mapsto x_{j-1}^i$ and $A_i x_1^i = 0$.

← extend this shift operator to a linear transformation of $M \subseteq \mathbb{C}H(w)^{\oplus n}$

Let $\Phi \in R(w, n)$. This yields the Jordan filtrations

$$M_i \supset \mathrm{Ker} A_i \supset \mathrm{Ker} A_i^2 \supset \cdots \supset \mathrm{Ker} A_i^q$$

with composition factors $\mathrm{Ker} A_i^j / \mathrm{Ker} A_i^{j-1}$ so that

$$X(\Phi) = \overline{\left\{ M : \dim \mathrm{Ker} A_i^j - \dim \mathrm{Ker} A_i^{j-1} = \Phi(x_j^i) \text{ for all } i \right\}}$$

is an irreducible component of $\mathrm{Irr}\mathrm{Gr}\mathbb{C}H(w)$.

Literature

Stembridge 1999: Minuscule elements of Weyl groups

- Clarify exactly which fully commutative w are λ-minuscule in terms of
  - reduced words
  - heaps
- Extend Proctor's classification of (dominant) λ-minuscule w (equivalently, their heaps) from simply-laced to symmetrizable Kac-Moody Weyl groups

$$\sum_{\rho} q^{|\rho|} = \prod_{u\in\lambda} \frac{1}{1-q^{h(u)}}$$

Literature

Garver, Patrias, Thomas 2018: Minuscule reverse plane partitions via quiver representations $kQ.$

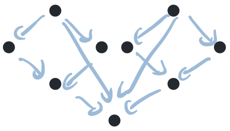
Generalize Hillman-Grassl (1976) bijective *proof* of Stanley's generating function (1971) for RPPs of shape λ, via multisets of rim hooks, to a proof of Proctor's generating function (1984) for RPPs of shape P a minuscule poset *using* isomorphism classes of representations of simply-laced Dynkin quivers.

Connection?

Consider in $R(32413524, 2)$ with $\Gamma = A_5$

$$\Phi = \begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 2 & & \end{array} = \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \otimes \begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

Regarding the heaps on the RHS as modules $N^1, N^2 \subset I(3)$ calculate $\mathrm{Ext}(N^1, N^2)$ to check that there is just one non-trivial extension, and verify that its socle does not contain $S(3)$



Another way to lower

Beta Testing

We could alternatively view the RPP as a filtration of a $\Pi(Q)$-module by $\mathbb{C}Q$-modules.

$$M \in X(\Phi)\,,\ \Phi = \begin{smallmatrix} & 1 & & 1 & \\ 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 2 & & \end{smallmatrix} \Rightarrow M|_{\underline{\mathbb{C}Q}} = \frac{2 \to 3 \leftarrow 4 = \tau^{-1}(\qquad)}{1 \to 2 \to 33 \leftarrow 4 \leftarrow 5} \nearrow$$

The upshot is that we can use the AR quiver to predict e.g. if $S(3)$ is a simple quotient.

Challenge: can we describe irreducible components of $\mathrm{Irr}\underline{\mathrm{Gr}T(\lambda)}$ using the AR quiver?

characterize M in $X(\Phi)$ using $M|_{\mathbb{C}Q}$ or the fact that Φ indexes nilpotent endomorphosim of an iso class $\mathbb{C}Q$ modules

Literature

Literature

Kleshchev, Ram 2008: Homogeneous representations of Khovanov-Lauda algebras

Explicit construction of irreducible graded representations of simply laced KLR algebras which are concentrated in one degree ("homogeneous") from $J(H(w))$.

Open questions

- The MV cycles associated to order ideals $J(H(w))$ are Schubert varieties in partial flag varieties G/P_λ
    - What about MV cycles associated to $R(w,n)$
- The homogeneous coordinate ring of G/P_λ is isomorphic to $\bigoplus V(n\lambda)$ and it carries a cluster algebra structure
    - How does the cluster combinatorics interact with the combinatorics of $\coprod_n R(w,n)$

;

Qn $M \in \Pi(Q)$-mod $\longleftarrow$ $N \in kQ$-mod

↑ choice of orientation of Q

$\mathrm{genJF}(M|_{kQ})$

isoclasses of kQ reps have "generic Jordan forms" (GJF)

rpp's.

minuscule = sink.

🕊️ Thank you for listening 🦜

Baumann-Kamnitzer: orientations adapted to reduced expression

Generalized preprojective algebras

Bernard Leclerc

Abstract. Classical preprojective algebras were introduced by Gelfand-Ponomarev and further studied by Dlab-Ringel and many others. Their connection with Lie theory was discovered by Lusztig, who realized the enveloping algebra of the positive part of a symmetric Kac-Moody Lie algebra as a convolution algebra of constructible functions on module varieties of a preprojective algebra. In joint work with Geiss and Schröer, we used Lusztig's construction to obtain categorifications of the cluster algebra structure on coordinate rings of certain finite-dimensional unipotent subgroups of symmetric Kac-Moody groups labelled by Weyl group elements. These results require the generalized Cartan matrix to be symmetric. So the question arises of a possible generalization to the symmetrizable case. This motivated a series of joint works with Geiss and Schröer on a new class of 1-Iwanaga-Gorenstein algebras associated with symmetrizable generalized Cartan matrices, and their generalized preprojective algebras. In these talks I will give a survey of this construction and its application to crystal bases, and I will explain how it relates to joint work with Hernandez on representations of quantum affine algebras. Finally if time allows, I will discuss some recent developments by Murakami and Fujita-Murakami.

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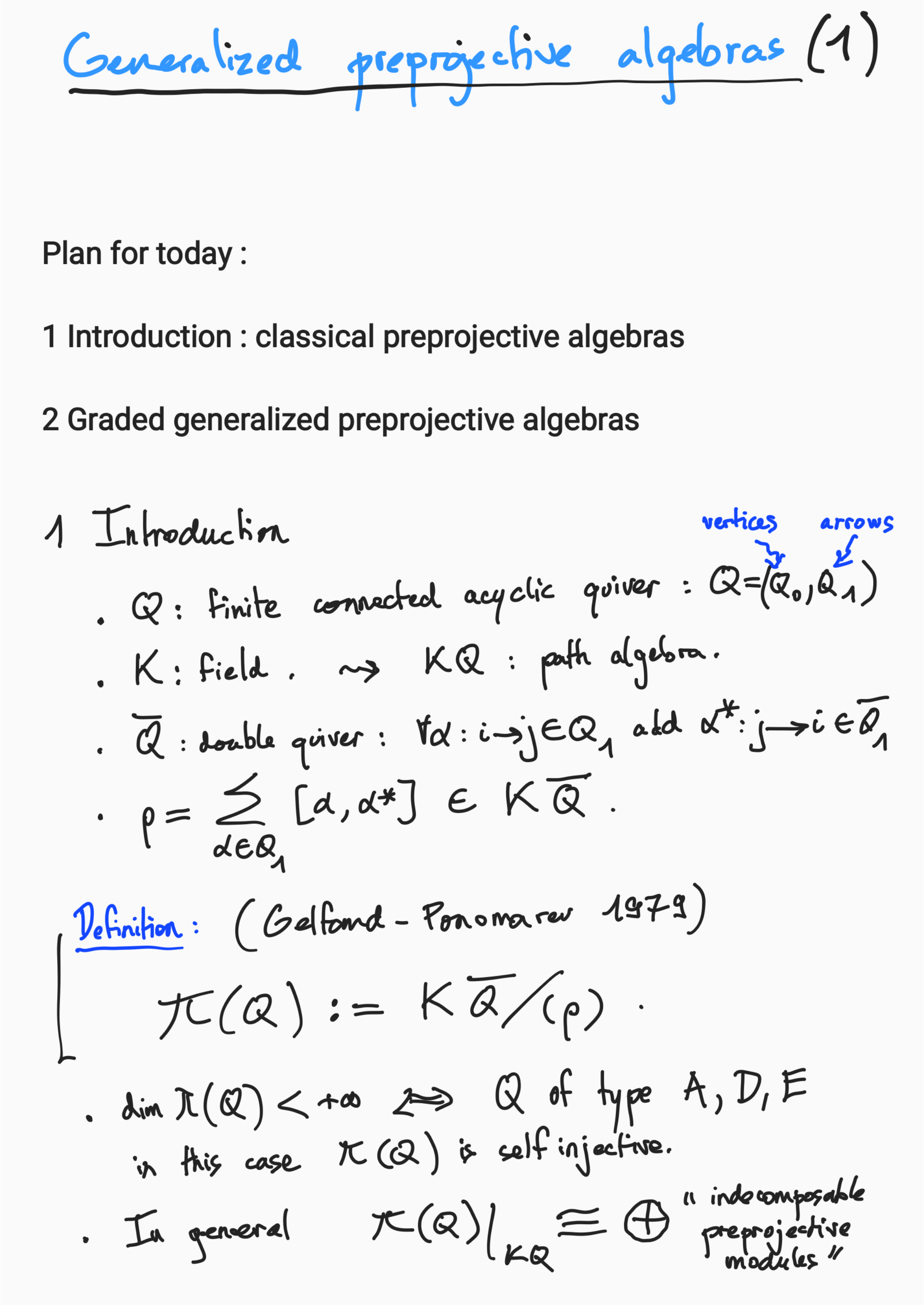


Generalized preprojective algebras (1)

Plan for today :

1 Introduction : classical preprojective algebras

2 Graded generalized preprojective algebras

1 Introduction

- Q : finite connected acyclic quiver : $Q=(\overset{\text{vertices}}{Q_0},\overset{\text{arrows}}{Q_1})$
- K : field. ⇝ KQ : path algebra.
- $\overline{Q}$: double quiver : $\forall \alpha : i \to j \in Q_1$ add $\alpha^* : j \to i \in \overline{Q}_1$
- $\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*] \in K\overline{Q}$.

Definition : (Gelfand - Ponomarev 1979)

$$\Pi(Q) := K\overline{Q}/(\rho).$$

- $\dim \Pi(Q) < +\infty \iff Q$ of type A, D, E

  in this case $\Pi(Q)$ is selfinjective.
- In general $\Pi(Q)|_{KQ} \cong \bigoplus$ " indecomposable preprojective modules "

preprojective : $\tau^{-k} P$

A-R transl. indec. proj KQ-module

. Why is $\pi(Q)$ important ?
relations with Lie theory.

Fix $K = \mathbb{C}$.

. 1990 Lusztig : "nilpotent varieties" = representation varieties of $\underline{\text{nilpotent}}$ modules :

all composition factors are 1-dimensional.

Let $\mathfrak{g}$ be the symmetric Kac-Moody algebra associated with Q

Let $U_q(\mathfrak{g})$ be the corresp. quantum group.

Thm Lusztig 1991 ~ Kashiwara-Saito 1997
2000

. Nilpotent varieties $\pi(Q)^{nil}_{\underline{d}}$ ← dim vector are of pure dimension :

all irred. components have the same dimension :

$$\dim\left(\mathrm{rep}(Q)_{\underline{d}}\right)$$ ← one of the irreducible components.

$$\# \mathrm{Irr}\left(\pi(Q)^{nil}_{\underline{d}}\right) = \dim U_q^+(\mathfrak{g})_{\underline{d}}$$

$\mathrm{Irr} := \bigsqcup_{\underline{d}} \mathrm{Irr}\left(\pi(Q)^{nil}_{\underline{d}}\right)$ is a labelling set for vertices of $\mathcal{B}(-\infty)$

$\rightsquigarrow$ geometric description of $\mathcal{B}(-\infty)$

- There is an associative algebra of constructible functions on nilpotent varieties isomorphic to $U^+(\mathfrak{g})$.

$\rightsquigarrow$ semicanonical basis of $U^+(\mathfrak{g})$

Nakajima varieties : V, W Q_0-graded vector spaces

$$\begin{tikzcd}
\mathfrak{M}(V,W) \arrow[d, "\pi"] \arrow[r, phantom, "\supset"] & \mathcal{L}(V,W) \arrow[d, "\pi"] \\
\mathfrak{M}_0(V,W) \arrow[r, phantom, "\ni"] & \{0\}
\end{tikzcd}$$

Lusztig: In type A-D-E

injective $\Pi(Q)$ module

$$\mathcal{L}(V,W) \cong \mathrm{Grass}_{\underline{e}_V}(I_W)$$

dimension vector

Let $L(W) = L(\lambda_W)$ highest-weight simple $\mathfrak{g}$-module:

$$\lambda_W = \sum_{i \in Q_0} \dim(W_i)\,\varpi_i$$

Thm (Nakajima)

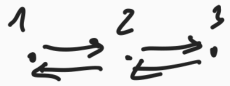
$$\mathrm{ch}(L(W)) = \sum_{V} \dim\left(H^{\mathrm{top}}\left(\mathrm{Grass}_{\underline{e}_V}(I_W)\right)\right) e^{\lambda_W - \alpha_V}$$

---

graded setting : $Q \rightsquigarrow \mathbb{Z}Q$ (repetition quiver)

$$(\mathbb{Z}Q)_0 = Q_0 \times \mathbb{Z}$$

Ex: $Q: 1 \leftarrow 2 \leftarrow 3$



$\mathbb{Z}Q:$

Nakajima theory: $(Q = A - D - E)$

$W^\bullet, V^\bullet$ $\mathbb{Z}Q_0$-graded f.d. vector-spaces

$$\mathcal{L}^\bullet(V^\bullet, W^\bullet) \cong \mathrm{Grass}_{e_{V^\bullet}}(I_{W^\bullet})$$

(Savage Tingley)

injective modules over $\pi^\bullet(Q)$

$$W^\bullet \rightsquigarrow \sum_{(i,r)\in(\mathbb{Z}Q)_0} \dim W^\bullet_{(i,r)} \cdot \varpi_{i,r} = \lambda_{W^\bullet}$$

"highest loop weight"

irreducible finite dim repr. of $U_q(L\mathfrak{g})$

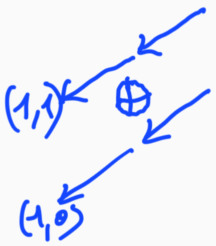
$$L(\lambda_{W^\bullet})$$

Nakajima: the homology of $\mathcal{L}^\bullet(W^\bullet)$ gives the character of the standard module

$$M(\lambda, \dot{w}) \twoheadrightarrow L(\dot{w})$$

Ex: $$W^\bullet = \underset{\dim 1}{W_{1,0}} \oplus \underset{\dim 1}{W_{1,1}}$$

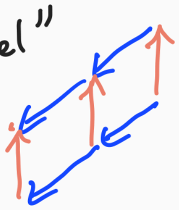
$$I_{W^\bullet} \simeq$$



$$\begin{tikzcd} M(\varpi_{1,0}+\varpi_{1,1}) \cong L(\varpi_{1,0}) \otimes L(\varpi_{1,1}) \arrow[d, two heads] & \dim 16 \\ L(\varpi_{1,0}+\varpi_{1,1}) & \dim 10 \end{tikzcd}$$

If we replace $I_{W^\bullet}$ by:

“generic kernel” ⟶



can check this gives the correct character for the simple module.

2 – Graded generalized preprojective algebras

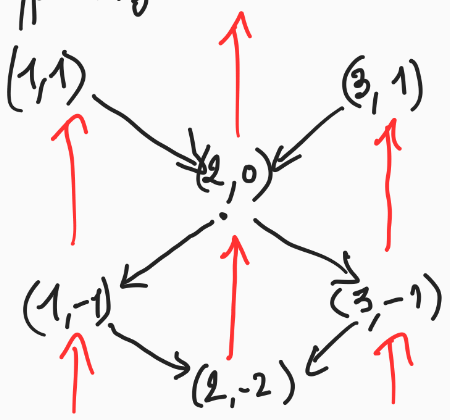
(jt. w. D. Hernandez)

- $C=(c_{ij})_{i,j\in I}$ indecomposable n x n Cartan matrix (finite type : $A_n, B_n, \dots, F_4, G_2$)
- $B = DC$ is symmetric $=(b_{ij})$ $\quad D=(d_i)_{i\in I}$ diagonal $\quad d_i\in\mathbb{Z}_{>0}, \ \min(d_i)=1$
- $\tilde{\Gamma}$ infinite quiver with vertex set : $\tilde{V}=I\times\mathbb{Z}$

arrows : $(i,r)\to(j,s) \Leftrightarrow \begin{pmatrix} b_{ij}\neq 0 \\ \text{and} \\ s=r+b_{ij}\end{pmatrix}$

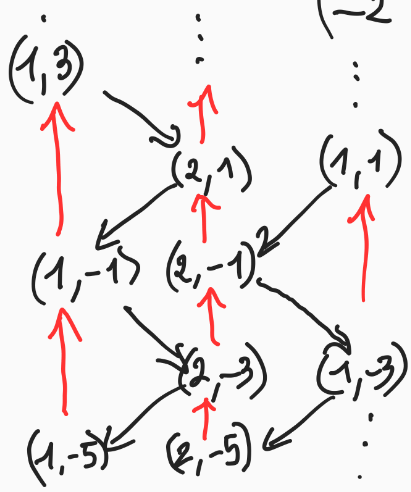
$\tilde{\Gamma}$ has two isomorphic connected components. Pick one and call it Γ.

Ex: type A_3, $\quad C=B=\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}$

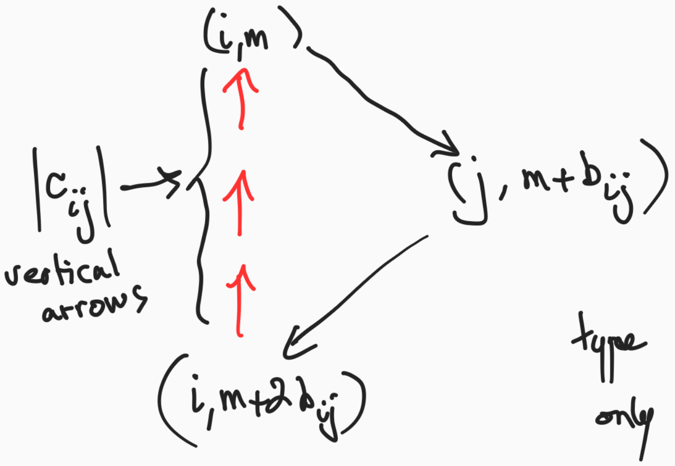


Type B2: $B = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$ $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$



Relations: For every $i \neq j$ s.t. $c_{ij} \neq 0$. For every $(i,m) \in V$ there is an oriented cycle:



type A D E
only triangles

Potential: S = formal sum of all these cycles.
Relations: all cyclic derivatives $\partial_\alpha S$ $\alpha \in$ arrow of Γ
$= 0$.

Definition: (Hernandez-L 2016)

$$\pi^{\bullet}(C) := K\Gamma \big/ \left(\partial_\alpha S, \ \alpha \in \{\text{arrows of } \Gamma\} \right)$$

- Let $(i,m) \in V$. Let $k \in \mathbb{Z}_{>0}$.

$(i,m) \rightsquigarrow S_{i,m}$ 1-dim simple $\pi^{\bullet}(C)$-mod.

$\rightsquigarrow I_{i,m}$ injective hull of $S_{i,m}$.

generic homomorphism

$$\begin{tikzcd} K^{(i)}_{k,m} \arrow[r, hook] & I_{(i,m)} \arrow[r, two heads] & I_{(i,m-kb_{ii})} \end{tikzcd}$$

generic kernel (finite-dimensional)

Thm: Let $U_q(\mathcal{L}\mathfrak{g})$ be the quantum loop algebra associated with C and (q not a root of unity).

$$i,m,k \rightsquigarrow L\left(\sum_{s=1}^{k} \varpi_{i,m-(2s-1)d_i}\right)$$

Kirillov-Reshetikhin modules.

. The q-character of this module is equal to the highest monomial times a Laurent polynomial equal (up to some explicit

monomial change of variables) to the
F-polynomial of $K^{(\dot{c})}_{k,m}$.

Ideas of the proof: Very indirect.

① Introduce the cluster algebra $\mathcal{A}$ with initial seed Γ.

② Prove that (truncations) of q-characters of KR-modules are "given" by certain cluster variables of $\mathcal{A}$.

③ Derksen - Weyman - Zelevinsky theory.

Remarks: ① Thm extends to tensor products of KR-modules and direct sum of generic kernels.

② In particular obtain a formula the q-char. of standard modules. In type ADE this recovers the formulas of Nakajima (using Lusztig, Savage-Tingley)

But our formula works also for BCFG.

⇝ "Nakajima type varieties" for BCFG ?

③ There are many more cluster variables in $\mathcal{A}$!!

Conjecture (HL 2016)

m "cluster monomial" of $\mathcal{A}$ $\xrightarrow{DWZ}$ $\dot{\pi}(C)_M$-mod

↓

affine highest-weight of an irreducible $U_q(Lg)$-module L ⇠ – – same connection – – ⇢

- Recently proved by Kashiwara-Kim-Oh-Park (2021)

<u>Example:</u> type A_3 $L(\varpi_{1,-6} + \varpi_{2,-3})$

The corresponding $\pi^\bullet(C)$-module:

$$M = \begin{tikzcd}
 & (2,-1) \arrow[dl] \arrow[dr] & \\
(1,-2) \arrow[dr] & & (3,-2) \arrow[dl] \\
 & (2,-3) & \\
 & & (3,-4) \arrow[uu, red] \arrow[dl] \\
 & (2,-5) \arrow[uu, red] \arrow[dl] & \\
(1,-6) & &
\end{tikzcd}$$

Generalized preprojective algebras (2)

Ungraded setting

Representation theory

Representation varieties and crystals

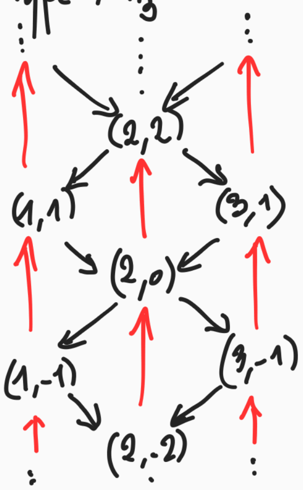
3 Ungraded setting (jt. w. Geiss - Schröer)

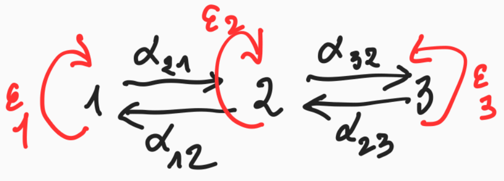
The quiver Γ has a natural $\mathbb{Z}$- action generated by:

$$\begin{aligned} V &\longrightarrow V \\ (i,r) &\longmapsto (i,r+2) \end{aligned}$$

This preserves the potential S. By moding out this action we obtain an algebra $\tilde{\Pi}(C)$.

Ex: type A_3.





$$(P_3)\left\{\begin{aligned} \alpha_{12}\alpha_{21} &= 0 \\ \alpha_{21}\alpha_{12} - \alpha_{23}\alpha_{22} &= 0 \\ \alpha_{32}\alpha_{23} &= 0 \end{aligned}\right.$$

$$(P_2)\begin{cases} \varepsilon_2\alpha_{21} = \alpha_{21}\varepsilon_1 \\ \alpha_{12}\varepsilon_2 = \varepsilon_1\alpha_{12} \\ \varepsilon_2\alpha_{23} = \alpha_{23}\varepsilon_3 \\ \alpha_{32}\varepsilon_2 = \varepsilon_3\alpha_{32} \end{cases}$$

In order to get a finite-dimensional algebra we add nilpotency relations on the ε_i :

$$(P_1)_k \qquad \varepsilon_1^k = \varepsilon_2^k = \varepsilon_3^k = 0 \quad (\text{some } k>0)$$

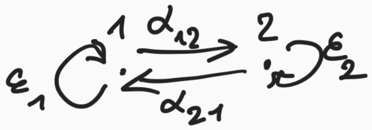
For other types $\underbrace{ADE}_{r=1}$ $\underbrace{BCF_4}_{r=2}$ $\underbrace{G_2}_{r=3}$

$$(P_1)_k \qquad \varepsilon_i^{k\cdot\frac{r}{d_i}} = 0 \ ; \ D = \operatorname{diag}(d_i)$$

minimal symmetrizer of C.

Ex: Type B_2: $\quad C = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$r=2$.



$$(P_1)_k: \ \varepsilon_1^k = \varepsilon_2^{2k} = 0$$

$$(P2) \quad \varepsilon_1\alpha_{12} = \alpha_{12}\varepsilon_2^2 \ , \ \varepsilon_2^2\alpha_{21} = \alpha_{21}\varepsilon_1$$

$$(P3) \quad \alpha_{12}\alpha_{21} = 0 \qquad \alpha_{21}\alpha_{12}\varepsilon_2 = \varepsilon_2\alpha_{21}\alpha_{12}$$

$k=1$. $\varepsilon_1=0$. $\varepsilon_2^2=0$

The indecomposable injective modules are:

$$I_1: \quad \begin{tikzcd} & 2 \arrow[dl, "\alpha_{12}"'] & \boxed{1} \arrow[dl, "\alpha_{21}"] \\ \boxed{1} & 2 \arrow[u, "\varepsilon_2"'] & \end{tikzcd} \quad = P_1$$

$$I_2: \quad \begin{tikzcd} & 2 \arrow[dr] & \\ & \boxed{2} \arrow[u] \arrow[dl] & 1 \arrow[dl] \\ 1 \arrow[dr] & \boxed{2} & \\ & 2 \arrow[u] & \end{tikzcd} \quad = P_2$$

Remarks: ① $\sum \operatorname{diag}\left(\frac{r}{d_i}\right)$ is a symmetrizer for ${}^tC$. In GLS we decided to take opposite convention to HL:

$$[HL] \longleftrightarrow (GLS)$$

$$C \longleftrightarrow {}^tC$$

"Langlands duality"

② In [GLS] we work with arbitrary symmetrizable generalized Cartan matrices.

. From now on I will switch to the convention of [GLS].

Def: C symmetrizable gen. Cartan matrix $=(c_{ij})_{1\leq i,j\leq n}$

$D = \mathrm{diag}(d_i)$, minimal symmetrizer

$d_i \in \mathbb{Z}_{>0}$, $\sum d_i$ minimal

If $c_{ij} < 0$

$g_{ij} = \gcd(c_{ij}, c_{ji})$; $f_{ij} = \frac{|c_{ij}|}{g_{ij}}$

. $\Omega \subseteq \{1, 2, \dots, n\}^2$, <u>acyclic</u> orientation :

(i) $\{(i,j), (j,i)\} \cap \Omega \neq \phi \Leftrightarrow c_{ij} < 0$

(ii) if $(i_1, i_2), (i_2, i_3), \dots, (i_t, i_{t+1}) \in \Omega$

then $i_1 \neq i_{t+1}$.

. $Q = (Q_0, Q_1)$ "simple quiver"

$Q_0 = \{1, \dots, n\}$

$$Q_1 = \{ \alpha_{ij}^{(g)} : j \longrightarrow i \mid (i,j) \in \Omega \; 1 \leqslant g < g_{ij} \}$$

$$\cup \{ \varepsilon_i : i \circlearrowleft \mid i \in Q_0 \}$$

. $\overline{Q} = (Q_0, \overline{Q_1})$ obtained by adding an arrow $\alpha_{ji}^{(g)} : i \to j$ for every $\alpha_{ij}^{(g)} \in Q_1$.

<u>Def: Algebra $H(C, kD, \Omega)$</u> $k \in \mathbb{Z}_{>0}$ K field

$$H_k = KQ / I_k$$

where I_k is the ideal generated by:

$$(H_1)_k : \quad \varepsilon_i^{kd_i} = 0 \qquad (i \in Q_0)$$

$$(H_2) : \quad \varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}$$

$$(\forall \alpha_{ij}^{(g)} \in Q_1)$$

Remarks: (1) H_k is finite-dimensional over K.

(2) If C is symmetric and $k=1$, then

$H_1 = KQ^\circ \leftarrow$ obtained by removing all ε_i.

More for any k, $H_k \cong K[X]/(X^k) \otimes_K KQ^\circ$

(3) In general H_K is similar a "species" where the fields are replaced by truncated polynomial rings.

Def Algebra $\Pi(C, kD)$

$\Pi_k = K\overline{Q}/\overline{I}_k$ where $\overline{I}_k$ is given by:

$$(P_1) \quad \varepsilon_i^{kd_i} = 0$$

$$(P_2) \quad \varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}} \quad (\forall \alpha_{ij}^{(g)} \in \overline{Q}_1),$$

(P3) $\forall i \in Q_0$:

$$\sum_{\substack{j \text{ s.t.} \\ c_{ij}<0}} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}-1} \mathrm{sgn}(i,j)\, \varepsilon_i^{f} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f} = 0$$

$$\mathrm{sgn}(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \Omega \\ -1 & \text{if } (i,j) \notin \Omega \end{cases}$$

__Note:__ (P2) and (P3) come from the potential:

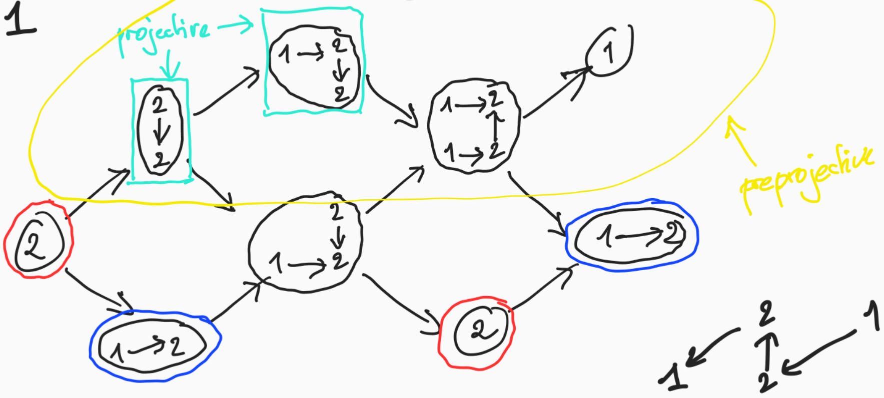
$$S(C,\Omega) = \sum_{\substack{i\to j \in \overline{Q}_1 \\ i \neq j}} \sum_{g=1}^{g_{ij}} \mathrm{sgn}(i,j)\, \varepsilon_i^{f_{ij}} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)}$$

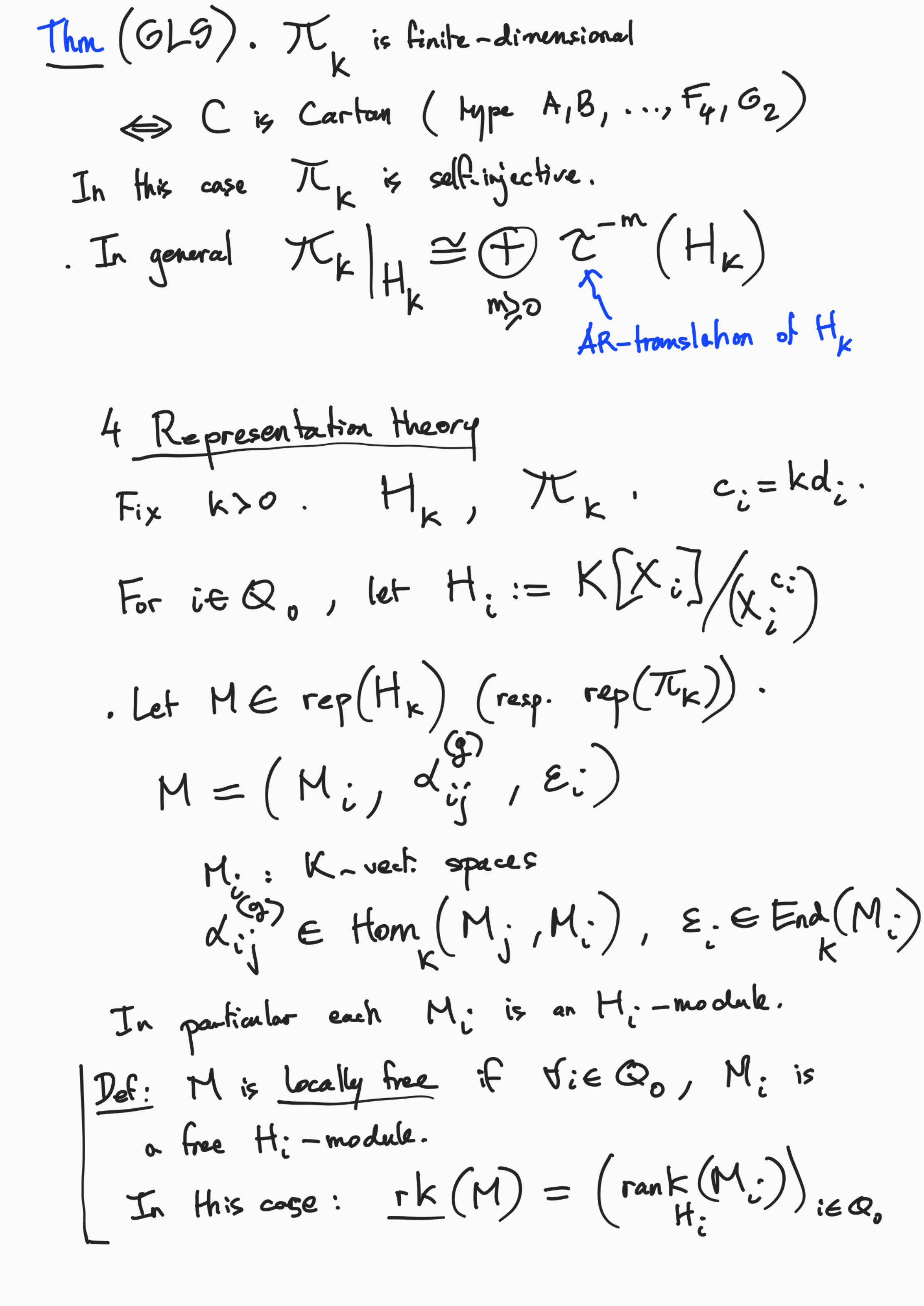
__Ex__ Type C2 $C = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\Omega = (2,1)$

$Q: \quad \varepsilon_1 \circlearrowright 1 \xrightarrow{\alpha_{21}} 2 \circlearrowleft \varepsilon_2$

$H_1: \quad 1 \xrightarrow{\alpha_{21}} 2 \circlearrowleft \varepsilon_2 \qquad \varepsilon_2^2 = 0$





Thm (GLS). Π_k is finite-dimensional

$\iff C$ is Cartan (type $A, B, \dots, F_4, G_2$)

In this case Π_k is self-injective.

. In general $\Pi_k|_{H_k} \cong \bigoplus_{m\geq 0} \tau^{-m}(H_k)$

AR-translation of H_k

4 Representation theory

Fix $k>0$. H_k, Π_k. $c_i = kd_i$.

For $i \in Q_0$, let $H_i := K[X_i]/(X_i^{c_i})$

. Let $M \in \mathrm{rep}(H_k)$ (resp. $\mathrm{rep}(\Pi_k)$).

$$M = (M_i, \alpha_{ij}^{(g)}, \varepsilon_i)$$

M_i : K-vect. spaces

$\alpha_{ij}^{(g)} \in \mathrm{Hom}_K(M_j, M_i)$, $\varepsilon_i \in \mathrm{End}_K(M_i)$

In particular each M_i is an H_i-module.

Def: M is locally free if $\forall i \in Q_0$, M_i is a free H_i-module.

In this case: $\underline{\mathrm{rk}}(M) = (\mathrm{rank}_{H_i}(M_i))_{i \in Q_0}$

4_1 Representations of H_K

Prop: Let $M \in \mathrm{rep}(H_K)$. Then M is locally free iff
$(\text{proj dim } M \leqslant 1) \Leftrightarrow (\text{inj dim } M \leqslant 1)$
$\Leftrightarrow (\text{proj dim } M < \infty) \Leftrightarrow (\text{inj dim } M < \infty)$

H_K is an Iwanaga-Gorenstein algebra of dim 1.

⇝ $\mathrm{rep}_{l.f.}(H_K)$ is "hereditary" but not abelian, only exact.

Def: M is rigid if $\mathrm{Ext}^1_H(M,M)=0$.

Thm: (GLS)

- There are finitely many isoclasses of indecomposable locally free rigid modules iff C is a Cartan matrix.
- In that case, $M \longmapsto \underline{rk}(M)$ gives a bijection with the positive roots $\Delta^+(C)$.
- In general there is a bijection between isoclasses of indec. locally free rigid modules of H_K and the of real Schur roots of (C, Ω).

4_2 Representations of $\Pi_K = \Pi$

Definition: $M \in \mathrm{rep}(\Pi)$. We say that M is E-filtered if it has a filtration where layers are isomorphic to

$$\underset{\overset{\nearrow}{\text{"generalized simples"}}}{E_i} := \text{unique loc. free } \pi\text{-module with rank } (0,\dots,0,\underset{\overset{\uparrow}{i}}{1},0,\dots,0)$$

This a generalization of "nilpotent modules"!

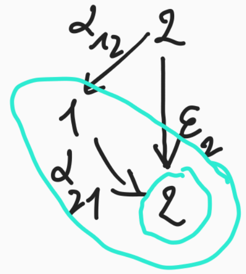
$\rightsquigarrow \mathrm{nil}_E(\pi)$

Clearly: if M is E-filtered, it is locally free.

The converse is false:

Ex: type C_2 minimal symmetrizer.

$X =$



is locally free

not E-filtered.

$(c_1=1, c_2=2)$

Def: Let $M \in \mathrm{rep}(\pi)$. Let $i \in Q_0$.

- $\mathrm{fac}_i(M)$: largest quotient module of M supported on vertex i for some k.

  We have a s.e.s. $K_i(M) \hookrightarrow M \twoheadrightarrow \mathrm{fac}_i(M)$

- $\mathrm{sub}_i(M)$: ——— submodule of M

$$\mathrm{sub}_i(M) \hookrightarrow M \twoheadrightarrow C_i(M).$$

Def: (Crystal module) $M \in \operatorname{rep}(\Pi)$ is a crystal module iff:

- $M = \{0\}$

  or

- $\operatorname{sub}_i(M)$ and $\operatorname{fac}_i(M)$ are free H_i-modules, $K_i(M)$ and $C_i(M)$ are crystal modules.

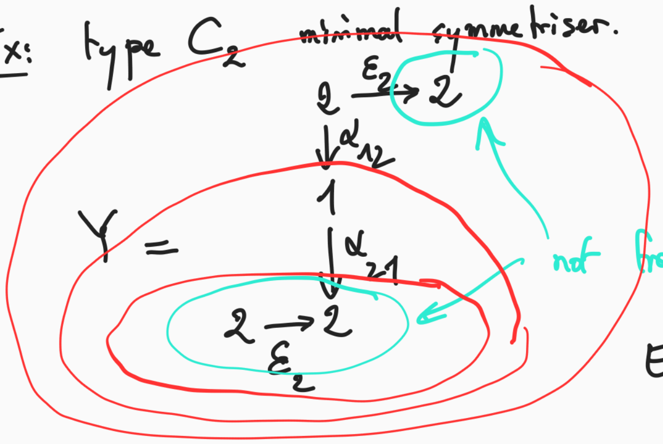
for every $i \in Q_0$

Rk: . E_i is crystal.

. If M is a crystal it is E-filtered.

But the converse is false :

Ex: type C_2 minimal symmetriser.



$\operatorname{sub}_1(Y) = \{0\}$
$\operatorname{fac}_1(Y) = \{0\}$

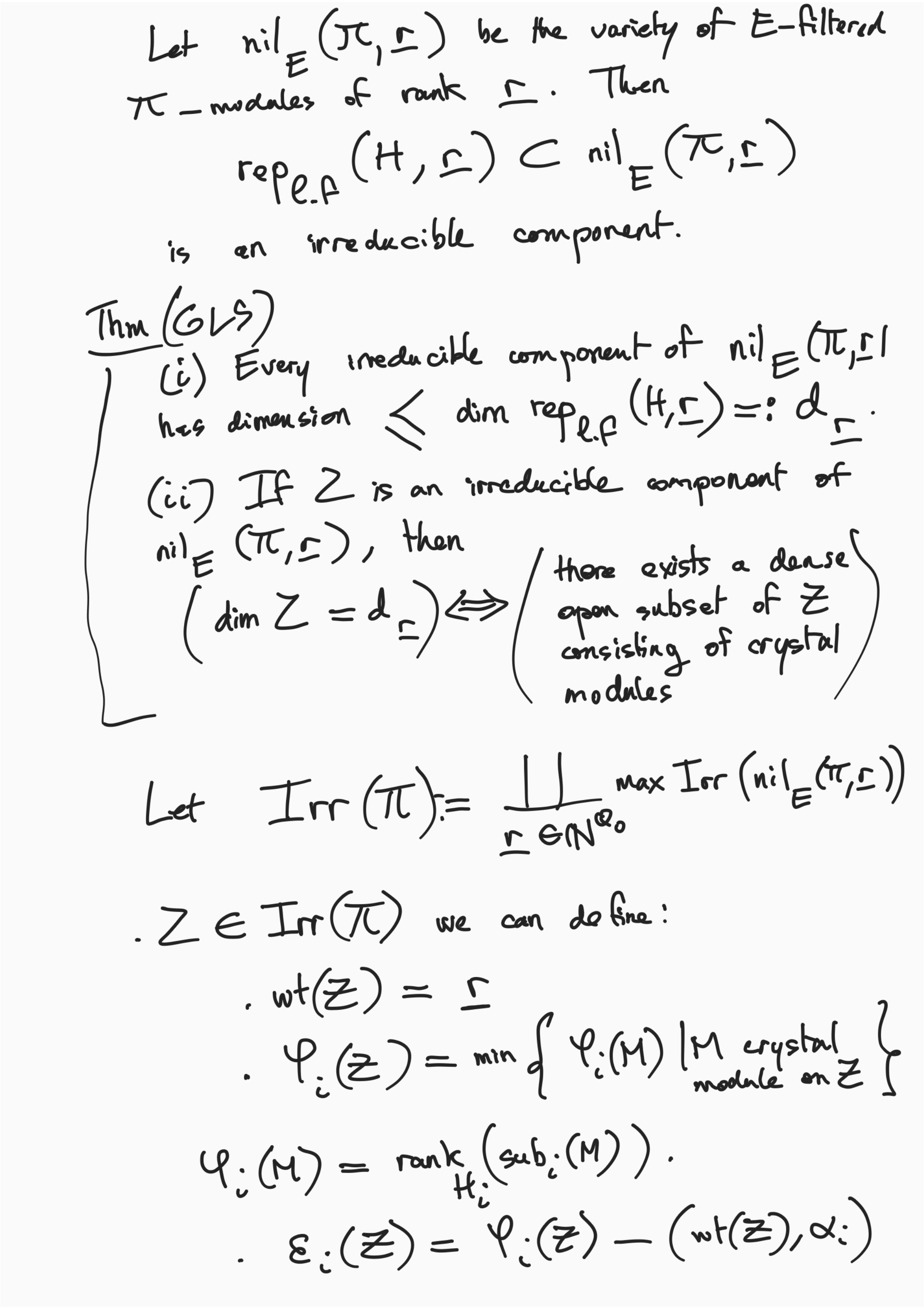
E-filtered.

4-3 Representation varieties

K is alg. closed.

$\underline{r} \in \mathbb{N}^{Q_0}$ a rank vector.

Prop: $\operatorname{rep}_{\text{l.f.}}(H, \underline{r})$ is smooth and irreducible.



Let $\mathrm{nil}_E(\pi,\underline{r})$ be the variety of E-filtered π-modules of rank $\underline{r}$. Then

$$\mathrm{rep}_{l.f}(H,\underline{r}) \subset \mathrm{nil}_E(\pi,\underline{r})$$

is an irreducible component.

Thm (GLS)

(i) Every irreducible component of $\mathrm{nil}_E(\pi,\underline{r}|$ has dimension $\leqslant \dim \mathrm{rep}_{l.f}(H,\underline{r}) =: d_{\underline{r}}$.

(ii) If Z is an irreducible component of $\mathrm{nil}_E(\pi,\underline{r})$, then

$$\left(\dim Z = d_{\underline{r}}\right) \Longleftrightarrow \left(\begin{array}{l}\text{there exists a dense}\\ \text{open subset of } Z\\ \text{consisting of crystal}\\ \text{modules}\end{array}\right)$$

Let $$\mathrm{Irr}(\pi) := \coprod_{\underline{r}\in\mathbb{N}^{Q_0}} \max \mathrm{Irr}\left(\mathrm{nil}_E(\pi,\underline{r})\right)$$

. $Z \in \mathrm{Irr}(\pi)$ we can define:

. $\mathrm{wt}(Z) = \underline{r}$

. $\varphi_i(Z) = \min\left\{ \varphi_i(M) \,\middle|\, \begin{array}{c}M \text{ crystal}\\ \text{module on } Z\end{array}\right\}$

$$\varphi_i(M) = \mathrm{rank}_{H_i}\left(\mathrm{sub}_i(M)\right).$$

. $\varepsilon_i(Z) = \varphi_i(Z) - \left(\mathrm{wt}(Z), \alpha_i\right)$

- $\left.\begin{array}{l}\tilde{e}_i(Z)\\ \tilde{f}_i(Z)\end{array}\right\} \in \mathrm{Irr}(\pi)$ defined by some bundle onstructions similur to Lusztig

Thm (GLS).

$$(\mathrm{Irr}(\pi), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i) \cong \mathcal{B}(-\infty) \text{ "crystal of } U_q^+(\mathfrak{g})\text{"}$$

Rk: if C is Cartan, more explicit description of $\mathrm{Irr}(\pi)$.

Calabi-Yau algebras and canonical bundles

Akishi Ikeda

Abstract. In this talk, we explore common features of Calabi-Yau algebras in representation theory and Calabi-Yau manifolds in geometry. First we see the role of inverse dualizing complexes in both sides and recall why Calabi-Yau algebras are called “Calabi-Yau” algebras. Next we see that the Calabi-Yau completions (derived preprojective algebras) can be interpreted asthe total spaces of canonical bundles of smooth varieties. If time permits, I explain the relationship between double gradings of Calabi-Yau completions and weights of torus actions on canonical bundles. This is a joint work with Yu Qiu.

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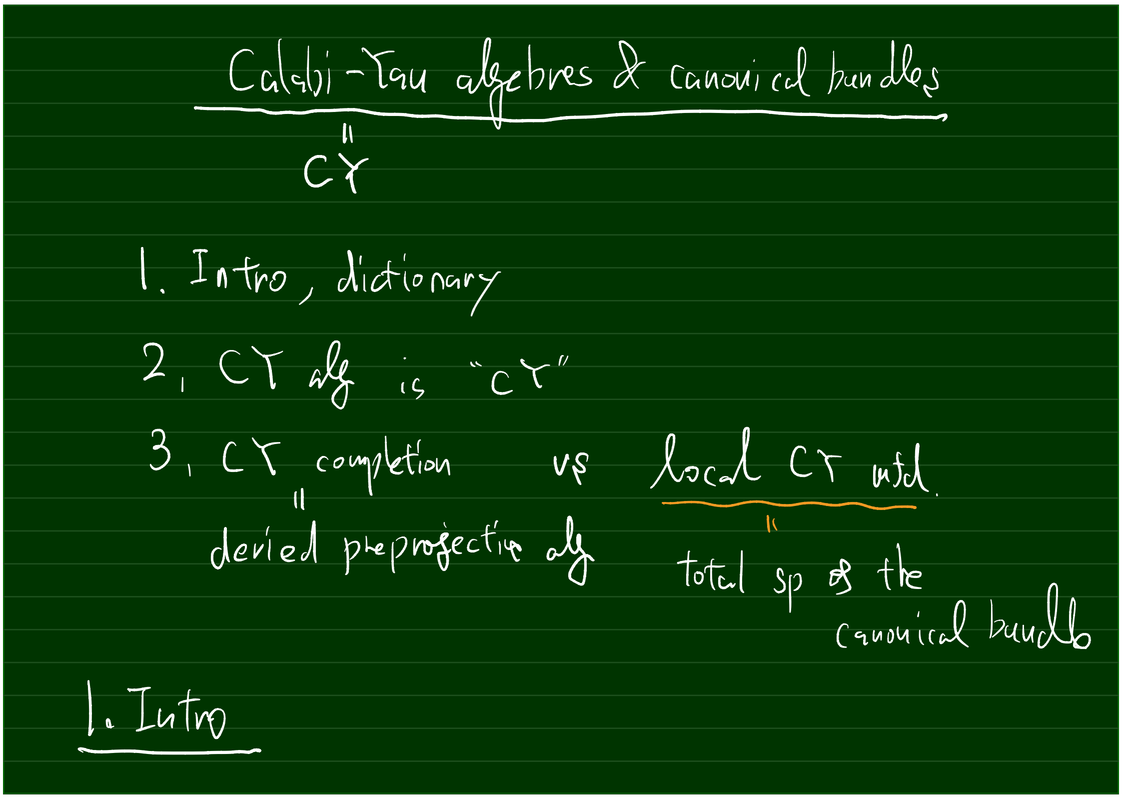
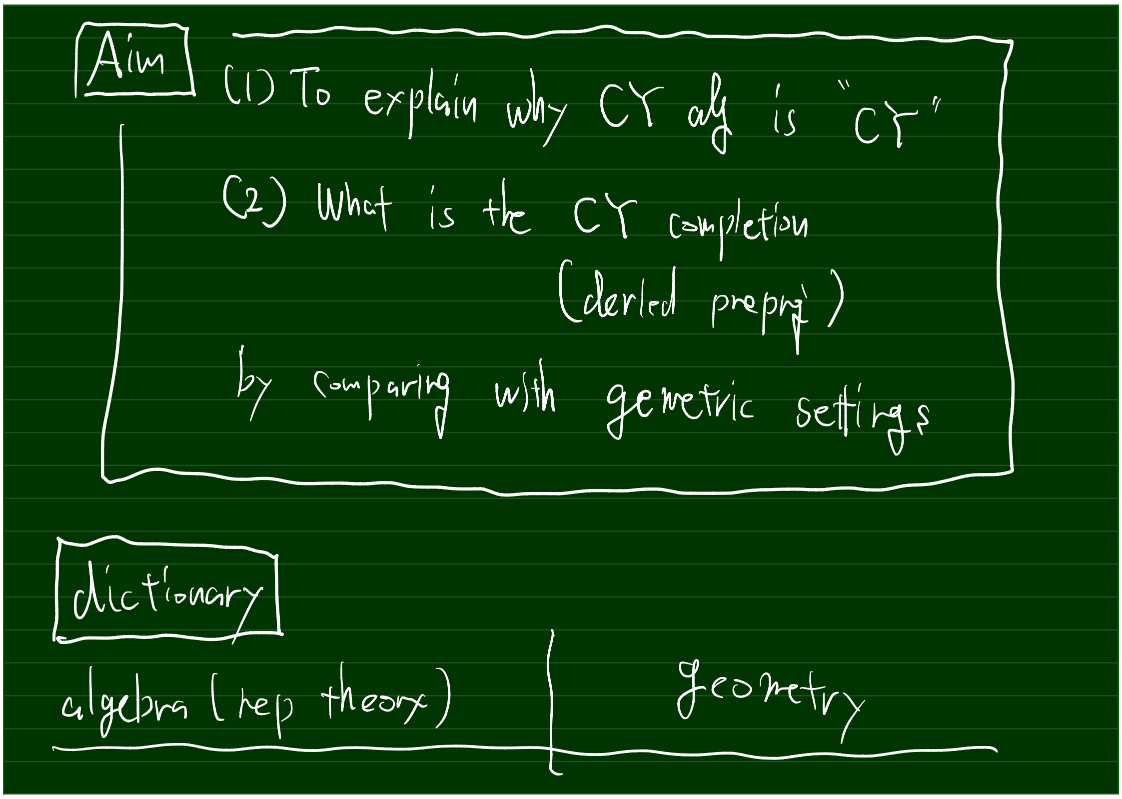


$\underset{\text{CY}}{\underset{\shortparallel}{\text{Calabi-Yau}}}$ algebras & canonical bundles

1. Intro, dictionary
2. CY alg is "CY"
3. $\underset{\text{derived preprojective alg}}{\underset{\shortparallel}{\text{CY completion}}}$ vs $\underset{\text{total sp of the canonical bundle}}{\underset{\shortparallel}{\text{local CY mfd.}}}$

1. Intro



Aim

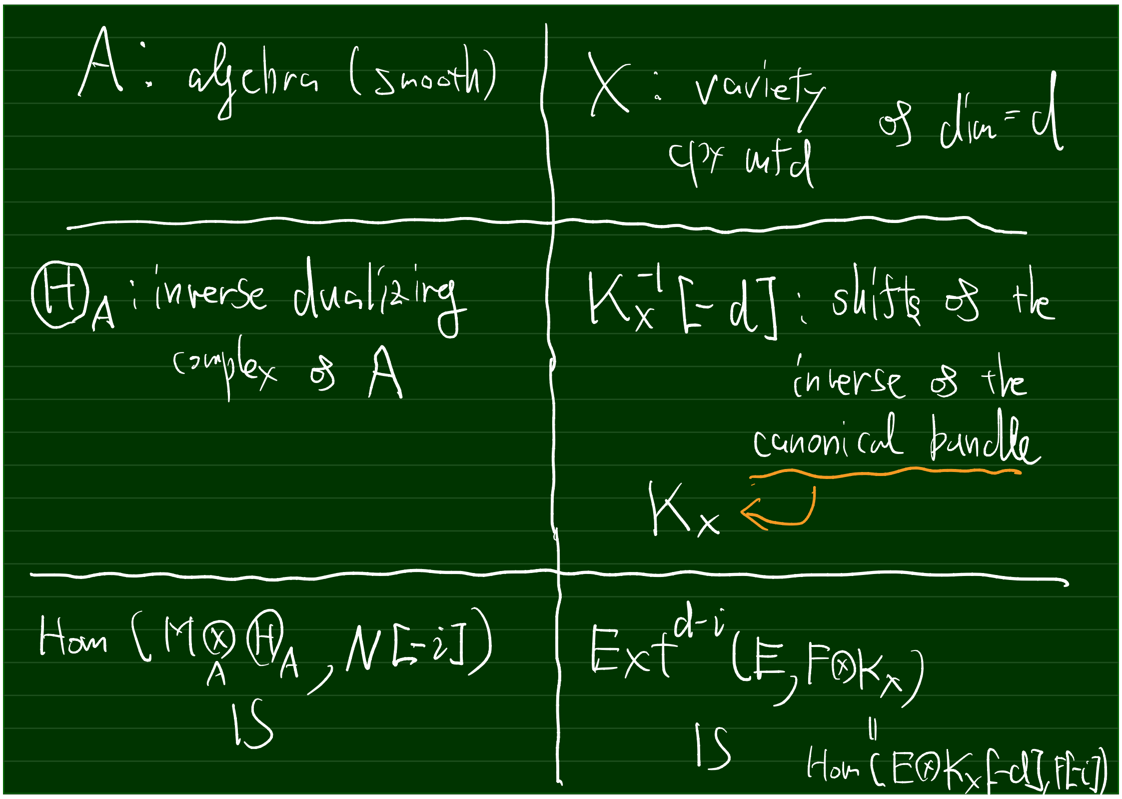
(1) To explain why CY alg is "CY"

(2) What is the CY completion (derived preproj)

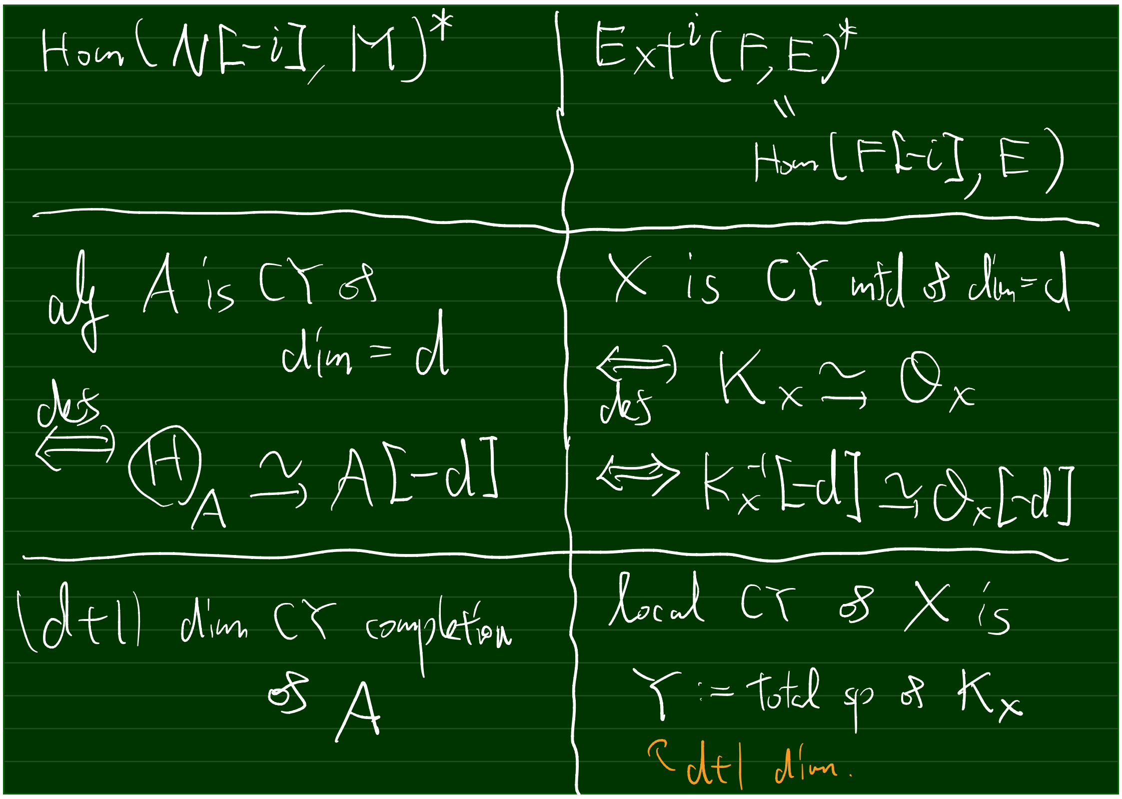
by comparing with geometric settings

dictionary

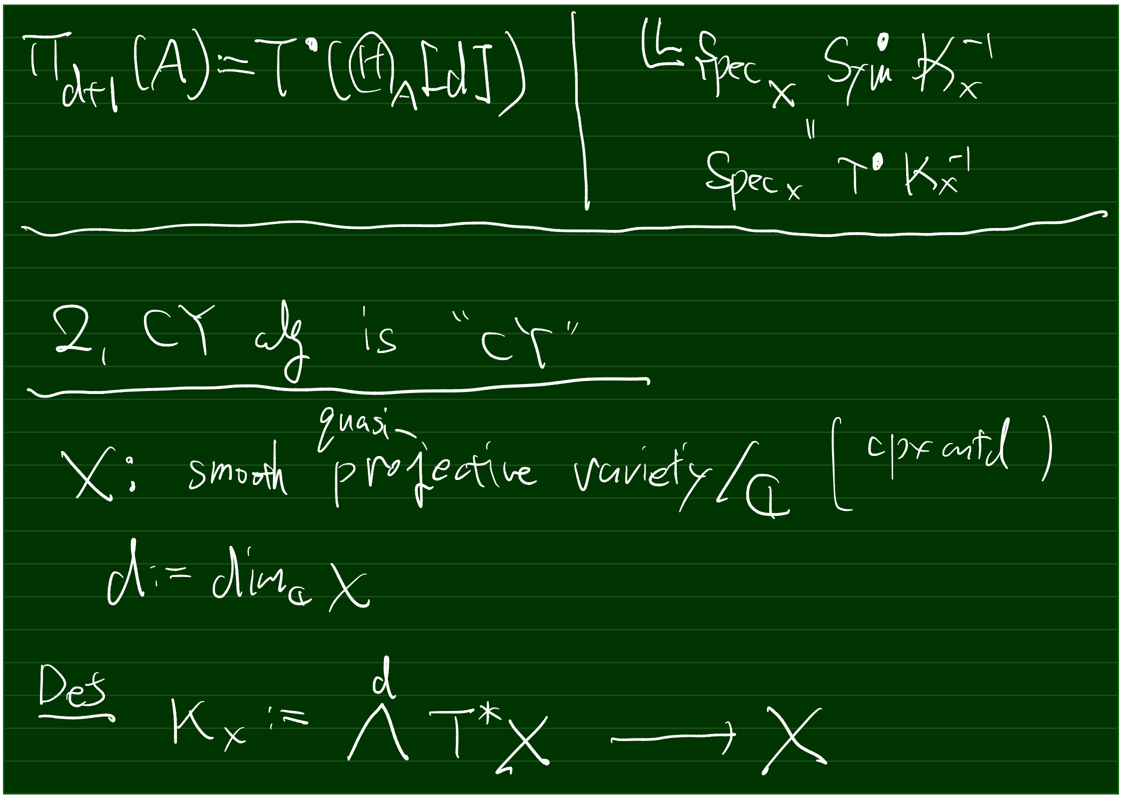
algebra (rep theory)	geometry



A: algebra (smooth)	X: variety of dim $= d$ cpx mfd
Θ_A: inverse dualizing complex of A	$K_X^{-1}[-d]$: shifts of the inverse of the $\underline{\text{canonical bundle}}$ K_X
$\mathrm{Hom}(M \otimes_A \Theta_A, N[-i])$ $\wr$	$\mathrm{Ext}^{d-i}(E, F \otimes K_X)$ $\wr \qquad | $ $\mathrm{Hom}(E \otimes K_X[-d], F[-i])$



$\mathrm{Hom}(N[-i], M)^*$	$\mathrm{Ext}^i(F, E)^*$ $\|$ $\mathrm{Hom}(F[-i], E)$
alg A is CY of dim $= d$ $\overset{\text{def}}{\Longleftrightarrow}$ $\Theta_A \xrightarrow{\sim} A[-d]$	X is CY mfd of dim $= d$ $\underset{\text{def}}{\Longleftrightarrow}$ $K_X \xrightarrow{\sim} \mathcal{O}_X$ $\Longleftrightarrow$ $K_X^{-1}[-d] \xrightarrow{\sim} \mathcal{O}_X[-d]$
$(d+1)$ dim CY completion of A	local CY of X is $Y :=$ total sp of K_X $d+1$ dim.



$\Pi_{d+1}(A) = T^\bullet(\Theta_A[d])$ | $\leftrightsquigarrow \mathrm{Spec}_X \mathrm{Sym}^\bullet \mathcal{K}_X^{-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

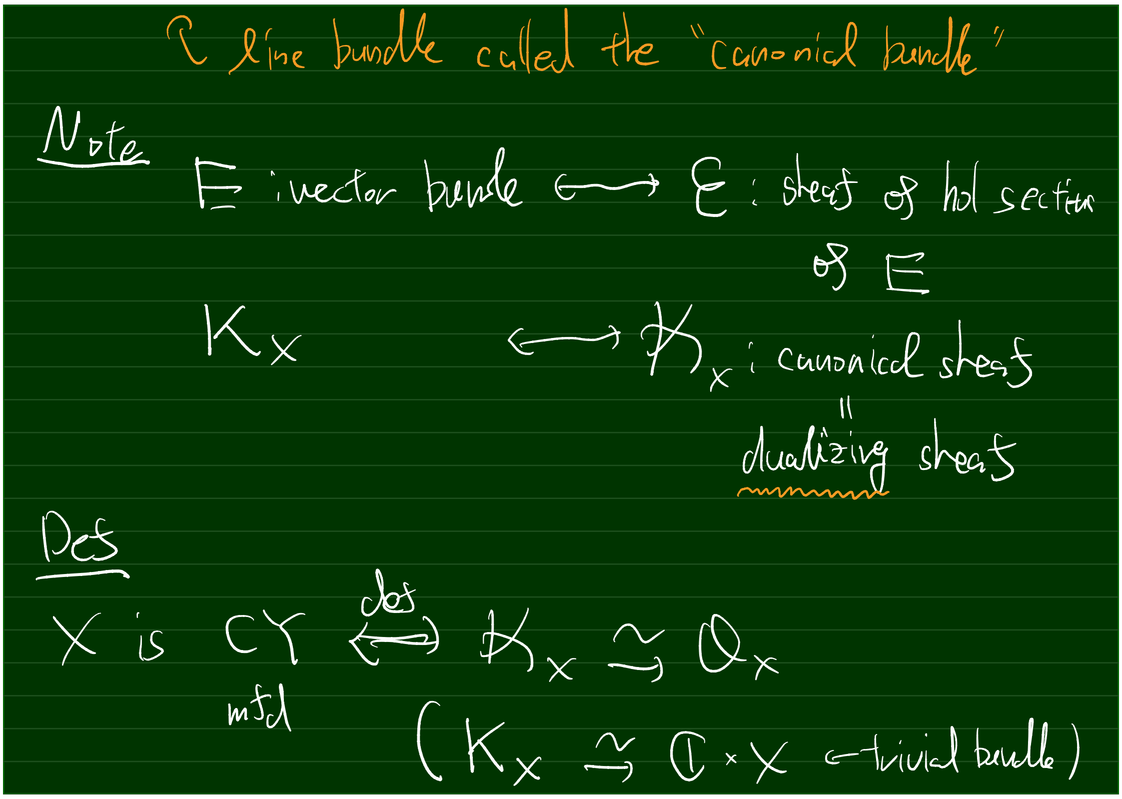
$\qquad\qquad\qquad\qquad\qquad \mathrm{Spec}_X T^\bullet K_X^{-1}$

2, CY alg is "CY"

X: smooth quasi-projective variety/$\mathbb{C}$ [cpx mfd]

$d := \dim_{\mathbb{C}} X$

Def $K_X := \bigwedge^d T^*X \longrightarrow X$



$\hookrightarrow$ line bundle called the "canonical bundle"

Note E: vector bundle $\longleftrightarrow$ $\mathcal{E}$: sheaf of hol section of E

$K_X \longleftrightarrow \mathcal{K}_X$: canonical sheaf

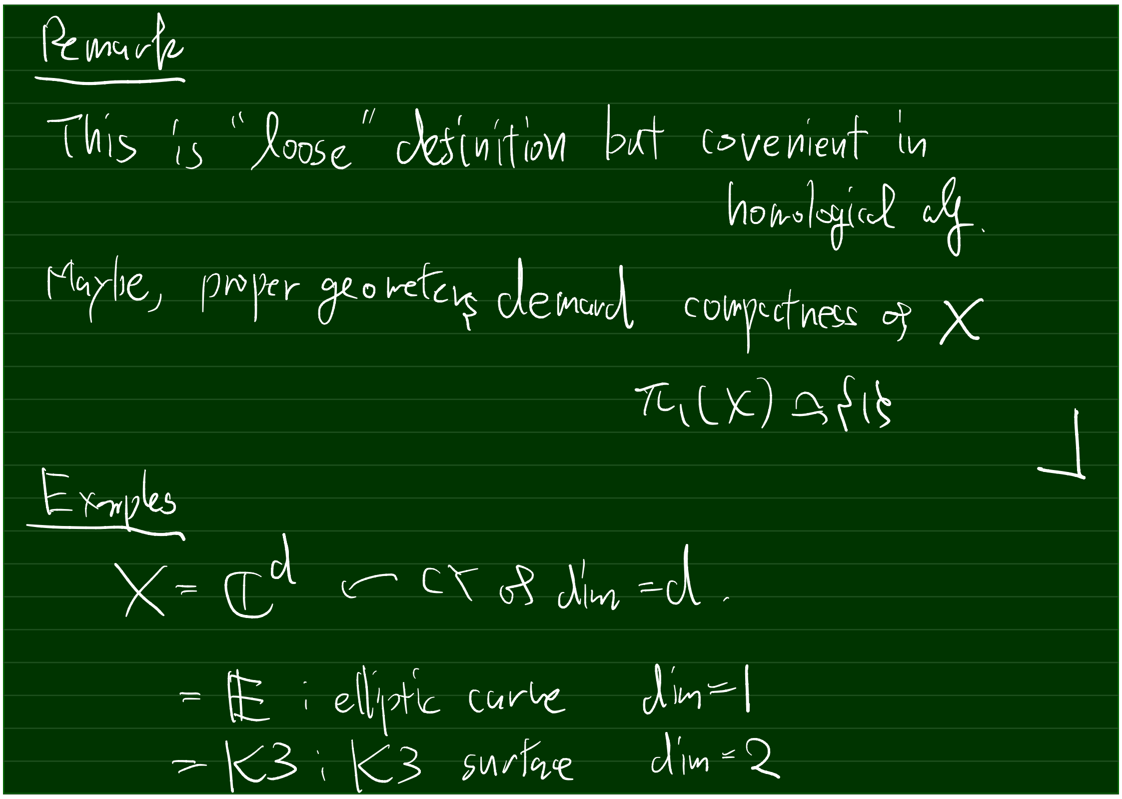
$\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad$ dualizing sheaf

Def

X is CY mfd $\overset{\text{def}}{\iff} \mathcal{K}_X \cong \mathcal{O}_X$

$(K_X \cong \mathbb{C} \times X \leftarrow$ trivial bundle$)$



Remark

This is "loose" definition but covenient in homological alg.

Maybe, proper geometers demand compactness of X

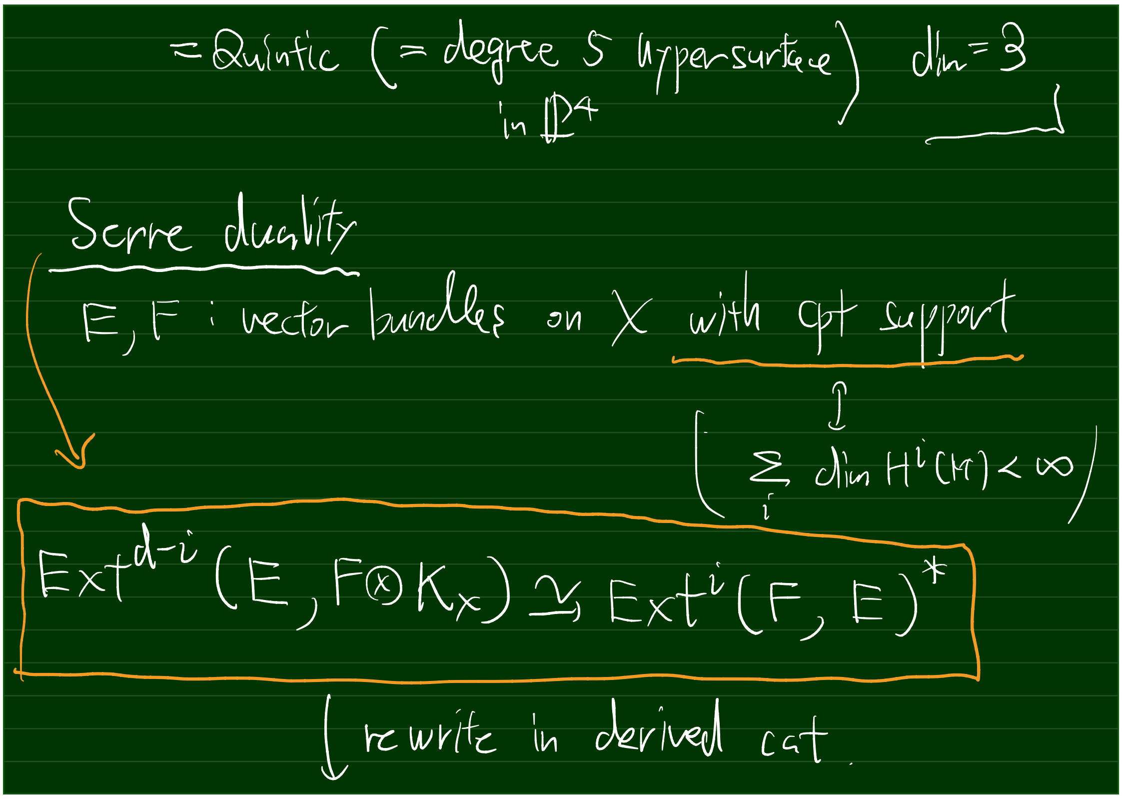
$\pi_1(X) \cong \{1\}$

Examples

$X = \mathbb{C}^d$ ← CY of dim $= d$.

$= \mathbb{E}$: elliptic curve dim$=1$

$= K3$: K3 surface dim$=2$



$=$ Quintic ($=$ degree 5 hypersurface in $\mathbb{P}^4$) dim$=3$

Serre duality

E, F : vector bundles on X with cpt support

$\Updownarrow$

$\left(\sum_i \dim H^i(M) < \infty \right)$

$$\boxed{\mathrm{Ext}^{d-i}(E, F \otimes K_X) \simeq \mathrm{Ext}^i(F, E)^*}$$

$\downarrow$ rewrite in derived cat.

$$\mathrm{Hom}(E, F[-i]\otimes K_X[d]) \xrightarrow{\sim} \mathrm{Hom}(F, E[i])^*$$

$$\mathrm{Hom}(E\otimes \underbrace{K_X^{-1}[-d]}, F[-i]) \xrightarrow{\sim} \mathrm{Hom}(\underbrace{F[-i]}_{\overset{\shortparallel}{F'}}, E)^*$$

$\boxed{\text{if } K_X \simeq \mathcal{O}_X}$)

↓ $\mathrm{Hom}(E[-d], F') \xrightarrow{\sim} \mathrm{Hom}(F', E)^*$

$\boxed{\text{Serre functor} = [d]}$

Inverse dualizing complex (≐ sheaf of the inverse of the canonical bundle)

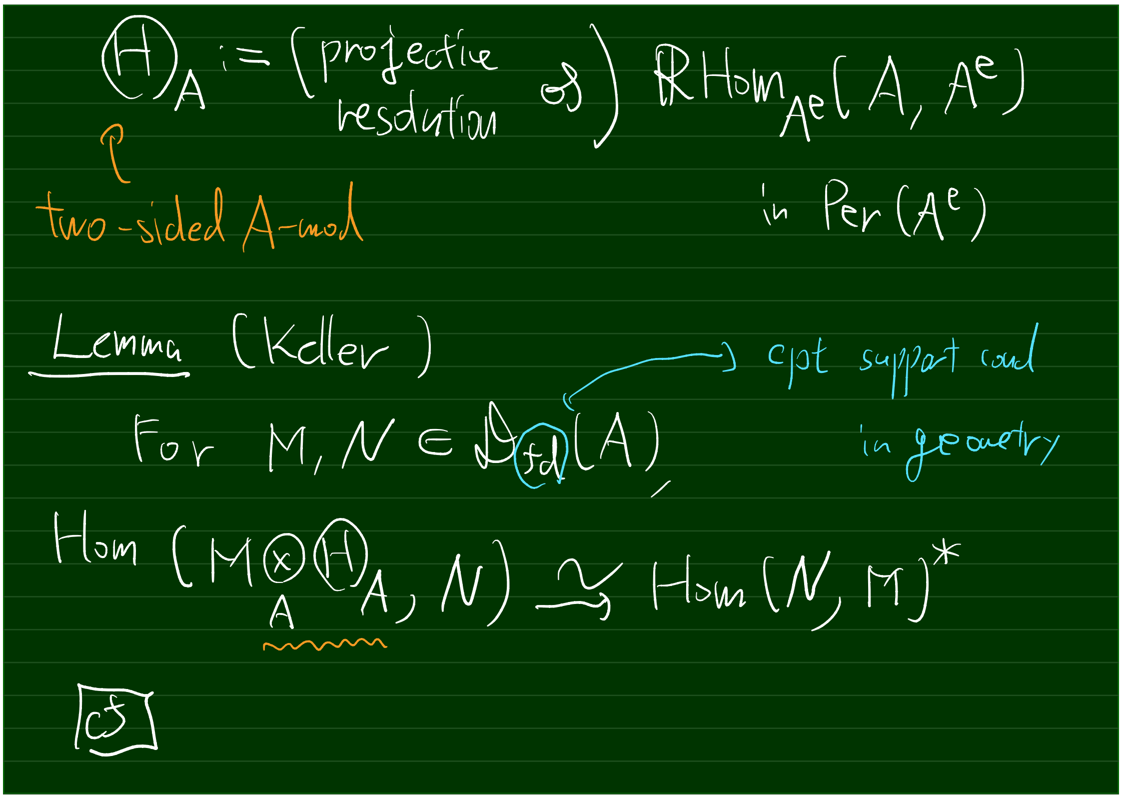
A: (smooth dg) algebra

(ex, A: alg with gdA $< \infty$)

$A^e := A^{op}\otimes A$

(A-mod = right A-modules
A^{op}-mod = left "
A^e-mod = two-sided ")

Def
The dualizing complex of A is defined by



$\textcircled{H}_A := (\text{projective resolution of}) \ \mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$

two-sided A-mod

in $\mathrm{Per}(A^e)$

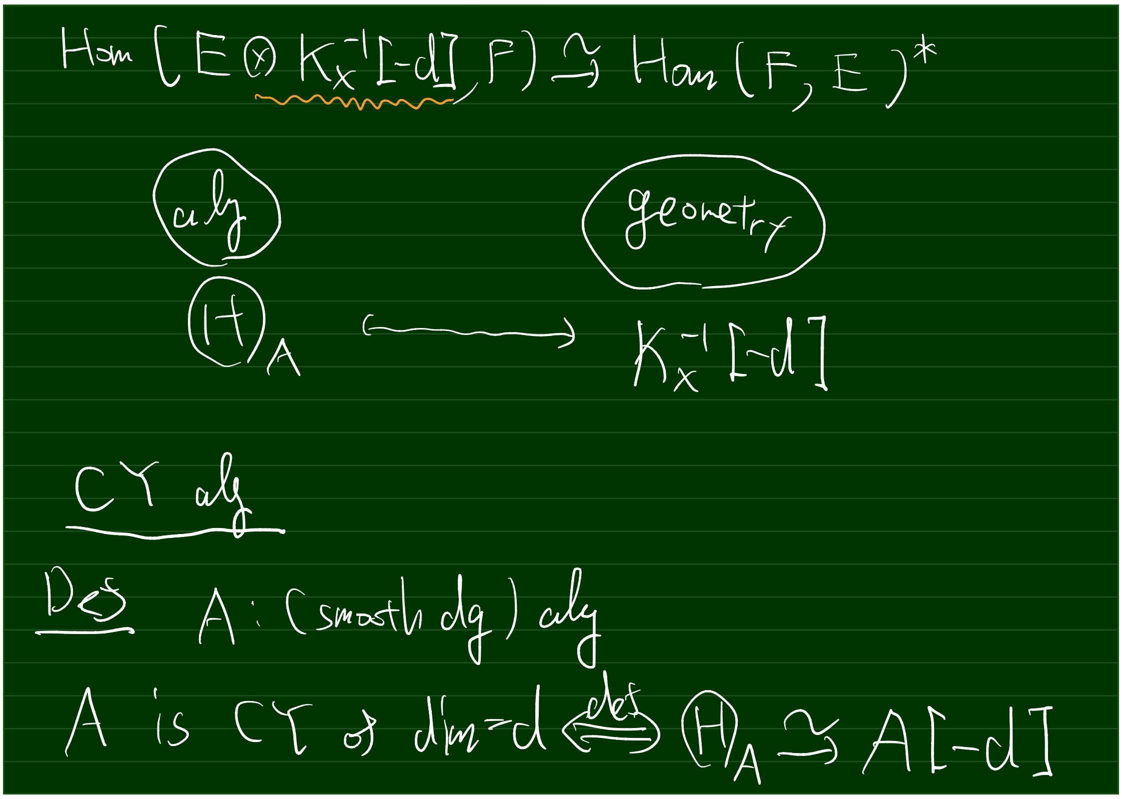
Lemma (Keller)

For $M, N \in \mathcal{D}_{fd}(A)$,

cpt support coh in geometry

$$\mathrm{Hom}(\underset{A}{M \otimes} \textcircled{H}_A, N) \xrightarrow{\sim} \mathrm{Hom}(N, M)^*$$

$\boxed{\text{cf}}$



$$\mathrm{Hom}(E \otimes K_X^{-1}[-d], F) \xrightarrow{\sim} \mathrm{Hom}(F, E)^*$$

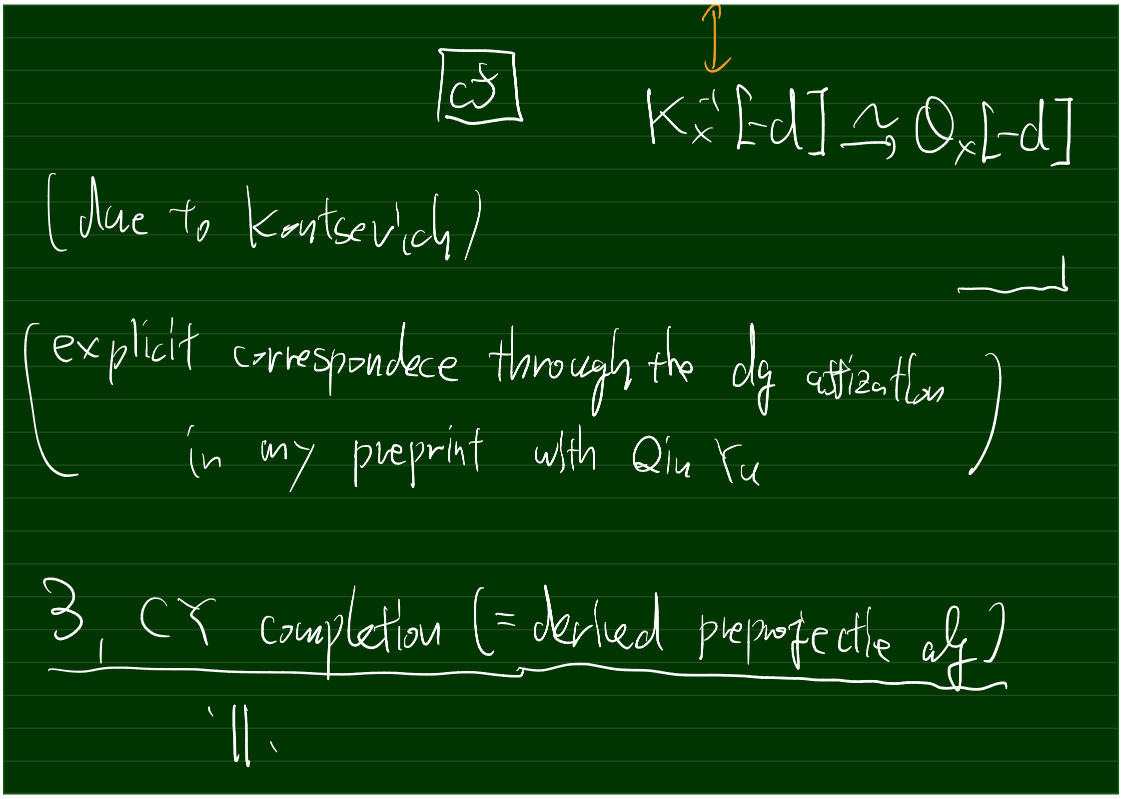
alg geometry

$$\textcircled{H}_A \longleftrightarrow K_X^{-1}[-d]$$

CY alg

Def A: (smooth dg) alg

A is CY of $\dim = d \overset{\text{def}}{\iff} \textcircled{H}_A \xrightarrow{\sim} A[-d]$



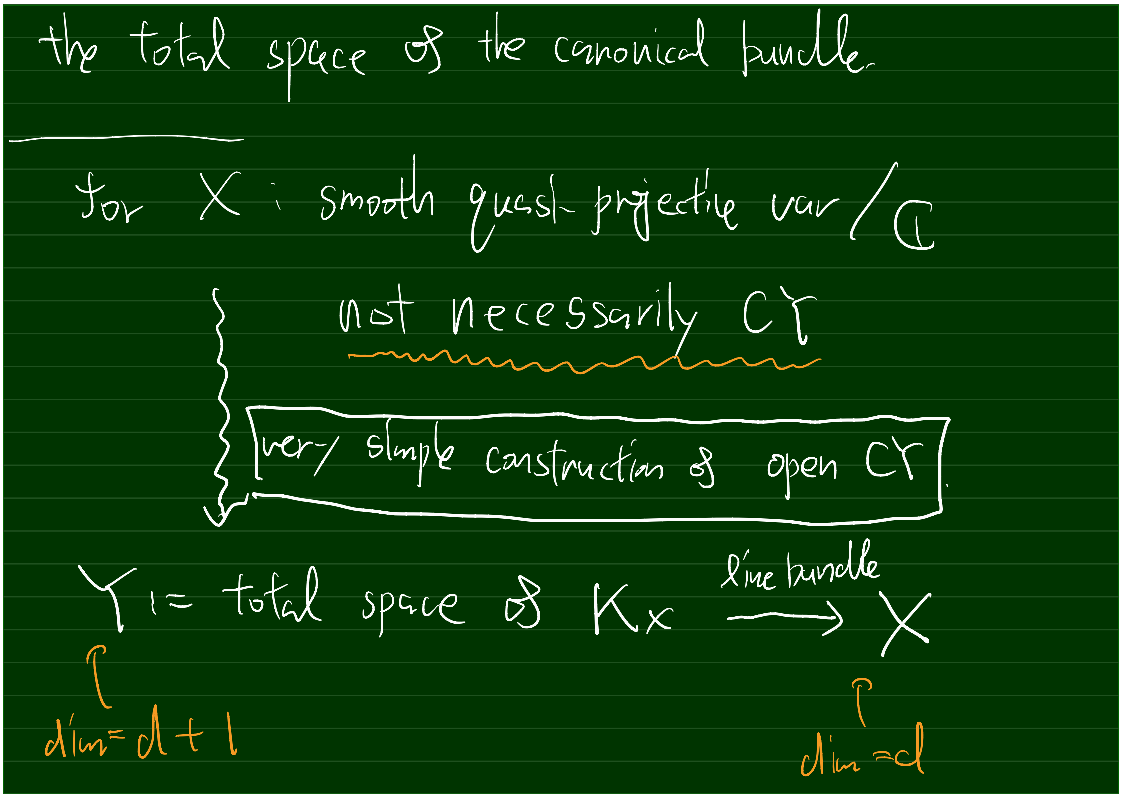
$\boxed{\text{cf}}$ $K_X^{-1}[-d] \xrightarrow{\sim} \mathcal{O}_X[-d]$

(due to Kontsevich)

(explicit correspondece through the dg affinization in my preprint with Qiu Yu)

3, CY completion (= derived preprojective alg)

$\parallel$



the total space of the canonical bundle.

for X : smooth quasi-projective var / $\mathbb{C}$

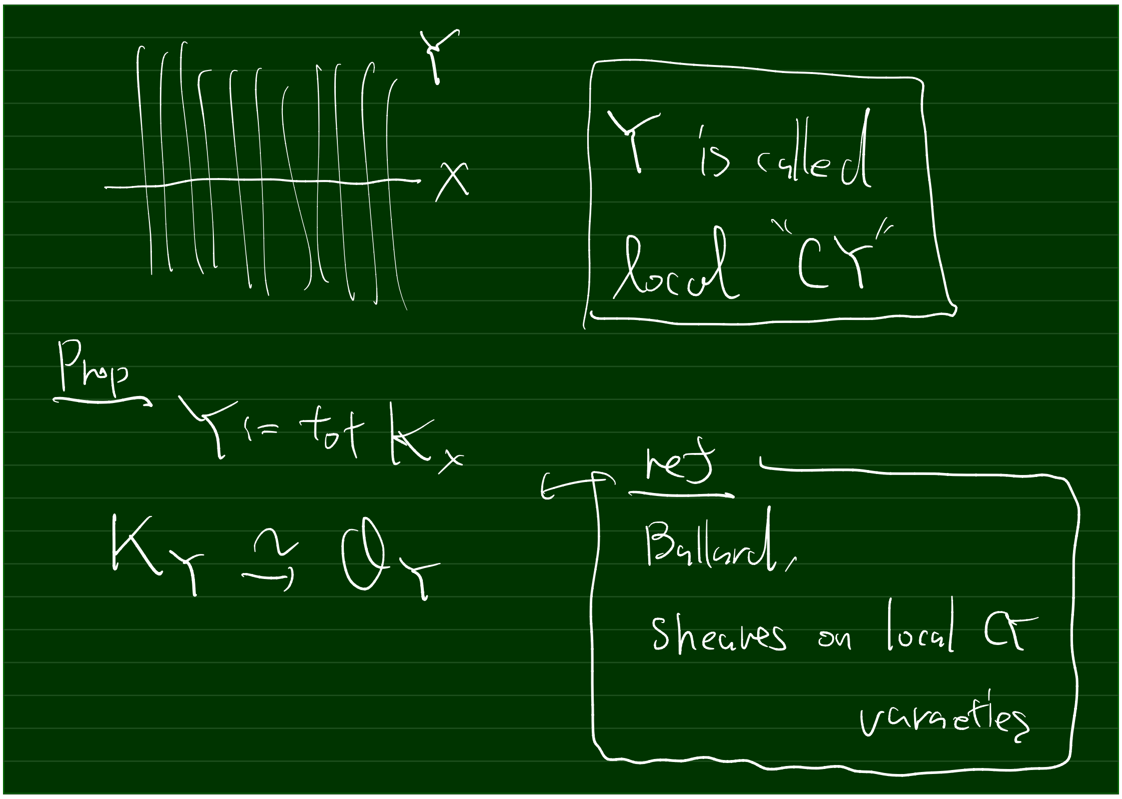
not necessarily CY

$\boxed{\text{very simple construction of open CY}}$

$Y :=$ total space of $K_X \xrightarrow{\text{line bundle}} X$

dim = d + 1

dim = d



Y is called local "CY"

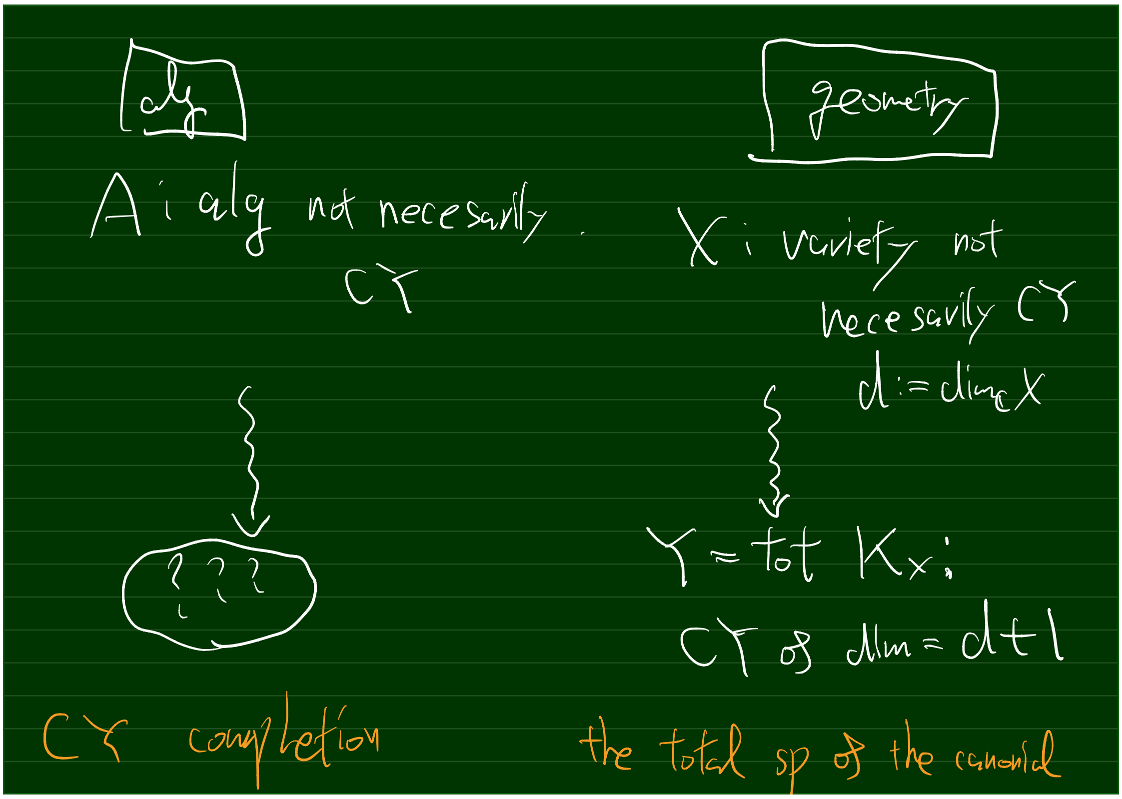
Prop

$Y := \mathrm{tot}\, K_X$

$K_Y \cong \mathcal{O}_Y$

ref

Ballard, Sheaves on local CY varieties



alg

A : alg not necesarily CY

???

geometry

X : variety not necesarily CY

$d := \dim_{\mathbb{C}} X$

$Y = \mathrm{tot}\, K_X$: CY of dim $= d+1$

CY completion

the total sp of the canonical

derived preprojective, bundle

Point

- V : vector sp $\rightsquigarrow \mathbb{V} := \operatorname{Spec} \operatorname{Sym}^\bullet \underset{\text{linear coord functions}}{\boxed{V^*}}$

$$V = \mathbb{C}^n \leftarrow \text{linear coord} = x_1, x_2, \cdots, x_n \qquad \mathbb{V} = \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \cdots, x_n]$$

- If $\dim V = 1$, $\operatorname{Sym}^\bullet V^* = T^\bullet V^*$ ← tensor alg.

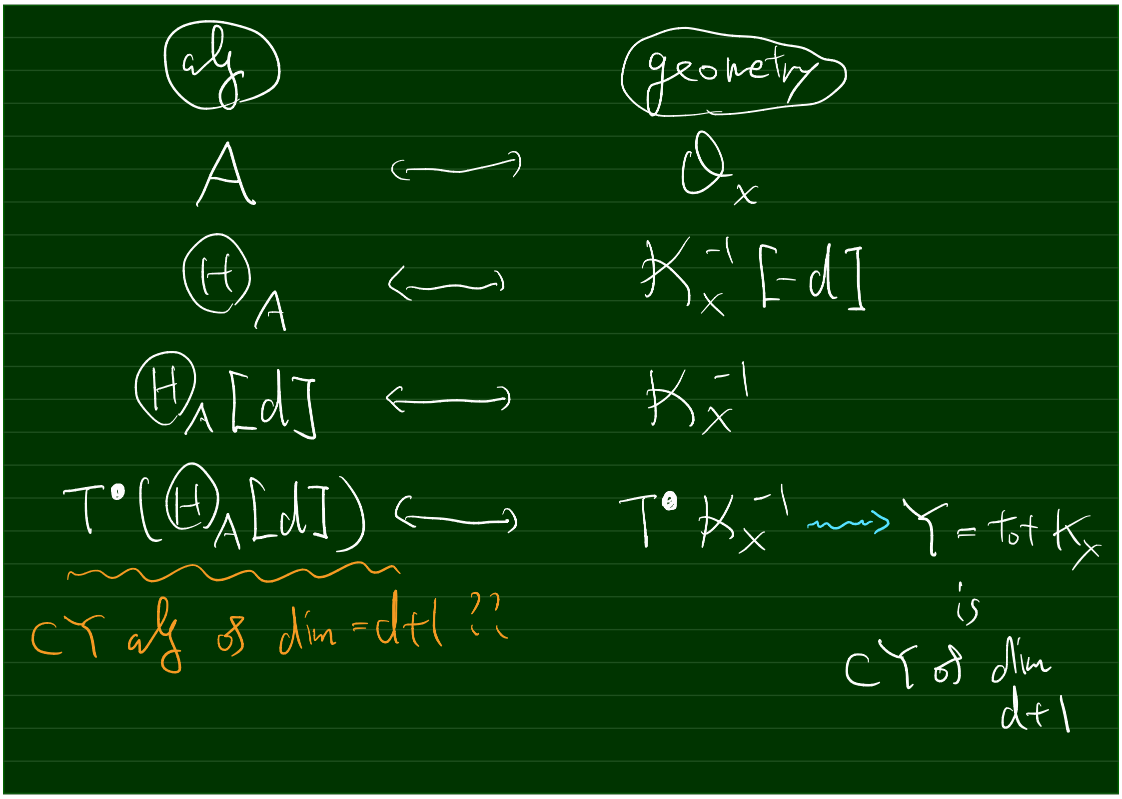
$E \longrightarrow X$: vector bundle, $\mathcal{E}$: sections.

$$\boxed{\operatorname{tot} E = \operatorname{Spec}_X \operatorname{Sym}^\bullet \mathcal{E}^*}$$

In particular,

$$\text{(local CY)}\ Y := \operatorname{tot} K_X = \operatorname{Spec}_X \operatorname{Sym}^\bullet \mathcal{K}_X^{-1} \simeq \operatorname{Spec}_X T^\bullet \mathcal{K}_X^{-1}$$

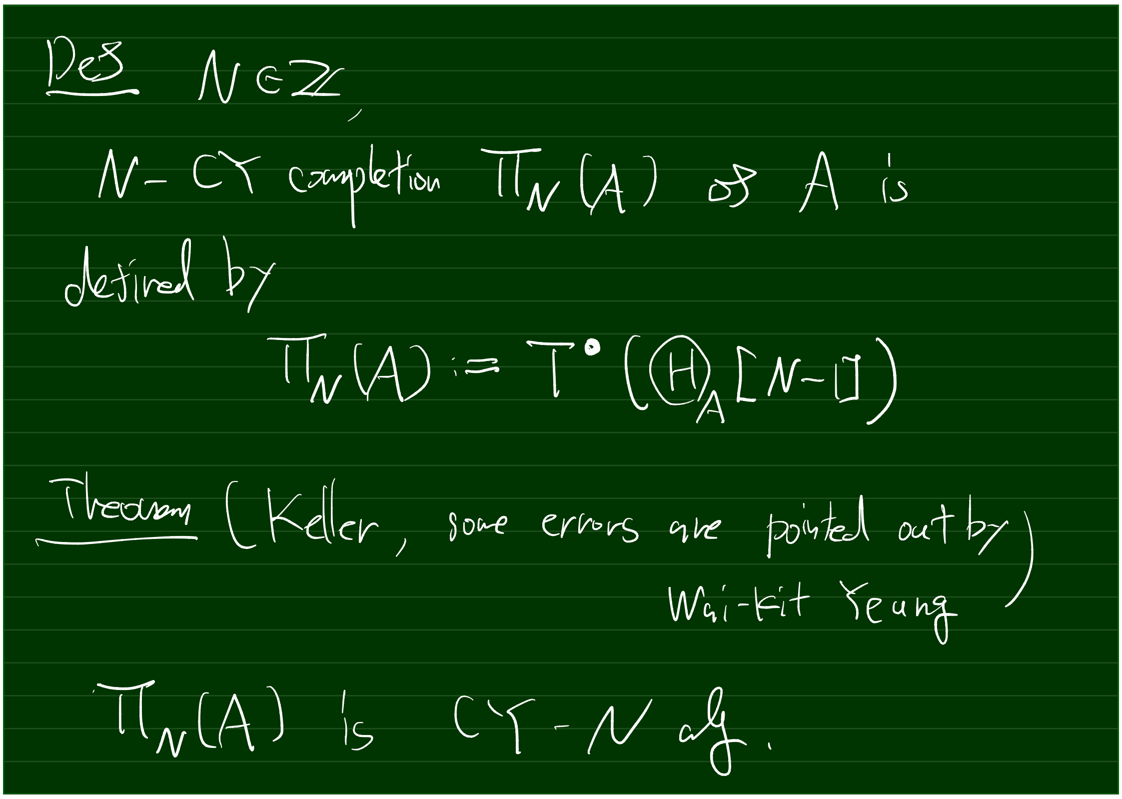
← K_X is rk $= 1$



alg		geometry
A	$\longleftrightarrow$	$\mathcal{O}_X$
Θ_A	$\longleftrightarrow$	$K_X^{-1}[-d]$
$\Theta_A[d]$	$\longleftrightarrow$	K_X^{-1}
$T^\bullet(\Theta_A[d])$	$\longleftrightarrow$	$T^\bullet K_X^{-1} \rightsquigarrow Y = \mathrm{tot}\, K_X$

$\underbrace{T^\bullet(\Theta_A[d])}$ CY alg of dim $= d+1$??

$Y = \mathrm{tot}\, K_X$ is CY of dim $d+1$



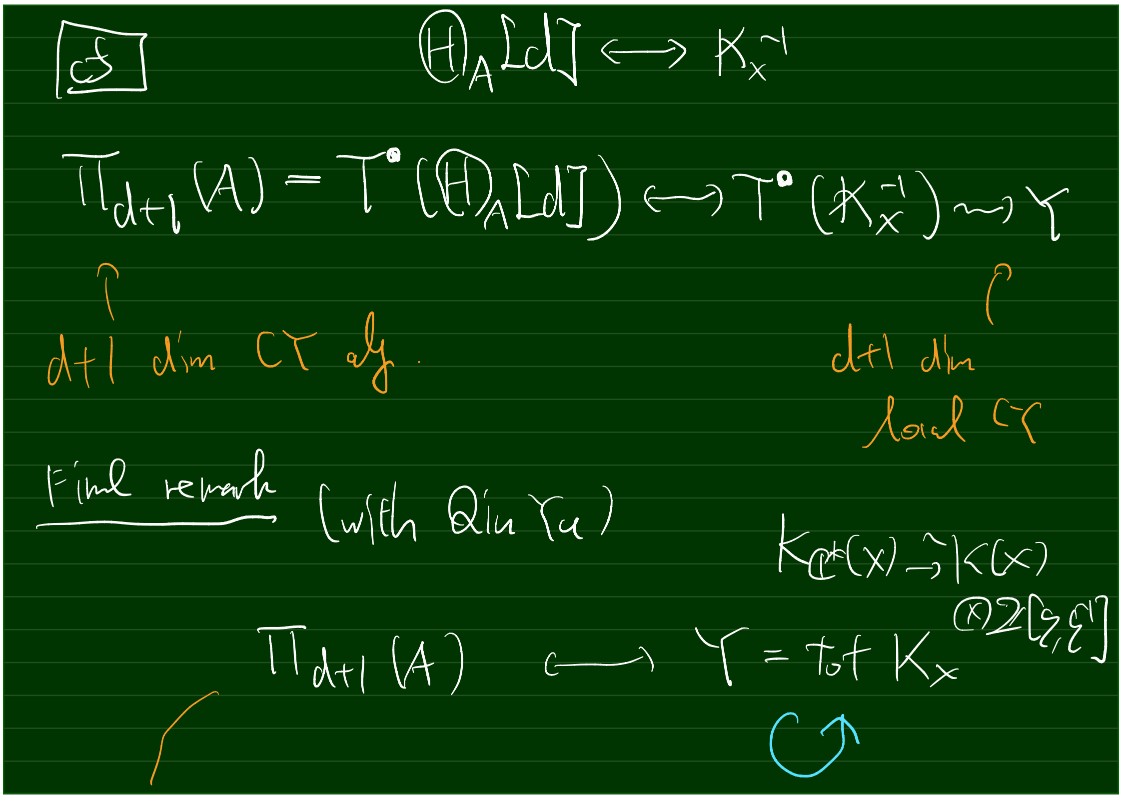
Def $N \in \mathbb{Z}$,

N-CY completion $\Pi_N(A)$ of A is defined by

$$\Pi_N(A) := T^\bullet(\Theta_A[N-1])$$

Theorem (Keller, some errors are pointed out by Wai-Kit Yeung)

$\Pi_N(A)$ is CY-N alg.



$\boxed{\text{cf}}$ $\Theta_A[d] \longleftrightarrow K_X^{-1}$

$\Pi_{d+1}(A) = T^\bullet(\Theta_A[d]) \longleftrightarrow T^\bullet(K_X^{-1}) \rightsquigarrow Y$

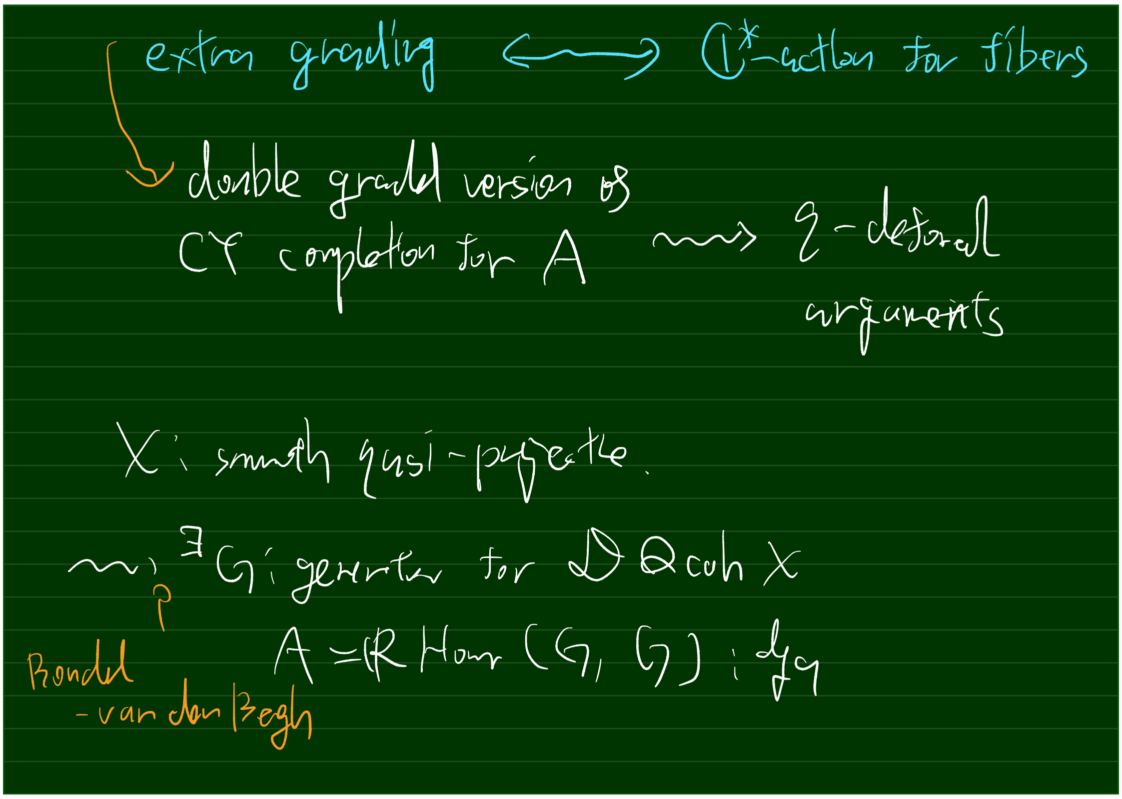
d+1 dim CY alg.

d+1 dim local CY

Final remark (with Qiu Yu)

$K_{\mathbb{C}^*}(X) \xrightarrow{\sim} K(X) \otimes \mathbb{Z}[q, q^{-1}]$

$\Pi_{d+1}(A) \longleftrightarrow Y = \mathrm{Tot}\, K_X \circlearrowleft$



extra grading ⟷ $\mathbb{C}^*$-action for fibers

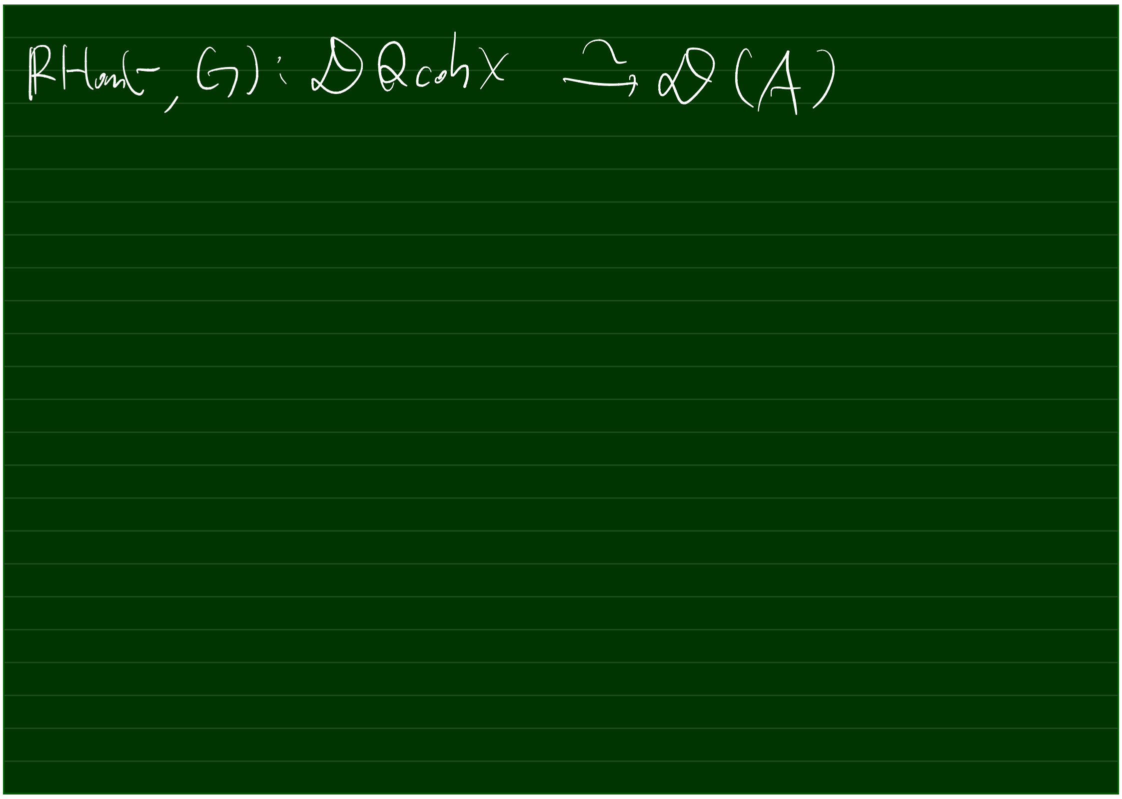
double graded version of CY completion for A ⇝ q-deformed arguments

X: smooth quasi-projective.

⇝ ∃G: generator for $D\,\mathrm{Qcoh}\,X$

Bondal - van den Bergh

$A = \mathbb{R}\mathrm{Hom}(G, G)$: dg



$$\mathbb{R}\mathrm{Hom}(-,G) : \mathcal{D}\,\mathrm{Qcoh}\,X \xrightarrow{\sim} \mathcal{D}(A)$$

Deformed Cartan matrices and generalized preprojective algebras

Kota Murakami

Abstract. E.Frenkel-Reshetikhin introduced a kind of quantization of the Cartan matrix, called the (q,t)-deformed Cartan matrix, which characterizes some aspects of the representation theory of affine quantum groups and deformed W-algebras with their specializations. In this talk, we will interpret the (q,t)-deformed Cartan matrix as an invariant from the representation theory of a bigraded version of the generalized preprojective algebra introduced by Geiss-Leclerc-Schröer. In particular, we will interpret several numerical properties of the (q,t)-deformed Cartan matrix as homological properties from the representation theory of the generalized preprojective algebra. This talk is based on a joint work with Ryo Fujita.

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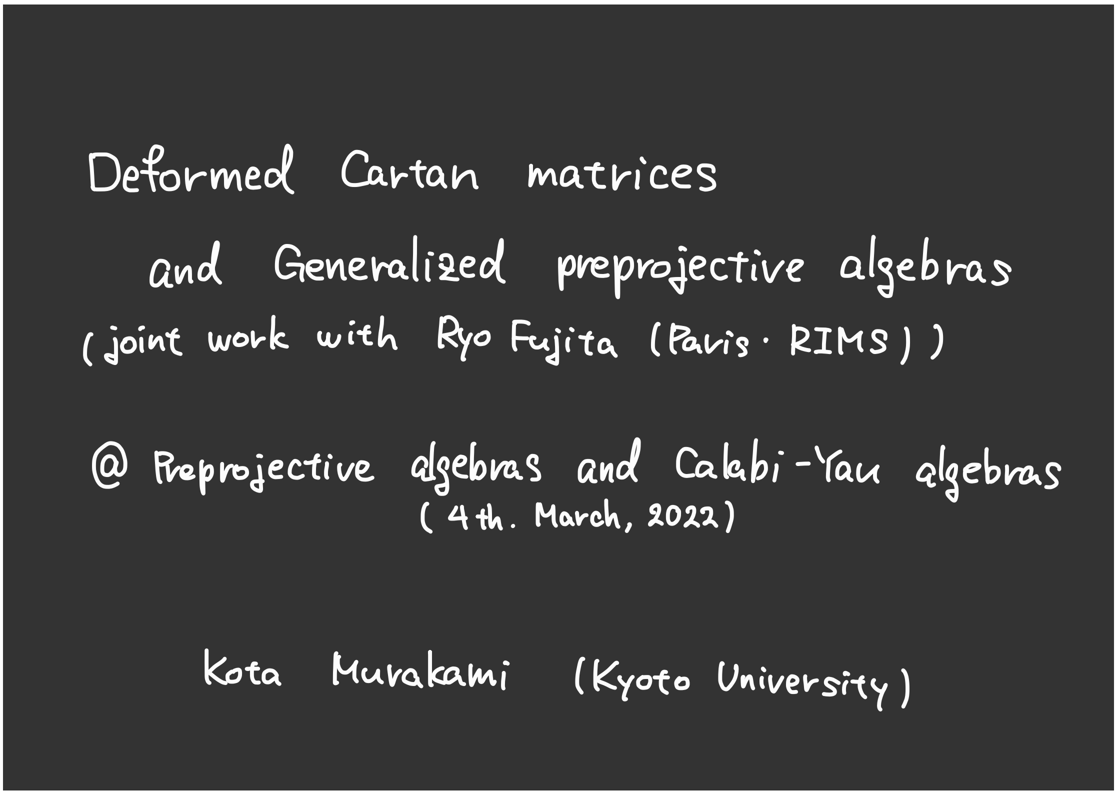
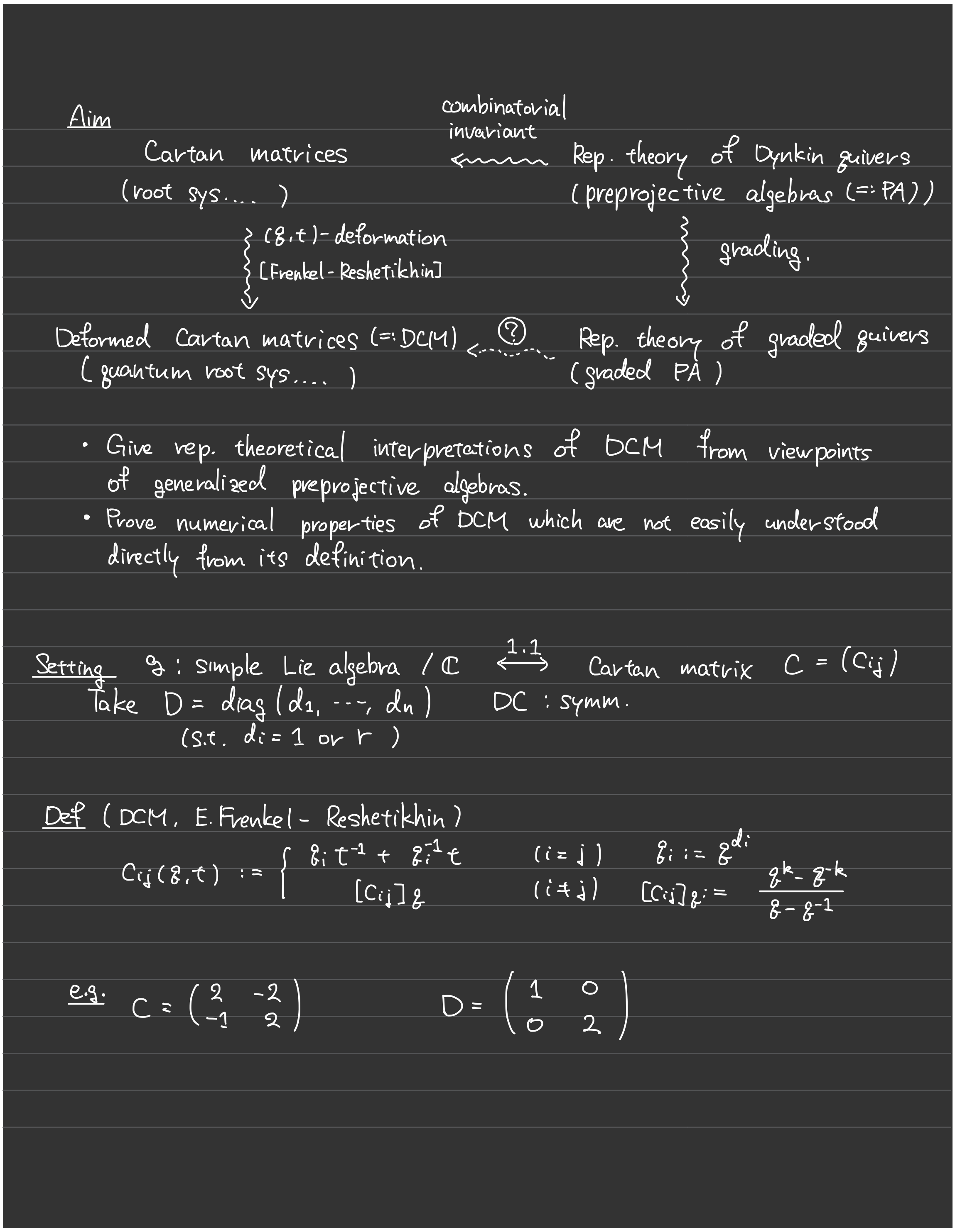


Deformed Cartan matrices and Generalized preprojective algebras

(joint work with Ryo Fujita (Paris · RIMS))

@ Preprojective algebras and Calabi-Yau algebras
(4th. March, 2022)

Kota Murakami (Kyoto University)



<u>Aim</u>

$$\begin{tikzcd}
\begin{array}{c}\text{Cartan matrices}\\ (\text{root sys}\ldots\ )\end{array} \arrow[d, squiggly, "\begin{array}{l}(q,t)\text{-deformation}\\ {[\text{Frenkel-Reshetikhin}]}\end{array}"] & & \begin{array}{c}\text{Rep. theory of Dynkin quivers}\\ (\text{preprojective algebras } (=: PA))\end{array} \arrow[ll, squiggly, "\begin{array}{c}\text{combinatorial}\\ \text{invariant}\end{array}"'] \arrow[d, squiggly, "\text{grading.}"] \\
\begin{array}{c}\text{Deformed Cartan matrices } (=: DCM)\\ (\text{quantum root sys}\ldots\ )\end{array} & & \begin{array}{c}\text{Rep. theory of graded quivers}\\ (\text{graded } PA\ )\end{array} \arrow[ll, dashed, "\textcircled{?}"']
\end{tikzcd}$$

- Give rep. theoretical interpretations of DCM from viewpoints of generalized preprojective algebras.
- Prove numerical properties of DCM which are not easily understood directly from its definition.

<u>Setting</u> $\mathfrak{g}$: simple Lie algebra / $\mathbb{C}$ $\xleftrightarrow{1:1}$ Cartan matrix $C = (C_{ij})$

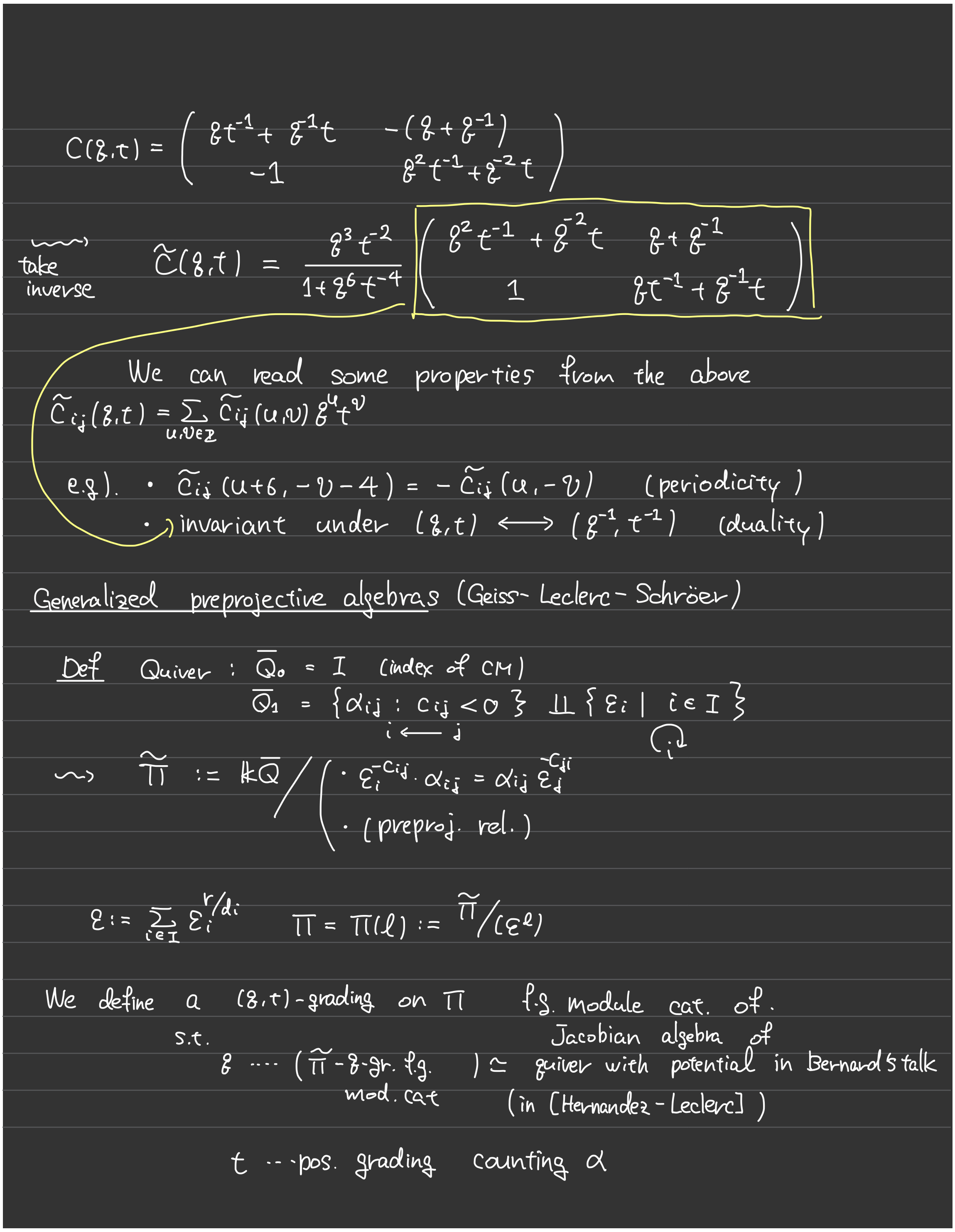
Take $D = \operatorname{diag}(d_1, \ldots, d_n)$ DC : symm.

(s.t. $d_i = 1$ or r)

<u>Def</u> (DCM, E. Frenkel - Reshetikhin)

$$C_{ij}(q,t) := \begin{cases} q_i t^{-1} + q_i^{-1} t & (i = j) \\ [C_{ij}]_q & (i \neq j) \end{cases} \qquad q_i := q^{d_i} \qquad [C_{ij}]_{q_i} := \frac{q^k - q^{-k}}{q - q^{-1}}$$

<u>e.g.</u> $C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\qquad$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$



$$C(q,t) = \begin{pmatrix} qt^{-1} + q^{-1}t & -(q+q^{-1}) \\ -1 & q^{2}t^{-1} + q^{-2}t \end{pmatrix}$$

$\rightsquigarrow$ take inverse

$$\widetilde{C}(q,t) = \frac{q^{3}t^{-2}}{1+q^{6}t^{-4}} \boxed{\begin{pmatrix} q^{2}t^{-1} + q^{-2}t & q + q^{-1} \\ 1 & qt^{-1} + q^{-1}t \end{pmatrix}}$$

We can read some properties from the above

$$\widetilde{C}_{ij}(q,t) = \sum_{u,v\in\mathbb{Z}} \widetilde{c}_{ij}(u,v)\, q^{u}t^{v}$$

e.g.).
- $\widetilde{c}_{ij}(u+6,-v-4) = -\widetilde{c}_{ij}(u,-v)$ (periodicity)
- invariant under $(q,t) \longleftrightarrow (q^{-1},t^{-1})$ (duality)

Generalized preprojective algebras (Geiss-Leclerc-Schröer)

Def Quiver : $\overline{Q}_0 = I$ (index of CM)

$$\overline{Q}_1 = \{\underset{i \longleftarrow j}{\alpha_{ij}} : c_{ij} < 0\} \amalg \{\underset{\circlearrowleft i}{\varepsilon_i} \mid i \in I\}$$

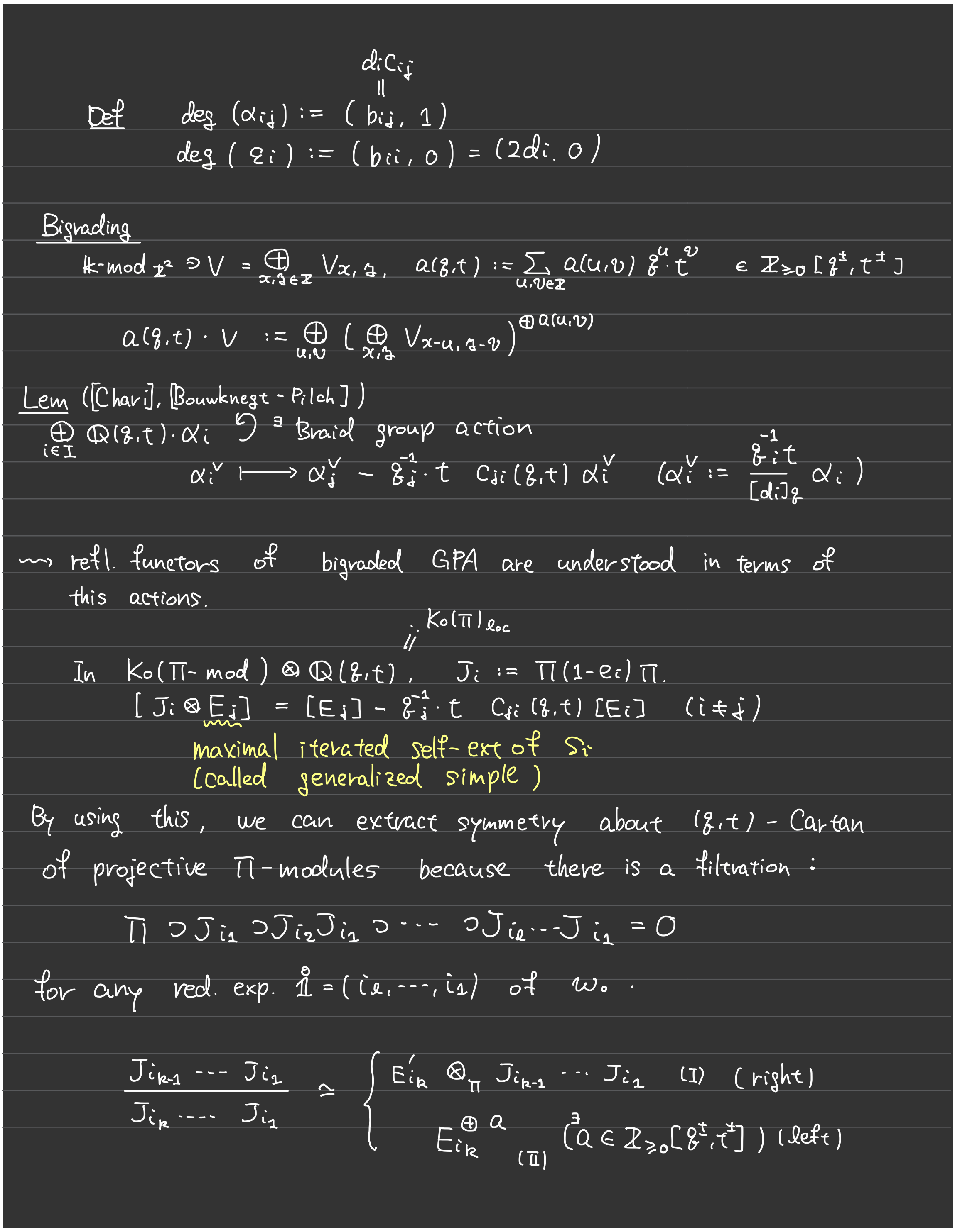
$$\rightsquigarrow \quad \widetilde{\Pi} := \Bbbk\overline{Q} \Big/ \left(\begin{array}{l} \cdot\ \varepsilon_i^{-c_{ij}} \cdot \alpha_{ij} = \alpha_{ij}\, \varepsilon_j^{-c_{ji}} \\ \cdot\ (\text{preproj. rel.}) \end{array} \right.$$

$$\varepsilon := \sum_{i\in I} \varepsilon_i^{r/d_i} \qquad \Pi = \Pi(\ell) := \widetilde{\Pi}/(\varepsilon^{\ell})$$

We define a (q,t)-grading on Π s.t.

q ⋯ ($\widetilde{\Pi}$-q-gr. f.g. mod. cat) $\simeq$ f.g. module cat. of Jacobian algebra of quiver with potential in Bernard's talk (in [Hernandez-Leclerc])

t ⋯ pos. grading counting α



Def $\deg(\alpha_{ij}) := (\underset{\overset{\|}{d_i c_{ij}}}{b_{ij}}, 1)$

$\deg(\varepsilon_i) := (b_{ii}, 0) = (2d_i. 0)$

Bigrading

$\Bbbk\text{-mod}_{\mathbb{Z}^2} \ni V = \bigoplus_{x,y \in \mathbb{Z}} V_{x,y}, \quad a(q,t) := \sum_{u,v \in \mathbb{Z}} a(u,v)\, q^u \cdot t^v \quad \in \mathbb{Z}_{\geq 0}[q^{\pm}, t^{\pm}]$

$$a(q,t) \cdot V := \bigoplus_{u,v} \Big(\bigoplus_{x,y} V_{x-u, y-v} \Big)^{\oplus a(u,v)}$$

Lem ([Chari], [Bouwknegt - Pilch])

$\bigoplus_{i \in I} \mathbb{Q}(q,t) \cdot \alpha_i \circlearrowleft {}^{\exists}$ Braid group action

$$\alpha_i^\vee \longmapsto \alpha_j^\vee - q_j^{-1} \cdot t \;\; C_{ji}(q,t)\, \alpha_i^\vee \qquad \Big(\alpha_i^\vee := \frac{q_i^{-1} t}{[d_i]_q} \alpha_i\Big)$$

$\rightsquigarrow$ refl. functors of bigraded GPA are understood in terms of this actions.

In $\overset{K_0(\Pi)_{loc}}{\overset{\|}{K_0(\Pi\text{-mod}) \otimes \mathbb{Q}(q,t)}}$, $\quad J_i := \Pi(1-e_i)\Pi$.

$$[J_i \otimes \underset{\sim\sim}{E_j}] = [E_j] - q_j^{-1} \cdot t \;\; C_{ji}(q,t)\, [E_i] \qquad (i \neq j)$$

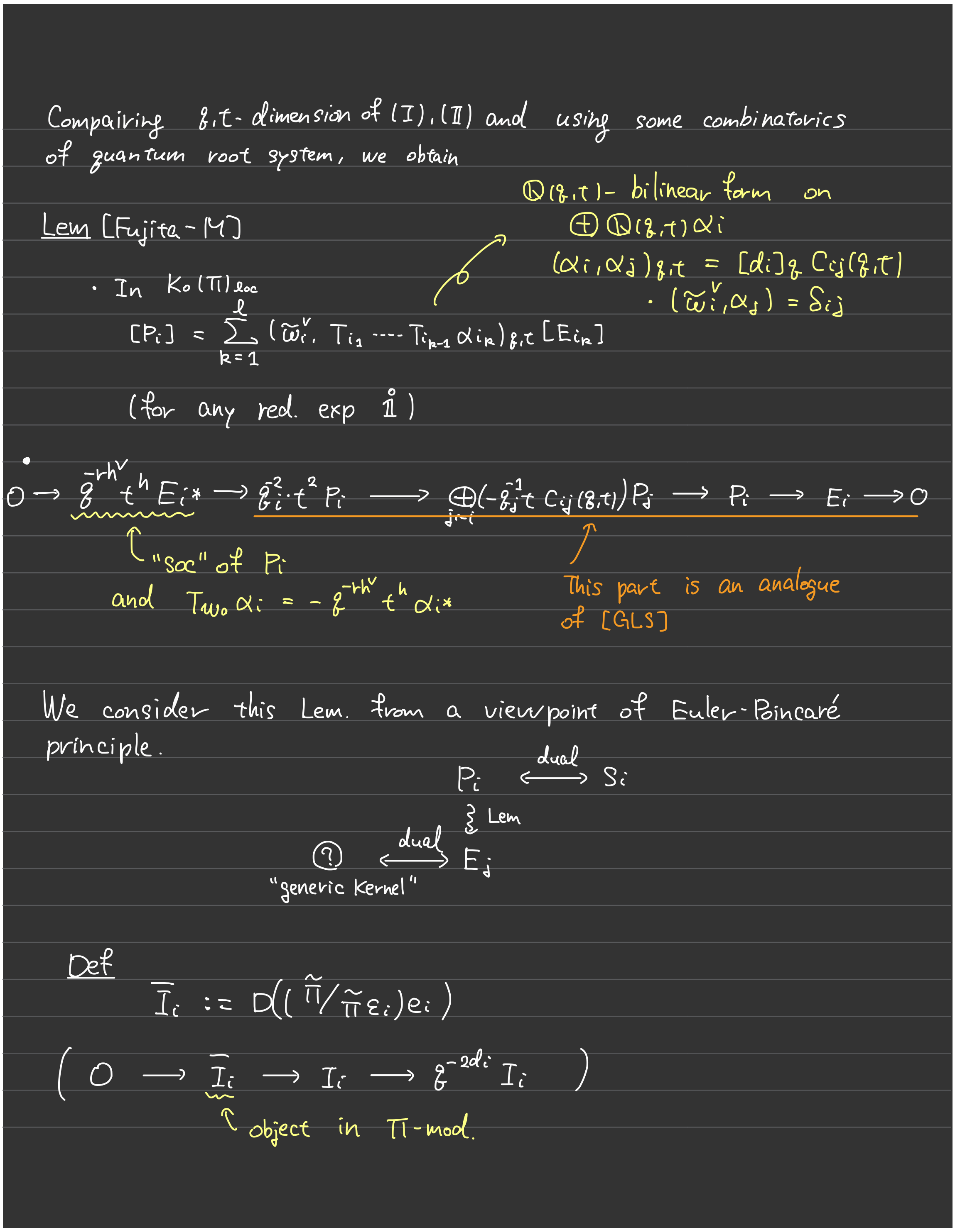
maximal iterated self-ext of S_i
(called generalized simple)

By using this, we can extract symmetry about (q,t) - Cartan of projective Π-modules because there is a filtration :

$$\Pi \supset J_{i_1} \supset J_{i_2} J_{i_1} \supset \cdots \supset J_{i_l} \cdots J_{i_1} = 0$$

for any red. exp. $\mathring{\mathbb{1}} = (i_l, \cdots, i_1)$ of w_0 .

$$\frac{J_{i_{k-1}} \cdots J_{i_1}}{J_{i_k} \cdots J_{i_1}} \simeq \begin{cases} E'_{i_k} \otimes_{\Pi} J_{i_{k-1}} \cdots J_{i_1} & \text{(I)} \quad (\text{right}) \\ E_{i_k}^{\oplus a} \;\; \text{(II)} & ({}^{\exists} a \in \mathbb{Z}_{\geq 0}[q^{\pm}, t^{\pm}]) \; (\text{left}) \end{cases}$$



Compairing q,t-dimension of (I), (II) and using some combinatorics of quantum root system, we obtain

$\mathbb{Q}(q,t)$- bilinear form on $\bigoplus \mathbb{Q}(q,t)\alpha_i$

$(\alpha_i, \alpha_j)_{q,t} = [d_i]_q C_{ij}(q,t)$

$\cdot\ (\tilde{\varpi}_i^\vee, \alpha_j) = \delta_{ij}$

Lem [Fujita-M]

- In $K_0(\Pi)_{loc}$

$$[P_i] = \sum_{k=1}^{\ell} (\tilde{\varpi}_i^\vee, T_{i_1} \cdots T_{i_{k-1}} \alpha_{i_k})_{q,t} [E_{i_k}]$$

(for any red. exp $\mathbf{i}$)

- $$0 \to \underbrace{q^{-rh^\vee} t^h E_{i^*}}_{} \to \underline{q_i^{-2} \cdot t^2 P_i \longrightarrow \bigoplus_{j \sim i} (-q_j^{-1} t\, C_{ij}(q,t)) P_j \longrightarrow P_i \longrightarrow E_i \longrightarrow 0}$$

"soc" of P_i

and $T_{w_0} \alpha_i = -q^{-rh^\vee} t^h \alpha_{i^*}$

This part is an analogue of [GLS]

We consider this Lem. from a viewpoint of Euler-Poincaré principle.

$$\begin{tikzcd} & P_i \arrow[r, leftrightarrow, "dual"] \arrow[d, phantom, "\wr\ \text{Lem}"] & S_i \\ \text{?} \arrow[r, leftrightarrow, "dual"] & E_j & \end{tikzcd}$$

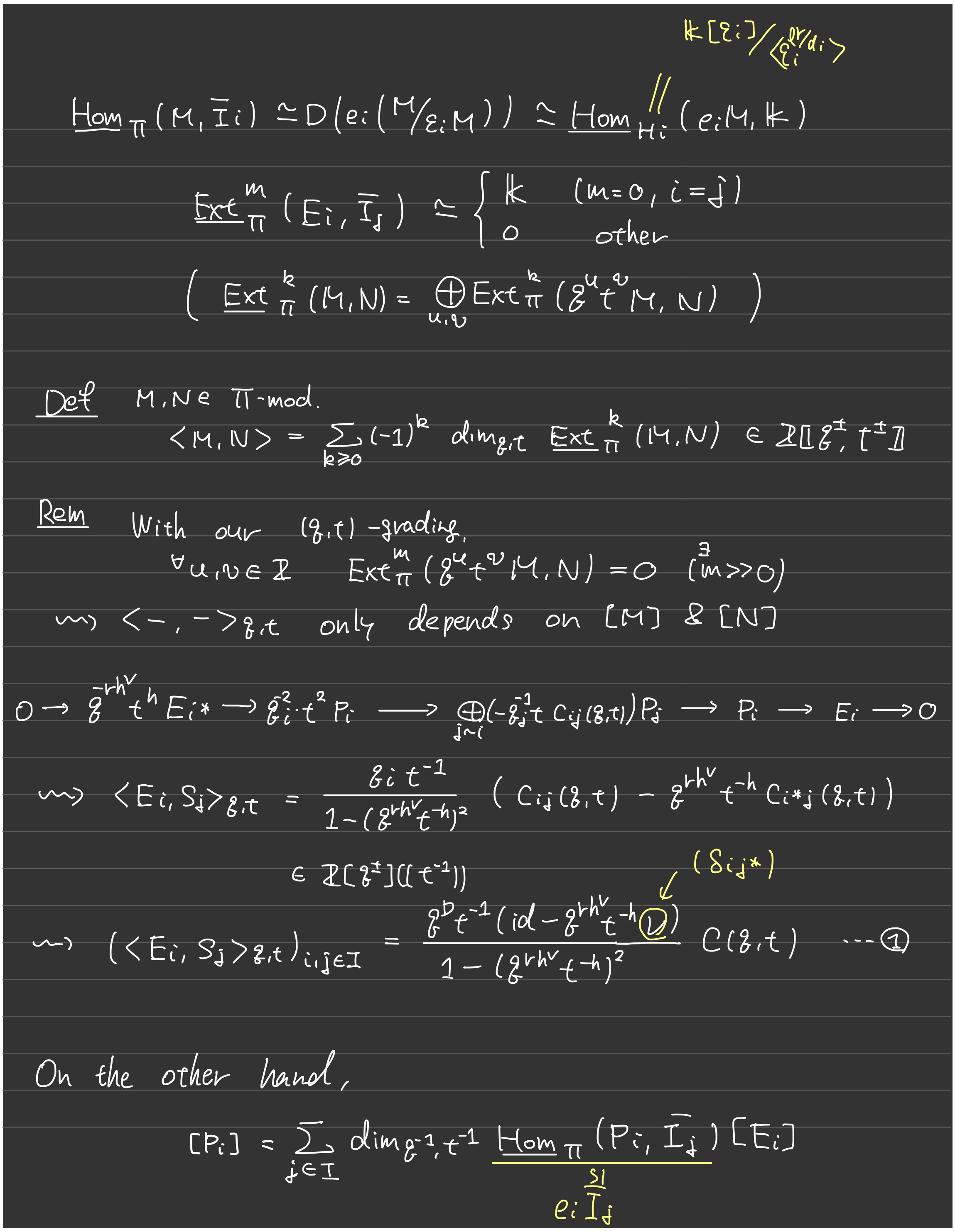
"generic Kernel"

Def

$$\overline{I}_i := D((\tilde{\Pi}/\tilde{\Pi}\varepsilon_i)e_i)$$

$$\left(0 \longrightarrow \overline{I}_i \longrightarrow I_i \longrightarrow q^{-2d_i} I_i \right)$$

object in Π-mod.



$$\underline{\mathrm{Hom}}_{\Pi}(M, I_i) \simeq D(e_i(M/\varepsilon_i M)) \simeq \underline{\mathrm{Hom}}_{\overset{\overset{\Bbbk[\varepsilon_i]/\langle \varepsilon_i^{r/d_i} \rangle}{\|}}{H_i}}(e_i M, \Bbbk)$$

$$\underline{\mathrm{Ext}}^m_{\Pi}(E_i, I_j) \simeq \begin{cases} \Bbbk & (m=0,\ i=j) \\ 0 & \text{other} \end{cases}$$

$$\left(\underline{\mathrm{Ext}}^k_{\Pi}(M,N) = \bigoplus_{u,v} \mathrm{Ext}^k_{\Pi}(q^u t^v M, N) \right)$$

<u>Def</u> $M, N \in \Pi$-mod.

$$\langle M, N \rangle = \sum_{k \geq 0} (-1)^k \dim_{q,t} \underline{\mathrm{Ext}}^k_{\Pi}(M,N) \in \mathbb{Z}[q^{\pm}, t^{\pm}]$$

<u>Rem</u> With our (q,t)-grading,

$$\forall u, v \in \mathbb{Z} \quad \mathrm{Ext}^m_{\Pi}(q^u t^v M, N) = 0 \quad (\exists\, m \gg 0)$$

$\rightsquigarrow$ $\langle -, - \rangle_{q,t}$ only depends on $[M]$ & $[N]$

$$0 \to q^{-rh^\vee} t^h E_{i^*} \to q_i^{-2} \cdot t^2 P_i \longrightarrow \bigoplus_{j \sim i} (-q_j^{-1} t\, c_{ij}(q,t)) P_j \longrightarrow P_i \longrightarrow E_i \longrightarrow 0$$

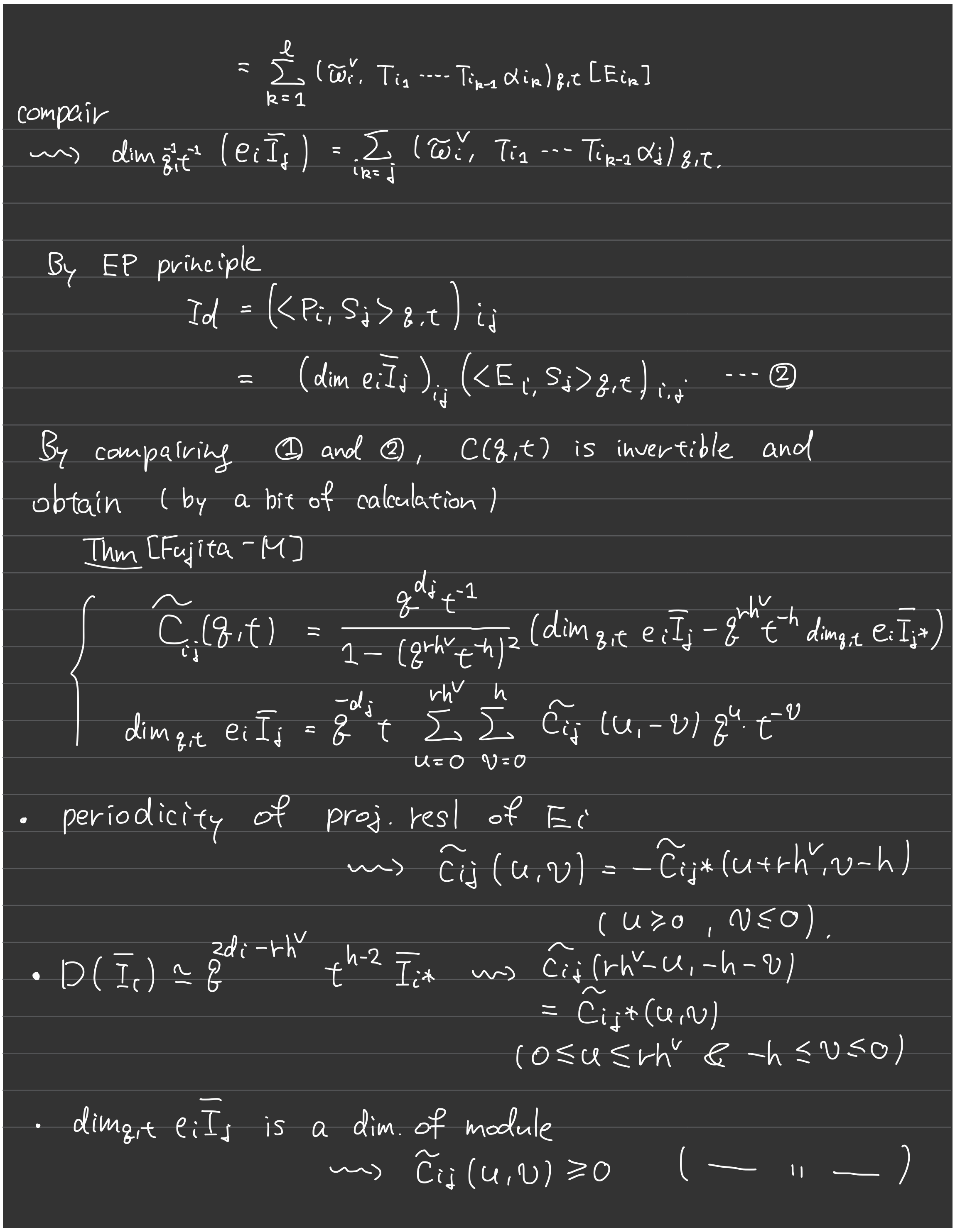
$$\rightsquigarrow \langle E_i, S_j \rangle_{q,t} = \frac{q_i t^{-1}}{1-(q^{rh^\vee} t^{-h})^2} \left(c_{ij}(q,t) - q^{rh^\vee} t^{-h} c_{i^* j}(q,t) \right)$$

$$\in \mathbb{Z}[q^{\pm}]((t^{-1}))$$

$$\rightsquigarrow (\langle E_i, S_j \rangle_{q,t})_{i,j \in I} = \frac{q^D t^{-1} (\mathrm{id} - q^{rh^\vee} t^{-h} \overset{(\delta_{ij^*})}{\boxed{\nu}})}{1-(q^{rh^\vee} t^{-h})^2} C(q,t) \quad \cdots ①$$

On the other hand,

$$[P_i] = \sum_{j \in I} \dim_{q^{-1}, t^{-1}} \underbrace{\underline{\mathrm{Hom}}_{\Pi}(P_i, I_j)}_{\overset{\rotatebox{90}{$\simeq$}}{e_i I_j}} [E_i]$$



$$= \sum_{k=1}^{\ell} (\tilde{\varpi}_i^\vee, T_{i_1} \cdots T_{i_{k-1}} \alpha_{i_k})_{q,t} [E_{i_k}]$$

compair

$$\rightsquigarrow \dim_{q^{-1}, t^{-1}} (e_i \overline{I}_j) = \sum_{i_k = j} (\tilde{\varpi}_i^\vee, T_{i_1} \cdots T_{i_{k-1}} \alpha_j)_{q,t}.$$

By EP principle

$$Id = (\langle P_i, S_j \rangle_{q,t})_{ij}$$

$$= (\dim e_i \overline{I}_j)_{ij} (\langle E_i, S_j \rangle_{q,t})_{i,j} \quad \cdots ②$$

By comparing ① and ②, $C(q,t)$ is invertible and obtain (by a bit of calculation)

Thm [Fujita-M]

$$\begin{cases} \tilde{C}_{ij}(q,t) = \dfrac{q^{d_j} t^{-1}}{1 - (q^{rh^\vee} t^{-h})^2} (\dim_{q,t} e_i \overline{I}_j - q^{rh^\vee} t^{-h} \dim_{q,t} e_i \overline{I}_{j^*}) \\ \dim_{q,t} e_i \overline{I}_j = q^{-d_j} t \sum_{u=0}^{rh^\vee} \sum_{v=0}^{h} \tilde{C}_{ij}(u, -v) q^u \cdot t^{-v} \end{cases}$$

- periodicity of proj. resl of E_i

$$\rightsquigarrow \tilde{C}_{ij}(u, v) = -\tilde{C}_{ij^*}(u + rh^\vee, v - h)$$

$$(u \geqq 0, v \leqq 0).$$

- $D(\overline{I}_i) \simeq q^{2d_i - rh^\vee} t^{h-2} \overline{I}_{i^*}$ $\rightsquigarrow$ $\tilde{C}_{ij}(rh^\vee - u, -h - v)$

$$= \tilde{C}_{ij^*}(u, v)$$

$$(0 \leqq u \leqq rh^\vee \ \& \ -h \leqq v \leqq 0)$$

- $\dim_{q,t} e_i \overline{I}_j$ is a dim. of module

$$\rightsquigarrow \tilde{C}_{ij}(u, v) \geqq 0 \quad (\text{—} \ \text{"} \ \text{—})$$

Cor (cf [Hernandez-Leclerc; ADE])

$$\widetilde{C}_{ij}(q,t) = q^{d_j} t^{-1} \sum_{k>0,\, i_k=j} (\widetilde{\varpi}_i^\vee, T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1} \alpha_j)_{q,t}.$$

$$(\underbrace{i_1, \cdots, i_\ell}_{\text{any red. word of } w_0}, \cdots) \quad \text{w/} \quad i_{k+\ell} = i_k^* \quad (k \in \mathbb{Z}_{>0})$$