A Reducibility Problem for Monodromy of Some Surface Bundles

(ある曲面バンドルのモノドロミーに関する可約性の問題)

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This Ph. D. thesis began as our paper Y. Imayoshi, M. Ito and H. Yamamoto "A Reducibility Problem for Monodromy of Some Surface Bundles," Journal of Knot Theory and Its Ramifications, to appear, which is a culmination of five years' work separately and jointly. I have revised the paper into the present form.

I would like to thank them for reading and criticizing earlier versions of the manuscript. However, any errors in the final version are certainly mine.

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Preface

A surface bundle determines a monodromy representation recording the twisting of the fiber under transport around a closed path in the base space. The fascinating relation between these monodromy representations and the Thurston classification of surface automorphisms will be studied. In this note we deal with a simple and interesting case: the fibrations of Fadell and Neuwirth.

A special case of our theorem has been obtained by Hiroshi Yamamoto in his thesis. This was announced at talks at the Fukuoka Conference in 1999 and at Ichinoseki 2000; a detailed treatment of this result appears in Y. Imayoshi, M. Ito and H. Yamamoto "On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces with two specified points," Osaka Journal of Mathematics **40**, 2003, pp. 659-685.

But it should be noted that these studies distinguished topologically the four types (elliptic, parabolic, hyperbolic, pseudohyperbolic) of Bers, rather than just the three types (reducible, pseudo-Anosov, of finite order) of Thurston. The precise relation between the elements of the monodromy group and the new Bers classification in the general case will be dealt with elsewhere.

Contents

1. Introduction and Summary of Results

Throughout this paper, X will be an oriented surface (two-manifold) of finite type (g, m), that is homeomorphic to a sphere with $g \geq 0$ handles from which one has removed $m \geq 0$ disjoint continua. To avoid special cases we assume X is of *non-excluded* type, i.e.,

$$2g - 2 + m > 0, \tag{1.1}$$

unless otherwise stated. Let $\{x_1^0, x_2^0, \dots\}$ be a sequence of distinct points of X. Define the surface X_n of finite type $(g, m+n)$ by setting

$$X_n = X \setminus \{x_1^0, \dots, x_n^0\} \tag{1.2}$$

for every n (X_0 is by convention X itself). The *mapping class group* $\mathrm{Mod}(X_n)$ is the group of isotopy classes of orientation preserving homeomorphism $f\colon X_n \to X_n$; the group law is composition. Whenever there can be no confusion, same notation is used for mappings and corresponding elements of $\mathrm{Mod}(X_n)$.

The *configuration space* of the surface X is the space F_nX of n-tuples of distinct points of X, that is,

$$F_nX = \left\{ (x_1, \cdots, x_n) \in \prod_{i=1}^{n} X; x_i \neq x_j \text{ for } i \neq j \right\}$$

where $\prod_{i=1}^{n} X$ denotes the n-fold product space. The fundamental group $\pi_1(F_nX)$ of the space F_nX is called the *pure braid group* on X (with base point $(x_1^0, \dots, x_n^0)$). Let

$$\pi\colon F_{n+1}X \to F_nX \tag{1.3}$$

be the projection of $F_{n+1}X$ onto F_nX which deletes the $(n+1)$st coordinate of $(x_1, \dots, x_{n+1})$. Then π exhibits $F_{n+1}X$ as a locally trivial fiber bundle over the base space F_nX, with fiber X_n (Theorem 3 in Fadell and Neuwirth [1]).

If an n-string braid on X is viewed as a motion of n points $x_1^0, \dots, x_n^0$ in X by means of a representative path in F_nX, then this motion extends to an isotopy $f_t\colon X \to X$ $(0 \leq t \leq 1)$ of X, where $f_0 = 1$ and f_1 fixes each point x_i^0. (1 denotes for us appropriate identity elements or identity maps, as will be clear from the context in which it occurs.) Geometric intuition thus suggests that the fiber bundle (1.3) determines a *monodromy map*

$$\iota\colon \pi_1(F_nX) \to \mathrm{Mod}(X_n) \tag{1.4}$$

recording the twisting of the fiber under transport around closed paths in the base space. The image of $\pi_1(F_nX)$ in $\mathrm{Mod}(X_n)$ under ι is called the *monodromy group*, and is denoted $\mathrm{Isot}(X,n)$.[1] For the convenience of the reader, we concisely explain this method in Section 2.

Our chief interest is to describe and classify the elements of the group $\mathrm{Isot}(X,n)$. When is such a class (or one of its members) of self maps reducible, or irreducible in the sense of Thurston [2]? What are the conditions on $\beta \in \pi_1(F_nX)$ so that $f \in \mathrm{Isot}(X,n)$ corresponding to β be reducible?

Now we are in a position to state

Theorem. *Let X be an oriented surface of non-excluded finite type (g,m); that is, $2g-2+m>0$. Then the element $1 \neq f \in \mathrm{Isot}(X,n)$ is not of finite order. Further, f is reducible if and only if it can be induced by $\beta \in \pi_1(F_nX)$ satisfying at least one of the following three conditions:*

(i) *β is non-spreading;*

(ii) *β has a boundary partition;*

(iii) *β has a tube structure over some subset in $\{1,2,\dots,n\}$ which is not a singleton.*

It is the proof of this theorem that will occupy all our energies for the rest of this paper. The proof is somewhat long and will be broken up into several lemmas. In Section 3 we have collected all the unconventional definitions.

The route we take is the following. In Section 2, we explore the algebraic nature of braid groups to see that the monodromy group $\mathrm{Isot}(X,n)$ has no finite order (i.e., torsion) elements. In Section 3, we show a mapping class $f \in \mathrm{Isot}(X,n)$ is reducible if it can be induced by a braid $\beta \in \pi_1(F_nX)$ satisfying condition (i) or (ii) (but the converse is false for $n>1$). In Section 4, to begin the transition to Teichmüller theory, we prepare the

[1] Clearly, $\mathrm{Isot}(X,n)$ consists of those elements $f \in \mathrm{Mod}(X_n)$ that are induced by self maps of X_n that fix each specified puncture x_i^0 on X_n and are isotopic to the identity as self maps of X. Isot stands for isotropy subgroup in $\mathrm{Mod}(X_n)$.

way by defining conformal structures σ on X, which is then applied, in Section 5, to pursue the relation of the foregoing conditions on β to the reducibility of the mapping class f. The background materials in obtaining these results are quasiconformal mapping and hyperbolic geometry. Finally, in Section 5, we show, with the help of Teichmüller theory, that a mapping class $f \in \mathrm{Isot}(X, n)$ is irreducible if it can be induced by a braid $\beta \in \pi_1(F_n X)$ satisfying none of conditions (i), (ii) and (iii), which will conclude the proof of the theorem. In an important sense our starting point is actually the fact that Thurston's classification can be obtained by looking at the element of the (Teichmüller) modular group induced by a self-map of a surface, cf. Theorem 4 in Bers [3]. We quote a fundamental result of Mumford [4] concerning the geometry of the moduli space $\mathcal{M}(X_n)$ in the course of the proof of Lemma 4.

The simplest case of our results was first obtained by Kra [5], where he established (among other things) that *the concept of a reducible element of* $\mathrm{Isot}(X, n)$ *can be described completely in terms of* $\pi_1(F_n X)$ *when* $n = 1$. Before writing the current version of the present paper, we showed that the statement of the theorem is true for $n = 2$. This was announced at talks at the Fukuoka Conference in 1999 (see Imayoshi-Ito-Yamamoto [6]) and at Ichinoseki 2000; a detailed treatment of this result appears in Imayoshi-Ito-Yamamoto [7]. It should be noted that these studies distinguished topologically the four types (elliptic, parabolic, hyperbolic, pseudohyperbolic) of Bers [3], rather than just the three types (reducible, pseudo-Anosov, of finite order) of Thurston [2]. The precise relation between the elements of $\mathrm{Isot}(X, n)$ and the new Bers classification in the general case of $n > 1$ will have to be clarified by further investigations. See also the note at the end of the last section.

Remark. The referee has pointed out that every essential (i.e., incompressible and non boundary-parallel) torus in the complement of n-string braid in $X \times S^1$ is either

(1) already essential in $X \times S^1$,

(2) parallel to the frontier in $X \times S^1$, or

(3) defining a solid torus.

These possibilities correspond to the conditions (i), (ii) and (iii) respectively. Moreover, the complement of n-string braid in $X \times S^1$ contains an essential torus if and only if the induced mapping class is reducible, as shown by Thurston [8]. Thus one is led to suspect that a purely topological attack on our theorem *will also* succeed. For an argument in the special case of $n = 1$ see Ichihara-Motegi [9],[2] based on three-manifold topology.

2. Start of the Proof of Theorem

We introduce now some notation that we will follow throughout this section. Let $\mathfrak{F}_n X$ be the group of orientation preserving homeomorphisms $f\colon X \to X$ such that f fixes $x_1^0, \ldots, x_n^0$ pointwise. $\mathfrak{F}_0 X$ denotes by convention the group of all orientation preserving homeomorphisms of X and is to be endowed with the compact-open topology. Provide $\mathfrak{F}_n X$ with subspace topology from $\mathfrak{F}_0 X$. The group $\pi_0(\mathfrak{F}_n X)$ of path components of $\mathfrak{F}_n X$ is called the *pure mapping class group* of X; it can be identified with a subgroup of index $(m+n)!/m!$ in $\mathrm{Mod}(X_n)$, since homeomorphisms of the punctured surfaces always extend to being homeomorphisms of the ambient compact surface.

The *evaluation map* $\varepsilon\colon \mathfrak{F}_0 X \to F_n X$ defined by

$$\varepsilon(f) = \big(f(x_1^0), \ldots, f(x_n^0)\big) \tag{2.1}$$

is a locally trivial fibering with fiber $\mathfrak{F}_n X$.[3] The exact sequence of the fibration then gives an exact sequence:

$$\pi_1(\mathfrak{F}_0 X) \xrightarrow{\varepsilon_*} \pi_1(F_n X) \xrightarrow{\partial_*} \pi_0(\mathfrak{F}_n X) \xrightarrow{i_*} \pi_0(\mathfrak{F}_0 X) \tag{2.2}$$

where the first homomorphism ε_* is induced by the evaluation map ε, the last i_*, by the inclusion map $i\colon \mathfrak{F}_n X \to \mathfrak{F}_0 X$.

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[2]Available from the first author: `http://vivaldi.ics.nara-wu.ac.jp/~ichihara/index.html`.

[3]The verification of the local triviality of ε is straightforward. Choose a small disk around $f(x_k)$ in X. We can find homeomorphisms of the disk which are identity on the boundary of the disk and move $f(x_k)$ to nearby interior points of the disk. These "little homeomorphisms" extend by the identity to all of X. Composing f with the "little homeomorphisms" clearly shows that, relative to ε, there exists a local cross section s of the homogeneous space $F_n X$ in $\mathfrak{F}_0 X$ at the point $(f(x_1), \cdots, f(x_n)) \in F_n X$. Then it is an easy matter to construct a "vertical tubular neighborhood" around s.

It is now convenient to review the explicit construction of ∂_*. Suppose $\beta \in \pi_1(F_nX)$, with β represented by a loop

$$(\beta_1, \cdots, \beta_n) \colon I \to F_nX. \tag{2.3}$$

Here (and hereafter) $I = [0,1]$. Then it is not difficult to construct an isotopy $f_t \colon X \to X$ ($t \in I$) such that $f_0 = 1$, $f_t(x_k^0) = \beta_k(t)$ and hence $f_1 \in \mathfrak{F}_nX$ (recall also that the array $(x_1^0, \ldots, x_n^0)$ was chosen to be the base point for the space F_nX). We may thus conclude that $\partial^*(\beta) = f_1$, or that the homomorphism ∂_* is nothing more than the monodromy map

$$\iota \colon \pi_1(F_nX) \to \mathrm{Isot}(X,n) \subset \mathrm{Mod}(X_n), \tag{2.4}$$

which was already introduced in the course of Section 1.

Proposition 1. *Let X be a surface of non-excluded type. Then the monodromy map $\iota \colon \pi_1(F_nX) \to \mathrm{Mod}(X_n)$ is injective. In particular, the elements of the monodromy group $\mathrm{Isot}(X,n)$ are classified by the pure braid group $\pi_1(F_nX)$ of X.*

This proposition has been proven implicitly many times. See, for example, Kra [5] and the papers quoted there. The proof we present below is motivated by Theorem 4.2 in Birman [10], where she proves the statement of the proposition is true for $g \geq 2$ and $m = 0$. The same argument, however, goes through in all cases $2g - 2 + m > 0$.

Proof. Outline of proof of Proposition To see the homomorphism ∂_* in (2.2) and hence the monodromy map ι is injective, we begin by showing $\pi_1(F_nX)$ is centerless. If $n = 1$, we merely note that $\pi_1(F_1X)$ is the fundamental group $\pi_1(X)$ of X. Assume inductively that $\pi_1(F_nX)$ is centerless. We first recall the exact sequence of our fibration $\pi \colon F_{n+1}X \to F_nX$ gives a short exact sequence:

$$\{1\} \longrightarrow \pi_1(X_n) \xrightarrow{i_*} \pi_1(F_{n+1}X) \xrightarrow{\pi_*} \pi_1(F_nX) \longrightarrow \{1\} \tag{2.5}$$

where the trivial group on the left is $\pi_2(F_nX) = \{1\}$, and the trivial group on the right is $\pi_0(X_n) = \{1\}$. (The first equality follows from the fact that $\pi_2(F_iX_j) \cong \pi_2(F_{i-1}X_{j+1})$ obtained by considering the exact sequence of the fibration $p \colon F_iX_j \to X_j$.) For any group G, let $\mathrm{center}\, G$ denote the center of G. Since π_* is surjective, it is clear that

$$\pi_*(\mathrm{center}\, \pi_1(F_{n+1}X)) \subseteq \mathrm{center}\, \pi_1(F_nX) = \{1\}. \tag{2.6}$$

Thus, center $\pi_1(F_{n+1}X)$ lies in the group

$$\ker \pi_* = \operatorname{im} i_* \cong \pi_1(X_n), \tag{2.7}$$

which is a free group of rank 2 or more, hence $\pi_1(F_{n+1}X)$ is also centerless.

We are therefore reduced to showing that

$$\ker \partial_* \subseteq \text{center}\, \pi_1(F_n X). \tag{2.8}$$

Given any loop class $\alpha \in \ker \partial_*$, there is an $(x_1^0, \dots, x_n^0)$-based loop

$$(\alpha_1, \cdots, \alpha_n) \colon I \to F_n X$$

which represents α. Then it is an easy matter to construct an isotopy $f_t \colon X \to X$ $(t \in I)$ with the properties $f_0 = f_1 = 1$, and $f_t(x_i^0) = \alpha_i(t)$ for $k = 1, \dots, n$ (because $\ker \partial_* = \operatorname{im} \varepsilon_*$). Let $\beta \in \pi_1(F_n X)$ be arbitrary, with β represented by an $(x_1^0, \dots, x_n^0)$-based loop

$$(\beta_1, \cdots, \beta_n) \colon I \to F_n X.$$

Define $H \colon I \times I \to F_n X$ by

$$H(t, s) = \left(f_t\left(\beta_1(s)\right), \dots, f_t\left(\beta_n(s)\right)\right), \ (t, s) \in I \times I. \tag{2.9}$$

Then H is continuous and $H|\partial(I \times I)$ represents the homotopy class $\alpha\beta\alpha^{-1}\beta^{-1}$. Since β was arbitrary, we may conclude that $\alpha \in \text{center}\, \pi_1(F_n X)$. □

An element of the mapping class group is *periodic* if and only if it can be induced by a conformal self-map with respect to some conformal structure (see Section 3 for definitions). The sufficiency statement is a rather deep result going back at least to Fenchel and Nielsen.

The next result, Lemma 1, may be thought of as saying that a non-trivial element $f \in \operatorname{Isot}(X, n)$ cannot be *elliptic* in the sense of Bers [3].

Lemma 1. *Under the hypothesis of Proposition* 1, *the element* $f \in \operatorname{Isot}(X, n)$ *is not of finite order unless the corresponding element of* $\pi_1(F_n X)$ *is unit.*

Proof. Suppose $f \colon X_n \to X_n$ represents an element of $\operatorname{Isot}(X, n)$ that has finite order, say ν. Applying Proposition 1, we need only show that f is, in fact, isotopic to the identity as a self map of X_n.

In view of the definition of monodromy group in Section 1, it is sufficient to assume that f is a self map of X that fixes each specified point x_i^0 and is isotopic to the identity as a self map of X (by abuse of language we do not distinguish between f and its continuous extension). We also note that f^ν is isotopic to the identity as a self map of

$$X_k = X \setminus \{x_1^0, \dots, x_k^0\} \tag{2.10}$$

for every $k \leq n$.

Observe now that (use the proposition with $X = X_k$ and $n = 1$, for example) for each non-negative integer k, a deformation of a pointed topological surface (X_k, x_{k+1}^0) can be recorded by the motion of x_{k+1}^0, giving an isomorphism between $\mathrm{Isot}(X_k, x_{k+1}^0)$ and the path classes:

$$\mathrm{Mod}(X_{k+1}) \supset \mathrm{Isot}(X_k, x_{k+1}^0) \cong \pi_1(X_k, x_{k+1}^0) \tag{2.11}$$

which is torsion free (where x_{k+1}^0 may serve as base point for the spaces involved).

Hence f must be isotopic to the identity as a self map of X_{k+1} once it has been shown that f is isotopic to the identity as a self map of X_k—because, as we already mentioned, f^ν is isotopic to the identity as a self map of X_{k+1}. Such an f is isotopic to the identity as a self map of X_n. Since this is established by an easy induction argument, we are done. □

Remark. The above proposition (and probably the lemma as well) are known results; we have included in this paper, for the sake of completeness, those results that are well known to the experts but are nevertheless absent from the literature.

3. The Nielsen-Thurston Classification and Some Types of Braids

The Nielsen-Thurston theory generalizes the well-known classification of torus automorphisms to the automorphisms of an arbitrary orientable surface S. We begin by briefly recalling this classification.

A *closed one-dimensional submanifold* of a surface S is a disjoint union of simple closed curves in S. A closed one-dimensional submanifold C will be called *essential* if no component is homotopic to a point, a boundary continuum of S, or another component of C. Following Thurston [2]; let us say that a mapping class $f \in \mathrm{Mod}(S)$ is *reducible* if there exists a closed one-dimensional submanifold C which is essential and respected by f (i.e., some element of f fixes C setwise), *irreducible* if it is not. A mapping class f is often called *periodic* if f is of finite order.

There exist irreducible classes which are periodic in the sense above. It should be mentioned that a non-periodic element $f \in \mathrm{Mod}(S)$ is irreducible if and only if the class contains a *pseudo-Anosov* diffeomorphism. In our approach, however, this concept will play no direct role: we refer the reader to Casson-Bleiler [11] for a definition of pseudo-Anosov diffeomorphism.

To relate the reducibility question in the monodromy group to braid properties, we first formalize an analogue of Thurston's "filling" curves in the spirit of Kra [5]. Recall that essential one-dimensional submanifolds C_1, C_2 of S *fill up* S (according to Thurston [2], and Casson-Bleiler [11]) if (a) they have minimal intersection, and (b) every component of $S \setminus (C_1 \cup C_2)$ is a disk or a half-open annulus whose ideal boundary lies on ∂S. Given essential one-dimensional submanifolds C_1 and C_2 of S, condition (b) is satisfied if and only if there does not exist an essential one-dimensional submanifold on S that is disjoint from $C_1 \cup C_2$.[4] In the braid category, the following is an analogous (although not identical) idea.

Definition 1. A *decomposition* for $\beta \in \pi_1(F_n X)$ is an essential one-dimensional submanifold on X which does not intersect β_i for $i = 1, \dots, n$, where a closed loop

$$(\beta_1, \beta_2, \cdots, \beta_n) \colon I \to F_n X \tag{3.1}$$

is a representative of β (up to free homotopy). A braid β *spreads* over X if β has no decompositions.

Trivial (*but important*) *Remarks.* (1) Although we fixed any sequence of distinct points $\{x_1^0, x_2^0, \dots\}$ in X so that the array $\{x_1^0, \dots, x_n^0\}$ may serve as base point for the pure braid group $\pi_1(F_n X)$, it is quite often more convenient to work with free homotopies without direct reference to base point for our purposes. The good thing is that if one considers a continuous deformation along the "deformation path" then the choice of base

[4] "Only if" is obvious, even without assuming C_1 and C_2 are essential.

point becomes immaterial! It will be of interest to note that Nielsen-Thurston-Bers type of surface automorphisms is preserved by conjugation. See also the proof of Lemma 2.

(2) Choose a torsion free Fuchsian group Γ uniformizing X. Kra [5] referred to an element $g \in \Gamma$ as "essential" if the axis of g projects to a curve that intersects every *admissible* curve on X. This idea is equivalent to Definition 1, as is to be expected; while the precise relation between spreading braids (defined as above) and "essential curves" in the language of Kra [5] is not completely transparent.

If a braid β of the braid group $\pi_1(F_nX)$ is represented by $(\beta_1(t), \cdots, \beta_n(t)), t \in I$, then each of the coordinate functions β_i defines (via its graph) an arc $b_i(t) = (\beta_i(t), t)$ in $X \times I$. Since $(\beta_1(t), \cdots, \beta_n(t)) \in F_nX$, the arcs $b_1, \cdots, b_n$ are disjoint. Their union $b = b_1 \cup \cdots \cup b_n$ is called a *geometric braid* and the arc b_i is called the ith *braid string.* This may be thought of as saying intuitively that the corresponding *open braid* is the class of deformations of the geometric braid holding endpoints fixed.

The wandering of the individual strings about on the surface X is easier to visualize in their 2-dimensional projection (when we represent β as paths in X with designated "over/under crossings"). However, we stress that Definition 1 was given without reference to these subtle geometric concepts.

A non-spreading braid $\beta \in \pi_1(F_nX)$ defines a reducible mapping class $f_\beta \in \mathrm{Isot}(X, n)$, since a decomposition C of β yields a reducing submanifold for an isotopy class f_β. Indeed, more explicitly, a representative of β is viewed as a motion of n "puncture points", then this motion extends to an isotopy of some element of f_β to the identity having arbitrarily small compact support that is disjoint from C (under free homotopy). Then each component of $X \setminus C$ is again of non-excluded finite type and is invariant under f_β. We can thus restrict attention in turn to each component of $X \setminus C$, and consider the *component* of the isotopy class f_β. But these considerations fall outside the scope of this paper and will be pursued elsewhere.

On the other hand, a spreading braid $\beta \in \pi_1(F_nX)$ can still define a reducible mapping class $f_\beta \in \mathrm{Isot}(X, n)$; in fact, if we think of our braid strings $b_1, \cdots, b_n$ as being made of elastic, one might readily imagine such types of phenomena in which we could freely homotope the configuration to an equivalent one so that the 2-dimensional projection of a non-empty set of strings is continuously deformed into a boundary continuum of X (whether β be spreading or not). This reflects, of course, the fact that an isotopy class f_β on X_n can be reduced by a curve contractible to a boundary component of X. Note that such a reducing curve is essential on X_n (but not on X). We formalize this notion.

Definition 2. A *boundary partition* for a braid $\beta \in \pi_1(F_nX)$ is a half-open annulus A embedded in X, whose ideal boundary lies on ∂X, and such that

(a) its boundary curve is disjoint from $\beta_1, \ldots, \beta_n$, and

(b) $(\beta_1 \cup \cdots \cup \beta_n) \cap A$ is nonempty,

where a closed loop

$$(\beta_1, \beta_2, \cdots, \beta_n) \colon I \to F_nX \tag{3.2}$$

is a representative of β (up to free homotopy).

Remark. There exists a braid $\beta \in \pi_1(F_nX)$ which has a boundary partition even when $n = 1$; however in this case it is trivially observed to be non-spreading (unless X is a thrice punctured sphere).

For the present we record the following

Lemma 2. *A mapping class* $f \in \mathrm{Isot}(X, n)$ *is reducible if it can be induced by a braid* $\beta \in \pi_1(F_nX)$ *satisfying condition* (i) *or* (ii) *of the theorem.*

Proof. Very little remains at issue in the light of the definitions above, which guarantees that an $(x_1^0, \ldots, x_n^0)$-based loop representing β as in (i) or (ii) can be deformed to avoid some essential one-dimensional submanifold $C \subset X_n$. (To see this, note that a boundary partition A defines an essential one-dimensional submanifold on X_n, by taking the boundary curves ∂A.)

Following a suitable representative of f_β by conjugation by a harmless continuous deformation along the deformation path of the base point $(x_1^0, \ldots, x_n^0)$, if necessary, we may assume that it fixes C setwise. This finishes the proof. □

Conversely, given a reducible mapping class $f_\beta \in \mathrm{Isot}(X, n)$, one may expect that it must be induced by a braid $\beta \in \pi_1(F_nX)$ satisfying condition (i) or (ii). Indeed, the content of Theorem 2 in Kra [5] tells us that f_β induced by a spreading element[5] β (which has no boundary partitions) is already irreducible in the simplest special case of $n = 1$. However, a counterexample is provided by taking a braid β whose strings, say b_1 and b_2, are forced to wrap around each other in a way which makes f_β reduced by the boundary of a small disk containing a pair of puncture points $\{x_1^0, x_2^0\}$, corresponding to the strings.

This example illustrates a general phenomenon. A set of adjacent strings which are not separated by any other string yields a "thin tube" $\mathcal{T}$ around itself partitioning the strings of β. The boundary $\partial\mathcal{T}$ is transverse to the foliation of $X \times I$ by $X \times \{t\}$, and implies a reducing curve for f_β. The above observation leads to a new concept, the "tube structure" for β.

Definition 3. Suppose $\Sigma \subseteq \{1, 2, \ldots, n\}$ is a subset of cardinality ≥ 2. A braid $\beta \in \pi_1(F_n X)$ *tubes* over Σ if there exists a sequence of closed loops

$$(\beta_1^s, \beta_2^s, \cdots, \beta_n^s) \colon I \to F_n X \tag{3.3}$$

representing β up to free homotopy, which converges to a closed loop

$$(\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_n) \colon I \to F_n^\Sigma X \tag{3.4}$$

with respect to the compact-open topology. Here we set

$$F_n^\Sigma X = \left\{ (x_1, \cdots, x_n) \in \prod_{i=1}^n X; x_i = x_j \text{ for } i, j \in \Sigma \text{ and } x_i \neq x_j \text{ otherwise} \right\}.$$

This is often called a *tube structure* for β (without reference to Σ).

Remarks. (1) In the simplest case of $n = 1$, any $\beta \in \pi_1(F_n X)$ cannot have a tube structure by the very definition.

(2) The definition is legitimate since the two spaces $F_n X$ and $F_n^\Sigma X$ are endowed with the subspace topology from the n-fold product space—namely, equations (3.3) and (3.4) are viewed as mappings from I to $\prod_{i=1}^n X$. We also note that the compact-open topology is the same as the usual *uniform convergence* topology in our discussion of tube structures; in fact, X carries a metric (a *hyperbolic* (*Poincaré*) *metric*, for example) which gives rise to the original topology on X.

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[5]called an "essential element" in Kra [5]

We end this section by stating the following

Lemma 3. *A mapping class* $f \in \mathrm{Isot}(X, n)$ *is reducible if it can be induced by a braid* $\beta \in \pi_1(F_n X)$ *satisfying condition* (iii) *of the theorem.*

Although the above lemma could be established by a purely topological argument, a hyperbolic-geometric proof, based on the conformal structures is more consistent with our approach. We therefore postpone the proof of Lemma 3 until after we have discussed some relevant aspects of the complex geometry of Teichmüller space. We shall return to this topic (including Lemma 4 for braids satisfying none of conditions (i), (ii) and (iii)) later.

4. Quasiconformal Deformation of Surfaces

Before proceeding to the rest of the proof of our main theorem, we must establish several technical propositions. As we mentioned in the course of the introduction, our starting point is the fact that Thurston's classification can be obtained by looking at the element of the Teichmüller modular group induced by a self-map of a surface, but Bers' theory of modular actions plays no part in this section; its role will become clearer in Section 5.

To begin the passage from topology to Teichmüller space theory, we introduce now some complex-analytic concepts. A *conformal structure* on a surface S is a topological mapping σ of S onto a Riemann surface (of non-excluded finite type). If $f : S \to S$ is an orientation preserving homeomorphism of S, and σ is a conformal structure on S, then the deviation of $\sigma \circ f \circ \sigma^{-1}$ from conformality is measured by the *dilatation*

$$K_\sigma(f) = K(\sigma \circ f \circ \sigma^{-1}). \tag{4.1}$$

We recall that $1 \le K_\sigma(f) \le +\infty$, with $K_\sigma(f) = 1$ signifying that $\sigma \circ f \circ \sigma^{-1}$ is *conformal*, and $K_\sigma(f) = +\infty$ signifying that this homeomorphism is not even *quasiconformal*. In this note, however, we may restrict ourselves to conformal structures σ on S for which $\sigma(S)$ has no ideal boundary curves (and hence every diffeomorphism is quasiconformal). Such conformal structures are called of the *first kind*, all others are called of the *second* *kind* in the language of Bers [3]. From now on, "conformal structure of the first kind" will by abuse of language be abbreviated by "conformal structure".

It is known, and easy to check, that S carries canonically a *hyperbolic metric* which is determined by its conformal structure σ. Riemannian concepts applied to this canonical hyperbolic metric on S become complex analytic invariants for the Riemann surface $\sigma(S)$. We use $d^S_\sigma(\ ,\ )$ to denote the hyperbolic distance on S (equipped with the hyperbolic metric induced from σ). If $p \in S$, we shall denote the distant set

$$\{q \in S; d^S_\sigma(p,q) < r\} \tag{4.2}$$

by the symbol $U^S_\sigma(p;r)$. The supremum of all r for which $U^S_\sigma(p;r)$ is isometric to a (Poincaré) disk is called the *injectivity radius of S at p*, and will be denoted by $r^S_\sigma(p)$. One can verify that

$$r^S_\sigma(p) = \frac{1}{2}\inf\left\{\ell^S_\sigma(\gamma); \gamma \text{ is a geodesic loop at } p\right\}, \tag{4.3}$$

where $\ell^S_\sigma(\gamma)$ is the length of a curve γ. For this and the next sections Buser [12] is being followed as reference for the necessary background material on the hyperbolic geometry of surfaces.

Proposition 2. *Let σ be a conformal structure on X. Suppose $\beta \in \pi_1(F_nX)$, with β represented by an $(x^0_1, \dots, x^0_n)$-based loop*

$$(\beta_1, \beta_2, \cdots, \beta_n)\colon I \to F_nX.$$

Assume that a K-quasiconformal isotopy $f_t\colon X \to X\ (t \in I)$ with respect to σ satisfies $f_0 = 1$, and $f_t(x^0_i) = \beta_i(t)$, $i = 1, \dots, n$. Then there exists a continuous real-valued function Φ defined on $(0,\infty)$ such that

(a) $\lim_{\zeta\to 0}\Phi(\zeta) = -\infty$, $\lim_{\zeta\to\infty}\Phi(\zeta) = \infty$,

(b) $\Phi(\zeta)$ *is strictly increasing, and*

(c) *for any index i,*

$$\operatorname{diam}\left\{\Phi\left(r^X_\sigma(\beta_i(t))\right); t \in I\right\} \le \log K. \tag{4.4}$$

Here an isotopy $F(t,x) = f_t(x)$ is *K-quasiconformal* if each intermediate map (t fixed) $f_t\colon X \to X$ is K-quasiconformal, i.e., $K_\sigma(f_t) \le K$ (any $t \in I$).

Proof. To see the existence of such a function, first recall the universal cover $\tilde{X}$ of X is conformally isomorphic to the unit disk $\Delta \subset \mathbb{C}$. Thus X is the quotient of Δ by a conformal action of $\pi_1(X) \subset \mathrm{Aut}(\Delta)$. Since $\mathrm{Aut}(\Delta)$ preserves the non-Euclidean metric d on Δ, we obtain a hyperbolic metric on X canonically determined by its conformal structure σ.

Lifting to the universal cover, we obtain a K-quasiconformal isotopy $\tilde{f}_t \colon \Delta \to \Delta$ $(t \in I)$. Just as conformal maps are hyperbolic isometries, quasiconformal maps distort lengths of closed geodesics by a bounded factor. More precisely, using the characterization of quasiconformality with the help of ring domains and an argument with Grötzsch's module theorem we see that

$$\mu\left(\tanh d(z, z')\right) \leq K\mu\left(\tanh d(\tilde{f}_t(z), \tilde{f}_t(z'))\right), \quad t \in I \tag{4.5}$$

for all $z, z' \in \Delta$. Here $\mu(r)$ denotes the module of the Grötzsch's extremal domain, i.e., the unit disk cut from 0 to r along the positive real axis (note that by a well-known formula in hyperbolic plane geometry, $\tanh d(0, r) = r$). We need the following properties of the function $\mu(r)$:

(i) $\mu(r)$ is continuous in the interval $0 < r < 1$,

(ii) $\lim_{r\to 0} \mu(r) = \infty$, $\lim_{r\to 1} \mu(r) = 0$, and

(iii) $\mu(r)$ decreases monotonically with increasing r.

For more details see Lehto-Virtanen [13, pp. 53-68].

Keeping the notation introduced so far in the course of the proof, and by (4.5), we have, setting $z' = \gamma(z)$, $\gamma \in \pi_1(X) \subset \mathrm{Aut}(\Delta)$,

$$\mu\left(\tanh d(z, \gamma(z))\right) \leq K\mu\left(\tanh d(\tilde{f}_t(z), \gamma(\tilde{f}_t(z)))\right), \quad t \in I \tag{4.6}$$

because $\tilde{f}_t$ is isotopic to the identity and hence commutes with every γ. (Same notation is used for loops and corresponding elements of $\mathrm{Aut}(\Delta)$.)

Combining these results with (4.3) we see that

$$\frac{1}{K} \leq \frac{\mu\left(\tanh 2r_\sigma^X(x)\right)}{\mu\left(\tanh 2r_\sigma^X(f_t(x))\right)} \leq K, \quad t \in I \tag{4.7}$$

for all $x \in X$. Notice that it is enough to prove only one of these inequalities, since the same proof, applied to the inverse of f_t, gives the other inequality. By setting $x = x_j^0 =$

$\beta_j(0)$, we clearly can obtain from (4.7) that

$$\left|\log \frac{1}{\mu\left(\tanh 2r_\sigma^X(\beta_j(t))\right)} - \log \frac{1}{\mu\left(\tanh 2r_\sigma^X(\beta_i(0))\right)}\right| \leq \log K, \quad t \in I$$

for $i = 1, \dots, n$.

From here it is easy to get the desired conclusion; the function involved,

$$\Phi(\zeta) = \frac{1}{2} \log \frac{1}{\mu\left(\tanh 2\zeta\right)}, \tag{4.8}$$

for example, satisfies the conditions of the proposition. □

The following similar proposition will also be needed in the next section.

Proposition 3. *Under the hypothesis of Proposition 2, there exists a continuous real-valued function Ψ defined on $(0, \infty)$ such that*

(a) $\lim_{\zeta \to 0} \Psi(\zeta) = -\infty$, $\lim_{\zeta \to \infty} \Psi(\zeta) = \infty$,

(b) $\Psi(\zeta)$ *is strictly increasing, and*

(c) *for any two distinct indices i, j,*

$$\operatorname{diam}\left\{\Psi\left(d_\sigma^X(\beta_i(t), \beta_j(t))\right); t \in I\right\} \leq \log K. \tag{4.9}$$

Remark. The proof below implies the connection between this proposition and the one preceding it. But this need not concern us too much here.

Proof. The idea is the same as utilized above. Consider geodesics joining any two points $x, x' \in X$ instead of geodesic loops at x. Using a lift of f_t to Δ and exactly the same argument with the monotonicity of the function $\mu(r)$, one can show, again by (4.5), that

$$\frac{1}{K} \leq \frac{\mu\left(\tanh d_\sigma^X(x, x')\right)}{\mu\left(\tanh d_\sigma^X(f_t(x), f_t(x'))\right)} \leq K, \quad t \in I$$

or that

$$\left|\log \frac{1}{\mu\left(\tanh d_\sigma^X(f_t(x), f_t(x'))\right)} - \log \frac{1}{\mu\left(\tanh d_\sigma^X(x, x')\right)}\right| \leq \log K, \quad t \in I.$$

In particular, it follows by letting $x = x_i^0 = \beta_i(0)$ and $x' = x_j^0 = \beta_j(0)$ that a function

$$\Psi(\zeta) = \frac{1}{2} \log \frac{1}{\mu\left(\tanh \zeta\right)} \tag{4.10}$$

can be chosen to satisfy all the desired conditions. □

We intend here to keep our promise and show that an easy hyperbolic-geometric argument provides us with a proof of Lemma 3.

Proof. Proof of Lemma 3 In the proof henceforth we will fix any conformal structure σ on X and the conformal structure σ will be dropped from the symbols, e.g., we will write simply $d^X(\ ,\ )$ instead of $d^X_\sigma(\ ,\ )$, etc., for brevity.

Suppose $\beta \in \pi_1(F_nX)$ tubes over some subset $\Sigma \subseteq \{1, 2, \dots, n\}$, and let

$$(\beta_1^s, \beta_2^s, \cdots, \beta_n^s) \colon I \to F_nX \tag{4.11}$$

be a sequence of closed loops representing β, which converges uniformly on I to a closed loop

$$(\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_n) \colon I \to F_n^\Sigma X \tag{4.12}$$

(see the remarks following Definition 3). The easiest way to check that $f_\beta \in \mathrm{Isot}(X, n)$ is reducible is to observe that some element of f_β is reduced by the boundary of a "small disk" around $\hat{\beta}_j(0)$ in X_n whenever $j \in \Sigma$.

To see this, first note that there exists a constant $r > 0$ such that unless both i and j belong to Σ,

$$U^X(\hat{\beta}_i(t); r) \cap U^X(\hat{\beta}_j(t); r) = \emptyset, \quad \text{all } t \in I. \tag{4.13}$$

This is so because $(\hat{\beta}_1(t), \hat{\beta}_2(t), \cdots, \hat{\beta}_n(t)) \in F_n^\Sigma X$. Without loss of generality we may assume that $U^X(\hat{\beta}_i(t); r)$ is isometric to a disk for $i = 1, \dots, n$.

Next, we fix any s in (4.11) so that for every i

$$\beta_i^s(t) \in U^X(\hat{\beta}_i(t); r/2), \quad \text{all } t \in I. \tag{4.14}$$

Finally, let $t_0 \in I$ be arbitrary. Then, if $|t - t_0|$ is small enough, it is an easy matter to construct a continuous deformation of the disjoint disks

$$h_t \colon \bigcup_{i=1}^n U^X(\hat{\beta}_i(t); r) \to \bigcup_{i=1}^n U^X(\hat{\beta}_i(t); r) \tag{4.15}$$

that fixes the boundaries of those $U^X(\hat{\beta}_i(t); r)$ and has the following properties:

(i) each h_t moves $\beta_i^s(t_0)$ to a nearby interior point $\beta_i^s(t)$ of the disk;

(ii) $h_t(\partial U^X(\hat{\beta}_i(t_0); r/2)) = \partial U^X(\hat{\beta}_i(t); r/2)$.

These "little diffeomorphisms" extend by the identity to all of X and are clearly homotopic to identity on X.

We view as before an n-string braid on X as a motion of n points $x_1^0, \ldots, x_n^0$ in X by means of a representative path in $F_n(X)$. This motion extends to a continuous deformation $f_t\colon X \to X$ of X, where $f_0 = 1$ and f_t moves $x_i^0 = \beta_i^s(0)$ to a point $\beta_i^s(t)$; the above observation tells us that the set

$$I' = \{t_0 \in I; f_t \text{ can be constructed}$$
$$\text{so that it sends } \partial U^X(\hat{\beta}_i(0); r/2) \text{ to } \partial U^X(\hat{\beta}_i(t); r/2) \text{ for } t \le t_0\}$$

is open in I. Precisely the same argument shows that I' is closed in I. In fact, if $t_0 \notin I'$, then nothing in a small neighborhood of t_0 could be in I', for otherwise, the existence of the "little diffeomorphisms" above would produce a contradiction. Hence $I = I'$, as required. □

5. Completion of the Proof of Theorem

The proof of Theorem will be complete when we have shown:

Lemma 4. *A mapping class $f \in \mathrm{Isot}(X, n)$ is irreducible if it can be induced by a braid $\beta \in \pi_1(F_nX)$ satisfying none of conditions* (i), (ii) *and* (iii) *of the theorem.*

Evidently, Lemmas 2, 3, and 4 together suffices to prove completely that $f \in \mathrm{Isot}(X, n)$ is reducible if and only if it can be induced by $\beta \in \pi_1(F_nX)$ satisfying at least one of the conditions (i), (ii), and (iii), and we are through.

Note that, as we already observed, every braid $\beta \in \pi_1(F_nX)$ satisfying at least one of the conditions (i), (ii), and (iii) defines a reducible mapping class $f_\beta \in \mathrm{Isot}(X, n)$; an element of such f_β can be chosen to leave invariant some essential closed one-dimensional submanifold of X_n (Lemmas 2 and 3).

We turn now to the remaining case—*any* element of $f_\beta \in \mathrm{Isot}(X, n)$ induced by a braid $\beta \in \pi_1(F_nX)$ satisfying none of conditions (i), (ii), and (iii) will have to be shown to leave invariant *no* essential closed one-dimensional submanifolds of X_n and thus, such a β must define an irreducible mapping class f_β (Lemma 4). To settle this obviously more cumbersome question, we use the theory of Teichmüller spaces, where the group of isotopy classes is represented via a natural discontinuous action as the (Teichmüller) modular group; which we concisely review to fix notation.

We start with an oriented surface S of non-excluded finite type (g, m) and define two conformal structures σ_1 and σ_2 on S to be *strongly equivalent* if $\sigma_2 \circ \sigma_1^{-1}$ is isotopic to a conformal map of $\sigma_1(S)$ onto $\sigma_2(S)$. The strong equivalence classes $[\sigma]$ of structures of the first kind are the points of *Teichmüller space* $T(S) = T(g, m)$ and the distance (*Teichmüller distance*) between two points $[\sigma_1]$ and $[\sigma_2]$ is defined as

$$\langle [\sigma_1], [\sigma_2] \rangle = \frac{1}{2} \log \inf K(f), \quad f \text{ is isotopic to } \sigma_2 \circ \sigma_1^{-1}. \tag{5.1}$$

With this metric $T(S)$ is a complete metric space homeomorphic to $\mathbb{R}^{6g-6+2m}$. (That $T(S)$ also has a natural complex structure which can be realized by embedding $T(S)$ as a bounded domain of holomorphy in $\mathbb{C}^{3g-3+m}$, and that, according to a theorem by Royden, the Teichmüller metric is the Kobayashi metric play virtually no part here.)

There is a natural action of the mapping class group $\mathrm{Mod}(S)$ on $T(S)$; that is, $f \in \mathrm{Mod}(S)$ acts by

$$f^*([\sigma]) = [\sigma \circ f]. \tag{5.2}$$

It is obvious that the *modular transformation* f^* is well defined, and that every f^* is an *isometry* of $T(S)$.

The moduli space of Riemann surfaces of type (g, m) is the quotient space:

$$\mathcal{M}(S) = T(S)/\,\mathrm{Mod}(S). \tag{5.3}$$

Clearly, in the light of the definitions above, two Riemann surfaces $\sigma_1(S)$ and $\sigma_2(S)$ are conformally equivalent if and only if $f^*([\sigma_1]) = [\sigma_2]$ for some $f \in \mathrm{Mod}(S)$.

Proof. Proof of Lemma 4 If $f_\beta \in \mathrm{Isot}(X, n) \subset \mathrm{Mod}(X_n)$ has a fixed-point $[\sigma] \in T(X_n)$, that is, $f_\beta^*([\sigma]) = [\sigma]$, then by definition the conformal structure σ on X_n transfers f_β to the isotopy class of a conformal automorphism of $\sigma(X_n)$. Since the group of conformal automorphisms is finite (a classical theorem of Hurwitz), f_β has finite order as well, so Lemma 1 gives that f_β is the identity.

The foregoing argument convinces the reader that the dynamics of f_β on Teichmüller space controls the reducibility of the mapping class f_β. This control is made precise by

Ber's fundamental theorem concerning an extremal problem for quasiconformal mappings; in fact, Bers' results tell us that a non-periodic mapping is irreducible if and only if there exist quasiconformal mappings whose dilatation cannot be decreased by varying the mapping within its homotopy class and by varying the conformal structure of the underlying surface.

Hence one is led to show that if $\beta \in \pi_1(F_nX)$ satisfies none of conditions of the theorem, then the *translation length* $a(f_\beta)$ of f_β defined by

$$a(f_\beta) = \inf_{[\sigma]\in T(X_n)} \langle [\sigma], f_\beta^*([\sigma]) \rangle \tag{5.4}$$

must be achieved, which completes the proof of the lemma (note that $f_\beta \neq 1$ acts without fixed points, as we have already seen).

Since $T(X_n)$ is simply-connected, we can identify $\mathrm{Mod}(X_n)$ with $\pi_1(\mathcal{M}(X_n))$, the orbifold fundamental group of $\mathcal{M}(X_n)$. The strategy of the proof below is to seek the shortest representative of the free homotopy class of closed loops on $\mathcal{M}(X_n)$ corresponding to f_β.

Assume that $\beta \in \pi_1(F_nX)$ satisfies none of conditions (i), (ii), and (iii). Consider a sequence $\{[\sigma_j]\} \in T(X_n)$ with

$$\lim_{j\to\infty} \langle [\sigma_j], f_\beta^*([\sigma_j]) \rangle = a(f_\beta). \tag{5.5}$$

To show the infimum in (5.4) is achieved, we begin by showing the minimizing sequence $\{[\sigma_j]\}$ must lie in a compact subset of moduli space; here we understand by $[\sigma_j]$ not the point in $T(X_n)$ but its projection in $\mathcal{M}(X_n) = T(X_n)/\mathrm{Mod}(X_n)$. By Mumford's compactness theorem, it suffices to find $\varepsilon > 0$ such that the shortest geodesic on $[\sigma_j]$ does not have length less than ε.

The next assertion is established without reference to condition (iii).

Assertion. Under the hypothesis of Proposition 2, for $\beta \in \pi_1(F_nX)$ satisfying none of conditions (i) and (ii), there exists a constant $\varepsilon_1 > 0$, that depends only on K and n, such that for any index i,

$$r_\sigma^X(\beta_i(t)) \geq \varepsilon_1, \quad \text{all } t \in I. \tag{5.6}$$

Proof. Proof of Assertion We continue to use the notation introduced in the previous section. Let ε_0 be a number with $0 < \varepsilon_0 \leq \operatorname{arcsinh} 1$. Assume for contradiction there exists an index, say k, such that

$$\Phi\left(r_\sigma^X(\beta_k(t_0))\right) < \Phi(\varepsilon_0) - n\log K, \tag{5.7}$$

for some $t_0 \in I$.

By Proposition 2 we clearly can choose a number ε_0' so that

$$\Phi\left(r_\sigma^X(\beta_k(t_0))\right) \leq \Phi(\varepsilon_0') \leq \Phi(\varepsilon_0), \tag{5.8}$$

and

$$\Phi\left(r_\sigma^X(\beta_i(t))\right) \neq \Phi(\varepsilon_0'), \quad t \in I, \quad i = 1, \ldots, n. \tag{5.9}$$

On the other hand, one knows, with the help of a version of the collar theorem which holds for non-compact hyperbolic surfaces with finite area (see Theorem 4.4.6 in Buser [12], for example), that each component of the set

$$\{x \in X; r_\sigma^X(x) < \varepsilon_0'\}$$

is either (i) an annulus contained in a collar, or (ii) a half-open annulus contained in a limiting case of a half-collar, called a cusp. This produces a contradiction (for which the reader may supply some details). □

Proof. Continuation of Proof of Lemma 4 Since the distance between two points $[\sigma_j]$ and $f_\beta^*([\sigma_j])$ is bounded, there is a uniform K' such that f_β is represented by a K'-quasiconformal map with respect to σ_j. Moreover a uniform bound $K \geq 1$ (depending only on K') can always be chosen so that for each σ_j, such a representative is isotopic to the identity by a K-quasiconformal isotopy. This is actually a general fact about quasiconformal mappings and is a known result of Earle and McMullen [14] in the theory of dynamics on surfaces. We may and *hereafter* do require that $\beta \in \pi_1(F_nX)$ is represented by a loop

$$(\beta_1, \beta_2, \cdots, \beta_n) \colon I \to F_nX$$

canonically determined by such K-quasiconformal isotopies.

We claim now that *there exists a constant* $\varepsilon > 0$, *that depends only on any* K *as above and* n, *such that the shortest geodesic* C *on* X_n *(with respect to all* σ_j*) does not have length less than* ε. There are two possibilities to consider.

Case I. C is *not* a null-homotopic curve on X. In this case we observe that C satisfies $\ell^{X_n}_{\sigma_j}(C) \geq 2\varepsilon_1$, where ε_1 is the number from the assertion above.

Otherwise, C is disjoint from each β_i, $i = 1, \ldots, n$ by (5.6), because $\ell^X_{\sigma_j}(C) \leq \ell^{X_n}_{\sigma_j}(C)$.[6] If $\beta_i \cap C = \emptyset$ for every i, then β is non-spreading unless C is non-essential on X. In this latter case C yields a half-open annulus in X containing at least one distinguished point, contrary to our assumption that β has no boundary partitions.

Case II. C is a null-homotopic curve on X. This time C bounds a disc D in X containing at least two distinguished points; it involves no loss of generality to assume $\{x^0_1, x^0_2\} \subset D$. We then show that $\ell^{X_n}_{\sigma_j}(C) \geq 2\varepsilon_2$, where ε_2 is a number with $\Psi(2\varepsilon_2) < \Psi(\varepsilon_1) - n \log K$ (see Proposition 3).

Indeed, if not, there exists a number ε'_1 satisfying

$$\Psi(2\varepsilon_2) \leq \Psi(\varepsilon'_1) \leq \Psi(\varepsilon_1), \tag{5.10}$$

and

$$\Psi\left(d^X_{\sigma_j}(\beta_1(t), \beta_i(t))\right) \neq \Psi(\varepsilon'_1), \quad t \in I, \quad i = 1, \ldots, n \tag{5.11}$$

by (4.9). Let Σ be the subset of $\{1, 2, \ldots, n\}$ so that

$$\left\{\Psi\left(d^X_{\sigma_j}(\beta_1(t), \beta_i(t))\right); t \in I\right\} \subseteq (-\infty, \Psi(\varepsilon'_1)) \tag{5.12}$$

if and only if $i \in \Sigma$. For notational and descriptive convenience the rest of the proof will be carried out only in the case of $\Sigma = \{1, 2, \cdots, k\}$ for some $k \geq 2$.[7]

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[6] Every complex structure in $T(X_n)$ extends to a complex structure on X. By abuse of language we do not distinguish between σ_j and its continuous extensions.

[7] In this case, the set $\beta_i \cap C$ is nonempty for some i establishing an obvious inequality $d^X_{\sigma_j}(x^0_1, x^0_2) < 2\varepsilon_2$ under the assumption that the shortest loop C on X_n has length less than $2\varepsilon_2$. Note that this inequality is valid even when the genus $g = 0$ (because a Poincaré disc of radius ε_2 about $x \in \beta_i \cap C$ contains D in the light of the assertion above).

Now we construct, noting (5.6), a sequence of closed loops

$$(\beta_1^s, \cdots, \beta_k^s, \beta_{k+1}^s, \cdots, \beta_n^s)\colon I \to F_n X \tag{5.13}$$

representing β (up to free homotopy) defined by

$$\beta_i^s(t) = \begin{cases} \text{the point dividing the geodesic segment} \\ \text{contained in a disc } U_{\sigma_j}^X(\beta_1(t); \varepsilon_1') \\ \text{between } \beta_1(t) \text{ and } \beta_i(t) \text{ in the ratio } 1 : (s-1), \text{ if } i \in \Sigma; \\ \beta_i(t), \text{ when } i \notin \Sigma \end{cases}$$

for all $t \in I$. Then the above sequence converges uniformly on I to a closed loop

$$(\beta_1, \cdots, \beta_k, \beta_{k+1}, \cdots, \beta_n)\colon I \to F_n^\Sigma X. \tag{5.14}$$

This is the final contradiction.

Let $\varepsilon > 0$ be so small that $\min(2\varepsilon_1, 2\varepsilon_2) > \varepsilon$. Since this ε has the required property, we have shown that the minimizing sequence $\{[\sigma_j]\}$ must lie in a compact subset of moduli space. We will *assume in the proof henceforth that* the sequence $\{[\sigma_j]\}$ converges when viewed in $\mathcal{M}(X_n)$.

To finish the proof of Lemma 4, one can then repeat the argument of Kra [5] (which Bers [3] has utilized originally) using the fact that there exists a sequence $\{\chi_j\}$ of elements of $\mathrm{Mod}(X_n)$ such that the sequence $\{\tau_j\}$, where $\tau_j = \chi_j([\sigma_j])$, converges. We set

$$\tau = \lim_{j\to\infty} \tau_j. \tag{5.15}$$

Since each χ_j is an isometry,

$$\left\langle \tau_j, \chi_j \circ f_\beta^* \circ \chi_j^{-1}(\tau_j) \right\rangle = \left\langle [\sigma_j], f_\beta^*([\sigma_j]) \right\rangle.$$

Thus by (5.5)

$$\lim_{j\to\infty} \left\langle \tau_j, \chi_j \circ f_\beta^* \circ \chi_j^{-1}(\tau_j) \right\rangle = a(f_\beta). \tag{5.16}$$

Together with (5.15) this implies that we may assume, by extracting a subsequence if necessary, that the sequence $\{\chi_j \circ f_\beta^* \circ \chi_j^{-1}(\tau)\}$ converges to some point of $T(X_n)$. Since $\mathrm{Mod}(X_n)$ acts properly discontinuously on $T(X_n)$, it involves no loss of generality to assume that $\chi_j \circ f_\beta^* \circ \chi_j^{-1}$ is constant (by passing to a subsequence). Setting $\chi_j = \chi$ we now conclude from (5.16) that

$$\langle \tau, \chi \circ f_\beta^* \circ \chi^{-1}(\tau) \rangle = a(f_\beta)$$

and therefore

$$\langle [\sigma], f_\beta^*([\sigma]) \rangle = a(f_\beta)$$

where $[\sigma] = \chi^{-1}(\tau)$. We are done. □

Note. After we obtained the results presented in this paper, we learned of an approach to some related problems by Bernardete-Nitecki-Gutierrez [15]. Using the epimorphism of the (*classical*) *braid group* $\mathfrak{B}_n$ onto the group $\mathfrak{U}_n$ of *isotopy classes*, or path components of the group of orientation-preserving self-homeomorphisms $f\colon (D_n, \partial D) \to (D_n, \partial D)$, where ∂D is the outer boundary of D_n (the complement of n open discs in the 2-disc D), they pull back the reducibility and periodicity questions in $\mathfrak{U}_n$ to questions in $\mathfrak{B}_n$.

As we remarked before, if a self map f of S is reduced by a closed one-dimensional submanifold C, we can restrict attention to each component S^i of $S \setminus C$, and consider the least iterate f^{ν_i} taking S^i into itself. Bers [3] has shown that every reducible mapping f is isotopic to a *completely reduced* mapping, and that f is *parabolic* if all the *component maps* of f are periodic (or trivial) and *pseudohyperbolic* if at least one component map is hyperbolic.

In discussing the Bers classification of elements of $\mathrm{Isot}(X, n)$, we shall make use of some arguments of Bernardete-Nitecki-Gutierrez [15] to study the component maps corresponding to the components of "exceptional" type with genus $= 0$.

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