The $AdS_5 \times S^5$ superstrings in the generalized light-cone gauge and the Nambu-Goto like action

$\left(\begin{array}{c}\text{一般化された光円錐ゲージに於ける}AdS_5 \times S^5\text{超弦} \\ \text{とそのNambu-Goto型作用}\end{array}\right)$

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Abstract

In this thesis, we consider the κ-symmetry-fixed Green-Schwarz action in the $AdS_5 \times S^5$ background in a version of the light-cone gauge [1]. After reviewing the generalized light-cone gauge for a bosonic sigma model, we first present the Hamiltonian dynamics of the Green-Schwarz action by using the transverse degrees of freedom. The remaining fermionic constraints are all second class. We convert the action in the phase space variables into the standard action written in terms of the fields and their derivatives [2]. We obtain a Nambu-Goto-type action which has the correct flat-space limit. In the latter half of this thesis, we consider the giant magnon in the flat space limit and near flat space limit [3]. Then, we obtain the scaled action that lead the giant magnon solution in the near flat space limit.

Contents

Chapter 1

Introduction

The AdS/CFT correspondence [4] shows that there exist the deep relation between the type IIB superstring theory in $AdS_5 \times S^5$ background and the 4-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Millls theory. Since the proposal of AdS/CFT correspondence, it has become an important issue to quantize the type IIB Green-Schwarz superstrings [5, 6] in the $AdS_5 \times S^5$ background [7]. One of the difficulties in quantizing the Green-Schwarz superstrings stems from the existence of the local κ symmetry, which halves the fermionic degrees of freedom. In the canonical Hamiltonian formalism, the local κ symmetry yields fermionic constraints. The half of these are first-class and the remaining half are second-class constraints. Covariant separation of the first and the second class constraints is a difficult task [5, 6, 8].

In the flat Minkowski target space, there was an attempt to quantize the action covariantly by introducing an infinite number of ghosts (see for example [9]). Other direction for covariant quantization is to add extra degrees of freedom in order to replace the second-class constraints with the first-class ones [10, 11, 12, 13, 14, 15, 16].

A less ambitious way to quantize the Green-Schwarz action is to abandon the covariance and to go to a non-covariant gauge. In flat target space, the Green-Schwarz action in the light-cone gauge becomes extremely simple [5, 6]. Light-cone quantization of quantum field theory was first recognized and developed in connection with the current algebra in the infinite momentum frame [17, 18, 19]. Light-cone quantization of (super)-strings played important roles in the development of string theory in seventies [20, 21, 22, 23] and that of superstring theory in eighties [5, 6]. Various gauges for the $AdS_5 \times S^5$ superstrings have been proposed [24, 25, 26, 27, 28, 29].

Recently, the Hofman-Maldacena limit [69] has attracted much attention [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]. It is a limit which takes the energy E and one of the angular momenta J infinite while keeping $E-J$ finite. E and J are eigenvalues of Cartan generators of $SO(2,4)$ and $SO(6)$ group respectively. In the Hofman-Maldacena limit, both the string and the dual gauge theory describe excitations called giant magnons and their generalizations. Good agreement in some physical quantities is found.

One way to take the Hofman-Maldacena limit is to employ a version of light-cone gauge in which $E-J$ appears as the light-cone energy. A sizable amount of literature has been accumulated which are devoted to the discussion of this gauge [58, 59, 60, 61, 29, 62, 53]. This gauge is sometimes referred to as the uniform light-cone gauge and is a generalization of that of [21] in the flat Minkowski space to the AdS background. The light-cone direction $X^{\pm}$ is chosen such that $X^{\pm}=(1/\sqrt{2})(t\pm\varphi)$, where t is the global time direction of AdS_5 and φ is a certain angle of S^5. The vectors $\partial/\partial X^{\pm}$ are Killing vectors of the target space geometry. The transverse direction manifestly keeps the covariance under a $SO(4)\times SO(4)$ subgroup of the local Lorentz group $SO(1,4)\times SO(5)$ [62]. The treatment of the fermionic second class constraints remains to be investigated however. In order to treat these remaining constraints, it is necessary to introduce the Dirac bracket.

It is a challenging problem to quantize the Green-Schwarz (GS) action [5, 6] in the $AdS_5\times S^5$ background [7]. Knowledge of the spectrum will reveal the strong coupling dynamics of the large N gauge theory through the AdS/CFT correspondence. One of the difficulties involved in the covariant quantization of the GS action stems from the existence of the local κ-symmetry, which halves the number of fermionic degrees of freedom [5, 6, 8]. One approach to solving this problem is to abandon the covariance and fix the κ-symmetry non-covariantly. But after the κ-symmetry fixing, the model is still a constrained system, due to the world-sheet diffeomorphism. Various gauges [24, 25, 26, 27, 28, 29, 57, 58, 59, 60, 61, 62] have been proposed to fix these symmetries. In particular, the uniform light-cone gauge, a generalization of the flat-space light-cone gauge [21] to a curved space background, has been extensively investigated in Refs. [57, 58, 59, 60, 61, 62].

It is natural for the Nambu-Goto-type action to appear when the world-sheet diffeomorphism is fixed by certain gauge conditions other than the conformal gauge. Solving the equations of motion for the world-sheet metric yields the Nambu-Goto-type action. For example, the Nambu-Goto-type action for the GS model in $AdS_5\times S^5$ in the static gauge can be found in Ref. [27].

This paper is organized as follows. In Chapter 2 and 3, we review minimally supersymmetry and superstring theory. In Chapter 4, we study the $AdS_5 \times S^5$ superstring in the generalized light-cone gauge as a constrained Hamiltonian system. Using the bosonic sigma model as an example, we explain also the procedure to obtain the standard Lagrangian in the generalized light-cone gauge. Next we start from the Lagrangian in the first-order formalism and arrive at the standard one and we show that the obtained Lagrangian reduces the correct form in flat-space limit. [2] In Chapter 5, we consider the giant magnon solution in the flat-space limit and near flat space limit. Then, we obtain the scaled action that lead the giant magnon solution in the near flat space limit. The chapter 5 is organized for the preparation for [3].

Chapter 2

Supersymmetry

2.1 Supersymmetry and Superspace

In this section, we introduce a superspace with x^μ, θ^α and $\bar{\theta}_{\dot{\alpha}}$ on which we can represent the algebra of $\mathcal{N}=1$ supersymmetry by using coset space method.

2.1.1 Supersymmetry

The algebra of $\mathcal{N}=1$ supersymmetry algebra is defined by

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \tag{2.1.1}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \tag{2.1.2}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = [P_\mu, P_\nu] = 0, \tag{2.1.3}$$

$$[Q_\alpha, J_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu}Q)_\alpha, \tag{2.1.4}$$

$$[\bar{Q}_{\dot{\alpha}}, J_{\mu\nu}] = -\frac{1}{2}(\bar{Q}\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}, \tag{2.1.5}$$

$$[J_{\mu\nu}, P_\lambda] = i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu), \tag{2.1.6}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}). \tag{2.1.7}$$

2.1.2 Superspace

We define the superspace as the coset space,

$$\text{superspace} = \frac{\text{super-Poincaré}}{\text{Lorentz}}$$

Introduce anti-commuting parameters θ^α and $\bar\theta^{\dot\alpha}$ which parameterize the superspace. The element of the super-Poincaré group,

$$g = \exp(-ix^\mu P_\mu + \frac{i}{2}\lambda^{\mu\nu}J_{\mu\nu}. + i\theta Q + i\bar\theta\bar Q) \tag{2.1.8}$$

Supercharges are realized as the translation generators of the anti-commuting parameters in the superspace. Coset element

$$L(x,\theta,\bar\theta) = \exp(-ix^\mu P_\mu + i\theta Q + i\bar\theta\bar Q) \tag{2.1.9}$$

Let derive the infinitesimal variation of the superspace parameters.

In the case of translation and super transformation,

$$\begin{aligned} g\cdot L(x,\theta,\bar\theta) &= e^{-iy\cdot P + i\zeta Q + i\bar\zeta\bar Q}\cdot e^{-ix\cdot P + i\theta Q + i\bar\theta\bar Q} \\ &= e^{-i(x+y)\cdot P + i(\zeta+\theta)Q + i(\bar\zeta+\bar\theta)\bar Q + \frac{1}{2}[-iy\cdot P + i\zeta Q + i\bar\zeta\bar Q, -ix\cdot P + i\theta Q + i\bar\theta\bar Q]} \end{aligned}$$

$$\begin{aligned} [-iy\cdot P + i\zeta Q + i\bar\zeta\bar Q, -ix\cdot P + i\theta Q + i\bar\theta\bar Q] &= -\zeta^\alpha\bar\theta^{\dot\alpha}\{Q_\alpha, \bar Q_{\dot\alpha}\} + \theta^\alpha\bar\zeta^{\dot\alpha}\{\bar Q_{\dot\alpha}, Q_\alpha\} \\ &= -2\zeta^\alpha(\sigma^{mu})_{\alpha\dot\alpha}P_\mu\bar\theta^{\dot\alpha} + 2\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}P_\mu\bar\zeta^{\dot\alpha} \\ &= -2\zeta\sigma^\mu\bar\theta P_\mu + 2\theta\sigma^\mu\bar\zeta P_\mu \end{aligned}$$

Hence, we find

$$\begin{aligned} g\cdot L(x,\theta,\bar\theta) &= e^{-i(x^\mu + y^\mu - i(\zeta\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\zeta))P_\mu + i(\theta+\zeta)Q + i(\bar\theta+\bar\zeta)\bar Q} \\ &= L(x + y - i\zeta\sigma\bar\theta + i\theta\sigma\bar\zeta, \theta+\zeta, \bar\theta+\bar\zeta). \end{aligned} \tag{2.1.10}$$

From that we can read off

$$\begin{aligned} x'^\mu &= x^\mu + y^\mu - i\zeta\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\zeta, \\ \theta'^\alpha &= \theta^\alpha + \zeta^\alpha, \\ \bar\theta'^{\dot\alpha} &= \bar\theta^{\dot\alpha} + \bar\zeta^{\dot\alpha}. \end{aligned} \tag{2.1.11}$$

Because supercharges Q and $\bar Q$ do not anti-commute each other, unlike P, the translations of the anti-commuting parameters θ and $\bar\theta$ contribute those of the commuting parameters x.

2.1.3 Superfields

The superfield ϕ is defined as the quantum field on superspace and can be written at each point of the superspace as

$$\phi(x,\theta,\bar\theta) = L(x,\theta,\bar\theta)\phi(0,0,0)L^{-1}(x,\theta,\bar\theta), \tag{2.1.12}$$

using the value at the origin. Infinitesimal transformation

$$\begin{aligned}\delta_\epsilon \phi &= \phi'(x', \theta', \bar{\theta}') - \phi(x, \theta, \bar{\theta}) \\ &= e^{i\epsilon I}\phi(x, \theta, \bar{\theta})e^{-i\epsilon I} - \phi(x, \theta, \bar{\theta}) \\ &= i[\epsilon I, \phi] \end{aligned} \tag{2.1.13}$$

Let us concretely construct the representation of each generator of the supersymmetry on the superspace. The translation P is defined as

$$\delta_\epsilon \phi = -i[\epsilon^\mu P_\mu, \phi] = i\epsilon^\mu \rho(P_\mu)\phi,$$

where $\rho(P_\mu)$ is a representation of P_μ. Then, we have

$$\begin{aligned}\delta_\epsilon \phi &= \phi(x + \epsilon, \theta, \bar{\theta}) - \phi(x, \theta, \bar{\theta}) \\ &= \epsilon^\mu \partial_\mu \phi \\ &= i\epsilon^\mu(-i\partial_\mu)\phi \end{aligned} \tag{2.1.14}$$

Hence, we obtain

$$\rho(P_\mu) = -i\partial_\mu \tag{2.1.15}$$

The supercharge Q is defined as

$$\delta_\zeta \phi = i[\zeta Q, \phi] = -i\zeta^\alpha \rho(Q_\alpha)\phi,$$

where $\rho(Q_\alpha)$ is a representation of Q_α on the superspace. Then, we have

$$\begin{aligned}\delta_\zeta \phi &= \phi(x - i\zeta\sigma\bar{\theta}, \theta + \zeta, \bar{\theta}) - \phi(x, \theta, \bar{\theta}) \\ &= -i\zeta\sigma^\mu\bar{\theta}\partial_\mu\phi + \zeta^\alpha \frac{\partial}{\partial\theta^\alpha}\phi \\ &= -i\zeta^\alpha \left\{ i\frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\bar{\theta})_\alpha \partial_\mu \right\} \phi \end{aligned} \tag{2.1.16}$$

Hence, we obtain

$$\rho(Q_\alpha) = i\frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\bar{\theta})_\alpha\partial_\mu \tag{2.1.17}$$

The supercharge $\bar{Q}$ conjugate to Q is defined as

$$\delta_\zeta \phi = i[\bar{\zeta}\bar{Q}, \phi] = i\bar{\zeta}^{\dot{\alpha}}\rho(\bar{Q}_{\dot{\alpha}})\phi,$$

where $\rho(\bar{Q}_\alpha)$ is a representation of $\bar{Q}_\alpha$ on the superspace. Then, we have

$$\begin{aligned}\delta_{\bar{\zeta}}\phi &= \phi(x + i\theta\sigma\bar{\zeta}, \theta, \bar{\theta} + \bar{\zeta}) - \phi(x, \theta, \bar{\theta}) \\ &= i\theta\sigma^\mu\bar{\zeta}\partial_\mu\phi + \bar{\zeta}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\phi \\ &= i\bar{\zeta}^{\dot{\alpha}}\left\{-i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\right\}\phi.\end{aligned} \tag{2.1.18}$$

Hence, we obtain

$$\rho(\bar{Q}_{\dot{\alpha}}) = -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu. \tag{2.1.19}$$

Indeed, the anti-commuting relation between $\rho(Q)$ and $\rho(\bar{Q})$ is calculated as

$$\begin{aligned}\{\rho(Q_\alpha), \rho(\bar{Q}_{\dot{\alpha}})\} &= \{i\frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\bar{\theta})_\alpha\partial_\mu, -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\} \\ &= -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu \\ &= 2(\sigma^\mu)_{\alpha\dot{\alpha}}\rho(P_\mu),\end{aligned} \tag{2.1.20}$$

which is certainly the anti-commuting relation between supercharges. Below we omit the symbol ρ.

General superfield is expanded in terms of θ and $\bar{\theta}$,

$$\begin{aligned}F(x, \theta, \bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) \\ &+ \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x)\end{aligned} \tag{2.1.21}$$

Therefore, the general superfield contains 16(bosonic) + 16(fermionic) field components.(see the table.2.1)

Table 2.1: The component fields of the general superfield

4 complex scalars	$f,\ m,\ n,\ d$
1 complex vector	v_μ
2 $(\frac{1}{2}, 0)$ spinors	$\phi,\ \psi$
2 $(0, \frac{1}{2})$ spinors	$\bar{\chi},\ \bar{\lambda}$

2.1.4 Covariant Spinor Derivatives

Notice the associativity of the group elements,

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3. \qquad g_1, g_2, g_3 \in G$$

This nature means that the right action commute with the left action,

$$[L_{g_1}, R_{g_3}] = 0$$

One can construct the covariant derivatives using this fact in the following. First we consider the left action of $L(\epsilon, \zeta, \bar{\zeta})$,

$$L(\epsilon, \zeta, \bar{\zeta})L(x, \theta, \bar{\theta}) \simeq [1 - i\epsilon^\mu P_\mu + i\zeta^\alpha Q_\alpha - i\bar{\zeta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}]L(x, \theta, \bar{\theta}) \tag{2.1.22}$$

On the other hand, the right action becomes

$$\begin{aligned} L(x, \theta, \bar{\theta})L(\epsilon, \zeta, \bar{\zeta}) &= L(x + \epsilon - i\theta\sigma\bar{\zeta} + i\zeta\sigma\bar{\theta}, \theta + \zeta, \bar{\theta} + \bar{\zeta}) \\ &\simeq \left[1 + (\epsilon^\mu - i\theta\sigma^\mu\bar{\zeta} + i\zeta\sigma^\mu\bar{\theta})\partial_\mu + \zeta^\alpha\frac{\partial}{\partial\theta^\alpha} + \bar{\zeta}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right] L(x, \theta, \bar{\theta}) \\ &= \left[1 + \epsilon^\mu\partial_\mu + \zeta^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\right) + \bar{\zeta}^{\dot{\alpha}}\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\right)\right] L(x, \theta, \bar{\theta}) \\ &\equiv (1 + \epsilon^\mu D_\mu + \zeta^\alpha D_\alpha - \bar{\zeta}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}})L(x, \theta, \bar{\theta}). \end{aligned} \tag{2.1.23}$$

We denote the covariant derivatives as

$$D_A = (D_\mu, D_\alpha, \bar{D}_{\dot{\alpha}}),$$

where

$$\begin{aligned} D_\mu &= \partial_\mu \\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \end{aligned} \tag{2.1.24}$$

By definition, it is obvious that

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\alpha\} = \{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{2.1.25}$$

Moreover the commutators for D's are

$$\begin{aligned} \{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= \{\frac{\partial}{\partial\theta^\alpha}, -i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\} + \{i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu, -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\} \\ &= -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu, \end{aligned} \tag{2.1.26}$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \tag{2.1.27}$$

2.1.5 Chiral Superfields

The chiral superfield is defined as the function which satisfies the condition,

$$\bar{D}_{\dot{\alpha}}\Phi = 0, \tag{2.1.28}$$

while the anti-chiral superfield is defined by the condition,

$$D_\alpha\bar{\Phi} = 0. \tag{2.1.29}$$

Introduce new variables $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then, because

$$\bar{D}_{\dot{\alpha}}(y^\mu) = 0 \quad \text{and} \quad \bar{D}_{\dot{\alpha}}\theta = 0,$$

any function written by only y^μ and θ automatically satisfy the condition (2.1.28). Using this fact, the chiral superfield is written by

$$\Phi = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{2.1.30}$$

The character of the component fields is summarized in the table 2.2.

Table 2.2: the component fields of chiral superfield

2 complex scalars	$A,\ F$
1 $(\frac{1}{2}, 0)$ spinor	ψ

Because of

$$\frac{\partial y^\mu}{\partial\theta^\alpha} = i(\sigma^\mu\bar{\theta})_\alpha, \tag{2.1.31}$$

$$\frac{\partial y^\mu}{\partial\bar{\theta}^{\dot{\alpha}}} = -i(\theta\sigma^\mu)_{\dot{\alpha}}, \tag{2.1.32}$$

$$\frac{\partial\Phi}{\partial\theta^\alpha} = \sqrt{2}\psi_\alpha(y) + 2\theta_\alpha F(y), \tag{2.1.33}$$

the variation of the chiral superfield under supersymmetric transformation is

$$\begin{aligned}
\delta\Phi &= \delta_\zeta\Phi + \delta_{\bar\zeta}\Phi \\
&= -i\zeta Q\Phi + i\bar\zeta^{\dot\alpha}\bar Q_{\dot\alpha}\Phi \\
&= \zeta^\alpha\left\{\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right\}\Phi + \bar\zeta^{\dot\alpha}\left\{\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\right\}\Phi \\
&= \zeta^\alpha\frac{\partial\Phi}{\partial y^\mu}\frac{\partial y^\mu}{\partial\theta^\alpha} + \zeta^\alpha\frac{\partial\Phi}{\partial\theta^\alpha} - i\zeta\sigma^\mu\bar\theta\frac{\partial\Phi}{\partial y^\mu} + \bar\zeta^{\dot\alpha}\frac{\partial\Phi}{\partial y^\mu}\frac{\partial y^\mu}{\partial\bar\theta^{\dot\alpha}} + i\theta\sigma^\mu\bar\zeta\frac{\partial\Phi}{\partial y^\mu} \\
&= (i\zeta\sigma^\mu\bar\theta - i\zeta\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\zeta + i\theta\sigma^\mu\bar\zeta)\partial_\mu\Phi + \sqrt{2}\zeta\psi(y) + 2\zeta\theta F(y) \\
&= 2i\theta\sigma^\mu\bar\zeta\partial_\mu\Phi + \sqrt{2}\zeta\psi(y) + 2\zeta\theta F(y) \\
&= \sqrt{2}\zeta\psi(y) + \sqrt{2}\theta^\alpha(i\sqrt{2}(\sigma^\mu\bar\zeta)_\alpha\partial_\mu A(y) + \sqrt{2}\zeta_\alpha F(y)) + 2i\theta\sigma^\mu\bar\zeta\sqrt{2}\theta^\alpha\partial_\mu\psi_\alpha(y) \\
&= \sqrt{2}\zeta\psi(y) + \sqrt{2}\theta^\alpha(i\sqrt{2}(\sigma^\mu\bar\zeta)_\alpha\partial_\mu A(y) + \sqrt{2}\zeta_\alpha F(y)) - 2\sqrt{2}i\theta^\beta\theta^\alpha(\sigma^\mu\bar\zeta)_\beta\partial_\mu\psi_\alpha(y) \\
&= \sqrt{2}\zeta\psi(y) + \sqrt{2}\theta^\alpha(i\sqrt{2}(\sigma^\mu\bar\zeta)_\alpha\partial_\mu A(y) + \sqrt{2}\zeta_\alpha F(y)) + \theta\theta i\sqrt{2}(\bar\zeta\bar\sigma^\mu)^\alpha\partial_\mu\psi_\alpha(y),
\end{aligned}$$

where we used the relation

$$\begin{aligned}
\theta\theta &= \theta^1\theta_1 + \theta^2\theta_2 \\
&= 2\theta^2\theta^1 \\
&\quad\to \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta
\end{aligned}$$

Hence, the infinitesimal variations of the component field for the chiral superfield are

$$\delta A = \sqrt{2}\zeta\psi, \tag{2.1.34}$$

$$\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar\zeta)_\alpha\partial_\mu A + \sqrt{2}\zeta_\alpha F, \tag{2.1.35}$$

$$\delta F = i\sqrt{2}(\bar\zeta\bar\sigma^\mu)^\alpha\partial_\mu\psi_\alpha. \tag{2.1.36}$$

Chapter 3

String Theory

3.1 Green-Schwarz Superstring

In this section, we review the superstring theory. As simple example, we review the case of particle, which is described by the action,

$$S = -m \int d\tau (\dot{X}^\mu \dot{X}_\mu)^{\frac{1}{2}}. \tag{3.1.1}$$

This action has Poincarè invariance and reparameterization invariance.

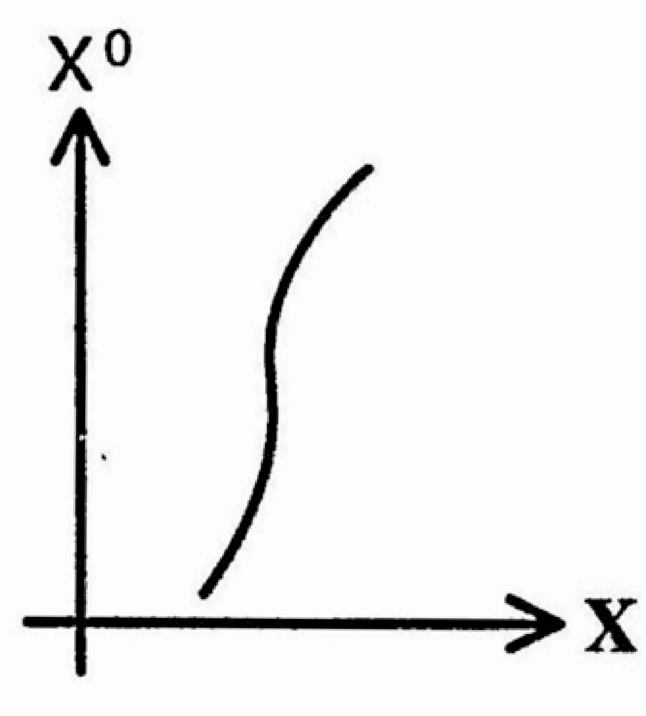


Figure 3.1: orbit of particle in Minkowski spacetime

The equation of motion for X^μ is written by

$$\dot{u}^\mu = 0, \tag{3.1.2}$$

where

$$u^\mu = \frac{1}{\sqrt{\dot{X}^\mu \dot{X}_\mu}} \dot{X}^\mu \tag{3.1.3}$$

is the normalized D-velocity. The action S_{pp} in the nonrelativistic limit, that is $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \ll 1$, has the usual nonrelativistic form,

$$\begin{aligned} S_{pp} &= -m \int \sqrt{dt^2 - d\boldsymbol{x}^2} \\ &= -m \int dt\sqrt{1-\boldsymbol{v}^2} \\ &\sim \int dt \left\{ \frac{m}{2}\boldsymbol{v}^2 - m \right\}. \end{aligned} \tag{3.1.4}$$

We can rewrite the action by introducing an auxiliary coordinate e as

$$S'_{pp} = \frac{1}{2} \int d\tau (e^{-1}\dot{X}^\mu \dot{X}_\mu - em^2). \tag{3.1.5}$$

The additional field e can be identified as the square root of a one-dimensional metric,

$$e = (-\gamma_{\tau\tau}(\tau))^{\frac{1}{2}}, \tag{3.1.6}$$

where $\gamma_{\tau\tau}(\tau)$ is the one-dimensional metric. In order to see that this action S'_{pp} is equivalent to the action S_{pp}, we have to eliminate the auxiliary field e by using that equation of motion. Taking the variation of S_{pp} for e

$$\delta_e S'_{pp} = \frac{1}{2} \int (-e^{-2}\dot{X}^2 - m^2)\delta e, \tag{3.1.7}$$

we can obtain its equation of motion

$$e^2 = -\frac{1}{m^2}\dot{X}_\mu \dot{X}^\mu. \tag{3.1.8}$$

Hence,

$$\begin{aligned} S'_{pp} &= \frac{1}{2} \int d\tau \left\{ \left(-\frac{1}{m^2}\dot{X}^2\right)^{-\frac{1}{2}} \dot{X}^2 - \left(-\frac{1}{m^2}\dot{X}^2\right)^{\frac{1}{2}} m^2 \right\} \\ &= -m \int d\tau \sqrt{-\dot{X}^2} \\ &= S_{pp}. \end{aligned} \tag{3.1.9}$$

Therefore, it is understood that the actions S'_{pp} and S_{pp} are classically equivalent at least. Moreover it is easy to see that the action S'_{pp} has the same symmetries as the earlier one S_{pp}, i.e. Poincarè and reparameterization symmetries. A point that is better than S_{pp} of

S'_{pp} is that it has a smooth massless limit $m \to 0$. In the following, to generalize to the action which describe the superparticle we set $m = 0$,

$$S'_{pp} = \frac{1}{2} \int d\tau e^{-1} \dot{X}^\mu \dot{X}_\mu \tag{3.1.10}$$

We can achieve the supersymmetric version of (3.1.10) by generalizing Minkowski spacetime with bosonic coordinates X^μ, to a superspace with fermionic coordinates as well as bosonic ones. If there are to be $\mathcal{N}$ supersymmetries, we introduce $\mathcal{N}$ anticommuting spinor coordinates $\theta^{Aa}(\tau)$, $A = 1, 2, \cdots, \mathcal{N}$. The indices a express that $\theta^{Aa}(\tau)$ is space-time spinor appropriate to D dimensions. As already mentioned In the last section, supersymmetry transformation is realized by the translation of the anticommutiong coordinates θ^A in the superspace

In this section, we introduce the supersymmetry whose infinitesimal variation has the following form:

$$\delta\theta^A = \epsilon^A, \qquad \delta X^\mu = i\bar{\epsilon}^A \Gamma^\mu \theta^A \tag{3.1.11}$$

$$\delta\bar{\theta}^A = \bar{\epsilon}^A, \qquad \delta e = 0 \tag{3.1.12}$$

with infinitesimal anticommuting parameters ϵ^A which are τ-independent spacetime spinors.

To make the supersymmetric action, we must construct the pieces of supersymmetric invariant. First we consider the element of the translation group of Minkowski spacetime,

$$G = \exp(iP_\mu X^\mu) \tag{3.1.13}$$

By using this, we construct the Cartan 1-form which is left invariant form by definition,

$$\begin{aligned} G^{-1}dG &= idX^\mu \exp(-iP_\nu X^\nu) P_\mu \exp(iP_\sigma X^\sigma) \\ &= idX^\mu P_\mu. \end{aligned} \tag{3.1.14}$$

So it is understood that the action (3.1.10) is constituted by the component of (3.1.14). Similarly, let us construct the supersymmetric action. In this case, the appropriate element of the symmetry group is

$$G = \exp(iP_\mu X^\mu + i\theta Q + i\bar{\theta}\bar{Q}) \tag{3.1.15}$$

The corresponding components of its Cartan 1-form are

$$\begin{aligned} G^{-1}dG|_P &= idX^\mu \exp(-iP_\nu X^\nu - i\theta Q - i\bar{\theta}\bar{Q}) P_\mu \exp(iP_\sigma X^\sigma + i\theta Q + i\bar{\theta}\bar{Q}) \\ &= idX^\mu P_\mu \end{aligned} \tag{3.1.16}$$

$$
\begin{aligned}
G^{-1}dG|_Q &= id\theta^{aA}\exp(-iP_\nu X^\nu - i\theta Q - i\bar{\theta}\bar{Q})Q_a^A \exp(iP_\sigma X^\sigma + i\theta Q + i\bar{\theta}\bar{Q}) \\
&= id\theta^{aA}\left(Q_a^A + \frac{1}{2}\bar{\theta}^{bB}\{Q_a^A, \bar{Q}_b^B\}\right) \\
&= id\theta^{aA}\bar{\theta}^{bB}\sigma^\mu_{ab}\delta^{AB}P_\mu + id\theta^{aA}Q_a^A \\
&= -i\bar{\theta}^A\sigma^\mu d\theta^A P_\mu + id\theta Q,
\end{aligned}
\tag{3.1.17}
$$

$$
G^{-1}dG|_{\bar{Q}} = i\theta^A\bar{\sigma}^\mu d\bar{\theta}^A P_\mu + id\bar{\theta}\bar{Q},
\tag{3.1.18}
$$

Finally we obtain

$$
G^{-1}dG = i(dX^\mu - \bar{\theta}^A\Gamma^\mu d\theta^A)P_\mu + id\theta Q + id\bar{\theta}\bar{Q}
\tag{3.1.19}
$$

Therefore the both $\dot{\theta}^A$ and $\dot{X}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A$ are invariant under supersymmetry. In fact, the term $\dot{\theta}^A$ is obviously invariant and because of

$$
\begin{aligned}
\delta(\dot{X}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A) &= i\bar{\epsilon}^A\Gamma^\mu\dot{\theta}^A - i\bar{\epsilon}^A\Gamma^\mu\dot{\theta}^A \\
&= 0,
\end{aligned}
\tag{3.1.20}
$$

so is the latter. The most straightforward generalization of (3.1.10) is simply given by

$$
S = \frac{1}{2}\int d\tau e^{-}(\dot{X}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A)^2
\tag{3.1.21}
$$

This action is Lorentz invariant and supersymmetric and so has the full super-Poincarè symmetry. The equations of motion for e and X^μ are given by, respectively

$$
p^2 = 0, \qquad \dot{p}^\mu = 0,
\tag{3.1.22}
$$

where

$$
p^\mu = \dot{X}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A.
\tag{3.1.23}
$$

On the other hand, the equation of motion for $\bar{\theta}$ is

$$
\Gamma \cdot p\,\dot{\theta} = 0.
\tag{3.1.24}
$$

Since $(\Gamma \cdot p)^2 = -p^2 = 0$, the matrix $\Gamma \cdot p$ has half the maximum possible rank. For this fact, half of its components are actually decoupled from the theory. This is a consequence of a far from obvious additional symmetry of (3.1.21). This symmetry is new fermionic symmetry, so called κ symmetry.

Suppose that $\kappa^{Aa}(\tau)$ denote $\mathcal{N}$ infinitesimal anti-commuting parameters. Note that κ^A has τ dependency. Its infinitesimal transformation is defined by

$$\delta\theta^A = i\Gamma \dot{p}\kappa^A, \tag{3.1.25}$$

$$\delta X^\mu = i\bar{\theta}^A\Gamma^\mu\delta\theta^A, \tag{3.1.26}$$

$$\delta e = 4e\dot{\bar{\theta}}^A\kappa^A. \tag{3.1.27}$$

3.1.1 Superstring action

It is easy that the point-particle action can be generalized to the string. Namely, the bosonic string action can be defined by

$$S_b = -\frac{1}{2\pi}\int d^2\xi\sqrt{-\gamma}\gamma^{ab}G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu. \tag{3.1.28}$$

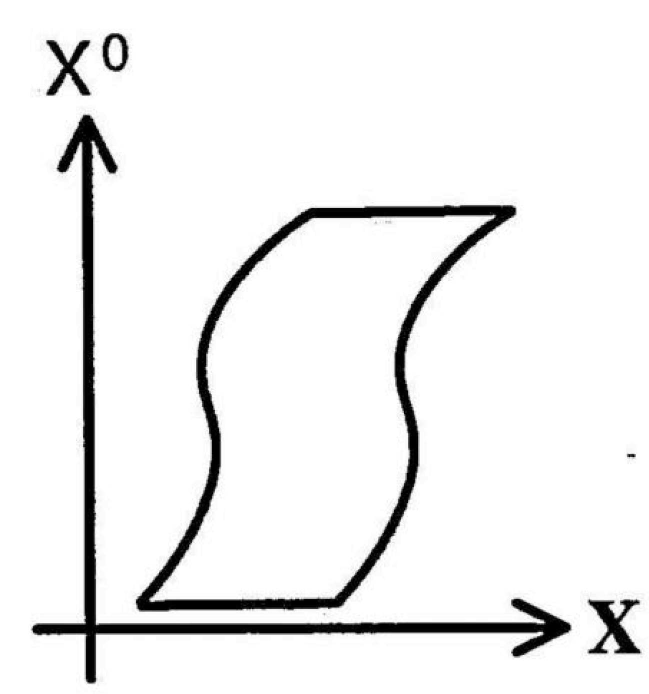


Figure 3.2: orbit of string in Minkowski spacetime

The action is diffeomorphism invariance which are the reparameterization of world-sheet coordinates, Weyl invariance,

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma),$$
$$\gamma'_{ab} = e^{2\omega(\tau,\sigma)}\gamma_{ab}, \tag{3.1.29}$$

and D-dimensional Poincarè invariance.

The energy momentum tensor is defined by the variation of the action with respect to the metric,

$$T_{ab} \sim \frac{\delta}{\delta\gamma^{ab}}S. \tag{3.1.30}$$

Similarly for the case of the point-particle, The supersymmetric action is obtained by replacing $\partial_a X^\mu$ to

$$\Pi^\mu_a = \partial_a X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_a\theta^A. \tag{3.1.31}$$

Hence the action is

$$S = -\frac{1}{2\pi}\int d^2\xi\sqrt{-\gamma}\gamma^{ab}G_{\mu\nu}\Pi_a^\mu\Pi_b^\nu. \qquad (3.1.32)$$

This obviously possesses local reparameterization invariance and $\mathcal{N}$ global supersymmetries and spacetime Lorentz symmetry. But the local κ symmetry which the superparticle action has is lost in this simple replacement. So we want to construct the κ symmetric action by add the extra terms,

$$S_{extra} = \frac{1}{\pi}\int d^2\xi(-i\epsilon^{ab}\partial_a X^\mu(\bar\theta^1\Gamma_\mu\partial_b\theta^1 - \bar\theta^2\Gamma_\mu\partial_b\theta^2) + \epsilon^{ab}\bar\theta^1\Gamma^\mu\partial_a\theta^1\bar\theta^2\Gamma^\mu\partial_b\theta^2) \qquad (3.1.33)$$

This extra term does not depend to the worldsheet metric γ^{ab} so that its term has no contributions to the energy-momentum tensor T_{ab}.

3.2 Two-Dimensional Sigma Model

3.2.1 O(N) Sigma Model

Suppose that two-dimensional surface is parameterized by $\{\sigma.\tau\}$, and the metric is $\eta = \text{diag}\{-1, 1\}$

In this section, we consider the two-dimensional theory with S^{N-1} as target space, so called $O(N)$ sigma model. The Lagrangian is defined as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i, \qquad (3.2.1)$$

with constraint

$$\phi_i\phi_i = 1, \qquad (3.2.2)$$

where $i = 1, 2, \cdots, N$. Here and in the following, the sum on repeated indices is implied.

To take the constraint into account , introduce the Lagrange Multiplier λ. Then the Lagrangian is written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \lambda(\phi_i\phi_i - 1) \qquad (3.2.3)$$

The equation of motion for ϕ is

$$-\partial_\mu\partial^\mu\phi_i + 2\lambda\phi_i = 0 \qquad (3.2.4)$$

The variation with respect to λ derives also the constraint (3.2.2). Multiplying (3.2.4) with ϕ_i and then summing up about the subscript i, we obtain

$$\lambda = \frac{1}{2}\phi_i\partial_\mu\partial^\mu\phi_i = -\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i, \qquad (3.2.5)$$

where we used (3.2.2). After eliminating λ the equation of motion (3.2.4) becomes, therefore,

$$\partial_\mu \partial^\mu \phi_i + \partial_\mu \phi_j \partial^\mu \phi_j \phi_i = 0. \tag{3.2.6}$$

Let consider the $O(3)$ sigma model as example. Introduce the light-cone variables,

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}(\tau + \sigma), \\ x^- &= \frac{1}{\sqrt{2}}(\tau - \sigma). \end{aligned} \tag{3.2.7}$$

In this coordinates, the metric and (3.2.6) become, respectively,

$$g = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}, \tag{3.2.8}$$

and

$$\partial_+ \partial_- \phi_i + (\partial_+ \phi_j \partial_- \phi_j) \phi_i = 0. \tag{3.2.9}$$

Because of

$$\phi_i \partial_+ \phi_i = \phi_i \partial_- \phi_i = 0, \tag{3.2.10}$$

the 3-vector $\vec{\phi}$ is perpendicular to both of $\partial_+\vec{\phi}$ and $\partial_-\vec{\phi}$. Moreover we can normalize $\partial_+\vec{\phi}$ and $\partial_-\vec{\phi}$ as follows;

$$(\partial_+ \phi_i)^2 = (\partial_- \phi_i)^2 = 1, \tag{3.2.11}$$

which are always possible if the theory is conformally invariance. In addition, suppose that the angle between $\partial_+\vec{\phi}$ and $\partial_-\vec{\phi}$ is α,

$$\cos\alpha = \frac{(\partial_+\vec{\phi}) \cdot (\partial_-\vec{\phi})}{|\partial_+\vec{\phi}||\partial_-\vec{\phi}|}. \tag{3.2.12}$$

Then we have the normal basis of S^2,

$$\{\vec{\phi}, \partial_+\vec{\phi}, \partial_-\vec{\phi}\} \tag{3.2.13}$$

Thus $\partial_+^2 \phi_i$ are written in terms of ϕ_i, $\partial_+\phi_i$ and $\partial_+\phi_i$,

$$\partial_+^2 \phi_i = A\phi_i + B\partial_- \phi_i + C\partial_+ \phi_i \tag{3.2.14}$$

The equations we must solve are

$$A = \phi_i \partial_+^2 \phi_i = -(\partial_+ \phi_i)^2 = -1, \tag{3.2.15}$$

$$B + C\cos\alpha = \partial_- \phi_i \partial_+^2 \phi_i = \partial_+ \cos\alpha + (\partial_+ \partial_- \phi_i)\partial_+ \phi_i = \partial_+ \cos\alpha, \tag{3.2.16}$$

$$B\cos\alpha + C = \partial_+ \phi_i \partial_+^2 \phi_i = -(\partial_+^2 \phi_i)\partial_+ \phi_i = 0. \tag{3.2.17}$$

Consequently we find (3.2.14) is

$$\partial_+^2 \phi_i = -\phi_i - \frac{\partial_+ \alpha}{\sin \alpha} \partial_- \phi_i + \cot \alpha \partial_+ \alpha \partial_+ \phi_i. \tag{3.2.18}$$

Similarly,

$$\partial_-^2 \phi_i = -\phi_i - \frac{\partial_- \alpha}{\sin \alpha} \partial_+ \phi_i + \cot \alpha \partial_- \alpha \partial_- \phi_i. \tag{3.2.19}$$

Differentiating (3.2.18) with respect to x^-, we obtain

$$\begin{aligned} \partial_- \partial_+^2 \phi_i &= \partial_+ (\partial_- \partial_+ \phi_i) \\ &= -\partial_+ (\cos \alpha \phi_i) \\ &= -\partial_+ \cos \alpha \phi_i - \cos \alpha \partial_+ \phi_i \end{aligned} \tag{3.2.20}$$

Hence,

$$\partial_- \phi_i \partial_- \partial_+^2 \phi_i = -\cos^2 \alpha \tag{3.2.21}$$

On the other hand,

$$\begin{aligned} \partial_- \phi_i \partial_- \partial_+^2 \phi_i &= \partial_- (\partial_- \phi_i \partial_+^2 \phi_i) - \partial_-^2 \phi_i \partial_+^2 \phi_i \\ &= \partial_- \partial_+ \cos \alpha - \left\{ 1 - \left(2 \frac{\cos \alpha}{\sin^2 \alpha} - \frac{\cos \alpha}{\sin^2 \alpha} - \frac{\cos^3 \alpha}{\sin^2 \alpha} \right) \partial_+ \alpha \partial_- \alpha \right\} \\ &= -\sin \alpha \partial_- \partial_+ \alpha - \cos \alpha \partial_+ \alpha \partial_- \alpha - \{ 1 - \cos \alpha \partial_+ \alpha \partial_- \alpha \} \\ &= -\sin \alpha \partial_- \partial_+ \alpha - 1 \end{aligned} \tag{3.2.22}$$

We get the sine-Gordon equation,

$$\partial_- \partial_+ \alpha + \sin \alpha = 0 \tag{3.2.23}$$

Now we consider the $O(N)$ sigma model with the coupling to fermions by using supersymmetry. The real superfields Φ_i are expanded in terms of θ as

$$\Phi_i(x, \theta) = \varphi_i(x) + \sqrt{2} \bar{\theta} \psi_i(x) + \bar{\theta} \theta F_i(x) \tag{3.2.24}$$

The superfields Φ_i satisfy the condition

$$\Phi_i \Phi_i = 1. \tag{3.2.25}$$

From the explicit calculation,

$$\begin{aligned} \Phi_i \Phi_i &= \left\{ \varphi_i + \sqrt{2} \bar{\theta} \psi_i + \bar{\theta} \theta F_i \right\} \left\{ \varphi_i + \sqrt{2} \bar{\theta} \psi_i + \bar{\theta} \theta F_i \right\} \\ &= \varphi_i \varphi_i + 2\sqrt{2} \bar{\theta} \psi_i \phi_i + 2\bar{\theta} \theta \varphi_i F_i + 2 \bar{\theta} \psi_i \bar{\theta} \psi_i \\ &= \varphi_i \varphi_i + 2\sqrt{2} \bar{\theta} \psi_i \phi_i + 2\bar{\theta} \theta \varphi_i F_i - \bar{\theta} \theta \bar{\psi}_i \psi_i, \end{aligned}$$

we find the conditions for component fields are

$$\varphi_i\varphi_i = 1, \quad \psi_i\varphi_i = 0, \quad \varphi_i F_i = \frac{1}{2}\bar{\psi}_i\psi_i. \tag{3.2.26}$$

The 2-dim. covariant spinor derivatives are given by

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu. \tag{3.2.27}$$

Hence operating D_α to the superfield Φ_i, we get

$$\begin{aligned}
D_\alpha\Phi_i &= \left\{\frac{\partial}{\partial\bar{\theta}^\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu\right\}\left\{\varphi_i + \sqrt{2}\bar{\theta}\psi_i + \bar{\theta}\theta F_i\right\} \\
&= \sqrt{2}\psi_{i\alpha} + 2\theta_\alpha F_i - i(\gamma^\mu\theta)_\alpha\partial_\mu\varphi_i - i\sqrt{2}(\gamma^\mu\theta)_\alpha(\bar{\theta}\partial_\mu\psi_i) \\
&= \sqrt{2}\psi_{i\alpha} + 2\theta_\alpha F_i - i(\gamma^\mu\theta)_\alpha\partial_\mu\varphi_i + \frac{i}{\sqrt{2}}\bar{\theta}\theta(\partial\!\!\!/\psi_i)_\alpha
\end{aligned}$$

and its conjugation,

$$\overline{D_\alpha\Phi_i} = (D_\beta\Phi_i)^\dagger(\gamma^0)_{\beta\alpha} \tag{3.2.28}$$

$$= \left\{\sqrt{2}\psi^\dagger_{i\beta} + 2\theta^\dagger_\beta F_i + i(\theta^\dagger\gamma^{\mu\dagger})_\beta\partial_\mu\varphi_i - \frac{i}{\sqrt{2}}\bar{\theta}\theta(\psi_i^\dagger\overleftarrow{\partial\!\!\!/}^\dagger)_\beta\right\}(\gamma^0)_{\beta\alpha} \tag{3.2.29}$$

$$= \sqrt{2}\bar{\psi}_{i\alpha} + 2\bar{\theta}_\alpha F_i + i(\bar{\theta}\gamma^\mu)_\alpha\partial_\mu\varphi_i - \frac{i}{\sqrt{2}}\bar{\theta}\theta(\bar{\psi}_i\overleftarrow{\partial\!\!\!/})_\alpha. \tag{3.2.30}$$

When we take these product, it becomes

$$\begin{aligned}
\overline{D_\alpha\Phi_i}D_\alpha\Phi_i\big|_{\bar{\theta}\theta} &= i\bar{\theta}\theta\bar{\psi}_i\partial\!\!\!/\psi_i + 4\bar{\theta}\theta F_iF_i + (\bar{\theta}\gamma^\mu\gamma^\nu\theta)\partial_\mu\varphi_i\partial_\nu\varphi_i - i\bar{\theta}\theta\bar{\psi}_i\overleftarrow{\partial\!\!\!/}\psi_i\Big|_{\bar{\theta}\theta} \\
&= \bar{\theta}\theta\left\{\partial_\mu\varphi_i\partial^\mu\varphi_i + i\bar{\psi}_i\overleftrightarrow{\partial\!\!\!/}\psi_i + 4F_iF_i\right\}
\end{aligned}$$

By using this, the supersymmetric action is given by

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8}\int d\bar{\theta}d\theta\overline{D_\alpha\Phi_i}D_\alpha\Phi_i \\
&= \frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i + \frac{i}{2}\bar{\psi}_i\overleftrightarrow{\partial\!\!\!/}\psi_i + 2F_iF_i
\end{aligned} \tag{3.2.31}$$

3.2.2 Bosonic sigma model in general background

Moreover we can consider the string theory in general background. The action is, of course

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\xi\,\mathcal{L}, \tag{3.2.32}$$

where the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{\underline{m}\underline{n}}(X)\partial_i X^{\underline{m}}\partial_j X^{\underline{n}}. \tag{3.2.33}$$

We assume that the target space is D-dimensional: $X^{\underline{m}} = X^{\underline{m}}(\xi)$, ($\underline{m} = 0, 1, \ldots, D-1$), and $G_{\underline{m}\underline{n}}(X)$ is the metric of the target space. Here

$$(\xi^0, \xi^1) = (\tau, \sigma), \qquad h^{ij} = \sqrt{-g}g^{ij}, \qquad i, j = 0, 1, \tag{3.2.34}$$

and λ is the coupling constant. h^{ij} is the Weyl-invariant combination of the world-sheet metric g_{ij}. Since $\det h^{ij} = -1$, we choose h^{00} and h^{01} as the independent Lagrange multipliers. We also use the following notation for the Lagrange multipliers: $e^0 = 1/(2h^{00})$, $e^1 = (h^{01}/h^{00})$.

Equations of motion for this model are given by

$$\partial_i(h^{ij}G_{\underline{m}\underline{n}}\partial_j X^{\underline{n}}) = \frac{1}{2}h^{ij}(\partial_{\underline{m}}G_{\underline{k}\underline{l}})\partial_i X^{\underline{k}}\partial_j X^{\underline{l}}. \tag{3.2.35}$$

Here $\partial_{\underline{m}} = \partial/\partial X^{\underline{m}}$.

Let us introduce the conjugate momenta by $P_{\underline{m}} := \partial\mathcal{L}/\partial\dot{X}^{\underline{m}}$.

$$P_{\underline{m}} = -\sqrt{\lambda}G_{\underline{m}\underline{n}}h^{0i}\partial_i X^{\underline{n}}. \tag{3.2.36}$$

Let $G^{\underline{m}\underline{n}}$ be the inverse of the metric $G_{\underline{m}\underline{n}}$. The equations of motion (3.2.35) and the definition of the conjugate momenta (3.2.36) can be converted into the equations of motion in the first-order form:

$$\begin{aligned}
\dot{X}^{\underline{m}} &= -\frac{1}{\sqrt{\lambda}h^{00}}G^{\underline{m}\underline{n}}P_{\underline{n}} - \left(\frac{h^{01}}{h^{00}}\right)\partial_1 X^{\underline{m}}, \\
\dot{P}_{\underline{m}} &= \partial_1\left[-\left(\frac{h^{01}}{h^{00}}\right)P_{\underline{m}} - \frac{\sqrt{\lambda}}{h^{00}}G_{\underline{m}\underline{n}}\partial_1 X^{\underline{n}}\right] \\
&\quad + \frac{\sqrt{\lambda}}{2h^{00}}\left[\frac{1}{\lambda}(\partial_{\underline{m}}G^{\underline{k}\underline{l}})P_{\underline{k}}P_{\underline{l}} + (\partial_{\underline{m}}G_{\underline{k}\underline{l}})\partial_1 X^{\underline{k}}\partial_1 X^{\underline{l}}\right].
\end{aligned} \tag{3.2.37}$$

The Hamiltonian density is given by

$$\mathcal{H} = P_{\underline{m}}\dot{X}^{\underline{m}} - \mathcal{L} = -e^0\Phi_0 - e^1\Phi_1, \tag{3.2.38}$$

where

$$\Phi_0 := \frac{1}{\sqrt{\lambda}}G^{\underline{m}\underline{n}}P_{\underline{m}}P_{\underline{n}} + \sqrt{\lambda}G_{\underline{m}\underline{n}}\partial_1 X^{\underline{m}}\partial_1 X^{\underline{n}}, \qquad \Phi_1 := P_{\underline{m}}\partial_1 X^{\underline{m}}. \tag{3.2.39}$$

The Virasoro constraints are given by $\Phi_0 \approx 0$, $\Phi_1 \approx 0$. The Hamiltonian density vanishes weakly: $\mathcal{H} \approx 0$. Using (3.2.37), we can check that the Virasoro constraints are consistent with the time evolution

$$\begin{aligned} \dot{\Phi}_0 &= -2(\partial_1 e^1)\Phi_0 - 8(\partial_1 e^0)\Phi_1 - \partial_1\Phi_0 - 2\partial_1\Phi_1, \\ \dot{\Phi}_1 &= -2(\partial_1 e^0)\Phi_0 - 2(\partial_1 e^1)\Phi_1 - e^0\partial_1\Phi_0 - e^1\partial_1\Phi_1. \end{aligned} \tag{3.2.40}$$

So we can see that there is no secondary constraint.

3.3 AdS String

The string living in $AdS_5 \times S^5$ spacetime is described by the action with the multipliers,

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d\xi^2(\mathcal{L}_{AdS} + \mathcal{L}_S), \tag{3.3.1}$$

where

$$\mathcal{L}_{AdS} = -\frac{1}{2}\eta_{PQ}\partial_a Y^P \partial^a Y^Q + \frac{1}{2}\tilde{\Lambda}(\eta_{PQ}Y^P Y^Q + 1) \tag{3.3.2}$$

$$\mathcal{L}_S = -\frac{1}{2}\eta_{PQ}\partial_a X^P \partial^a X^Q + \frac{1}{2}\Lambda(\eta_{PQ}Y^P Y^Q + 1) \tag{3.3.3}$$

Here X^M, $M = 1, 2, \cdots, 6$ and Y^P, $P = 0, 1, \cdots, 5$ are the embedding coordinate of $\mathbb{R}^6$ and $\mathbb{R}^{2,4}$ and the Lagrange multipliers $\tilde{\Lambda}$ and Λ are introduced to constrain that the string coordinates X's and Y's live in S^5 and AdS_5 respectively as in the last section. There exist the constraint expressing the vanishing of the total 2-d energy-momentum tensor,

$$T_{ab} \sim \frac{\delta\mathcal{L}}{\delta h^{ab}}. \tag{3.3.4}$$

The constraints are

$$\eta^{PQ}(\dot{Y}_P\dot{Y}_Q + Y'_P Y'_Q) + \dot{X}_M\dot{X}_M + X'_M X'_M = 0 \tag{3.3.5}$$

$$\eta^{PQ}\dot{Y}_P Y'_Q + \dot{X}_M X'_M = 0. \tag{3.3.6}$$

Similar for $O(N)$ sigma model, the equations of motion are

$$\partial^a\partial_a Y_P - \tilde{\Lambda}Y_P = 0, \quad \tilde{\Lambda} = \eta^{PQ}\partial^a Y_P \partial_a Y_Q, \tag{3.3.7}$$

$$\partial^a\partial_a X_M - \Lambda X_M = 0, \quad \Lambda = \partial^a X_M \partial_a X_M \tag{3.3.8}$$

3.3.1 The rotating point-particle

Here we introduce the global coordinates

$$Y_1 + iY_2 = \sinh\rho\sin\theta e^{i\phi_1}, \quad Y_3 + iY_4 = \sinh\rho\cos\theta e^{i\phi_2}, \quad Y_5 + iY_6 = \cosh\rho e^{it}, \tag{3.3.9}$$

$$X_1 + iX_2 = \sin\gamma\cos\psi e^{i\varphi_1}, \quad X_3 + iX_4 = \sin\gamma\sin\psi e^{i\varphi_2}, \quad X_5 + iX_6 = \cos\gamma e^{i\varphi_3}, \tag{3.3.10}$$

which variables satisfy automatically the constraint Then the spacetime metric (the line element) is written by

$$\begin{aligned} ds^2_{AdS_5} &= d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho d\Omega_3, \\ ds^2_{S^5} &= d\gamma^2 + \cos^2\gamma d\varphi_3^2 + \sin^2\gamma d\tilde{\Omega}_3, \end{aligned} \tag{3.3.11}$$

where Ω_3 and $\tilde{\Omega}_3$ are the solid angle of mutually different 3-spere,

$$d\Omega_3 = d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2, \tag{3.3.12}$$

$$d\tilde{\Omega}_3 = d\psi^2 + \cos^2\psi d\varphi_1^2 + \sin^2\psi d\varphi_2^2. \tag{3.3.13}$$

The action is invariant under the $SO(2,4)$ and $SO(6)$ global symmetries which are generated by the conserved charges,

$$S_{PQ} = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma(Y_P\dot{Y}_Q - Y_Q\dot{Y}_P), \tag{3.3.14}$$

$$J_{MN} = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma(X_M\dot{X}_N - X_N\dot{X}_M), \tag{3.3.15}$$

Because both $SO(2,4)$ and $SO(6)$ have the Cartan algebra of rank 3, the strings on $AdS_5 \times S^5$ has 6 conserved charges $(E, S_1, S_2; J_1, J_2, J_3)$ corresponding to the energy E and two spins (S_1, S_2) on AdS_5 and three angular momenta (J_1, J_2, J_3),

$$\begin{aligned} &E = S_{50}, \quad S_1 = S_{12}, \quad S_2 = S_{34} \\ &J_1 = J_{12}, \quad J_2 = J_{34}, \quad J_3 = J_{56}. \end{aligned} \tag{3.3.16}$$

These quantities correspond to the isometry of $AdS_5 \times S^5$ and therefore these corresponding parameters do not appear in the coefficients of the metric.

Now we consider the simplest solution, the rotating point-particle on S^5,

$$\begin{aligned} &t = \kappa\tau, \quad \rho = 0, \quad \gamma = \frac{\pi}{2}, \\ &\varphi_1 = \kappa\tau, \quad \varphi_2 = \varphi_3 = \psi = 0. \end{aligned} \tag{3.3.17}$$

It is easy to see that this configuration is the solution of the equation of motion. The non vanishing charges which this solution has are the energy E and the angular momentum J_1,

$$\begin{aligned} E &= \frac{i\sqrt{\lambda}}{4\pi}\int d\sigma\left\{(Y_5+iY_0)\frac{\partial}{\partial\tau}(Y_5-iY_0)-(Y_5-iY_0)\frac{\partial}{\partial\tau}(Y_5+iY_0)\right\}\Bigg| \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma\dot{t}\cosh^2\rho\Bigg| \\ &= \sqrt{\lambda}\kappa, \end{aligned} \tag{3.3.18}$$

$$\begin{aligned} J_1 &= \frac{i\sqrt{\lambda}}{4\pi}\int d\sigma\left\{(X_1+iX_2)\frac{\partial}{\partial\tau}(X_1-iX_2)-(X_1-iX_2)\frac{\partial}{\partial\tau}(X_1+iX_1)\right\}\Bigg| \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma\dot{\varphi}_1\sin^2\gamma\cos^2\psi\Bigg| \\ &= \sqrt{\lambda}\kappa \end{aligned} \tag{3.3.19}$$

So because the energy and the momentum are related by

$$E = J_1, \tag{3.3.20}$$

it suggest that the energy is written as a function of momentum.

Chapter 4

The $AdS_5 \times S^5$ superstrings in the generalized light-cone gauge and the Nambu-Goto-like action

4.1 string/gauge correspondence

In order to see the gauge/string correspondence, we consider the following action as an example,

$$S = \frac{1}{g_{\text{YM}}^2} \int d^4x \left[\text{tr}(\partial_\mu \phi_i \partial^\mu \phi_i) + c^{ijk} \, \text{tr}(\phi_i \phi_j \phi_k) + d^{ijkl} \, \text{tr}(\phi_i \phi_j \phi_k \phi_l) \right], \tag{4.1.1}$$

where the matrices $(\phi_i)_{ab}$ express $N \times N$ hermitian matrix. This theory has $U(N)$ gauge invariance. The quadratic term is written by

$$\frac{1}{g_{\text{YM}}^2} \text{tr}(\partial_\mu \phi_i \partial^\mu \phi_i) = -\frac{1}{g_{\text{YM}}} ((\phi_i)_{ab} \partial_\mu \partial^\mu (\phi_j)_{cd}) \delta_{ij} \delta_{ad} \delta_{bc} \tag{4.1.2}$$

We can understand that the propagators of the matrix valued fields is

$$\langle (\phi_i)_{ab}(x)(\phi_j)_{cd}(0) \rangle \sim \frac{g_{\text{YM}}^2}{x^2} \delta_{ij} \delta_{ad} \delta_{bc} \tag{4.1.3}$$

from (4.1.2). In order to express its matrix indices explicitly, it is useful to draw the Feynman diagram using double line notation as in Fig.4.1.

Similarly, the vertices can be also read off from the Lagrangian (4.1.1), that is, there exist 3-point and 4-point vertices (Fig.4.2). The both of the vertices have the same contribution whose scale is g_{YM}^{-2}.

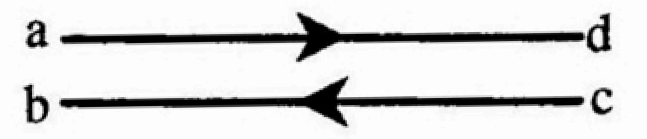


Figure 4.1: propagator in double line form

Figure 4.2: 3-point vertex and 4-point vertex in double line form

By making use of the double line notation for propagators any Feynman diagram in the perturbative expansion may be viewed as a simplicial decomposition of a surface. Let us confirm this fact. First we regard all closed index loops of any Feynman diagrams as the edges of simply connected surface elements. That is, we can now see that the surface elements are connected by each propagator and then construct a original diagram (surface). Let us consider the surface has F faces (surface elements), E propagators (edges) and V vertices which are the total numbers of 3-point vertices and 4-point ones. Here F counts the number of index loops occurring in the diagram. When the index loops appear, the number N will enter into the amplitude as its contribution. in general we can understand, therefore, that such a diagram contributes a factor,

$$r = N^F (g_{\mathrm{YM}}^2)^{E-V}, \tag{4.1.4}$$

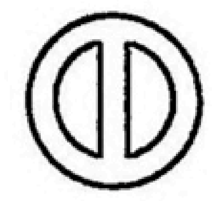
to the amplitude. Recalling that the Eular number χ of a surface is given by

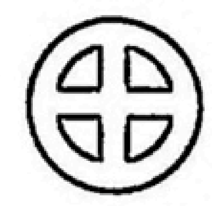
$$\chi = E - V - F, \tag{4.1.5}$$

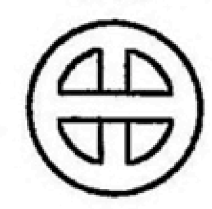
we obtain

$$r = N^{V-E+F} (g_{\mathrm{YM}}^2 N)^{E-V} = N^{\chi} (g_{\mathrm{YM}}^2 N)^{E-V} \tag{4.1.6}$$

Now the combination $\lambda = g_{\mathrm{YM}}^2 N$ emerge as a quantum loop counting parameter constant. We can classify the diagrams according to their order in λ and N. Let us see some examples,

 $\sim (g_{\rm YM}^2)^{(3-2)} N^3 = N^2 \lambda$

 $\sim (g_{\rm YM}^2)^{(8-5)} N^5 = N^2 \lambda^3$

 $\sim (g_{\rm YM}^2)^{(6-4)} N^2 = \lambda^2$

We observe that the first two graphs are planar, i.e. they may without crossing of propagators, whereas the last one is non-planar. Non-planar graphs are suppressed by powers $1/N^2$ with respect to planar ones. Moreover, because the Eular number is also given by

$$\chi = 2 - 2g, \tag{4.1.7}$$

finally,

$$r = N^{2-2g} (g_{\rm YM}^2 N)^{E-V}, \tag{4.1.8}$$

where g denotes the genus of the surface. Hence the perturbation expansion can be organized as a double expansion in λ and $1/N^2$. The free energy F becomes

$$N^2 F = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} c_{g,N} \lambda^n, \tag{4.1.9}$$

where $c_{g,N}$ is a expansion constant corresponding to a certain g and N. This suggests that a large N limit keeping λ fixed is a very interested limit ('t Hooft limit).

The structure of the genus expansion looks like the perturbative expansion of string theory which are a sum over worldsheets with genus g. The role of $1/N^2$ are played by the string coupling constants g_s. So it suggests that the 't Hooft limit ($N \to \infty$) corresponds to free string theory.

4.2 $AdS_5 \times S^5$ string

4.2.1 Introduction to AdS/CFT correspondence

The first concrete proposal of string/gauge correspondence is AdS/CFT correspondence conjecture due to Maldacena in 1997. In its simplest form it states that the maximally supersymmetric gauge theory in four dimensions corresponds to type IIB superstring in $AdS_5 \times S^5$. The boundary of $AdS_5 \times S^5$ is four dimensional and this is space-time where the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory lives. The central relations in the AdS/CFT correspondence relate the gauge theory parameters g_{YM} and N to the string theory parameters α', g_s and the radius R of the $AdS_5 \times S^5$ as

$$\frac{R^4}{\alpha'^2} = g_{\text{YM}}^2 N, \quad 4\pi g_s = g_{\text{YM}}^2. \tag{4.2.1}$$

Hence investigating the string theory in $AdS_5 \times S^5$ are very important. Unfortunately though, even the free ($g_s = 0$) $AdS_5 \times S^5$ string is a rather complicated two dimensional field theory, whose quantization remains a very challenging open problem. Hence the string theory side of the duality conjecture could so far only be addressed by studying its low energy effective description in terms of type IIB supergravity. This approximation to string theory is only meaningful as long as the curvature of the background is small compared to the string scale, i.e. the radius R in string units needs to be very large.

4.2.2 Light-cone gauge

The action of the string theory in $AdS_5 \times S^5$ space-time has appeared in the last chapter. In this section, we choose the light-cone gauge. In order to do this, we introduce the light-cone coordinates. First of all, let us decompose the target space index $\underline{m}$ into $\underline{m} = (\mathfrak{a}, m)$, $\mathfrak{a} = \pm$, $m = 1, 2, \ldots, D-2$. We assume that the target space metric takes the form

$$G_{\underline{m}\underline{n}} \mathrm{d}X^{\underline{m}} \mathrm{d}X^{\underline{n}} = G_{\mathfrak{a}\mathfrak{b}} \mathrm{d}X^{\mathfrak{a}} \mathrm{d}X^{\mathfrak{b}} + G_{mn} \mathrm{d}X^m \mathrm{d}X^n, \tag{4.2.2}$$

and $\partial/\partial X^{\pm}$ are Killing vectors, i.e.

$$\partial_{\pm} G_{\mathfrak{a}\mathfrak{b}} = \partial_{\pm} G_{mn} = 0. \tag{4.2.3}$$

We first recall the procedure of the light-cone gauge fixing in the flat target space. In this case, the world-sheet diffeomorphism is fixed by setting the world-sheet metric conformally flat. The residual symmetry is used to set $X^+ = \kappa\tau$.

But in a certain curved target space such as AdS space-time, there is an obstacle in making the world-sheet metric be conformally flat and obey the light-cone gauge condition [29]. Instead, in the generalized light-cone approach, the world-sheet diffeomorphism is fixed by imposing the following two conditions*:

$$X^+ = \kappa\tau, \qquad \dot{P}_- = 0. \tag{4.2.4}$$

*Here for simplicity we consider the sector with vanishing winding number.

Equation of motion (3.2.37) for X^+

$$\dot{X}^+ = \kappa = -\frac{1}{\sqrt{\lambda}h^{00}}(G^{++}P_+ + G^{+-}P_-) \tag{4.2.5}$$

determines the Lagrange multiplier h^{00} as

$$h^{00} = -\frac{1}{\sqrt{\lambda}\kappa}(G^{++}P_+ + G^{+-}P_-). \tag{4.2.6}$$

Equation of motion for P_-

$$0 = \dot{P}_- = \partial_1 \left[-\left(\frac{h^{01}}{h^{00}}\right) P_- - \frac{\sqrt{\lambda}}{h^{00}} G_{--}\partial_1 X^- \right] \tag{4.2.7}$$

fixes h^{01} up to an arbitrary function of τ:

$$h^{01} = -\frac{\sqrt{\lambda}}{P_-} G_{--}\partial_1 X^- - \frac{f(\tau)}{P_-} h^{00}. \tag{4.2.8}$$

The function $f(\tau)$ is arises from the residual symmetry The residual symmetry is fixed by setting $f(\tau) = 0$. Moreover we have the Virasoro constraints, of course. Solving the Virasoro constraint $\Phi_1 = 0$ gives the relation $\partial_1 X^- = -(1/P_-)P_m\partial_1 X^m$.

Therefore, the worldsheet metric is fixed as

$$h^{00} = -\frac{1}{\sqrt{\lambda}\kappa}(G^{++}P_+ + G^{+-}P_-), \qquad h^{01} = \frac{\sqrt{\lambda}}{P_-^2} G_{--}P_m\partial_1 X^m. \tag{4.2.9}$$

The Virasoro constraint $\Phi_0 = 0$ gives a quadratic equation for P_+:

$$G^{++}P_+^2 + 2P_-G^{+-}P_+ + P_-^2 G^{--} + G^{mn}P_mP_n + \frac{\lambda}{P_-^2}G_{--}(P_m\partial_1 X^m)^2 + \lambda G_{mn}\partial_1 X^m \partial_1 X^n = 0. \tag{4.2.10}$$

The equations of motion for the dynamical variables in the reduced phase space are given by

$$\begin{aligned}
\dot{X}^m &= -\frac{\sqrt{\lambda}}{h^{00}}\left[\frac{1}{\lambda}G^{mn}P_n + \frac{G_{--}}{P_-^2}(P_n\partial_1 X^n)\partial_1 X^m\right], \\
\dot{P}_m &= \partial_1 \left[-\left(\frac{h^{01}}{h^{00}}\right) P_m - \frac{\sqrt{\lambda}}{h^{00}} G_{mn}\partial_1 X^n \right] \\
&\quad + \frac{1}{2\sqrt{\lambda}h^{00}} \Big[(\partial_m G^{++})P_+^2 + 2(\partial_m G^{+-})P_+P_- + (\partial_m G^{--})P_-^2 + (\partial_m G^{kl})P_kP_l \\
&\qquad + \frac{\lambda}{P_-^2}(\partial_m G_{--})(P_n\partial_1 X^n)^2 + \lambda(\partial_m G_{kl})\partial_1 X^k \partial_1 X^l \Big].
\end{aligned} \tag{4.2.11}$$

Using the Poisson bracket $\{X^m(\tau,\sigma), P_n(\tau,\sigma')\}_{\text{P.B.}} = 2\pi\delta^m_n\delta(\sigma-\sigma')$, the equations o motion can be rewritten as

$$\dot{X}^m(\tau,\sigma) = \{X^m(\tau,\sigma), H_{\text{LC}}\}_{\text{P.B.}}, \qquad \dot{P}_m(\tau,\sigma) = \{P_m(\tau,\sigma), H_{\text{LC}}\}_{\text{P.B.}}. \tag{4.2.12}$$

The light cone Hamiltonian is found to be[†]

$$H_{\text{LC}} := -\frac{\kappa}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma\, P_+. \tag{4.2.13}$$

Here P_+ is a solution to the quadratic equation (4.2.10).

In the Hamilton formalism, the light-cone Hamiltonian H_{LC} can be understood by using a canonical transformation $(X^{\underline{m}}, P_{\underline{m}}) \to (\tilde{X}^{\underline{m}}, \tilde{P}_{\underline{m}})$

$$\tilde{X}^+ = X^+ - \kappa\tau, \qquad \tilde{X}^- = X^-, \qquad \tilde{X}^m = X^m, \qquad \tilde{P}_{\underline{m}} = P_{\underline{m}}, \tag{4.2.14}$$

whose generating functional is given by

$$W(X,\tilde{P},\tau) = \int \frac{\mathrm{d}\sigma}{2\pi}\Big[(X^+(\tau,\sigma) - \kappa\tau)\tilde{P}_+(\tau,\sigma) + X^-(\tau,\sigma)\tilde{P}_-(\tau,\sigma) + X^m(\tau,\sigma)\tilde{P}_m(\tau,\sigma)\Big]. \tag{4.2.15}$$

The transformed Hamiltonian is given by

$$\tilde{H} = H + \frac{\partial W}{\partial\tau} = H_{LC}. \tag{4.2.16}$$

4.2.3 $AdS_5 \times S^5$ string in the generalized light-cone gauge

The bosonic part of the Green-Schwarz model for the $AdS_5 \times S^5$ background is a special case of the sigma model (3.2.33). The coordinates for $D = 10$-dimensional target space is chosen as

$$X^{\underline{m}} = (X^+, X^-, X^a, X^{4+s}), \qquad X^a = z^a, \quad a = 1,2,3,4, \qquad X^{4+s} = y^s, \quad s = 1,2,3,4, \tag{4.2.17}$$

and the $AdS_5 \times S^5$ metric is given by

$$G_{\underline{mn}}(X)\mathrm{d}X^{\underline{m}}\mathrm{d}X^{\underline{n}} = G_{\mathfrak{ab}}\mathrm{d}X^{\mathfrak{a}}\mathrm{d}X^{\mathfrak{b}} + G_z\sum_{a=1}^{4}(\mathrm{d}z^a)^2 + G_y\sum_{s=1}^{4}(\mathrm{d}y^s)^2, \tag{4.2.18}$$

where

$$G_{++} = G_{--} = -\frac{1}{2}\left(\frac{1+(z^2/4)}{1-(z^2/4)}\right)^2 + \frac{1}{2}\left(\frac{1-(y^2/4)}{1+(y^2/4)}\right)^2, \tag{4.2.19}$$

[†]In addition, we must examine the open string and the closed string boundary conditions and the level-matching condition. We will not dwell upon these in this paper.

$$G_{+-} = G_{-+} = -\frac{1}{2}\left(\frac{1+(z^2/4)}{1-(z^2/4)}\right)^2 - \frac{1}{2}\left(\frac{1-(y^2/4)}{1+(y^2/4)}\right)^2, \tag{4.2.20}$$

$$G_z = \frac{1}{(1-(z^2/4))^2}, \qquad G_y = \frac{1}{(1+(y^2/4))^2}. \tag{4.2.21}$$

Here

$$z^2 = \sum_{a=1}^{4}(z^a)^2, \qquad y^2 = \sum_{s=1}^{4}(y^s)^2. \tag{4.2.22}$$

The coupling constant λ is related to the radius R of the AdS_5 and S^5 as follows: $\sqrt{\lambda} = R^2/\alpha'$.

In the generalized light-cone gauge, P_+ is determined by the following equation

$$G^{++}P_+^2 + 2BP_+ + C = 0, \tag{4.2.23}$$

where $B = G^{+-}P_-$,

$$\begin{aligned} C = {} & G^{--}P_-^2 + \frac{1}{G_z}\sum_{a=1}^{4}P_a^2 + \frac{1}{G_y}\sum_{s=1}^{4}P_{4+s}^2 \\ & + \frac{\lambda}{P_-^2}G_{--}(P_a\partial_1 z^a + P_{4+s}\partial_1 y^s)^2 + \lambda G_z\sum_{a=1}^{4}(\partial_1 z^a)^2 + \lambda G_y\sum_{s=1}^{4}(\partial_1 y^s)^2. \end{aligned} \tag{4.2.24}$$

For $AdS_5 \times S^5$, we can take the flat Minkowski limit $R \to \infty$. In this case,

$$G^{++} = 0 + O(R^{-2}), \qquad G^{+-} = -1 + O(R^{-2}). \tag{4.2.25}$$

Therefore, in order to have a finite Minkowski limit, the sign for P_+ must be chosen as

$$P_+ = \frac{1}{G^{++}}(-B + \epsilon_B\sqrt{B^2 - G^{++}C}), \tag{4.2.26}$$

where ϵ_B is 1 for $B > 0$ and -1 for $B < 0$.

4.3 The $AdS_5 \times S^5$ Green-Schwarz superstring in the generalized light-cone gauge

4.3.1 The Green-Schwarz action in the $AdS_5 \times S^5$ background

The Green-Schwarz superstring in the flat target space was proposed in [5, 6]. Generalization to the action for the curved supergravity background was done in [63].

More explicit Green-Schwarz action in the $AdS_5 \times S^5$ background was constructe in [7] based on the coset superspace $PSU(2,2|4)/(SO(1,4) \times SO(5))$. (See also [64, 65]). Originally, the Wess-Zumino term is written in the three-dimensional form. The manifestly two-dimensional form of the Wess-Zumino term was presented in [66, 11, 67].

The Green-Schwarz action for the $AdS_5 \times S^5$ is given by

$$S_{\rm GS} = \frac{1}{2\pi}\int d^2\xi\, \mathcal{L}_{\rm GS}, \tag{4.3.1}$$

$$\mathcal{L}_{\rm GS} = -\frac{1}{2}\sqrt{\lambda}h^{ij}\,\eta_{\underline{a}\underline{b}}E_i^{\underline{a}}E_j^{\underline{b}} + \sqrt{\lambda}\epsilon^{ij}\left(E_i^{\underline{\alpha}}\varrho_{\underline{\alpha}\underline{\beta}}E_j^{\underline{\beta}} - \overline{E}_i^{\underline{\bar\alpha}}\varrho_{\underline{\bar\alpha}\underline{\bar\beta}}\overline{E}_j^{\underline{\bar\beta}}\right). \tag{4.3.2}$$

Here E_i^A is the induced vielbein for the type IIB superspace:

$$E_i^A = E^A{}_M\partial_i Z^M = E^A{}_{\underline{m}}\partial_i X^{\underline{m}} + E^A{}_{\underline{\mu}}\partial_i\theta^{\underline{\mu}} + \overline{E}^A{}_{\underline{\bar\mu}}\partial_i\bar\theta^{\underline{\bar\mu}}. \tag{4.3.3}$$

The local Lorentz index $A = (\underline{a}, \underline{\alpha}, \underline{\bar\alpha})$ take values in the following way: $\underline{a} = (\mathfrak{a}, a, 4+s)$, $\mathfrak{a} = \pm$, $a = 1,2,3,4$, $s = 1,2,3,4$, $\underline{\alpha} = 1,2,\ldots,16$ and $\underline{\bar\alpha} = \bar1,\bar2,\ldots,\overline{16}$. We use the 16-component notation for Weyl spinors. The constant matrix ϱ in the Wess-Zumino term is given by

$$C\Gamma^{01234} = \begin{pmatrix} \varrho_{\underline{\alpha}\underline{\beta}} & 0 \\ 0 & \varrho^{\underline{\alpha}\underline{\beta}} \end{pmatrix}, \qquad \Gamma^{\underline{a}} = \begin{pmatrix} 0 & (\gamma^{\underline{a}})^{\underline{\alpha}\underline{\beta}} \\ (\gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}} & 0 \end{pmatrix}. \tag{4.3.4}$$

It is related to the existence of the self-dual Ramond-Ramond 5-form flux.

In the large radius limit, (4.3.2) goes to the Lagrangian in the flat Minkowski space up to (divergent) surface terms‡.

Let us decompose each of the two 16-component Weyl spinors into two 8-component $SO(4) \times SO(4)$ spinors:

$$\theta^{\underline{\alpha}} = \begin{pmatrix} \theta^{+\alpha} \\ \theta^{-\dot\alpha} \end{pmatrix}, \qquad \bar\theta^{\underline{\bar\alpha}} = \begin{pmatrix} \bar\theta^{+\bar\alpha} \\ \bar\theta^{-\dot{\bar\alpha}} \end{pmatrix}, \tag{4.3.5}$$

where $\alpha = 1,2,\ldots,8$, $\dot\alpha = \dot1,\dot2,\ldots,\dot8$, $\bar\alpha = \bar1,\bar2,\ldots,\bar8$ and $\dot{\bar\alpha} = \dot{\bar1},\dot{\bar2},\ldots,\dot{\bar8}$.

We first fix the κ-symmetry by setting $\theta^{-\dot\alpha} = \bar\theta^{-\dot{\bar\alpha}} = 0$. In the 32-component notation, these conditions are equivalent to the condition $\Gamma^+\Theta = 0$. In the large radius limit, it directly goes to the κ-symmetry fixing condition for the flat Minkowski target space.

To simplify expressions, we combine the remaining fermionic coordinates into $\Psi^{\hat\alpha}$:

$$(\Psi^{\hat\alpha}) = \begin{pmatrix} \theta^{+\alpha} \\ \bar\theta^{+\bar\alpha} \end{pmatrix}, \qquad \hat\alpha = \hat1,\hat2,\ldots,\widehat{16}. \tag{4.3.6}$$

‡The surface terms purely come from the Wess-Zumino term.

he coordinates for the reduced type IIB superspace is given by $Z^M = (X^{\underline{m}}, \Psi^{\hat{\alpha}}) = (X^+, X^-, X^m, \theta^{+\alpha}, \bar{\theta}^{+\bar{\alpha}})$. We further decompose $X^m = (X^a, X^{4+s}) = (z^a, y^s)$ and choose a representative of the coset superspace as follows

$$G(Z) = \exp\left(X^+\widehat{P}_+ + X^-\widehat{P}_-\right)\exp\left(\theta^{+\alpha}\widehat{Q}^+_\alpha + \bar{\theta}^{+\bar{\alpha}}\widehat{\overline{Q}}{}^+_{\bar{\alpha}}\right) g_z g_y. \tag{4.3.7}$$

Here $\widehat{P}_\pm$, $\widehat{Q}^+_\alpha$ and $\widehat{\overline{Q}}{}^+_{\bar{\alpha}}$ belong to $psu(2,2|4)$ generators. The vielbeins $E^A{}_M$ can be read from the Cartan one-form $G^{-1}\mathrm{d}G$. See Appendix C for details.

The κ-symmetry fixed action for $AdS_5 \times S^5$ can be written as

$$\mathcal{L}_{\rm GS} = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{\underline{mn}}(X)\mathcal{D}_iX^{\underline{m}}\mathcal{D}_jX^{\underline{n}} + \frac{1}{2}\sqrt{\lambda}\epsilon^{ij}B_{\hat{\alpha}\hat{\beta}}\mathcal{D}_i\Psi^{\hat{\alpha}}\mathcal{D}_j\Psi^{\hat{\beta}}. \tag{4.3.8}$$

The target space metric $G_{\underline{mn}}$ is the same as the bosonic one (4.2.18), $B_{\hat{\alpha}\hat{\beta}} = B_{\hat{\alpha}\hat{\beta}}(Z)$, and Λ's are introduced through

$$\begin{aligned}
\mathcal{D}_iX^+ &= \partial_iX^+,\\
\mathcal{D}_iX^- &= \partial_iX^- + \Lambda^-{}_{\hat{\alpha}}\mathcal{D}_i\Psi^{\hat{\alpha}},\\
\mathcal{D}_iX^m &= \partial_iX^m + (\Lambda^m{}_{n\hat{\alpha}}\mathcal{D}_i\Psi^{\hat{\alpha}})X^n,\\
\mathcal{D}_i\Psi^{\hat{\alpha}} &= \partial_i\Psi^{\hat{\alpha}} + (\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}\partial_iX^+)\Psi^{\hat{\beta}}.
\end{aligned} \tag{4.3.9}$$

Here $\Lambda^-{}_{\hat{\alpha}}$ and $\Lambda^m{}_{n\hat{\alpha}}$ depend only on fermionic variables $\Psi^{\hat{\gamma}}$ and $\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}$ is a constant. See Appendix C for details.

The conjugate momenta are given by

$$\begin{aligned}
P_+ &= -\sqrt{\lambda}h^{0i}G_{+,a}\mathcal{D}_iX^a + \mathcal{P}_{\hat{\alpha}}\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}\Psi^{\hat{\beta}},\\
P_- &= -\sqrt{\lambda}h^{0i}G_{-,a}\mathcal{D}_iX^a,\\
P_m &= -\sqrt{\lambda}h^{0i}G_{mn}\mathcal{D}_iX^n,\\
P_{\hat{\alpha}} &= -\sqrt{\lambda}B_{\hat{\alpha}\hat{\beta}}\mathcal{D}_1\Psi^{\hat{\beta}} + P_-\Lambda^-{}_{\hat{\alpha}} + P_m\Lambda^m{}_{n\hat{\alpha}}X^n.
\end{aligned} \tag{4.3.10}$$

We have fermionic primary constraints:

$$\Phi_{\hat{\alpha}} = P_{\hat{\alpha}} + \sqrt{\lambda}B_{\hat{\alpha}\hat{\beta}}\mathcal{D}_1\Psi^{\hat{\beta}} - P_-\Lambda^-{}_{\hat{\alpha}} - P_m\Lambda^m{}_{n\hat{\alpha}}X^n \approx 0. \tag{4.3.11}$$

The Hamiltonian density is given by

$$\mathcal{H} = P_{\underline{m}}\dot{X}^{\underline{m}} + P_{\hat{\alpha}}\dot{\Psi}^{\hat{\alpha}} - \mathcal{L} = -e^0\,\Phi_0 - e^1\Phi_1, \tag{4.3.12}$$

where

$$\Phi_0 = \frac{1}{\sqrt{\lambda}}G^{ab}\Pi_a\Pi_b + \sqrt{\lambda}G_{ab}\mathcal{D}_1X^a\mathcal{D}_1X^b + \frac{1}{\sqrt{\lambda}}G^{mn}P_mP_n + \sqrt{\lambda}G_{mn}\mathcal{D}_1X^m\mathcal{D}_1X^n, \tag{4.3.13}$$

$$\Phi_1 = \Pi_a \mathcal{D}_1 X^a + P_m \mathcal{D}_1 X^m. \tag{4.3.14}$$

Here $\Pi_+ := P_+ - P_{\hat{\alpha}} \Lambda^{\hat{\alpha}}{}_{\hat{\beta}} \Psi^{\hat{\beta}}$ and $\Pi_- := P_-$.

Since the action (4.3.8) is a singular system, it is necessary to introduce fermionic Lagrange multipliers $\chi^{\hat{\alpha}}$ for (4.3.11). The Hamilton form of the equations of motion are given by

$$\dot{Z}^M(\tau,\sigma) = \{Z^M(\tau,\sigma), H\}_{\text{P.B.}} + \frac{1}{2\pi}\int d\sigma' \{Z^M(\tau,\sigma), \Phi_{\hat{\alpha}}(\tau,\sigma')\}_{\text{P.B.}}\, \chi^{\hat{\alpha}}(\tau,\sigma'), \tag{4.3.15}$$

$$\dot{P}_M(\tau,\sigma) = \{P_M(\tau,\sigma), H\}_{\text{P.B.}} + \frac{1}{2\pi}\int d\sigma' \{P_M(\tau,\sigma), \Phi_{\hat{\alpha}}(\tau,\sigma')\}_{\text{P.B.}}\, \chi^{\hat{\alpha}}(\tau,\sigma'), \tag{4.3.16}$$

where

$$H = \frac{1}{2\pi}\int d\sigma\, \mathcal{H}. \tag{4.3.17}$$

The singularity of the action comes from the fact that $\dot{\Psi}^{\hat{\alpha}}$ or equivalently $\mathcal{D}_0\Psi^{\hat{\alpha}}$ does not appear in (4.3.10); $\dot{\Psi}^{\hat{\alpha}}$ can not be expressed by the phase space variables. The equations of motion (4.3.15) for $Z^M = \Psi^{\hat{\alpha}}$ can be rewritten as $\mathcal{D}_0\Psi^{\hat{\alpha}} = \chi^{\hat{\alpha}}$. Therefore, the introduction of the fermionic Lagrange multipliers $\chi^{\hat{\alpha}}$ is eventually equivalent to converting $\mathcal{D}_0\Psi^{\hat{\alpha}}$ into $\chi^{\hat{\alpha}}$.

4.3.2 Generalized light-cone gauge

We first reduce the phase space from $\Gamma = \{(X^{\underline{m}}, P_{\underline{m}}, \Psi^{\hat{\alpha}}, P_{\hat{\alpha}})\}$ to $\Gamma^* = \{(X^m, P_m, \Psi^{\hat{\alpha}}, P_{\hat{\alpha}})\}$ by taking the generalized light-cone gauge and by solving the Virasoro constraints $\Phi_0 = 0$ and $\Phi_1 = 0$.

Let us take the generalized light-cone gauge:

$$X^+ = \kappa\tau, \qquad \dot{P}_- = 0. \tag{4.3.18}$$

The Virasoro constraint $\Phi_1 = 0$ is solved by setting $\mathcal{D}_1 X^- = -(1/P_-)P_m\mathcal{D}_1X^m$. The bosonic Lagrange multipliers are determined as

$$h^{00} = -\frac{1}{\sqrt{\lambda}\kappa}(G^{++}\Pi_+ + G^{+-}P_-), \qquad h^{01} = \frac{\sqrt{\lambda}}{P_-^2}G_{--}P_m\mathcal{D}_1X^m. \tag{4.3.19}$$

The Virasoro constraint $\Phi_0 = 0$ yields the following quadratic equation

$$\begin{aligned}&(G^{++}\Pi_+^2 + 2G^{+-}P_-\Pi_+ + G^{--}P_-^2) + G^{mn}P_mP_n \\ &+ \frac{\lambda}{P_-^2}G_{--}(P_m\mathcal{D}_1X^m)^2 + \lambda G_{mn}\mathcal{D}_1X^m\mathcal{D}_1X^n = 0,\end{aligned} \tag{4.3.20}$$

which gives a solution

$$P_+ = P_{\hat\alpha}\Lambda^{\hat\alpha}{}_{\hat\beta}\Psi^{\hat\beta} + \Pi_+^{(\mathrm{sol})}. \tag{4.3.21}$$

Here

$$\Pi_+^{(\mathrm{sol})} = \frac{1}{G^{++}}\left(-B + \epsilon_B\sqrt{B^2 - G^{++}\tilde{C}}\right), \tag{4.3.22}$$

with $B = G^{+-}P_-$, $\epsilon_B = \mathrm{sign}(B)$,

$$\begin{aligned}\tilde{C} &= G^{--}P_-^2 + \frac{1}{G_z}\sum_{a=1}^{4}P_a^2 + \frac{1}{G_y}\sum_{s=1}^{4}P_{4+s}^2 \\ &+ \frac{\lambda}{P_-^2}G_{--}(P_a\mathcal{D}_1 z^a + P_{4+s}\mathcal{D}_1 y^s)^2 + \lambda G_z\sum_{a=1}^{4}(\mathcal{D}_1 z^a)^2 + \lambda G_y\sum_{s=1}^{4}(\mathcal{D}_1 y^s)^2.\end{aligned} \tag{4.3.23}$$

The time evolution for the reduced phase variables is given by

$$\dot{F}(\tau,\sigma) = \{F(\tau,\sigma), H_{\mathrm{LC}}\}^*_{\mathrm{P.B.}} + \frac{1}{2\pi}\int \mathrm{d}\sigma'\{F(\tau,\sigma), \Phi_{\hat\alpha}(\tau,\sigma')\}^*_{\mathrm{P.B.}}\,\chi^{\hat\alpha}(\tau,\sigma'). \tag{4.3.24}$$

Here $\{F,G\}^*_{\mathrm{P.B.}}$ is the Poisson bracket in the reduced phase space Γ^*. The light-cone Hamiltonian is given by

$$H_{\mathrm{LC}} = -\frac{\kappa}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\sigma\, P_+, \tag{4.3.25}$$

and $\Phi_{\hat\alpha}$ is the fermionic constraints in the reduced phase space:

$$\Phi_{\hat\alpha} = P_{\hat\alpha} + \sqrt{\lambda}B_{\hat\alpha\hat\beta}\partial_1\Psi^{\hat\beta} - P_-\Lambda^-{}_{\hat\alpha} - P_m\Lambda^m{}_{n\hat\alpha}X^n. \tag{4.3.26}$$

As in the bosonic case, the light-cone Hamiltonian can be understood by canonical transformation. It can be also explained by using the first-order form of the action:

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\xi\left(P_+\dot{X}^+ + P_-\dot{X}^- + P_m\dot{X}^m + P_{\hat\alpha}\dot\Psi^{\hat\alpha} - \mathcal{H} - \Phi_{\hat\alpha}\chi^{\hat\alpha}\right). \tag{4.3.27}$$

By taking the generalized light-cone gauge and by substituting the solutions of the Virasoro constraints into the action, we have

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\xi\left(P_m\dot{X}^m + P_{\hat\alpha}\dot\Psi^{\hat\alpha} - \mathcal{H}_{\mathrm{LC}} - \Phi_{\hat\alpha}\chi^{\hat\alpha}\right). \tag{4.3.28}$$

Here $\mathcal{H}_{\mathrm{LC}} = -\kappa P_+$ and we have dropped the total τ-derivative term $P_-\dot{X}^-$.

We can see that the remaining fermionic constraints $\Phi_{\hat\alpha} \approx 0$ are second class:

$$\{\Phi_{\hat\alpha}(\tau,\sigma), \Phi_{\hat\beta}(\tau,\sigma')\}^*_{\mathrm{P.B.}} = -2\pi\mathcal{C}_{\hat\alpha\hat\beta}(\tau,\sigma)\delta(\sigma-\sigma'), \tag{4.3.29}$$

where

$$\mathcal{C}_{\hat{\alpha}\hat{\beta}} = P_-(\partial\Lambda^-{}_{\hat{\alpha}}/\partial\Psi^{\hat{\beta}}) + P_-(\partial\Lambda^-{}_{\hat{\beta}}/\partial\Psi^{\hat{\alpha}})$$
$$- P_m X^n \left(\Lambda^m{}_{k\hat{\alpha}}\Lambda^k{}_{n\hat{\beta}} + \Lambda^m{}_{k\hat{\beta}}\Lambda^k{}_{n\hat{\alpha}} - (\partial\Lambda^m{}_{n\hat{\alpha}}/\partial\Psi^{\hat{\beta}}) - (\partial\Lambda^m{}_{n\hat{\beta}}/\partial\Psi^{\hat{\alpha}})\right) + \sqrt{\lambda}(\partial_1 B_{\hat{\alpha}\hat{\beta}})$$
$$- \sqrt{\lambda}\left[(\partial_m B_{\hat{\alpha}\hat{\gamma}})\Lambda^m{}_{k\hat{\beta}}X^k + (\partial_m B_{\hat{\beta}\hat{\gamma}})\Lambda^m{}_{k\hat{\alpha}}X^k - (\partial B_{\hat{\alpha}\hat{\gamma}}/\partial\Psi^{\hat{\beta}}) - (\partial B_{\hat{\beta}\hat{\gamma}}/\partial\Psi^{\hat{\alpha}})\right]\partial_1\Psi^{\hat{\gamma}}.$$

We assume that $\mathcal{C}$ is invertible. For $AdS_5 \times S^5$, this is indeed the case since the terms in the first line of the above equation start with an invertible matrix:

$$P_-(\partial\Lambda^-{}_{\hat{\alpha}}/\partial\Psi^{\hat{\beta}}) + P_-(\partial\Lambda^-{}_{\hat{\beta}}/\partial\Psi^{\hat{\alpha}}) = 2\sqrt{2}\,\mathrm{i}\,P_-\,(\gamma_+)_{\hat{\alpha}\hat{\beta}} + \mathcal{O}(\Psi^2), \tag{4.3.30}$$

$$(\gamma_+)_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & (\gamma_+)_{\alpha\bar{\beta}} \\ (\gamma_+)_{\bar{\alpha}\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix}. \tag{4.3.31}$$

The consistency of the time evolution of the fermionic constraints ($\dot{\Phi}_{\hat{\alpha}} = 0$) determines the fermionic Lagrange multipliers as follows

$$\chi^{\hat{\alpha}}(\tau,\sigma) = \left(\mathcal{C}^{-1}(\tau,\sigma)\right)^{\hat{\alpha}\hat{\beta}}\mathcal{X}_{\hat{\beta}}(\tau,\sigma). \tag{4.3.32}$$

Here $\mathcal{X}_{\hat{\beta}}(\tau,\sigma) = \{\Phi_{\hat{\beta}}(\tau,\sigma), H_{\mathrm{LC}}\}^*_{\mathrm{P.B.}}$. Since the explicit form of $\mathcal{X}_{\hat{\beta}}$ is rather lengthy and is not necessary here, we do not write it here.

The (equal τ) Dirac bracket is given by

$$\{F,G\}_{\mathrm{D.B.}} = \{F,G\}^*_{\mathrm{P.B.}} + \frac{1}{2\pi}\int d\sigma\{F,\Phi_{\hat{\alpha}}(\tau,\sigma)\}^*_{\mathrm{P.B.}}\left(\mathcal{C}^{-1}(\tau,\sigma)\right)^{\hat{\alpha}\hat{\beta}}\{\Phi_{\hat{\beta}}(\tau,\sigma),G\}^*_{\mathrm{P.B.}}. \tag{4.3.33}$$

Using the Dirac bracket we can choose $(X^m, P_m, \Psi^{\hat{\alpha}})$ as dynamical variables and $P_{\hat{\alpha}}$ can be treated as the solution of the fermionic constraints:

$$P_{\hat{\alpha}} = -\sqrt{\lambda}B_{\hat{\alpha}\hat{\beta}}\partial_1\Psi^{\hat{\beta}} + P_-\Lambda^-{}_{\alpha} + P_m\Lambda^m{}_{n\hat{\alpha}}X^n. \tag{4.3.34}$$

The time evolution of the dynamical variables are now given by

$$\dot{F} = \{F, H_{\mathrm{LC}}\}_{\mathrm{D.B.}}. \tag{4.3.35}$$

Let

$$\mathcal{U}^m{}_{\hat{\alpha}} = \Lambda^m{}_{n\hat{\alpha}}X^n, \qquad \mathcal{V}_{m\hat{\alpha}} = \sqrt{\lambda}(\partial_m B_{\hat{\alpha}\hat{\beta}})\partial_1\Psi^{\hat{\beta}} - P_n\Lambda^n{}_{m\hat{\alpha}}. \tag{4.3.36}$$

The Dirac bracket among the dynamical variables are given by

$$\begin{aligned}
\{X^m(\tau,\sigma), X^n(\tau,\sigma')\}_{\text{D.B.}} &= -2\pi\,\mathcal{U}^m{}_{\hat\alpha}\big(\mathcal{C}^{-1}\big)^{\hat\alpha\hat\beta}\mathcal{U}^n{}_{\hat\beta}\,\delta(\sigma-\sigma'),\\
\{X^m(\tau,\sigma), P_n(\tau,\sigma')\}_{\text{D.B.}} &= 2\pi\left(\delta^m_n - \mathcal{U}^m{}_{\hat\alpha}\big(\mathcal{C}^{-1}\big)^{\hat\alpha\hat\beta}\mathcal{V}_{n\hat\beta}\right)\,\delta(\sigma-\sigma'),\\
\{X^m(\tau,\sigma), \Psi^{\hat\alpha}(\tau,\sigma')\}_{\text{D.B.}} &= -2\pi\,\mathcal{U}^m{}_{\hat\beta}\big(\mathcal{C}^{-1}\big)^{\hat\beta\hat\alpha}\delta(\sigma-\sigma'),\\
\{P_m(\tau,\sigma), P_n(\tau,\sigma')\}_{\text{D.B.}} &= -2\pi\,\mathcal{V}_{m\hat\alpha}\big(\mathcal{C}^{-1}\big)^{\hat\alpha\hat\beta}\mathcal{V}_{n\hat\beta}\,\delta(\sigma-\sigma'),\\
\{P_m(\tau,\sigma), \Psi^{\hat\alpha}(\tau,\sigma')\}_{\text{D.B.}} &= -2\pi\,\mathcal{V}_{m\hat\beta}\big(\mathcal{C}^{-1}\big)^{\hat\beta\hat\alpha}\,\delta(\sigma-\sigma'),\\
\{\Psi^{\hat\alpha}(\tau,\sigma), \Psi^{\hat\beta}(\tau,\sigma')\}_{\text{D.B.}} &= 2\pi\big(\mathcal{C}^{-1}\big)^{\hat\alpha\hat\beta}\delta(\sigma-\sigma').
\end{aligned} \tag{4.3.37}$$

The quantization of these transverse degrees of freedom is then a straightforward task: to replace $i\hbar$ times the Dirac bracket by the graded commutator. Because of the fermionic constraints, all corresponding quantum operators become non-commutative.

4.4 Generalized light-cone gauge and uniform light-cone gauge

In this section, we show that the generalized light-cone gauge reduce the uniform light-cone gauge, in which $P_- = \text{const.}$ by rescaling the worldsheet coordinates. In the section 4.4.1, we restrict ourselves to the bosonic sector of the theory. In the section 4.4.2, we discuss the fermionic sector.

4.4.1 Bosonic Part

The bosonic action of $AdS_5 \times S^5$ superstring in first-order formalism is

$$\mathcal{L} = P_{\underline{m}}\dot{X}^{\underline{m}} + e^0\Phi_0 + e^1\Phi_1, \tag{4.4.1}$$

where

$$\Phi_0 := \frac{1}{\sqrt{\lambda}}G^{\underline{m}\underline{n}}P_{\underline{m}}P_{\underline{n}} + \sqrt{\lambda}G_{\underline{m}\underline{n}}\partial_1 X^{\underline{m}}\partial_1 X^{\underline{n}}, \tag{4.4.2}$$

$$\Phi_1 := P_{\underline{m}}\partial_1 X^{\underline{m}} \tag{4.4.3}$$

and e^0 and e^1 are the Lagrange multipliers that work to impose Virasoro constraints $\Phi_0 = \Phi_1 = 0$ on the theory. Here set

$$P_{\underline{m}} \equiv \sqrt{\lambda}P^{(r)}_{\underline{m}}. \tag{4.4.4}$$

Then the Lagrangian becomes

$$\mathcal{L} = \sqrt{\lambda}(P^{(r)}_{\underline{m}}\dot{X}^{\underline{m}} + e^0\Phi^{(r)}_0 + e^1\Phi^{(r)}_1) \equiv \sqrt{\lambda}\mathcal{L}^{(r)}, \qquad S = \frac{\sqrt{\lambda}}{2\pi}\int d^2\xi \mathcal{L}^{(r)} \tag{4.4.5}$$

where the superscript (r) represents that the parameter λ is scaled out and of course, the constraint functions becomes

$$\Phi^{(r)}_0 := G^{\underline{mn}}P^{(r)}_{\underline{m}}P^{(r)}_{\underline{n}} + G_{\underline{mn}}\partial_1 X^{\underline{m}}\partial_1 X^{\underline{n}}, \tag{4.4.6}$$

$$\Phi^{(r)}_1 := P^{(r)}_{\underline{m}}\partial_1 X^{\underline{m}}. \tag{4.4.7}$$

Now we choose the generalized light-cone gauge as in the last chapter,

$$X^+ = \kappa\tau, \quad \dot{P}_- = 0 \quad (\to P_- = P_-(\sigma)). \tag{4.4.8}$$

From the constraints $\Phi^{(r)}_0 = \Phi^{(r)}_1 = 0$, we obtain

$$P^{(r)}_+ = \frac{G^{+-}P^{(r)}_-}{G^{++}}\left(\sqrt{1 - \frac{G^{++}C^{(r)}}{(G^{+-}P^{(r)}_-)^2}} - 1\right), \tag{4.4.9}$$

in this gauge. Here

$$C^{(r)} = (P^{(r)}_-)^2 G^{--} + G^{mn}P^{(r)}_m P^{(r)}_n + \frac{1}{(P^{(r)}_-)^2}G_{--}(P^{(r)}_m\partial_1 X^m)^2 + G_{mn}\partial_1 X^m \partial_1 X^n. \tag{4.4.10}$$

Then the Lagrangian becomes

$$\mathcal{L}^{(r)}_{LC} = P^{(r)}_+\dot{X}^+ + P^{(r)}_m\dot{X}^m \tag{4.4.11}$$

The light-cone Hamiltonian is defined by

$$\mathcal{H}_{LC} = -\kappa P_+, \tag{4.4.12}$$

because of $X^+ = \kappa\tau$. Now we consider the following redefinition:

$$\frac{1}{\omega}P^{(r)}_-(\sigma)d\sigma = d\tilde{\sigma}, \tag{4.4.13}$$

$$\tau = \tilde{\tau}, \tag{4.4.14}$$

where ω is a constant. Then we obtain the reduced action

$$S_{\text{red}} = \frac{\sqrt{\lambda}}{2\pi}\int d^2\tilde{\xi}\left[\tilde{P}_m\tilde{\partial}_0 X^m + \kappa\tilde{P}_+\right] \equiv \frac{\sqrt{\lambda}}{2\pi}\int d^2\tilde{\xi}\tilde{\mathcal{L}}^{(r)}_{LC}, \tag{4.4.15}$$

where

$$\tilde{P}_{\underline{m}} \equiv \omega(P_-^{(r)})^{-1}P_{\underline{m}}^{(r)} = \omega P_-^{-1}P_{\underline{m}}. \tag{4.4.16}$$

Thus in the worldsheet coordinates with $\tilde{}$,

$$\tilde{P}_- \equiv \omega = \text{const}. \tag{4.4.17}$$

Therefore we can conclude that to set $P_- = \text{const}$ is always possible in this gauge as long as we treat only the bosonic sector at least. It means that the generalized light-cone gauge can reduce the uniform light-cone gauge with $P_- \equiv \text{const}$ under the consideration of only the bosonic sector. Explicitly, the Lagrangian in first-order formalism can be written by

$$\begin{aligned}\tilde{\mathcal{L}}_{LC} &\equiv \sqrt{\lambda}\tilde{\mathcal{L}}_{LC}^{(r)} \\ &= \sqrt{\lambda}\left[\tilde{P}_m\tilde{\partial}_0 X^m + \kappa\frac{G^{+-}\omega}{G^{++}}\left(\sqrt{1-\frac{G^{++}\tilde{C}}{(G^{+-}\omega)^2}}-1\right)\right]\end{aligned} \tag{4.4.18}$$

with

$$\tilde{C} = \omega^2 G^{--} + G^{mn}\tilde{P}_m\tilde{P}_n + \frac{1}{\omega^2}G_{--}(\tilde{P}_m\tilde{\partial}_1 X^m)^2 + G_{mn}\tilde{\partial}_1 X^m\tilde{\partial}_1 X^n. \tag{4.4.19}$$

Summary

$$P_{\underline{m}}^{(r)} = \lambda^{-\frac{1}{2}}P_{\underline{m}}$$

$$\tilde{\tau} = \tau$$

$$d\tilde{\sigma} = \frac{1}{\omega}P_-^{(r)}(\sigma)d\sigma = \frac{1}{\omega}\lambda^{-\frac{1}{2}}P_-(\sigma)d\sigma$$

$$\tilde{P}_{\underline{m}} = \omega(P_-^{(r)})^{-1}P_{\underline{m}}^{(r)} = \omega(P_-)^{-1}P_{\underline{m}}$$

These suggest that we can regard the rescaling of the conjugate momentum (4.4.4) as that of the worldsheet coordinate σ.

4.4.2 Fermionic Part

In the last section, although we restricted ourselves only the bosonic sector, we can also practically include the fermionic sector. So we turn to the case which include the fermionic part as well as the bosonic one. Rescaling (4.4.4), we also obtain

$$\mathcal{L} = \sqrt{\lambda}(P_{\underline{m}}^{(r)}\dot{X}^{\underline{m}} + P_{\hat{\alpha}}^{(r)}\dot{\Psi}^{\hat{\alpha}} + e^0\Phi_0^{(r)} + e^1\Phi_1^{(r)} - \Phi_{\hat{\alpha}}^{(r)}\chi^{\hat{\alpha}}), \tag{4.4.20}$$

where

$$\Phi_0^{(r)} = G^{ab}\Pi_a^{(r)}\Pi_b^{(r)} + G_{ab}\mathcal{D}_1X^a\mathcal{D}_1X^b + G^{mn}P_m^{(r)}P_n^{(r)} + G_{mn}\mathcal{D}_1X^m\mathcal{D}_1X^n, \tag{4.4.21}$$

$$\Phi_1^{(r)} = \Pi_a^{(r)}\mathcal{D}_1X^a + P_m^{(r)}\mathcal{D}_1X^m. \tag{4.4.22}$$

$$\Phi_{\hat{\alpha}}^{(r)} = P_{\hat{\alpha}}^{(r)} + B_{\hat{\alpha}\hat{\beta}}\mathcal{D}_1\Psi^{\hat{\beta}} - P_-^{(r)}\Lambda^-{}_{\hat{\alpha}} - P_m^{(r)}\Lambda^m{}_{n\hat{\alpha}}X^n, \tag{4.4.23}$$

Here $\Pi_+^{(r)} := P_+^{(r)} - P_{\hat{\alpha}}^{(r)}\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}\Psi^{\hat{\beta}}$ and $\Pi_-^{(r)} := P_-^{(r)}$. The generalized light-cone gauge is

$$X^+ = \kappa\tau, \quad \dot{P}_- = 0 \quad (\rightarrow P_- = P_-(\sigma)). \tag{4.4.24}$$

The Virasoro constraint $\Phi_0^{(r)} = 0$ yields the following quadratic equation

$$\begin{aligned}&\left\{G^{++}(\Pi_+^{(r)})^2 + 2G^{+-}P_-^{(r)}\Pi_+^{(r)} + G^{--}(P_-^{(r)})^2\right\} + G^{mn}P_m^{(r)}P_n^{(r)}\\ &+\frac{1}{(P_-^{(r)})^2}G_{--}(P_m^{(r)}\mathcal{D}_1X^m)^2 + G_{mn}\mathcal{D}_1X^m\mathcal{D}_1X^n = 0,\end{aligned} \tag{4.4.25}$$

which gives a solution

$$P_+^{(r)} = P_{\hat{\alpha}}^{(r)}\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}\Psi^{\hat{\beta}} + \Pi_+^{(r)}. \tag{4.4.26}$$

Here

$$\Pi_+^{(r)} = \frac{G^{+-}P_-^{(r)}}{G^{++}}\left\{\epsilon_B\sqrt{1 - \frac{G^{++}\tilde{C}^{(r)}}{(G^{+-}P_-^{(r)})^2}} - 1\right\} \tag{4.4.27}$$

with $\epsilon_B = \text{sign}(G^{+-}P_-)$ and

$$\begin{aligned}\tilde{C}^{(r)} &= G^{--}(P_-^{(r)})^2 + \frac{1}{G_z}\sum_{a=1}^{4}(P_a^{(r)})^2 + \frac{1}{G_y}\sum_{s=1}^{4}(P_{4+s}^{(r)})^2\\ &+\frac{1}{(P_-^{(r)})^2}G_{--}(P_a^{(r)}\mathcal{D}_1z^a + P_{4+s}^{(r)}\mathcal{D}_1y^s)^2 + G_z\sum_{a=1}^{4}(\mathcal{D}_1z^a)^2 + G_y\sum_{s=1}^{4}(\mathcal{D}_1y^s)^2.\end{aligned} \tag{4.4.28}$$

By taking the generalized light-cone gauge and by substituting the solutions of the Virasoro constraints into the action, we have

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d^2\xi\left(P_m^{(r)}\dot{X}^m + P_{\hat{\alpha}}^{(r)}\dot{\Psi}^{\hat{\alpha}} + \kappa P_+^{(r)} - \Phi_{\hat{\alpha}}^{(r)}\chi^{\hat{\alpha}}\right), \tag{4.4.29}$$

with

$$\Phi_{\hat{\alpha}}^{(r)} = P_{\hat{\alpha}}^{(r)} + B_{\hat{\alpha}\hat{\beta}}\partial_1\Psi^{\hat{\beta}} - P_-^{(r)}\Lambda^-{}_{\hat{\alpha}} - P_m^{(r)}\Lambda^m{}_{n\hat{\alpha}}X^n. \tag{4.4.30}$$

Let us consider the redefinition of the worldsheet coordinates,

$$\frac{1}{\omega}P_-^{(r)}(\sigma)d\sigma = d\tilde{\sigma}, \tag{4.4.31}$$

$$\tau = \tilde{\tau}. \tag{4.4.32}$$

Then

$$S=\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}^2\tilde{\xi}\left(\tilde{P}_m\tilde{\partial}_0X^m+\tilde{P}_{\hat{\alpha}}\tilde{\partial}_0\Psi^{\hat{\alpha}}+\kappa\tilde{P}_+-\tilde{\Phi}_{\hat{\alpha}}\chi^{\hat{\alpha}}\right), \tag{4.4.33}$$

with

$$\tilde{P}_+=\tilde{P}_{\hat{\alpha}}\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}\Psi^{\hat{\beta}}+\tilde{\Pi}_+, \tag{4.4.34}$$

$$\tilde{\Phi}_{\hat{\alpha}}=\tilde{P}_{\hat{\alpha}}+B_{\hat{\alpha}\hat{\beta}}\tilde{\partial}_1\Psi^{\hat{\beta}}-\omega\Lambda^{-}{}_{\hat{\alpha}}-\tilde{P}_m\Lambda^m{}_{n\hat{\alpha}}X^n, \tag{4.4.35}$$

where

$$\tilde{\Pi}_+=\frac{G^{+-}\omega}{G^{++}}\left\{\epsilon_B\sqrt{1-\frac{G^{++}C'}{(G^{+-}\omega)^2}}-1\right\}, \tag{4.4.36}$$

with

$$C'=G^{--}\omega^2+\frac{1}{G_z}\sum_{a=1}^{4}(\tilde{P}_a)^2+\frac{1}{G_y}\sum_{s=1}^{4}(\tilde{P}_{4+s})^2$$
$$+\frac{1}{\omega^2}G_{--}(\tilde{P}_a\tilde{\mathcal{D}}_1z^a+\tilde{P}_{4+s}\tilde{\mathcal{D}}_1y^s)^2+G_z\sum_{a=1}^{4}(\tilde{\mathcal{D}}_1z^a)^2+G_y\sum_{s=1}^{4}(\tilde{\mathcal{D}}_1y^s)^2. \tag{4.4.37}$$

Here because of (4.4.31),

$$\mathcal{D}_1=\frac{1}{\omega}P_-^{(r)}\tilde{\mathcal{D}}_1. \tag{4.4.38}$$

Thus we can set $\tilde{P}_-=\omega$ in the rescaled theory and therefore we conclude that the generalized light-cone gauge reduces the uniform light-cone gauge.

In particular, because we do not specify the form of the metric we find that the discussion here is general not restricted to the $AdS_5\times S^5$.

4.5 The Light-cone Lagrangian in the Lagrange formalism

In the section 4.3, we adopted the light-cone gauge and considered the Hamiltonian dynamics of the GS action by using the physical degrees of freedom. The action is formulated in the first-order formalism, i.e., is written in terms of the phase space variables. This first-order formulation is suited for considering the problem of canonical quantization.

Unfortunately, the reduced action in the generalized light-cone gauge still has an involved form. Before considering the quantization problem of the action, we would like to investigate quantum fluctuation in various limits. Extensive study around the plane-wave region was done in [57]. But, in general, the first-order Lagrangian is not so convenient study the quantum spectrum in various limits, such as the BMN limit [68], the Hofman-Maldacena limit [69], or Maldacena-Swanson limit [70]. They can be better investigated by using the Lagrangian written in terms of the fields and their derivatives. Therefore, in this section, we reformulate the GS action to the standard form in the generalized light-cone gauge.

4.5.1 The Lagranginan for the bosonic sigma model in standard formalism

The action for the bosonic sigma model in the D-dimensional curved target space is given by

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\xi\, \mathcal{L}, \tag{4.5.1}$$

where

$$\mathcal{L} = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{\underline{m}\underline{n}}\partial_i X^{\underline{m}}\partial_j X^{\underline{n}}. \tag{4.5.2}$$

Here, we have $\underline{m},\underline{n} = 0, 1, \ldots, D-1$, $\xi^0 = \tau$, $\xi^1 = \sigma$, $\partial_i = \partial/\partial\xi^i$ and $h^{ij} = \sqrt{-g}g^{ij}$ $(i, j = 0, 1)$.

The conjugate momenta are given by

$$P_{\underline{m}} = -\sqrt{\lambda}h^{0j}G_{\underline{m}\underline{n}}\partial_j X^{\underline{n}}. \tag{4.5.3}$$

The target space metric is assumed to have the form

$$G_{\underline{m}\underline{n}}\mathrm{d}X^{\underline{m}}\mathrm{d}X^{\underline{n}} = G_{\mathfrak{a}\mathfrak{b}}\mathrm{d}X^{\mathfrak{a}}\mathrm{d}X^{\mathfrak{b}} + G_{mn}\mathrm{d}X^m\mathrm{d}X^n. \tag{4.5.4}$$

Here, $\mathfrak{a}, \mathfrak{b} = \pm$ denote the longitudinal directions and $m, n = 1, 2, \ldots, D-2$ denote the transverse directions. We assume that $\partial/\partial X^{\pm}$ is a Killing vector.

Let us decompose the Lagrangian into two pieces as follows:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \tag{4.5.5}$$

$$\mathcal{L}_1 = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{\mathfrak{a}\mathfrak{b}}\partial_i X^{\mathfrak{a}}\partial_j X^{\mathfrak{b}}, \tag{4.5.6}$$

$$\mathcal{L}_2 = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{mn}\partial_i X^m\partial_j X^n. \tag{4.5.7}$$

The first part, $\mathcal{L}_1$, and the second part, $\mathcal{L}_2$, are related to the metrics for the longitudinal and transverse directions, respectively.

Under the target space metric ansatz (4.5.4), the momenta P_- conjugate to X^- is given by

$$P_- = -\sqrt{\lambda} h^{0j} G_{-,a} \partial_j X^a. \tag{4.5.8}$$

The generalized light-cone gauge is given by the following two conditions:§

$$X^+ = \kappa\tau, \qquad P_- = \sqrt{\lambda}\omega = \text{const} \tag{4.5.9}$$

which fix the world-sheet diffeomorphism.

4.5.2 From the first-order form to the standard Lagrangian

The reduced action in the generalized light-cone gauge is given by (using the notation of Ref. [1])

$$S_{\text{red}} = \frac{1}{2\pi} \int \mathrm{d}^2\xi \big(P_m \dot{X}^m - \mathcal{H}_{\text{LC}}\big), \tag{4.5.10}$$

where

$$\mathcal{H}_{\text{LC}} = -\kappa P_+. \tag{4.5.11}$$

This is the first-order Lagrangian, $\mathcal{L} = \mathcal{L}(X^m, P_m)$, written in terms of the transverse coordinates X^m and their conjugate momenta P_m. Here, P_+ is a solution of the equation

$$G^{++}P_+^2 + 2\sqrt{\lambda}\omega G^{+-} P_+ + C = 0, \tag{4.5.12}$$

where

$$C = \lambda\omega^2 G^{--} + \lambda G_{mn} \partial_1 X^m \partial_1 X^n + K^{mn} P_m P_n, \tag{4.5.13}$$

$$K^{mn} := G^{mn} + \frac{1}{\omega^2} G_{--} \partial_1 X^m \partial_1 X^n. \tag{4.5.14}$$

Explicitly, P_+ is given by

$$P_+ = \frac{\varepsilon}{G^{++}} \sqrt{\lambda (G^{+-})^2 \omega^2 - G^{++} C} - \sqrt{\lambda}\omega \frac{G^{+-}}{G^{++}}, \tag{4.5.15}$$

where $\varepsilon = \pm 1$.

§Originally, the second condition was given by $\partial_1(P_-) = 0$ in the generalized light-cone gauge. The most general solution is $P_- = P_-(\sigma)$. However, without loss of generality, we can set P_- to a constant by redefining the world-sheet space variable σ and the conjugate momenta such that $P_-(\sigma)\mathrm{d}\sigma = P'_-\mathrm{d}\sigma'$ with P'_- constant, as shown in the section 4.4. Therefore, we adopt the condition $P_- = $ const. as one of the gauge conditions.

Now, let us convert this first-order Lagrangian into the standard form. The equations of motion for P_m,

$$\dot{X}^m + \kappa \frac{\partial P_+}{\partial P_m} = 0, \tag{4.5.16}$$

yield the following relations:

$$\varepsilon \sqrt{\lambda\omega^2(G^{+-})^2 - G^{++}C}\, \dot{X}^m = \kappa K^{mn} P_n. \tag{4.5.17}$$

It is convenient to introduce $\mathcal{J}_{ij}$ and $\mathcal{G}_{ij}$ as

$$\mathcal{J}_{00} := \frac{\kappa^2}{G^{++}}, \qquad \mathcal{J}_{11} := \frac{\omega^2}{G_{--}}, \qquad \mathcal{J}_{01} = \mathcal{J}_{10} := 0, \tag{4.5.18}$$

$$\mathcal{G}_{ij} := G_{mn}\partial_i X^m \partial X^n, \qquad i,j = 0,1. \tag{4.5.19}$$

Let K_{mn} be the inverse of K^{mn}:

$$K_{mn} = G_{mn} - \frac{(G_{mm'}\partial_1 X^{m'})(G_{nn'}\partial_1 X^{n'})}{\mathcal{J}_{11} + \mathcal{G}_{11}}. \tag{4.5.20}$$

Then, we have

$$P_m = \frac{\varepsilon}{\kappa}\sqrt{\lambda\omega^2(G^{+-})^2 - G^{++}C}\, K_{mn}\dot{X}^n. \tag{4.5.21}$$

By substituting this relation into (4.5.13), we obtain

$$C = \lambda\omega^2 G^{--} + \lambda\mathcal{G}_{11} + \frac{1}{\kappa^2}\left(\lambda\omega^2(G^{+-})^2 - G^{++}C\right) K_{mn}\dot{X}^m\dot{X}^n. \tag{4.5.22}$$

Then, we have

$$C = \frac{\lambda\omega^2 G^{--} + \lambda\mathcal{G}_{11} + (\lambda\omega^2/\kappa^2)(G^{+-})^2 K_{mn}\dot{X}^m\dot{X}^n}{1 + (1/\kappa^2)G^{++}K_{mn}\dot{X}^m\dot{X}^n}. \tag{4.5.23}$$

Note that the relation

$$K_{mn}\dot{X}^m\dot{X}^n = \frac{\mathcal{J}_{11}\mathcal{G}_{00} + \det(\mathcal{G}_{ij})}{\mathcal{J}_{11} + \mathcal{G}_{11}} \tag{4.5.24}$$

holds. Then, after some calculations, we find

$$\lambda\omega^2(G^{+-})^2 - G^{++}C = -\frac{\lambda\kappa^2}{\det(\mathcal{J}_{ij} + \mathcal{G}_{ij})}(\mathcal{J}_{11} + \mathcal{G}_{11})^2. \tag{4.5.25}$$

Assuming $\det(\mathcal{J}_{ij} + \mathcal{G}_{ij}) < 0$, we have

$$\sqrt{\lambda\omega^2(G^{+-})^2 - G^{++}C} = \varepsilon' \sqrt{-\frac{\lambda}{\det(\mathcal{J}_{ij} + \mathcal{G}_{ij})}}\,\kappa(\mathcal{J}_{11} + \mathcal{G}_{11}), \tag{4.5.26}$$

where $\varepsilon' = \mathrm{sign}(\kappa(\mathcal{J}_{11} + \mathcal{G}_{11}))$.

Let J_{ij} be the matrix defined by

$$J_{ij} := \mathcal{J}_{ij} + \mathcal{G}_{ij}, \tag{4.5.27}$$

and J^{ij} be its inverse. It is seen that

$$K_{mn}\dot{X}^n = \frac{1}{\mathcal{J}_{11} + \mathcal{G}_{11}} G_{mn}\left[(\mathcal{J}_{11} + \mathcal{G}_{11})\dot{X}^n - \mathcal{G}_{01}\partial_1 X^n\right] = \frac{\det(\mathcal{J}_{ij} + \mathcal{G}_{ij})}{\mathcal{J}_{11} + \mathcal{G}_{11}} G_{mn} J^{0j}\partial_j X^n. \tag{4.5.28}$$

Then, we finally obtain

$$P_m = -\varepsilon\varepsilon'\sqrt{-\lambda\det(J_{ij})}\, G_{mn} J^{0j}\partial_j X^n. \tag{4.5.29}$$

By substituting this expression into the first-order form of the action, we find the reduced Lagrangian in the generalized light-cone gauge.

Let us summarize the above results. The Lagrangian of the bosonic sigma model in the generalized light-cone gauge is given by

$$S_{\text{red}} = \frac{1}{2\pi}\int \mathrm{d}^2\xi\, \mathcal{L}_{\text{LC}}, \tag{4.5.30}$$

where

$$\mathcal{L}_{\text{LC}} = -\varepsilon\varepsilon'\sqrt{-\lambda\det(\mathcal{J}_{ij} + \mathcal{G}_{ij})} + \sqrt{\lambda}\kappa\omega\frac{G_{+-}}{G_{--}}. \tag{4.5.31}$$

Here, we have

$$\mathcal{J}_{00} = \frac{\kappa^2}{G^{++}}, \qquad \mathcal{J}_{11} = \frac{\omega^2}{G_{--}}, \qquad \mathcal{J}_{01} = \mathcal{J}_{10} = 0, \qquad \mathcal{G}_{ij} = G_{mn}\partial_i X^m \partial_j X^n. \tag{4.5.32}$$

4.5.3 Rederivation of the reduced Lagrangian

In this subsection, we rederive the reduced Lagrangian (4.5.31) without employing the first-order formalism.

Here, let us start from the Lagrangian (4.5.2). We decompose it as follows:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \tag{4.5.33}$$

$$\mathcal{L}_1 = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{\mathfrak{a}\mathfrak{b}}\partial_i X^{\mathfrak{a}}\partial_j X^{\mathfrak{b}}, \qquad \mathcal{L}_2 = -\frac{1}{2}\sqrt{\lambda}h^{ij}G_{mn}\partial_i X^m \partial_j X^n. \tag{4.5.34}$$

The generalized light-cone gauge conditions are given by

$$X^+ = \kappa\tau, \qquad P_- = -\sqrt{\lambda}h^{0j}G_{-,\mathfrak{a}}\partial_j X^{\mathfrak{a}} = \sqrt{\lambda}\omega = \text{const}. \tag{4.5.35}$$

We interpret the second condition as the following relation for $\dot{X}^-$:

$$\dot{X}^- = -\left(\frac{h^{01}}{h^{00}}\right)\partial_1 X^- - \frac{\omega}{G_{--}} - \kappa h^{00}\left(\frac{G_{+-}}{G_{--}}\right). \tag{4.5.36}$$

Then, after some calculations, we find

$$\mathcal{L}_1 = \mathcal{L}_1' + P_-\dot{X}^-, \tag{4.5.37}$$

where

$$\begin{aligned}\mathcal{L}_1' &= \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) + \frac{1}{2}\sqrt{\lambda}\frac{\omega^2}{h^{00}G_{--}} \\ &- \frac{1}{2}\sqrt{\lambda}h^{00}\frac{\kappa^2}{G^{++}} + \frac{1}{2}\sqrt{\lambda}\frac{G_{--}}{h^{00}}(\partial_1 X^-)^2 + \sqrt{\lambda}\omega\left(\frac{h^{01}}{h^{00}}\right)\partial_1 X^-.\end{aligned} \tag{4.5.38}$$

Note that $P_-\dot{X}^-$ is a total τ-derivative term. Then, we use $\mathcal{L}_1'$ as the Lagrangian in the generalized light-cone gauge.

In $\mathcal{L}_1'$, the field X^- appears only through the form of $\partial_1 X^-$. The field $\partial_1 X^-$ plays the role of an auxiliary field. The equations of motion for $\partial_1 X^-$ give

$$\partial_1 X^- = -\frac{\omega h^{01}}{G_{--}}. \tag{4.5.39}$$

Substituting this solution into $\mathcal{L}_1'$, we obtain

$$\mathcal{L}_1' = \sqrt{\lambda}\kappa\omega\left(\frac{G_{++}}{G_{--}}\right) - \frac{1}{2}\sqrt{\lambda}h^{00}\frac{\kappa^2}{G^{++}} - \frac{1}{2}\sqrt{\lambda}h^{11}\frac{\omega^2}{G_{--}}. \tag{4.5.40}$$

Let us introduce a world-sheet symmetric tensor $\mathcal{J}_{ij}$ as

$$\mathcal{J}_{00} := \frac{\kappa^2}{G^{++}}, \qquad \mathcal{J}_{11} := \frac{\omega^2}{G_{--}}, \qquad \mathcal{J}_{01} = \mathcal{J}_{10} := 0. \tag{4.5.41}$$

The reduced action now has the form

$$\begin{aligned}\mathcal{L}' &= \mathcal{L}_1' + \mathcal{L}_2 \\ &= \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) - \frac{1}{2}\sqrt{\lambda}h^{ij}(\mathcal{J}_{ij} + \mathcal{G}_{ij}),\end{aligned} \tag{4.5.42}$$

where

$$\mathcal{G}_{ij} = G_{mn}\partial_i X^m \partial_j X^n. \tag{4.5.43}$$

Because the world-sheet diffeomorphism is fixed by the light-cone gauge conditions (4.5.35), the quantities h^{ij} are determined by solving the equations of motion for h^{ij}:

$$h^{ij} = \pm\sqrt{-\det(J_{ij})}\, J^{ij}, \tag{4.5.44}$$

where $J_{ij} = \mathcal{J}_{ij} + \mathcal{G}_{ij}$, and J^{ij} is the inverse of J_{ij}. Then, we finally obtain the reduced Lagrangian in the generalized light-cone gauge,

$$\mathcal{L}' = \pm\sqrt{-\lambda \det(\mathcal{J}_{ij} + \mathcal{G}_{ij})} + \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right). \qquad (4.5.45)$$

4.6 The GS action

Now let us consider the GS action in the $AdS_5 \times S^5$ background. The GS action in the flat target space [5, 6] is generalized in the curved supergravity background in Ref. [63]. A more explicit GS action in the $AdS_5 \times S^5$ background is constructed in Ref. [7]. (see also Ref. [64] and [65]). Originally, the Wess-Zumino term is written in the three-dimensional form. The manifestly two-dimensional form of the Wess-Zumino term is presented in Ref. [66, 11, 67].

We write the GS action in the $AdS_5 \times S^5$ background as follows:

$$S_{\mathrm{GS}} = \frac{1}{2\pi}\int \mathrm{d}^2\xi\, \mathcal{L}_{\mathrm{GS}}, \qquad (4.6.1)$$

$$\mathcal{L}_{\mathrm{GS}} = -\frac{1}{2}\sqrt{\lambda}h^{ij}\eta_{\underline{a}\underline{b}}E^{\underline{a}}_i E^{\underline{b}}_j + \sqrt{\lambda}\epsilon^{ij}(E^{\underline{\alpha}}_i \varrho_{\underline{\alpha}\underline{\beta}} E^{\underline{\beta}}_j - \overline{E}^{\underline{\bar{\alpha}}}_i \varrho_{\underline{\bar{\alpha}}\underline{\bar{\beta}}} \overline{E}^{\underline{\bar{\beta}}}_j). \qquad (4.6.2)$$

Here, we have $\underline{a}, \underline{b} = 0, 1, \ldots, 9$, $\underline{\alpha}, \underline{\beta}, \underline{\bar{\alpha}}, \underline{\bar{\beta}} = 1, 2, \ldots, 16$, $h^{ij} = \sqrt{-g}g^{ij}$ $(i, j = 0, 1)$, $\epsilon^{01} = 1$ and

$$\eta_{\underline{a}\underline{b}} = \mathrm{diag}(-1, 1, \ldots, 1). \qquad (4.6.3)$$

The induced vielbein for the type IIB superspace ${E^A}_i$ is denoted by

$$E^A_i = {E^A}_M \partial_i Z^M = {E^A}_{\underline{m}}\partial_i X^{\underline{m}} + {E^A}_{\underline{\alpha}}\partial_i\theta^{\underline{\alpha}} + {\overline{E}^A}_{\underline{\bar{\alpha}}}\partial_i\bar{\theta}^{\underline{\bar{\alpha}}}, \qquad A = (\underline{a}, \underline{\alpha}, \underline{\bar{\alpha}}). \qquad (4.6.4)$$

We use the Majorana-Weyl representation for the Gamma matrices:

$$\Gamma^{\underline{a}} = \begin{pmatrix} 0 & (\gamma^{\underline{a}})^{\underline{\alpha}\underline{\beta}} \\ (\gamma^{\underline{a}})_{\underline{\alpha}\underline{\beta}} & 0 \end{pmatrix}, \qquad (\Gamma^{\underline{a}})^* = \Gamma^{\underline{a}}, \qquad \{\Gamma^{\underline{a}}, \Gamma^{\underline{b}}\} = 2\eta^{\underline{a}\underline{b}}1_{32}, \qquad (4.6.5)$$

$\underline{a} = 0, 1, \ldots, 9$, $\underline{\alpha}, \underline{\beta} = 1, 2, \ldots, 16$. We denote the $n \times n$ identity matrix by 1_n. (For our specific choice of the Gamma matrices, see the appendix.)

A 32-component Weyl spinor Θ with positive chirality has the following form in the Majorana-Weyl representation:

$$\Theta = \begin{pmatrix} \theta^{\underline{\alpha}} \\ 0 \end{pmatrix}. \qquad (4.6.6)$$

Above, we have used the 16-component notation for the Weyl spinors. A spinor wit upper index $\underline{\alpha}$ represents a Weyl spinor with positive chirality.

The constant matrix ϱ in the Wess-Zumino term is given by

$$C\Gamma^{01234} = \begin{pmatrix} \varrho_{\underline{\alpha}\underline{\beta}} & 0 \\ 0 & \varrho^{\underline{\alpha}\underline{\beta}} \end{pmatrix}. \tag{4.6.7}$$

Here, C is the charge conjugation matrix.

4.6.1 κ-symmetry fixing

Let us decompose each of the two 16-component Weyl spinors into two 8-component $SO(4) \times SO(4)$ spinors:

$$\theta^{\underline{\alpha}} = \begin{pmatrix} \theta^{+\alpha} \\ \theta^{-\dot{\alpha}} \end{pmatrix}, \qquad \bar{\theta}^{\underline{\bar{\alpha}}} = \begin{pmatrix} \bar{\theta}^{+\bar{\alpha}} \\ \bar{\theta}^{-\dot{\bar{\alpha}}} \end{pmatrix}, \tag{4.6.8}$$

where $\alpha = 1, 2, \ldots, 8$, $\dot{\alpha} = \dot{1}, \dot{2}, \ldots, \dot{8}$, $\bar{\alpha} = \bar{1}, \bar{2}, \ldots, \bar{8}$ and $\dot{\bar{\alpha}} = \dot{\bar{1}}, \dot{\bar{2}}, \ldots, \dot{\bar{8}}$.

We first fix the κ-symmetry by setting $\theta^{-\dot{\alpha}} = \bar{\theta}^{-\dot{\bar{\alpha}}} = 0$. In the 32-component notation, these conditions are equivalent to the condition $\Gamma^{+}\Theta = 0$. To simplify expressions, we combine the remaining fermionic coordinates as

$$(\Psi^{\hat{\alpha}}) = \begin{pmatrix} \theta^{+\alpha} \\ \bar{\theta}^{+\bar{\alpha}} \end{pmatrix}, \qquad \hat{\alpha} = \hat{1}, \hat{2}, \ldots, \hat{16}. \tag{4.6.9}$$

Let $\mathcal{M}^2$ be a 16×16 matrix,

$$\mathcal{M}^2 = \begin{pmatrix} (\mathcal{M}^2)^{\alpha}{}_{\beta} & (\mathcal{M}^2)^{\alpha}{}_{\bar{\beta}} \\ (\mathcal{M}^2)^{\bar{\alpha}}{}_{\beta} & (\mathcal{M}^2)^{\bar{\alpha}}{}_{\bar{\beta}} \end{pmatrix}, \tag{4.6.10}$$

whose elements are constructed only from the fermionic variables:

$$\begin{aligned}
(\mathcal{M}^2)^{\alpha}{}_{\beta} &= \frac{1}{2}(\theta^{+}\gamma_{ab})^{\alpha}(\bar{\theta}^{+}\gamma^{ab}\varrho)_{\beta} - \frac{1}{2}(\theta^{+}\gamma_{a'b'})^{\alpha}(\bar{\theta}^{+}\gamma^{a'b'}\varrho)_{\beta}, \\
(\mathcal{M}^2)^{\alpha}{}_{\bar{\beta}} &= -\frac{1}{2}(\theta^{+}\gamma_{ab})^{\alpha}(\theta^{+}\gamma^{ab}\varrho)_{\bar{\beta}} + \frac{1}{2}(\theta^{+}\gamma_{a'b'})^{\alpha}(\theta^{+}\gamma^{a'b'}\varrho)_{\bar{\beta}}, \\
(\mathcal{M}^2)^{\bar{\alpha}}{}_{\beta} &= \frac{1}{2}(\bar{\theta}^{+}\gamma_{ab})^{\bar{\alpha}}(\bar{\theta}^{+}\gamma^{ab}\varrho)_{\beta} - \frac{1}{2}(\bar{\theta}^{+}\gamma_{a'b'})^{\bar{\alpha}}(\bar{\theta}^{+}\gamma^{a'b'}\varrho)_{\beta}, \\
(\mathcal{M}^2)^{\bar{\alpha}}{}_{\bar{\beta}} &= -\frac{1}{2}(\bar{\theta}^{+}\gamma_{ab})^{\bar{\alpha}}(\theta^{+}\gamma^{ab}\varrho)_{\bar{\beta}} + \frac{1}{2}(\bar{\theta}^{+}\gamma_{a'b'})^{\bar{\alpha}}(\theta^{+}\gamma^{a'b'}\varrho)_{\bar{\beta}}.
\end{aligned} \tag{4.6.11}$$

Here, we have $a, b = 1, 2, 3, 4$, $a', b' = 5, 6, 7, 8$.

For later convenience, let us introduce the following matrices:

$$\frac{\cosh \mathcal{M} - 1_{16}}{\mathcal{M}^2} = \begin{pmatrix} (K_{11})^{\alpha}{}_{\beta} & (K_{12})^{\alpha}{}_{\bar{\beta}} \\ (K_{21})^{\bar{\alpha}}{}_{\beta} & (K_{22})^{\bar{\alpha}}{}_{\bar{\beta}} \end{pmatrix}, \qquad \frac{\sinh \mathcal{M}}{\mathcal{M}} = \begin{pmatrix} (L_{11})^{\alpha}{}_{\beta} & (L_{12})^{\alpha}{}_{\bar{\beta}} \\ (L_{21})^{\bar{\alpha}}{}_{\beta} & (L_{22})^{\bar{\alpha}}{}_{\bar{\beta}} \end{pmatrix}. \tag{4.6.12}$$

Now, the κ-symmetry fixed action in the $AdS_5 \times S^5$ can be written as [1]

$$\mathcal{L}_{\rm GS} = -\frac{1}{2}\sqrt{\lambda} h^{ij} G_{\underline{m}\underline{n}} \mathcal{D}_i X^{\underline{m}} \mathcal{D}_j X^{\underline{n}} + \frac{1}{2}\sqrt{\lambda}\epsilon^{ij} B_{\hat{\alpha}\hat{\beta}} \mathcal{D}_i \Psi^{\hat{\alpha}} \mathcal{D}_j \Psi^{\hat{\beta}}, \tag{4.6.13}$$

where $\underline{m}, \underline{n} = 0, 1, \ldots, 9$, $\hat{\alpha}, \hat{\beta} = \hat{1}, \hat{2}, \ldots, \hat{16}$. The target space metric $G_{\underline{m}\underline{n}}$ is the bosonic $AdS_5 \times S^5$ metric, chosen as follows:

$$\mathrm{d}s^2 = \mathrm{d}s^2_{AdS_5} + \mathrm{d}s^2_{S^5} = G_{\underline{m}\underline{n}} \mathrm{d}X^{\underline{m}} \mathrm{d}X^{\underline{n}} = G_{\mathfrak{a}\mathfrak{b}} \mathrm{d}X^{\mathfrak{a}} \mathrm{d}X^{\mathfrak{b}} + G_{mn} \mathrm{d}X^m \mathrm{d}X^n. \tag{4.6.14}$$

The AdS_5-part metric is chosen as

$$\mathrm{d}s^2_{AdS_5} = -\left(\frac{1 + (z^2/4)}{1 - (z^2/4)}\right)^2 \mathrm{d}t^2 + G_z \sum_{a=1}^{4} (\mathrm{d}z^a)^2, \tag{4.6.15}$$

and the S^5-part metric is chosen as

$$\mathrm{d}s^2_{S^5} = \left(\frac{1 - (y^2/4)}{1 + (y^2/4)}\right)^2 \mathrm{d}\varphi^2 + G_y \sum_{s=1}^{4} (\mathrm{d}y^s)^2, \tag{4.6.16}$$

where

$$G_z = \frac{1}{(1 - (z^2/4))^2}, \qquad G_y = \frac{1}{(1 + (y^2/4))^2}. \tag{4.6.17}$$

Here, we have

$$z^2 = \sum_{a=1}^{4} (z^a)^2, \qquad y^2 = \sum_{s=1}^{4} (y^s)^2. \tag{4.6.18}$$

We choose the coordinates $X^{\underline{m}}$ as follows:

$$X^{\pm} = \frac{1}{\sqrt{2}}(t \pm \varphi), \qquad X^a = z^a, \qquad X^{4+s} = y^s. \tag{4.6.19}$$

The metric for the longitudinal directions $G_{\mathfrak{a}\mathfrak{b}}$ ($\mathfrak{a}, \mathfrak{b} = \pm$) is given by

$$G_{++} = G_{--} = -\frac{1}{2}\left(\frac{1 + (z^2/4)}{1 - (z^2/4)}\right)^2 + \frac{1}{2}\left(\frac{1 - (y^2/4)}{1 + (y^2/4)}\right)^2, \tag{4.6.20}$$

$$G_{+-} = G_{-+} = -\frac{1}{2}\left(\frac{1 + (z^2/4)}{1 - (z^2/4)}\right)^2 - \frac{1}{2}\left(\frac{1 - (y^2/4)}{1 + (y^2/4)}\right)^2. \tag{4.6.21}$$

For G_{mn} $(m,n=1,2,\dots,8)$, we have

$$G_{ab}=G_z\delta_{ab},\qquad G_{4+s,4+s'}=G_y\delta_{s,s'},\qquad G_{a,4+s}=0, \tag{4.6.22}$$

where $a,b=1,2,3,4$ and $s,s'=1,2,3,4$. Let us denote the inverse of G_{mn} by $G^{\underline{mn}}$. Not that $\partial/\partial X^{\pm}$ are Killing vectors.

The derivatives $\mathcal{D}_j$ in (4.6.13) are given by

$$\begin{aligned}
\mathcal{D}_iX^+&=\partial_iX^+,\\
\mathcal{D}_iX^-&=\partial_iX^-+\Lambda^-{}_{\hat\alpha}\mathcal{D}_i\Psi^{\hat\alpha},\\
\mathcal{D}_iX^m&=\partial_iX^m+(\Lambda^m{}_{n\hat\alpha}\mathcal{D}_i\Psi^{\hat\alpha})X^n,\\
\mathcal{D}_i\Psi^{\hat\alpha}&=\partial_i\Psi^{\hat\alpha}+(\Lambda^{\hat\alpha}{}_{\hat\beta}\partial_iX^+)\Psi^{\hat\beta},
\end{aligned} \tag{4.6.23}$$

where $\hat\alpha=(\alpha,\bar\alpha)$, $\hat\alpha=1,2,\dots,16$, and $\alpha,\bar\alpha=1,2,\dots,8$. The terms $\Lambda^-{}_{\hat\alpha}$ in $\mathcal{D}_iX^-$ are given by

$$\begin{aligned}
\Lambda^-{}_{\alpha}&=2\sqrt{2}\mathrm{i}\Big[(\bar\theta^+\gamma_+K_{11})_\alpha+(\theta^+\gamma_+K_{21})_\alpha\Big],\\
\Lambda^-{}_{\bar\alpha}&=2\sqrt{2}\mathrm{i}\Big[(\bar\theta^+\gamma_+K_{12})_{\bar\alpha}+(\theta^+\gamma_+K_{22})_{\bar\alpha}\Big],
\end{aligned} \tag{4.6.24}$$

where

$$(\gamma_+)_{\underline{\alpha\beta}}=\frac{1}{2}\Big((\gamma_0)_{\underline{\alpha\beta}}+(\gamma_9)_{\underline{\alpha\beta}}\Big)=\begin{pmatrix}1_8&0\\0&0\end{pmatrix}. \tag{4.6.25}$$

The terms $\Lambda^m{}_{n\hat\alpha}$ in $\mathcal{D}_iX^m=(\mathcal{D}_iz^a,\mathcal{D}_iy^s)$ are given by

$$\begin{aligned}
\Lambda^a{}_{b\alpha}&=-2\left[(\bar\theta^+\gamma^{ab}\varrho K_{11})_\alpha-(\theta^+\gamma^{ab}\varrho K_{21})_\alpha\right],\\
\Lambda^{\bar a}{}_{b\bar\alpha}&=-2\left[(\bar\theta^+\gamma^{ab}\varrho K_{12})_{\bar\alpha}-(\theta^+\gamma^{ab}\varrho K_{22})_{\bar\alpha}\right],\\
\Lambda^a{}_{(4+s)\hat\alpha}&=0,
\end{aligned} \tag{4.6.26}$$

$$\begin{aligned}
\Lambda^{4+s}{}_{a\hat\alpha}&=0,\\
\Lambda^{4+s}{}_{(4+s')\alpha}&=2\left[(\bar\theta^+\gamma^{4+s,4+s'}\varrho K_{11})_\alpha-(\theta^+\gamma^{4+s,4+s'}\varrho K_{21})_\alpha\right],\\
\Lambda^{4+s}{}_{(4+s')\bar\alpha}&=2\left[(\bar\theta^+\gamma^{4+s,4+s'}\varrho K_{12})_{\bar\alpha}-(\theta^+\gamma^{4+s,4+s'}\varrho K_{22})_{\bar\alpha}\right].
\end{aligned} \tag{4.6.27}$$

The terms $\Lambda^{\hat\alpha}{}_{\hat\beta}$ in $\mathcal{D}_i\Psi^{\hat\alpha}=(\mathcal{D}_i\theta^{+\alpha},\mathcal{D}_i\bar\theta^{+\bar\alpha})$ are given by

$$\Lambda^\alpha{}_\beta=-\frac{\mathrm{i}}{\sqrt{2}}(\gamma_+\varrho)_\beta{}^\alpha,\qquad \Lambda^{\bar\alpha}{}_{\bar\beta}=\frac{\mathrm{i}}{\sqrt{2}}(\gamma_+\varrho)_{\bar\beta}{}^{\bar\alpha},\qquad \Lambda^\alpha{}_{\bar\beta}=\Lambda^{\bar\alpha}{}_\beta=0. \tag{4.6.28}$$

The fields $B_{\hat{\alpha}\hat{\beta}}$ in the Wess-Zumino term in (4.6.13) are defined by

$$\begin{aligned}
B_{\alpha\beta} &= 2W_{\gamma\delta}\left((L_{11})^{\gamma}{}_{\alpha}(L_{11})^{\delta}{}_{\beta} - (L_{21})^{\bar{\gamma}}{}_{\alpha}(L_{21})^{\bar{\delta}}{}_{\beta}\right), \\
B_{\alpha\bar{\beta}} &= 2W_{\gamma\delta}\left((L_{11})^{\gamma}{}_{\alpha}(L_{12})^{\delta}{}_{\bar{\beta}} - (L_{21})^{\bar{\gamma}}{}_{\alpha}(L_{22})^{\bar{\delta}}{}_{\bar{\beta}}\right), \\
B_{\bar{\alpha}\beta} &= 2W_{\gamma\delta}\left((L_{12})^{\gamma}{}_{\bar{\alpha}}(L_{11})^{\delta}{}_{\beta} - (L_{22})^{\bar{\gamma}}{}_{\bar{\alpha}}(L_{21})^{\bar{\delta}}{}_{\beta}\right), \\
B_{\bar{\alpha}\bar{\beta}} &= 2W_{\gamma\delta}\left((L_{12})^{\gamma}{}_{\bar{\alpha}}(L_{12})^{\delta}{}_{\bar{\beta}} - (L_{22})^{\bar{\gamma}}{}_{\bar{\alpha}}(L_{22})^{\bar{\delta}}{}_{\bar{\beta}}\right),
\end{aligned} \tag{4.6.29}$$

where

$$W_{\alpha\beta} = \frac{\left[(1+(z^2/4))(1-(y^2/4))\varrho_{\alpha\beta} - z^a y^s (\gamma_a \gamma_{4+s}\varrho)_{\alpha\beta}\right]}{(1-(z^2/4))(1+(y^2/4))}. \tag{4.6.30}$$

4.6.2 Generalized light-cone gauge

Now let us consider the relevant part of (4.6.13),

$$\mathcal{L}_1 := -\frac{1}{2}\sqrt{\lambda} h^{ij} G_{\mathfrak{a}\mathfrak{b}} \mathcal{D}_i X^{\mathfrak{a}} \mathcal{D}_j X^{\mathfrak{b}}. \tag{4.6.31}$$

The generalized light-cone gauge is chosen as

$$X^+ = \kappa\tau, \qquad P_- = -\sqrt{\lambda} h^{0i} G_{-,\mathfrak{a}} \mathcal{D}_i X^{\mathfrak{a}} =: \sqrt{\lambda}\omega = \text{const.} \tag{4.6.32}$$

From the second equation, we have

$$\mathcal{D}_0 X^- = -\frac{\omega}{h^{00}G_{--}} - \kappa\left(\frac{G_{+-}}{G_{--}}\right) - \left(\frac{h^{01}}{h^{00}}\right)\mathcal{D}_1 X^-, \tag{4.6.33}$$

or

$$\dot{X}^- = -\Lambda^-{}_{\hat{\alpha}}\mathcal{D}_0\Psi^{\hat{\alpha}} - \frac{\omega}{h^{00}G_{--}} - \kappa\left(\frac{G_{+-}}{G_{--}}\right) - \left(\frac{h^{01}}{h^{00}}\right)\mathcal{D}_1 X^-, \tag{4.6.34}$$

$$\mathcal{L}_1 = \mathcal{L}_1' + P_- \dot{X}^-, \tag{4.6.35}$$

$$\begin{aligned}
\mathcal{L}_1' = {} & \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) + \sqrt{\lambda}\omega\Lambda^-{}_{\hat{\alpha}}\mathcal{D}_0\Psi^{\hat{\alpha}} \\
& -\frac{1}{2}\sqrt{\lambda}h^{00}\frac{\kappa^2}{G^{++}} + \frac{1}{2}\sqrt{\lambda}\frac{\omega^2}{h^{00}G_{--}} \\
& +\frac{1}{2}\sqrt{\lambda}\frac{G_{--}}{h^{00}}(\mathcal{D}_1 X^-)^2 + \sqrt{\lambda}\omega\left(\frac{h^{01}}{h^{00}}\right)\mathcal{D}_1 X^-.
\end{aligned} \tag{4.6.36}$$

Now, $\mathcal{D}_1 X^-$ is an auxiliary field. By solving the equations of motion, we obtain

$$\mathcal{D}_1 X^- = -\frac{\omega h^{01}}{G_{--}}. \tag{4.6.37}$$

Substitution of this relation into $\mathcal{L}_1'$ yields

$$\mathcal{L}_1' = +\sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) + \sqrt{\lambda}\omega\Lambda^-{}_{\hat{\alpha}}\mathcal{D}_0\Psi^{\hat{\alpha}} - \frac{1}{2}\sqrt{\lambda}h^{00}\left(\frac{\kappa^2}{G^{++}}\right) - \frac{1}{2}\sqrt{\lambda}h^{11}\left(\frac{\omega^2}{G_{--}}\right). \tag{4.6.38}$$

We define

$$\mathcal{J}_{00} := \frac{\kappa^2}{G^{++}}, \qquad \mathcal{J}_{11} := \frac{\omega^2}{G_{--}}, \qquad \mathcal{J}_{01} = \mathcal{J}_{10} := 0. \tag{4.6.39}$$

The GS action in the generalized light-cone gauge is given by

$$\mathcal{L}_{\text{GS}}' = -\frac{1}{2}\sqrt{\lambda}h^{ij}(\mathcal{J}_{ij} + \mathcal{G}_{ij}) + \frac{1}{2}\sqrt{\lambda}\epsilon^{ij}B_{\hat{\alpha}\hat{\beta}}\mathcal{D}_i\Psi^{\hat{\alpha}}\mathcal{D}_j\Psi^{\hat{\beta}} + \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) + \sqrt{\lambda}\omega\Lambda^-{}_{\hat{\alpha}}\mathcal{D}_0\Psi^{\hat{\alpha}}, \tag{4.6.40}$$

where

$$\mathcal{G}_{ij} = G_{mn}\mathcal{D}_iX^m\mathcal{D}_jX^n. \tag{4.6.41}$$

Then, removing h^{ij}, we finally obtain the GS Lagrangian in the $AdS_5 \times S^5$ in the generalized light-cone gauge:

$$\mathcal{L}_{\text{GS}}' = \sqrt{-\lambda\det(\mathcal{J}_{ij} + \mathcal{G}_{ij})} + \frac{1}{2}\sqrt{\lambda}\epsilon^{ij}B_{\hat{\alpha}\hat{\beta}}\mathcal{D}_i\Psi^{\hat{\alpha}}\mathcal{D}_j\Psi^{\hat{\beta}} + \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) + \sqrt{\lambda}\omega\Lambda^-{}_{\hat{\alpha}}\mathcal{D}_0\Psi^{\hat{\alpha}}. \tag{4.6.42}$$

Here, we have

$$\mathcal{D}_i\theta^{+\alpha} = \partial_i\theta^{+\alpha} - \frac{\mathrm{i}\kappa}{\sqrt{2}}\delta_{i,0}(\varrho\gamma_+\theta^+)^\alpha, \qquad \mathcal{D}_i\bar{\theta}^{+\bar{\alpha}} = \partial_i\theta^{+\bar{\alpha}} + \frac{\mathrm{i}\kappa}{\sqrt{2}}\delta_{i,0}(\varrho\gamma_+\bar{\theta}^+)^{\bar{\alpha}}, \tag{4.6.43}$$

$$\mathcal{D}_iX^m = \partial_iX^m + (\Lambda^m{}_{n\hat{\alpha}}\mathcal{D}_i\Psi^{\hat{\alpha}})X^n. \tag{4.6.44}$$

This is the main result of this paper. It can be rewritten as follows:

$$\mathcal{L}_{\text{GS}}' = \sqrt{-\lambda\det(\mathcal{J}_{ij} + \mathcal{G}_{ij})} + \sqrt{\lambda}\kappa\omega\left(\frac{G_{+-}}{G_{--}}\right) + \sqrt{\lambda}\epsilon^{ij}\mathcal{D}_i\Psi^{\hat{\alpha}}\left[\frac{\sinh\mathcal{M}^T}{\mathcal{M}^T}\mathcal{W}\frac{\sinh\mathcal{M}}{\mathcal{M}}\right]_{\hat{\alpha}\hat{\beta}}\mathcal{D}_j\Psi^{\hat{\beta}} + \mathrm{i}\,2\sqrt{2\lambda}\,\omega\Psi^{\hat{\alpha}}(\gamma_+)_{\hat{\alpha}\hat{\beta}}\left(\frac{\cosh\mathcal{M} - 1_{16}}{\mathcal{M}^2}\right)_{\hat{\beta}\hat{\gamma}}\mathcal{D}_0\Psi^{\hat{\gamma}}, \tag{4.6.45}$$

where

$$\left[\frac{\sinh \mathcal{M}^T}{\mathcal{M}^T}\mathcal{W}\frac{\sinh \mathcal{M}}{\mathcal{M}}\right]_{\hat{\alpha}\hat{\beta}} = \left[\frac{\sinh \mathcal{M}^T}{\mathcal{M}^T}\right]_{\hat{\alpha}}^{\ \hat{\gamma}} \mathcal{W}_{\hat{\gamma}\hat{\delta}} \left[\frac{\sinh \mathcal{M}}{\mathcal{M}}\right]^{\hat{\delta}}_{\ \hat{\beta}}, \tag{4.6.46}$$

$$\left[\frac{\sinh \mathcal{M}^T}{\mathcal{M}^T}\right]_{\hat{\alpha}}^{\ \hat{\gamma}} = \left[\frac{\sinh \mathcal{M}}{\mathcal{M}}\right]^{\hat{\gamma}}_{\ \hat{\alpha}}, \qquad \mathcal{W}_{\hat{\gamma}\hat{\delta}} = \begin{pmatrix} W_{\gamma\delta} & 0 \\ 0 & W_{\bar{\gamma}\bar{\delta}} \end{pmatrix}, \tag{4.6.47}$$

with $W_{\bar{\gamma}\bar{\delta}} = W_{\gamma\delta}$, and

$$(\gamma_+)_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & (\gamma_+)_{\alpha\bar{\beta}} \\ (\gamma_+)_{\bar{\alpha}\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix}. \tag{4.6.48}$$

In contract to ordinary Nambu-Goto actions, we have chosen the sign before the square root term to be positive. This comes from the requirement that the action must have the correct flat-space limit. Indeed, the Lagrangian (4.6.45) goes to the correct κ-symmetry fixed light-cone gauge Lagrangian in the limit.

The Lagrangian (4.6.45) will serves as a starting point for developing the various limits and investigating the quantum fluctuations.

4.7 Flat-limit of the Lagrangian

4.7.1 Flat-space limit

In the subsection 4.5.2 and 4.5.3, we found the bosonic Lagrangian in the generalized light-cone gauge in the standard form, which is written by

$$\mathcal{L}_{\rm LC} = \sqrt{\lambda}\kappa\omega\frac{G^{+-}}{G^{++}}\sqrt{1-\frac{G^{++}C}{(G^{+-})^2\omega^2}}\left(1+\frac{1}{\kappa^2}G^{++}\Lambda\right) - \sqrt{\lambda}\kappa\frac{G^{+-}\omega}{G^{++}}, \tag{4.7.1}$$

where

$$\Lambda = K_{mn}\dot{X}^m\dot{X}^n \tag{4.7.2}$$

with

$$K_{mn} = G_{mn} - \frac{(1/\omega^2)G_{--}(G_{mm'}\partial_1 X^{m'})(G_{nn'}\partial_1 X^{n'})}{1+(1/\omega^2)G_{--}G_{kl}\partial_1 X^k\partial_1 X^l}. \tag{4.7.3}$$

In this section, we treat the flat-space limit. To do this, we consider the rescaling coordinates

$$\zeta = \frac{\zeta_{sc}}{R}, \quad \eta = \frac{\eta_{sc}}{R}, \tag{4.7.4}$$

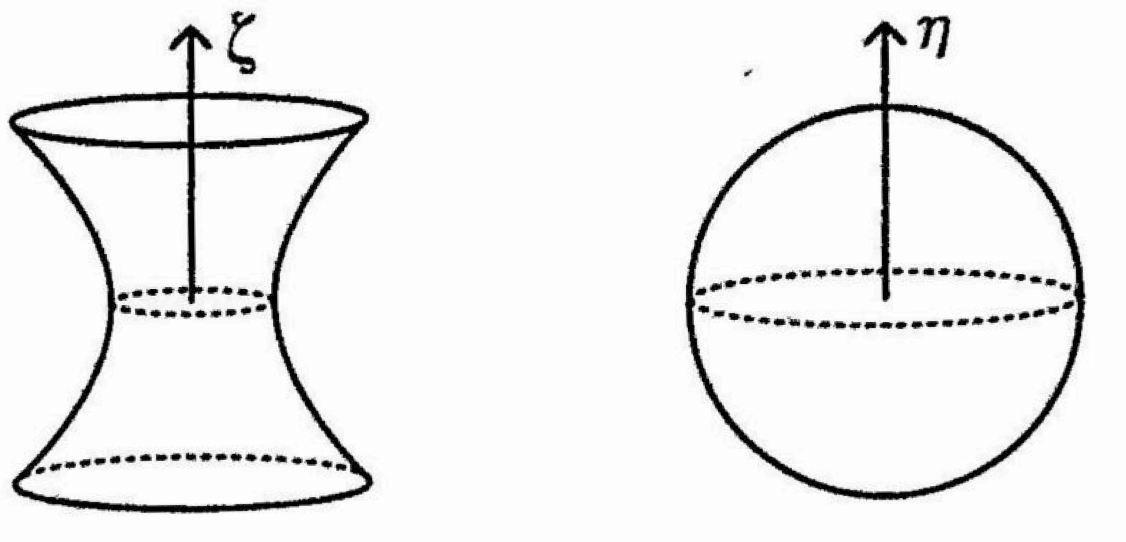


Figure 4.3: $AdS_5 \times S^5$

where ζ and η express the height from the equatorial plane which are respectively described by t and φ_3 of AdS_5 and S^5 as show in Figure 4.3. Then some elements of the target space metric is rescaled as

$$G_{++} = G_{--} = \frac{G^{sc}_{++}}{R^2}, \quad G_y = \frac{G^{sc}_y}{R^2}, \quad G_z = \frac{G^{sc}_y}{R^2} \tag{4.7.5}$$

and Λ and C are respectively

$$\begin{aligned}\Lambda &= \frac{1}{R^2}\left[G^{sc}_y(\dot{y}^s)^2 + G^{sc}_z(\dot{z}^a)^2 - \frac{G^{sc}_{--}\frac{1}{R^4\omega^2}\left\{G^{sc}_y\dot{y}^s y'^s + G^{sc}_z\dot{z}^a z'^a\right\}^2}{1+\frac{G^{sc}_{--}}{R^4\omega^2}\left\{G^{sc}_{--}(y'^s)^2 + G^{sc}_{--}(z'^a)^2\right\}}\right] \\ &\equiv R^{-2}\Lambda_{sc}\end{aligned} \tag{4.7.6}$$

and

$$\begin{aligned}C &= \frac{1}{R^2}\frac{G^{--}_{sc}\omega^2 + G^{sc}_y(y'^s)^2 + G^{sc}_z(z'^a)^2 + \frac{\omega^2}{\kappa^2}(G^{+-})^2\Lambda_{sc}}{1+\frac{1}{R^4\kappa^2}G^{++}_{sc}\Lambda_{sc}} \\ &\equiv R^{-2}C_{sc}.\end{aligned} \tag{4.7.7}$$

$$\sqrt{1-\frac{G^{++}C}{(G^{+-})^2\omega^2}} = \sqrt{1-\frac{1}{R^4}\frac{G^{++}_{sc}C_{sc}}{(G^{+-})^2\omega^2}} \tag{4.7.8}$$

Thus we obtain the following bosnic light-cone Lagrangian,

$$\begin{aligned}\mathcal{L}_{\rm LC} &= \sqrt{\lambda}\left\{\frac{\kappa\omega G^{+-}}{G^{++}}\sqrt{1-\frac{G^{++}C}{(G^{+-})^2}}\left(1+\frac{1}{\kappa^2}G^{++}\Lambda\right) - \frac{\kappa\omega G^{+-}}{G^{++}}\right\} \\ &\sim \frac{R^2}{\alpha'}\left[R^2\kappa\omega\frac{G^{+-}}{G^{++}_{sc}}\left\{1-\frac{1}{2R^4}\frac{G^{++}_{sc}}{(G^{+-})^2\omega^2}C_{sc}\right\}\left\{1+\frac{1}{R^4}\frac{G^{++}_{sc}}{\kappa^2}\Lambda_{sc}\right\} - R^2\kappa\omega\frac{G^{+-}}{G^{++}_{sc}}\right] \\ &\sim -\frac{1}{\alpha'}\kappa\omega G^{+-}\left[\frac{1}{2\omega^2}\frac{1}{(G^{+-})^2}C_{sc} - \frac{1}{\kappa^2}\Lambda_{sc}\right],\end{aligned} \tag{4.7.9}$$

where

$$\Lambda_{sc} \sim G^{sc}_y(\dot{y}^s)^2 + G^{sc}_z(\dot{z}^a)^2, \tag{4.7.10}$$

$$C_{sc} \sim \omega^2 G^{--}_{sc} + G^{sc}_y(y'^s)^2 + G^{sc}_z(z'^a)^2 + \frac{\omega^2}{\kappa^2}(G^{+-})^2\Lambda_{sc}. \tag{4.7.11}$$

Now the remaining elements of the target space metric and its determinant can be expanded with respect to $R \sim \lambda^{\frac{1}{4}}$ as

$$G_{+-} = -1 - \frac{1}{2R^2}(\zeta^2_{sc} - \eta^2_{sc}) \sim -1, \tag{4.7.12}$$

$$\det(G_{\mathfrak{ab}}) = (1 + \frac{\zeta^2_{sc}}{R^2})(-1 + \frac{\eta^2_{sc}}{R^2}) \sim -1, \tag{4.7.13}$$

$$G^{+-} = -\frac{G_{+-}}{\det(G_{\mathfrak{ab}})} \sim -1 \tag{4.7.14}$$

Hence we obtain finally the Lagrangian under $R \to \infty$

$$\mathcal{L}_{\rm LC} \sim -\frac{1}{\alpha'}\kappa\omega\left[\frac{1}{2\kappa^2}\{G^{sc}_y(\dot{y}^s)^2 + G^{sc}_z(\dot{z}^a)^2\} - \frac{1}{2\omega^2}\{G^{sc}_y(y'^s)^2 + G^{sc}_z(z'^a)^2\} - \frac{1}{2}G^{--}_{sc}\right] \tag{4.7.15}$$

The last term in eq.(4.7.15) yields the mass terms. Now note that there exist the parameters κ and ω which were introduced on the occasion when gauge-fixing (4.4.8). So let choose these parameters as

$$\kappa = -\lambda^{-\frac{1}{4}}\tilde{\kappa}, \quad \omega = \lambda^{-\frac{1}{4}}\tilde{\kappa}. \tag{4.7.16}$$

That is,

$$P_- = \sqrt{\lambda}\omega = \lambda^{\frac{1}{4}}\tilde{\kappa} \to \infty. \tag{4.7.17}$$

Then we obtain the following Lagrangian in this limit:

$$\begin{aligned}\mathcal{L}_{\rm LC} &\sim \frac{1}{2\alpha'}\left\{G^{sc}_y(\dot{y}^s)^2 + G^{sc}_z(\dot{z}^a)^2 - G^{sc}_y(y'^s)^2 - G^{sc}_z(z'^a)^2\right\} \\ &= -\frac{1}{2\alpha'}\eta^{ij}G^{sc}_{mn}\partial_i X^m \partial_j X^n,\end{aligned} \tag{4.7.18}$$

in whice we introduced 2-d Minkowski metric

$$(\eta_{ij}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.7.19}$$

Moreover, we can get the Lagrangians in various flat-like limit by considering rescaling of string coordinates. As follows, it is shown that we obtain the correct Lagrangian in the flat-space limit from (4.7.18).

Flat limit

In this case, we have to do the rescaling of string coordinates,

$$X^m = \lambda^{-\frac{1}{4}}\tilde{X}^m. \tag{4.7.20}$$

Then the scaled metric becomes

$$G^{sc}_{mn} = \lambda^{\frac{1}{2}}\begin{pmatrix} \frac{\eta_{sc}^2}{\tilde{y}^2}\mathbf{1}_4 & 0 \\ 0 & \frac{\zeta_{sc}^2}{\tilde{z}^2}\mathbf{1}_4 \end{pmatrix} \sim \lambda^{\frac{1}{2}}\delta_{mn}, \tag{4.7.21}$$

where we used

$$\eta_{sc}^2 \sim \tilde{y}^2, \qquad \zeta_{sc}^2 \sim \tilde{z}^2. \tag{4.7.22}$$

Hence the light-cone Lagrangian becomes

$$\mathcal{L}_{LC} \sim -\frac{1}{2\alpha'}\eta^{ij}\delta_{mn}\partial_i\tilde{X}^m\partial_j\tilde{X}^n. \tag{4.7.23}$$

Also let us rescale the fermionic fields as follows:

$$\theta^{+\alpha} = \frac{\lambda^{-1/8}}{\sqrt{2}}(S_1^\alpha + \mathrm{i}S_2^\alpha), \qquad \bar{\theta}^{+\bar{\alpha}} = \frac{\lambda^{-1/8}}{\sqrt{2}}(S_1^\alpha - \mathrm{i}S_2^\alpha), \qquad \alpha = 1, 2, \ldots, 8. \tag{4.7.24}$$

Taking $\lambda \to \infty$, and ignoring (divergent) surface terms, the GS Lagrangian in the generalized light-cone gauge arrives at

$$\mathcal{L}'^{\text{flat}}_{\text{GS}} = \frac{1}{2}\sum_{m=1}^{8}\left[(\partial_0\tilde{X}^m)^2 - (\partial_1\tilde{X}^m)^2\right] + \sqrt{2}\mathrm{i}\tilde{\kappa}\sum_{\alpha=1}^{8}\left[S_1^\alpha(\partial_0+\partial_1)S_1^\alpha + S_2^\alpha(\partial_0-\partial_1)S_2^\alpha\right]. \tag{4.7.25}$$

This is the correct form of the Green-Schwarz action in the light-cone gauge.

Chapter 5

Scaling limit to the giant magnon solution

5.1 Giant magnon

Th giant magnon solution [69] is the string moving on $\mathbb{R}_t \times S^2$ which is the subspace of $AdS_5 \times S^5$. Especially, the both ends of the string are attached on a big circle of S^2 as in Fig.2.

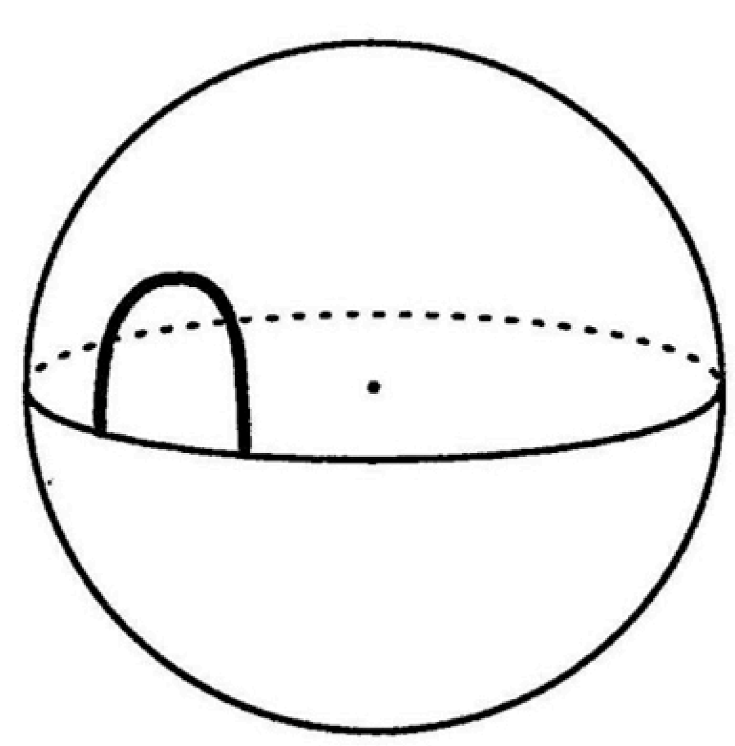


Figure 5.1: giant magnon solution on $S^2 \subset S^5$

5.1.1 Constructing the giant magnon

The situation of the giant magnon looks strange because of the type IIB string theory that is important in study of AdS/CFT correspondence has usually only closed strings. But we can also consider open strings in that framework under a limiting set-up. In the following, let us see its special circumstance in flat-space. We consider the closed string living in the cylinder with radius 1 in Minkowski space as in Fig.5.2

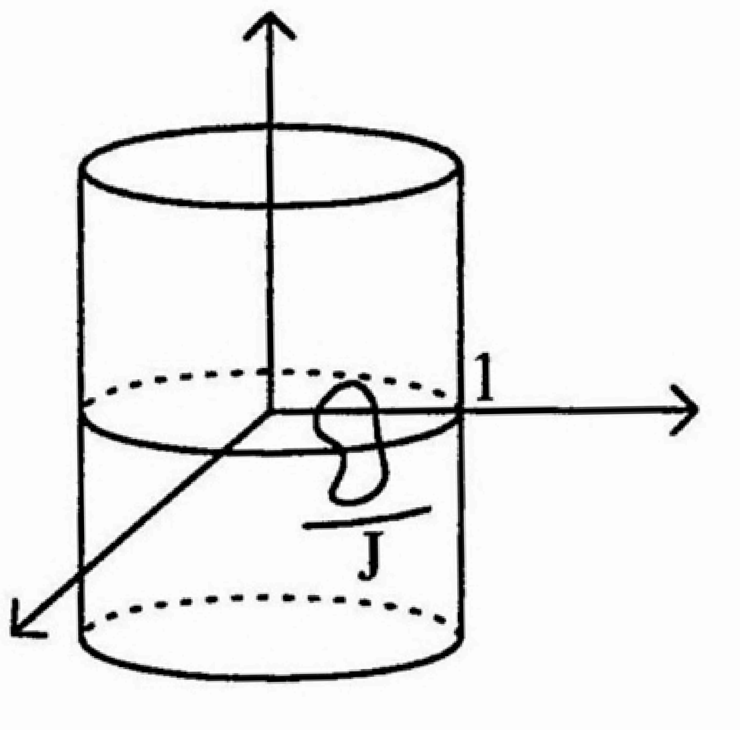


Figure 5.2: orbit of particle in Minkowski spacetime

To treat this string, we introduce the coordinate $X^1 + iX^2 = e^{i\theta}$. Then, the action is

$$\begin{aligned} S &= -\frac{\sqrt{\lambda}}{2\pi}\int d\xi^2 \frac{1}{2} h^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\xi^2 h^{ab}\left(\partial_a X^+ \partial_b X^- - \frac{1}{2}\partial_a \boldsymbol{X}\cdot\partial_b \boldsymbol{X}\right) \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\xi^2 \mathcal{L}, \end{aligned} \tag{5.1.1}$$

where we introduced the light-cone coordinate $X^{\pm} = 1/\sqrt{2}(X^0 \pm \theta)$ in the second line and $\boldsymbol{X} = (X^3, \cdots, X^9)^t$. We choose the light-cone gauge, $X^+ = \tau$ and $p_- = \partial\mathcal{L}/\partial\dot{X}^- =$ const $= \omega$. Now in this gauge, the worldsheet Hamiltonian is equal to p_+,

$$\mathcal{H} = p_+ = i\partial_+ = i\sqrt{2}(\partial_0 + \partial_\theta) = \sqrt{2}(E - J), \tag{5.1.2}$$

while

$$p_- = i\partial_- = i\sqrt{2}(\partial_0 - \partial_\theta) = \sqrt{2}(E + J), \tag{5.1.3}$$

where E and J are the spacetime energy and momentum, respectively. The positivity of the light-cone Hamiltonian is assured by BPS condition $E \geq J$. First of all, we consider the ground state which satisfies

$$0 = p_+ = \dot{X}^-. \tag{5.1.4}$$

Hence, the ground state corresponds to a lightlike trajectory with $X^- =$ const. the string on the θ line is folded as in Fig.(5.3).

On the other hand, we can not generically consider one localized excitation because of the level-matching condition of the closed string. So we must always put pair of two

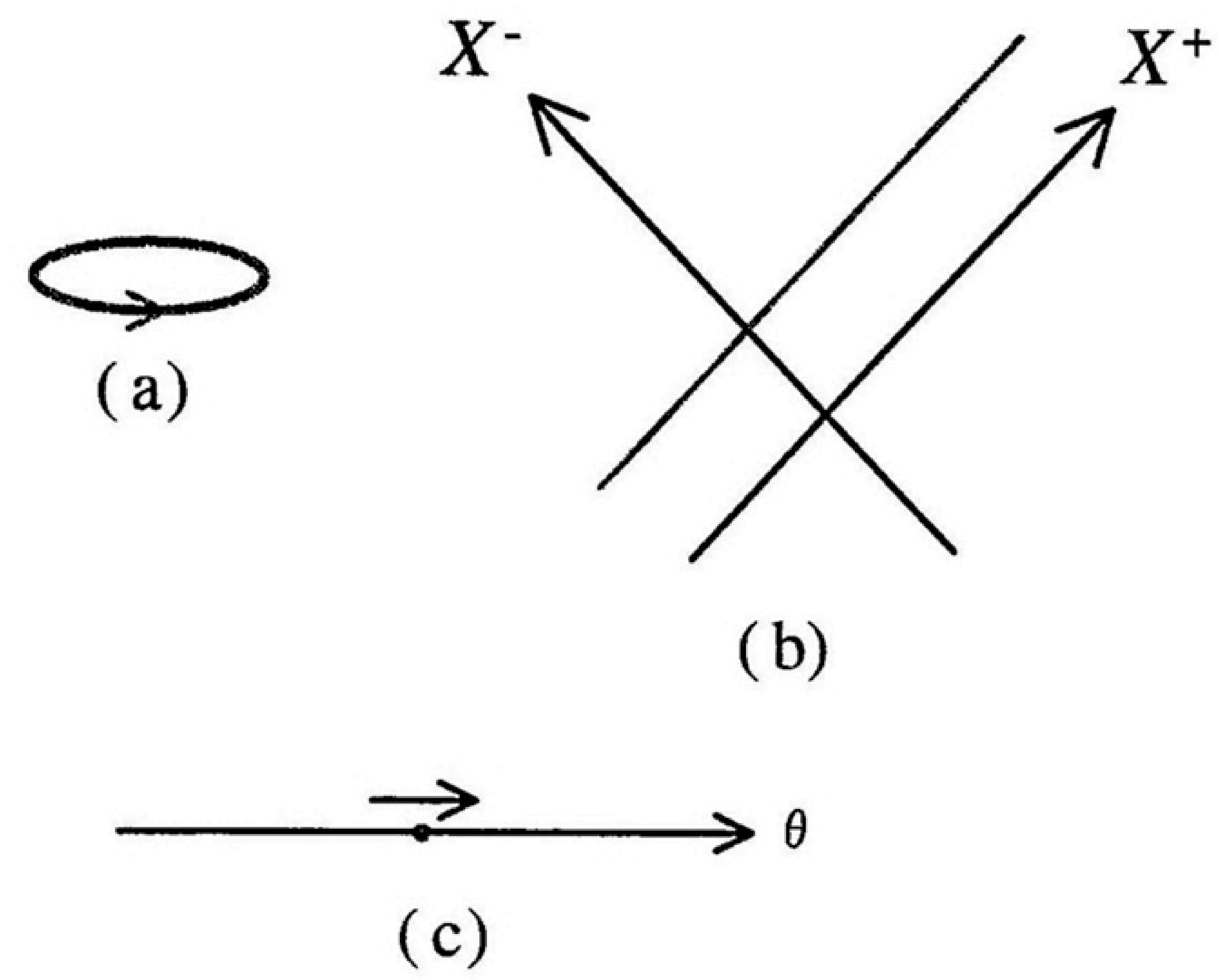


Figure 5.3: A light-cone ground state of the closed string, $P_+ = 0$, which has no excitations. (a) Worldsheet picture of the ground state. (b) Spacetime description of the configurations. (c) The shape of the configuration at a given time t. The string is folded along θ line.

exciations carring worldsheet momentum with the same value but opposite sign. Therefore the next simplest state is to put two excitations with the momentums, so called p and $-p$. Integrating the element of the energy momentum tensor,

$$T_{\tau\sigma} \sim \partial_\tau X^+ \partial_\sigma X^- = \partial_\sigma X^-, \tag{5.1.5}$$

along the region where there exists the local exicitation with momentum p, we obtain

$$\Delta X^- \sim \int d\sigma T_{\tau\sigma} = p \tag{5.1.6}$$

Thus the spacetime description at a time X^0 becomes two particles moving at the speed of light separated by

$$\Delta\theta|_{X^0} \sim \Delta X^-|_{X^+} \sim p. \tag{5.1.7}$$

and joined by a string. Essencially, the string of the first local excitation has such spacetime picture. However, we can give up the level-matching condition as follows: We consider the limit of an infinite string and Then the two excitation can not encounter each other forever and we can get a single excitation with momentum p along the infinite string. By the above consideration, it is easy to see the spacetime picture is to have two lightlike trajectories which seprated by $\Delta X^- \sim p$ and these particles are joined by a string. So we got an open string.

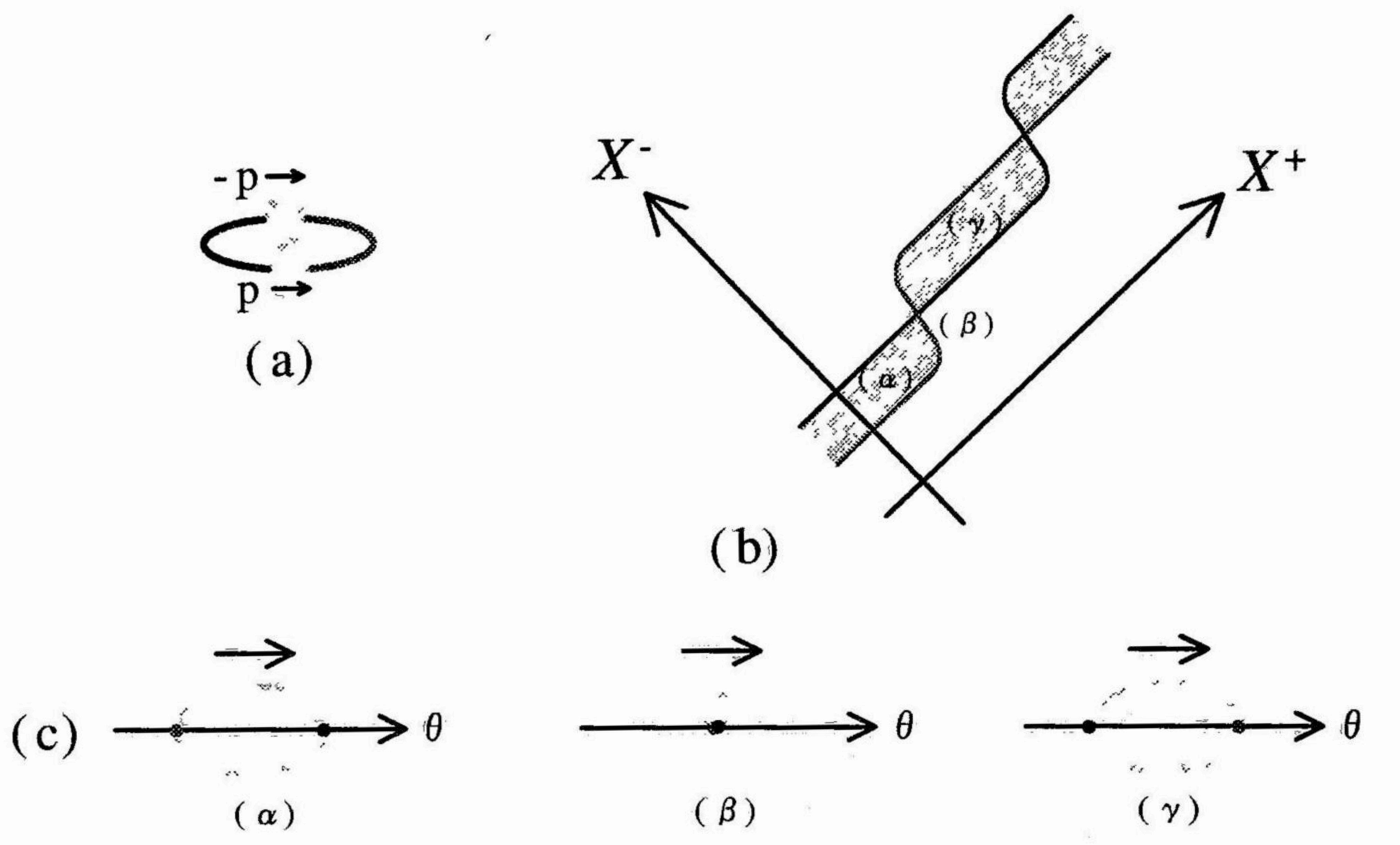


Figure 5.4: A state of the closed string, which has two localized excitations with opposite momenta propagating aling the string. (a) Worldsheet picture of the two localized excitations. (b) Spacetime description of the configurations. The symbols (α), (β) and (γ) correspond to these in (c). (c) The shape of the configuration at a given time t. the red and blue parts in (a) are locally the same as in Fig.5.3; i.e. the parts are folded and connected by two lines. Althought at (α) and (γ), the two excitations exist at a mutually different position of the string worldsheet, at (β), these exist at the same position.

Now because of

$$P_- = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_- \\ = \frac{\sqrt{\lambda}\omega}{\pi} r, \tag{5.1.8}$$

where we took the range of σ to be from $-r$ to r. we obtain

$$r = \frac{\pi}{\sqrt{\lambda}\omega} P_-. \tag{5.1.9}$$

Thus it is understood that the size of the worldsheet is related to the momentum and the energy by

$$r \sim E + J. \tag{5.1.10}$$

Hence, we can understand that the limit of an infinite string correspond to the limit taking the energy and the momentum to infinity. But the fact that J is very large means that the light-cone Hamiltonian also becomes very large by eq(5.1.2) if we do not impose any condition to the manner of the limit. We are not interested in this case physically. In order to avoid this circumstance, we take $E, J \rightarrow \infty$ while $E - J$ is fixed. In summary,

$$E, J \rightarrow \infty \quad \text{with} \quad E - J, \;\; \lambda = g_{\rm YM}^2 N, \;\; p \;\; \text{fixed}. \tag{5.1.11}$$

Now let us apply the above consideration to $AdS_5 \times S^5$. First we pick a generator J of the isometry $SO(6)$ of S^5. The angle parameter corresponding to the generator specifies the above mentioned big circle of S^2 and its eigenvalue gives a conserved quantity of the string. In the other words, J is a U(1) charge and belongs to the Cartan subalgebra of $\mathfrak{so}(6)$. Of course, the generator corresponding to the time translation, i.e. the energy E is also a conserved quantity. Then we consider the limit that E and J are very large while $E - J$ keeps finite. At the same moment, we take the standard 'tHooft limit (planar limit),

$$N \rightarrow \infty, \quad g_{\rm YM} \rightarrow 0 \quad \text{with} \;\; \lambda \;\; \text{to be fixed}, \tag{5.1.12}$$

where N and $g_{\rm YM}$ are the rank of the gauge group and the coupling constant at gauge theory side and can also be rewritten to the radius of the AdS_5 and S^5 spaces compared to string scale and string coupling constant from central relation in AdS/CFT correspondence, respectively.

Following [38], let us construct concretely the giant magnon solution in the generalized light-cone gauge. First of all, we start out from the parameterization of the metric of S^5:

$$(ds^2)_{S^5} = \cos^2\gamma d\varphi_3^2 + d\gamma^2 + \sin^2\gamma \sum_{i=1}^{4} \left[d\left(\frac{y_i}{y}\right)\right]^2, \tag{5.1.13}$$

where

$$\sum_{s=1}^{4} x_s^2 + x_5^2 + x_6^2 = 1$$

$$\eta \equiv \sqrt{\sum_{i=1}^{4} x_i^2} = \sin\gamma, \quad x_5 + i x_6 = e^{i\varphi_3}\sqrt{1-\eta^2},$$

$$x_i = \frac{y_i}{y}\sin\gamma, \quad y^2 \equiv \sum_{i=1}^{4} y_i^2. \tag{5.1.14}$$

In order to specify the S^2 part in S^5 space, we choose the following coordinates

$$(\varphi_3, y_1, y_2, y_3, y_4) = (\varphi_3, y, 0, 0, 0). \tag{5.1.15}$$

Then the S^2 part of the metric can be written as

$$(ds^2)_{S^2} = (1-\eta^2)d\varphi_3^2 + \frac{d\eta^2}{1-\eta^2}. \tag{5.1.16}$$

Choosing $\frac{y}{2} = \tan\frac{\gamma}{2}$ or

$$\cos\gamma = \frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}, \quad \sin\gamma = \frac{y}{1+\frac{y^2}{4}}. \tag{5.1.17}$$

Additionally, we also take the parameterization of the metric of AdS_5

$$(ds^2)_{AdS_5} = \cosh^2\gamma dt^2 + d\gamma^2 + \sinh^2\gamma \sum_{i=1}^{4}\left[d\left(\frac{y_i}{y}\right)\right]^2, \tag{5.1.18}$$

From AdS_5 coordinates, only time coordinate is chosen,

$$(t, z_1, z_2, z_3, z_4) = (t, 0, 0, 0, 0). \tag{5.1.19}$$

Then the $\mathbb{R}_t$ part of the metric can be written as

$$(ds^2)_{AdS_5} \to (ds^2)_{\mathbb{R}_t} = -dt^2 \tag{5.1.20}$$

We obtain the parametrization of the metric quoted in the text. The giant magnon solution is respected by $\eta = \eta_{gm}(\sigma - v\tau)$, or $y^s = \delta_{s,1}y^1$, $y^1 = y_{gm}(\sigma - v\tau)$ and its energy integal is given below in eq.(5.1.33).

In the classical analysis, it is regitimate to restrict ourselves to $y^s = \delta_{s,1}y^1 = \delta_{s,1}y$, $P_{y,s} = \delta_{s,1}P_y$, $z^a = 0$. From the relevant part of the light-cone Hamiltonian density $\mathcal{H}_{LC}$ in Σ 2.3 of [1], we obtain the classical Hamiltonian density $\mathcal{H}_{LC,c}$

$$P_{+,c} = \frac{-G_c^{+-}\omega + \epsilon_B\sqrt{(G_c^{+-}\omega)^2 - G_c^{++}C_c}}{G_c^{++}}, \qquad \mathcal{H}_{LC,c} = -\kappa P_{+,c}, \tag{5.1.21}$$

where

$$C_c = \frac{\eta^2\omega^2}{2(1-\eta)^2} + \frac{y^2}{\eta^2}P_y^2 - \frac{1}{2\omega^2}\eta^2(P_y\partial_1 y)^2 + \frac{\eta^2}{y^2}(\partial_1 y)^2 \tag{5.1.22}$$

and

$$G_c = \begin{pmatrix} G_{++,c} & G_{+-,c} \\ G_{-+,c} & G_{--,c}\end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \eta^2 & 2-\eta^2 \\ 2-\eta^2 & \eta^2\end{pmatrix}. \tag{5.1.23}$$

After some calculation, (5.1.21) reads

$$P_{+,c} = -\omega\frac{\eta^2-2}{\eta^2} + \epsilon_B\frac{\sqrt{1-\eta^2}}{\eta^2}\sqrt{\left(2-\frac{y^2}{\omega^2}P_y^2\right)\left(2\omega^2 - \frac{\eta^4}{y^2}y'^2\right)}. \tag{5.1.24}$$

In order to find the corresponding reduced action functional of y, y', $\dot{y}$ via the first-order formalism, we go back to express P_y in terms of y, y', $\dot{y}$, via

$$\begin{aligned}\dot{y} &= \frac{\partial \mathcal{H}_{LC,c}}{\partial P_y} \\ &= \kappa\epsilon_B \frac{\sqrt{1-\eta^2}\sqrt{2\omega^2 - \eta^4\left(\frac{y'}{y}\right)^2 y^2 P_y}}{\omega^2\eta^2\sqrt{2-\frac{1}{\omega^2}y^2P_y^2}}\end{aligned} \tag{5.1.25}$$

Inverting this, we obtain

$$P_y^2 = \frac{2\omega^4\dot{y}^2}{2\omega^2\kappa^2(1-\eta^2)\frac{y^4}{\eta^4} + y^2\left(\omega^2\dot{y}^2 - \kappa^2(1-\eta^2)y'^2\right)}, \tag{5.1.26}$$

and

$$\begin{aligned}S_c[y,y',\dot{y}] &= -\frac{\sqrt{\lambda}}{2\pi}\int d^2\xi(P_y\dot{y} - \mathcal{H}_{LC}) \\ &= \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau \left\{-\kappa\omega\frac{\eta^2-2}{\eta^2} \pm \frac{\sqrt{2}}{|y|\eta^2}\sqrt{2\kappa^2\omega^2(1-\eta^2)y^2 + \eta^4\left(\omega^2\dot{y}^2 - \kappa^2(1-\eta^2)y'^2\right)}\right\} \\ &= \frac{1}{2\pi}\int d\sigma d\tau L(y,y',\dot{y}).\end{aligned} \tag{5.1.27}$$

Substituting the ansatz $y = y(\sigma - v\tau)$ and hence $\dot{y} = -vy'$ into eq.(5.1.27), we obtain a mechanical Lagrangian $L^{mec}(y, y'; v)$, regarding y' as time derivative of y;

$$L^{mec}(y,y';v) = \sqrt{\lambda}\left\{-\kappa\omega\frac{\eta^2-2}{2} \pm \frac{\sqrt{2}|\kappa|}{|y|\eta^2}\sqrt{2\omega^2(1-\eta^2)y^2 - (1-\tilde{v}^2-\eta^2)\eta^4 y'^2}\right\}, \tag{5.1.28}$$

where

$$\tilde{v}^2 = \frac{\omega^2}{\kappa^2}v^2. \tag{5.1.29}$$

Because we regarded the space coordinate σ of the string worldsheet as a new "time", the Hamiltonian is conserved with respect of σ. From the mechanical Lagrangian $\mathcal{L}^{mec}$, we get

$$\begin{aligned}H^{mec} &= y'\Pi^y - L^{mec} \\ &= -\kappa\omega\frac{2-\eta^2}{\eta^2} \\ &\quad \mp \frac{2\sqrt{2}|\kappa|(1-\eta^2)|y|\omega^2}{\eta^2}\frac{1}{\sqrt{2\omega^2(1-\eta^2)y^2 - (1-\tilde{v}^2-\eta^2)y'^2\eta^4}},\end{aligned} \tag{5.1.30}$$

where Π^y is the "conjugate momentum" of y', given by

$$\Pi^y = \frac{\partial L^{mec}}{\partial y'} = \mp\frac{\sqrt{2}|\kappa|}{|y|}\frac{(1-\tilde{v}^2-\eta^2)\eta^4 y'}{2\omega^2(1-\eta^2)y^2-(1-\tilde{v}^2-\eta^2)y'\eta^4}. \tag{5.1.31}$$

Note that the mechanical Hamiltonian H^{mec} is constant along the string and the end points of the giant magnon solution lie in the equator of S^2. Thus we can conclude , by the estimation of the Hamiltonian at the end points of the solution,

$$H^{mec} \equiv 0, \tag{5.1.32}$$

for the giant magnon. From this equation we obtain

$$y'^2 = \frac{2\omega^2(1-\eta^2)}{(\eta^2-2)^2}\frac{y^2}{(1-\tilde{v}^2-\eta^2)}, \tag{5.1.33}$$

and

$$P_y = \pm\frac{\tilde{v}}{\sqrt{2}\,y\sqrt{(1-\eta^2)(1-\tilde{v}^2-\eta^2)}}\eta^2. \tag{5.1.34}$$

Then the light-cone Hamiltonian become

$$\begin{aligned}\mathcal{H}_{LC} &= P_y\dot{y} - \Pi^y y' + H^m ec \\ &= |\kappa||\omega|\frac{\eta^2(1-\eta^2)}{(2-\eta^2)(1-\tilde{v}^2-\eta^2)}\end{aligned} \tag{5.1.35}$$

Coming back to the original light-cone, it is now possible to the light-cone energy such that dispersion relation,

$$E_{LC} = \frac{\sqrt{\lambda}}{2\pi}\int_{-\infty}^{\infty} d\sigma \mathcal{H}_{LC}\Big|_{g.m.} = \frac{\sqrt{\lambda}}{\pi}\int \frac{dz}{|\partial_y z||y'}\mathcal{H}_{LC}\Big|_{g.m.} = \sqrt{2\lambda}\frac{\kappa}{2\pi}\sqrt{1-\tilde{v}^2}, \tag{5.1.36}$$

$$P_{sol} \equiv -\frac{\sqrt{2}}{|\omega|}\int d\sigma P_y y' = 2\arccos\tilde{v}. \tag{5.1.37}$$

Hence

$$E_{LC} = \sqrt{2\lambda}\frac{\kappa}{2\pi}\left|\sin\frac{P_{sol}}{2}\right|. \tag{5.1.38}$$

5.1.2 Giant magnon solution in "flat space" limit

From eq.(5.1.33), the giant magnon equation can be written to that with only η as

$$(\partial_1\eta)^2 = \frac{\omega^2\eta^2(1-\eta^2)^2}{2(1-\frac{\eta^2}{2})}\frac{1}{1-\tilde{v}^2-\eta^2} \tag{5.1.39}$$

In order to consider the near flat space limit, we introduce the scaled variables η_{sc}. Then, the equation (5.1.39) becomes

$$(\partial_1 \eta_{sc})^2 = \frac{\tilde{\kappa}^2 \eta_{sc}^2}{2} \frac{1}{\lambda^2(1-v^2) - \eta_{sc}^2}. \tag{5.1.40}$$

Taking simply the limit, $\lambda \to \infty$, we have

$$\partial_1 \eta_{sc} = 0. \tag{5.1.41}$$

Then, we obtain, by using $\dot{y} = -vy'$

$$\eta_{sc} = \eta_0 = \text{const.} \tag{5.1.42}$$

Because the strings which describe giant magnon solution must satisfy $\eta = 0$ at the edges, it is found that the solution is

$$\eta_{sc} \equiv 0. \tag{5.1.43}$$

Hence, the giant magnon solution has the whole on the line corresponding to the equator of S^2 in the flat-space limit.

On the other hand, notice that there is another non-trivial limit; i.e. we are interested in the limit,

$$\lambda \to \infty \qquad \text{with } \lambda^2(1-v^2) \text{ fixed}, \tag{5.1.44}$$

in which the right hand side of (5.1.40) does not vanish. Now, we define $\tilde{k}$ as

$$\lambda^{\frac{1}{2}}(1-v^2) = \tilde{k}^2. \tag{5.1.45}$$

Then, the eq.(5.1.40) with $\lambda \to \infty$ becomes

$$\partial \eta_{sc} = \epsilon \frac{\tilde{\kappa}\eta_{sc}}{\sqrt{2}} \sqrt{\frac{1}{\tilde{k}^2 - \eta_{sc}^2}}. \tag{5.1.46}$$

Therfore, we obtain

$$\int \sqrt{\tilde{k}^2 - \eta_{sc}^2} \frac{d\eta_{sc}}{\eta_{sc}} = \int \epsilon \frac{\tilde{\kappa}}{\sqrt{2}} d\sigma. \tag{5.1.47}$$

After the replacements,

$$\eta_{sc} = \tilde{k} \sin\theta \qquad \text{and} \qquad t = \tan\frac{\theta}{2}, \tag{5.1.48}$$

the left hand side of (5.1.47) becomes

$$\tilde{k}\int dt\frac{(1-t^2)^2}{t(1+t^2)^2}=\tilde{k}\left(\log t-\frac{2}{1+t^2}\right). \tag{5.1.49}$$

Hence, we get

$$\tilde{k}\left(\log t-\frac{2}{1+t^2}\right)=\epsilon\frac{\tilde{\kappa}}{\sqrt{2}}\sigma. \tag{5.1.50}$$

By using the solution $t=t(\sigma)$ of this equation, the gaint magnon solution in the limit is written by

$$\eta_{sc}=\tilde{k}\sin\theta=2\tilde{k}\frac{t}{1+t^2}. \tag{5.1.51}$$

5.2 The Scaling Limit to the Giant Magnon Solution

There are several approaches to simplify the theory. One of the most famous method is so called BMN (Berenstein Maldacena Nastase) limit.[68] They showed how certain operators in SYM theory can be related to string theory in the pp-wave limit in a certain way. Namely, it was suggested that string theory in pp-wave background gives good approximation of a certain class of " nearly " chiral operators and vice versa. In this section, we investigate the limit found out in the last section. A main goal of the present section is to deduce an appropriate (double) scaling limit of the $AdS_5\times S^5$ superstring Hamiltonian in the light-cone gauge which contains the scaled giant magnon as a classical solution. Now we restrict ourselves to the bosonic part of the $AdS_5\times S^5$ light-cone Hamiltonian.

Let us recall the dispersion relation of the giant magnon, which are (5.1.36), (5.1.37), (5.1.38). The light-cone energy and the worldsheet momentum are parameterized by the phase velocity $\tilde{v}$ (5.1.29) of the giant magnon. The scaling limit (5.1.44) implies an approach to the singularity, which, in the present case, is $\tilde{v}\longrightarrow 1$. From the worldsheet momentum of the giant magnon solution,

$$P_{sol}=2\arccos\tilde{v}, \tag{5.2.1}$$

it is understood that the limit $\tilde{v}\longrightarrow 1$ corresponds to $P_{sol}\longrightarrow 0$. When $P_{sol}\ll 1$, then $\tilde{v}$ can be expanded as

$$\tilde{v}=1-\frac{1}{2}\left(\frac{P_{sol}}{2}\right)^2+\frac{1}{4!}\left(\frac{P_{sol}}{2}\right)^4+\mathcal{O}(\lambda^{-\frac{3}{2}}), \tag{5.2.2}$$

and therefore

$$\lambda^{\frac{1}{2}}(1-\tilde{v}^2) = \left(\frac{\lambda^{\frac{1}{4}}P_{sol}}{2}\right)^2 - \frac{\lambda^{\frac{1}{2}}}{3}\left(\frac{P_{sol}}{2}\right)^4 + \cdots . \tag{5.2.3}$$

Hence noticing the leading term of (5.2.3), in the word of the worldsheet momentum, the limit (5.1.44) means that

$$\lambda \to \infty, \quad P_{sol} \to 0, \quad \lambda^{\frac{1}{4}}P_{sol} \equiv k = \text{finite}. \tag{5.2.4}$$

Now k and $\tilde{k}$ which was defined in eq.(5.1.45) are related by

$$k = \frac{\tilde{k}}{2}. \tag{5.2.5}$$

On the other hand, this is followed by the subtraction of the "non universal part" of the energy from the energy formula of the giant magnon solution (5.1.36), i.e.

$$\lim_{\lambda\to\infty} \lambda^{\frac{1}{4}}\left(E_{LC} - \frac{\sqrt{2}}{2}\lambda^{\frac{1}{4}}\frac{\kappa}{2\pi}k\right) = -\frac{\sqrt{2}}{48}\frac{\kappa}{2\pi}k^3, \tag{5.2.6}$$

This limit similar to near flat space limit of type IIB string theory on $AdS_5 \times S^5$ space suggested in [70]. Here, let us review the procedure of taking the near flat space limit according to the authors. The authors discussed the near flat space limit via Penrose limit. The idea is to weaken the Penrose limit and then it is expected that more structures of the original theory get preserved. The first step is to find the null geodesics with respect to which the pp-wave limit can be taken. We start with the standard metric of $AdS_5 \times S^5$ in global coordinates

$$ds^2 = R^2\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\tilde{\Omega}_3^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\Omega_3^2\right] \tag{5.2.7}$$

where each of the unit 3-spheres, we take below S^3 from the spherical part, can be parameterized by

$$d\Omega_3^2 = d\phi_1^2 + \cos^2\phi_1\, d\phi_2 + \sin^2\phi_2\, d\phi_3. \tag{5.2.8}$$

The light-like directions are characterized by covariantly constant null vectors

$$\nabla_\mu v_\nu = 0, \quad v_\mu v^\mu = 0. \tag{5.2.9}$$

The S^5 part of the null geodesics is parameterized by $\theta = 0$ and $\rho = 0$, i.e. in our case this is the equator of the five-sphere. Here we choose this particle-like solution as

$$t = \psi = \sigma^+, \tag{5.2.10}$$

where σ^+ is light-cone direction of the worldsheet coordinates. The plane wave limit corresponds to $\lambda \to \infty$ with the product $P\sqrt{\lambda}$ fixed and the giant magnon sector (Hofman-Maldacena limit) corresponds to $\lambda \to \infty$ with P fixed. Compared these limits, Maldacena and Swanson proposed the interpolate region, that is the limit $\lambda \to \infty$ with the product $P\lambda^{\frac{1}{4}}$ fixed. To treat this, consider to promote the order of the worldsheet momentum P to the approapriate order by using the boost transformation. Now we boost the worldsheet coordinates. Because the boost transformation of the worldsheet is defined by

$$\begin{aligned} \sigma' &= \gamma(\sigma + v\tau), \\ \tau' &= \gamma(\tau + v\sigma), \end{aligned} \tag{5.2.11}$$

where $\gamma = 1/\sqrt{1-v^2}$ and an arbitrary constant $v < 1$, the light-cone directions σ^+ and σ^- are transformed as

$$\begin{aligned} \sigma'^+ &= 2\gamma\sigma^+, \\ \sigma'^- &= 0. \end{aligned} \tag{5.2.12}$$

Additionally, since the 4-d momentum is also the same property under the boost transformation, of course,

$$\begin{aligned} E'_{LC} &= \gamma(E_{LC} + vP), \\ P' &= \gamma(P + vE_{LC}), \end{aligned} \tag{5.2.13}$$

where E_{LC} and P are the worldsheet energy and momentum, respectively. But because the light-cone energy E_{LC} equals to 0 as seen in the section 3.3.1, the transformation (5.2.13) becomes

$$\begin{aligned} E'_{LC} &= \gamma v P, \\ P' &= \gamma P. \end{aligned} \tag{5.2.14}$$

Hence, in order to lower the order of P by the order $\lambda^{\frac{1}{4}}$, we understand

$$\gamma \sim \lambda^{\frac{1}{4}}. \tag{5.2.15}$$

Now we expand about the solution (5.2.10) boosted. This amounts to a redefinition of the fields as follows

$$\begin{aligned} t &= \sqrt{g}\sigma^+ + \frac{\tau}{\sqrt{g}}, \quad \psi = \sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}} \\ \rho &= \frac{z}{\sqrt{g}}, \quad \theta = \frac{y}{\sqrt{g}}. \end{aligned} \tag{5.2.16}$$

where the parameter g is related to λ via $g = \lambda^{\frac{1}{4}}/4\pi$.

The near flat space limit is, as in the Penrose limit, taking $g \to \infty$. In the Lagrangian the leading terms are divergent, $g \cdot \partial_-(\tau - \chi)$, but since they are total derivatives they can be dropped. The relevant (bosonic) part of the Lagrangian thus becomes

$$S = 4\left[-\partial_+\tau\,\partial_-\tau + \partial_+\chi\,\partial_-\chi + \partial_+\vec{z}\,\partial_-\vec{z} + \partial_+\vec{y}\,\partial_-\vec{y} - \vec{y}^2\,\partial_-\chi - \vec{z}^2\,\partial_-\tau\right] \tag{5.2.17}$$

The reduced model has two conserved chiral currents

$$\begin{aligned} j_+^\chi &= \partial_+\chi - \frac{\vec{y}^2}{2} + \text{fermions}, \quad \partial_-\chi_+^\chi = 0 \\ j_+^\tau &= \partial_+\tau - \frac{\vec{z}^2}{2} + \text{fermions}, \quad \partial_-\chi_+^\tau = 0. \end{aligned} \tag{5.2.18}$$

The right-moving conformal invariance in preserved since its generator

$$T_{--} = -(\partial_\tau)^2 + (\partial_\chi)^2 + (\partial_-\vec{z})^2 + (\partial_-\vec{y})^2 + \text{fermions} \tag{5.2.19}$$

is conserved, i.e. $\partial_+ T_{--} = 0$. The left-moving conformal invariance however is broken, since $T_{++} \propto (j_+^\chi - j_+^\tau)$. One can still impose the Virasoro constraints requiring

$$\begin{aligned} & j_+^\chi + j_+^\tau = \partial_+(\tau + \chi) + \frac{z^2 - y^2}{2} = 0, \\ & T_{--} = 0. \end{aligned} \tag{5.2.20}$$

This can also be considered as a gauge fixing condition.

It is useful now to make a change of variables

$$x^+ = \sigma^+, \quad x^- = 2(\tau + \chi). \tag{5.2.21}$$

For completeness we write down the complete gauge fixed Lagrangian

$$\begin{aligned} \mathcal{L} =& 4\Big\{\partial_+\vec{z}\,\partial_-\vec{z} + \partial_+\vec{y}\,\partial_-\vec{y} - \frac{1}{4}(\vec{z}^2 + \vec{y}^2) + (\vec{y}^2 - \vec{x}^2)[(\partial_-\vec{z})^2 + (\partial_-\vec{y})^2] \\ & + i\psi_+\partial_-\psi_+ + i\psi_-\partial_-\psi_- + i\psi_-\Pi\psi_+ + i(\vec{y}^2 - \vec{x}^2)\psi_-\partial_-\psi_- \\ & - \psi_-(\partial_- z^j\Gamma^j + \partial_- y^{j'}\Gamma^{j'})(z^i\Gamma^i - y^{i'}\Gamma^{i'})\psi_- \\ & + \frac{1}{24}\left[\psi_-\Gamma^{ij}\psi_-\psi_-\Gamma^{ij}\psi_-\psi_-\Gamma^{i'j'}\psi_-\psi_-\Gamma^{i'j'}\psi_-\right]\Big\}. \end{aligned} \tag{5.2.22}$$

In the last expression a simple rescaling of $\psi_\pm$ was used. The indices i and i' correspond to the transverse directions in the anti-de Sitter and the spherical parts, respectively.

Let us turn to the rescaling of the bosonic field variables. Noticing the $1-\tilde{v}^2-z$ factor which is present in , we rescale

$$\eta = \eta_{sc}\lambda^{-\frac{1}{4}}, \tag{5.2.23}$$

$$y = y_{sc}\lambda^{-\frac{1}{4}}, \tag{5.2.24}$$

where y_{sc} and η_{sc} are related by

$$\eta_{sc} \sim y_{sc} - \frac{1}{4}\lambda^{-\frac{1}{2}}y_{sc}^3. \tag{5.2.25}$$

Indeed,

$$\begin{aligned}
\lambda^{\frac{1}{4}}P_{sol} &= -\sqrt{2}\int_{-\infty}^{\infty} d\sigma P_y y' \\
&= 2\sqrt{2}\int_0^{\sqrt{1-\tilde{v}^2}} \frac{dz}{|\partial_y z|}|P_y| \\
&\sim 2\int_0^{\sqrt{1-\tilde{v}^2}\lambda^{\frac{1}{4}}} \frac{\eta_{sc}dz_{sc}}{\sqrt{\lambda^{\frac{1}{2}}(1-\tilde{v}^2)-\eta_{sc}^2}} \\
&= k,
\end{aligned} \tag{5.2.26}$$

reproducing the scaling limit proposed above. Let us now obtain the light-cone Hamiltonian density in the scaling variables, we obtain

$$\begin{aligned}
\mathcal{H}_{LC} &= \frac{1}{2}\left[\frac{1}{G_y^{sc}}(P_y^{sc})^2 + G_y^{sc}(y_{sc}')^2\right] + \frac{1}{4\lambda^{\frac{1}{2}}}\eta_{sc}^2\left[\tilde{\kappa}^2 - \frac{1}{G_y^{sc}}(P_y^{sc})^2 - G_y^{sc}(y_{sc}')^2\right] \\
&\quad + \frac{1}{16\lambda^{\frac{1}{2}}}\eta_{sc}^2\left[-\frac{1}{G_y^{sc}}(P_y^{sc})^2 + G_y^{sc}(y_{sc}')^2\right]^2 + \mathcal{O}\left(\frac{1}{\lambda}\right) \\
&= \mathcal{H}_{LC}^{(-\frac{1}{2})} + \mathcal{O}\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{5.2.27}$$

Hence, we find

$$\dot{y}_{sc} = \frac{\partial\mathcal{H}_{LC}^{(-\frac{1}{2})}}{\partial P_y^{sc}} \tag{5.2.28}$$

$$= \frac{1}{G_y^{sc}}P_y^{sc} - \frac{1}{2\lambda^{\frac{1}{2}}}\eta_{sc}^2\frac{1}{G_y^{sc}}P_y^{sc} - \frac{1}{4\lambda^{\frac{1}{2}}}\eta_{sc}^2\frac{1}{G_y^{sc}}\left[-\frac{1}{G_y^{sc}}(P_y^{sc})^2 + G_y^{sc}(y_{sc}')^2\right]P_y^{sc}. \tag{5.2.29}$$

Inverting this expression iteratively, we obtain

$$P_y^{sc} = G_y^{sc}\dot{y}_{sc} + \frac{1}{2\lambda^{\frac{1}{2}}}\eta_{sc}^2 G_y^{sc}\dot{y}_{sc}\left[1 - \frac{1}{2}G_y^{sc}(\dot{y}_{sc})^2 + \frac{1}{2}G_y^{sc}(y_{sc}')^2\right] + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{5.2.30}$$

We now show that the above rescaled Hamiltonian density leads to a classical solution where dispersion relation takes the form eq.(5.2.6). This naturally leads to a relation in the derivatives of the scaling variables Substituting $\dot{y} = -\tilde{v}y'$ into (5.2.27), we obtain

$$\mathcal{H}_{LC}^{(-\frac{1}{2})}\Big| = \frac{\kappa}{2} y_{sc}'^2 (1+\tilde{v}^2) + \frac{\kappa y_{sc}^2}{4\lambda^{\frac{1}{2}}}(1-2y_{sc}'^2). \tag{5.2.31}$$

Substituting $\dot{y} = -\tilde{v}y'$ into (5.2.30), we obtain , up to $\mathcal{O}(1/\lambda^{\frac{1}{2}})$,

$$P_y| = -\tilde{v}y_{sc}' \tag{5.2.32}$$

The Hamiltonian dynamics of the scaling theory is summarized by the first-order action, given by

$$S_{sc} \equiv \frac{1}{2\pi}\int d\xi^2 \left(P_y \dot{y}_{sc} - \mathcal{H}_{LC}^{(-\frac{1}{2})}\right), \tag{5.2.33}$$

which is written to the following standard form with respect to y and its derivatives by using (5.2.30) and (5.2.27):

$$S = \frac{1}{2\pi}\int d^2\xi \left[\frac{1}{2}G_y^{sc}\{y_{sc}'^2 - \dot{y}_{sc}^2\} - \frac{\tilde{\kappa}^2}{4\lambda^{\frac{1}{2}}}\eta_{sc}^2 + \frac{1}{4\lambda^{\frac{1}{2}}}\eta_{sc}^2 G_y^{sc}\{y_{sc}'^2 + \dot{y}_{sc}^2\}\right. \tag{5.2.34}$$

$$\left. - \frac{\eta_{sc}^2}{16\lambda^{\frac{1}{2}}\tilde{\kappa}^2}(G_y^{sc})^2\{y_{sc}'^2 - \dot{y}_{sc}^2\}^2\right]. \tag{5.2.35}$$

Indeed, substituting $\dot{y} = -\tilde{v}y'$ into (5.2.35), we obtain

$$\begin{aligned} S_{sc}| &= \frac{1}{2\pi}\int d\sigma d\tau \frac{1}{\sqrt{\lambda}}\left(-\frac{1}{2}\frac{y_{sc}'^2}{y_{sc}^2}(\sqrt{\lambda}(1-\tilde{v}^2) - \eta_{sc}^2) - \frac{\tilde{\kappa}^2}{4}\right)\eta_{sc}^2 \\ &\equiv \frac{1}{2\pi}\int d\sigma d\tau \frac{1}{\sqrt{\lambda}} L_{sc}^{(sol)} \end{aligned} \tag{5.2.36}$$

Regarding the σ_r derivative as time, we can treat $L_{sc}^{(sol)}$ as a mechanical Lagrangian. From its energy integral and the condition of the giant magnon ($\dot{y}_{sc} = 0$ at the turning point $y_{sc}' = 0$), we obtain

$$y_{sc}'^2 = \frac{y_{sc}^2}{2(\sqrt{\lambda}(1-\tilde{v}^2) - \eta_{sc}^2)}. \tag{5.2.37}$$

Using this equation, we can compute the light-cone energy,

$$\begin{aligned} E_{LC} &= \frac{1}{2\pi}\int d\sigma \mathcal{H}_{LC} \\ &\sim \frac{1}{2\pi}\int d\sigma_r \lambda^{\frac{1}{4}}\, \mathcal{H}_{LC}^{(-\frac{1}{2})}\Big| \\ &= \lambda^{\frac{1}{4}}\frac{\sqrt{2}}{2}\frac{\kappa}{2\pi}k - \lambda^{-\frac{1}{4}}\frac{\sqrt{2}}{48}\frac{\kappa}{2\pi}k^3, \end{aligned} \tag{5.2.38}$$

which satisfies the limit (5.2.6). Similarly, we obtain

$$
\begin{aligned}
P_{sol} &= -\sqrt{2}\int d\sigma P_y^{sc} y'_{sc} \\
&\sim -2\sqrt{2}\lambda^{-\frac{1}{4}}\int \frac{d\eta_{sc}}{|\partial_y \eta_{sc}|} P_y^{sc}| \\
&= 2\sqrt{2}\lambda^{-\frac{1}{4}}\int \frac{d\eta_{sc}}{|\partial_y \eta_{sc}|} \tilde{v} y'_{sc} \\
&= \lambda^{-\frac{1}{4}} k.
\end{aligned} \tag{5.2.39}
$$

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Appendix A

Metrics of Various Spaces

A.1 S^2 Metric

In this section, let us reproduce the S^2 metric by considering S^2 space as that embedded into $\mathbb{R}^3$. The $\mathbb{R}^3$ metric is

$$(ds^2)_{\mathbb{R}^3} = dx_1^2 + dx_2^2 + dx_3^2. \tag{A.1.1}$$

On the other hand, the S^2 is defined by

$$x_1^2 + x_2^2 + x_3^2 = 1. \tag{A.1.2}$$

Now we introduce η and φ_3 as follows:

$$\begin{aligned} \sqrt{1-\eta^2}\epsilon^{i\varphi_3} &= x_1 + ix_2, \\ \eta &= x_3. \end{aligned} \tag{A.1.3}$$

Then eq.(A.1.2) is automatically satisfied. Noting that

$$\left(-\frac{\eta}{\sqrt{1-\eta^2}}d\eta + i\sqrt{1-\eta^2}d\varphi_3\right)\epsilon^{i\varphi_3} = dx_1 + idx_2, \tag{A.1.4}$$

$$\left(-\frac{\eta}{\sqrt{1-\eta^2}}d\eta - i\sqrt{1-\eta^2}d\varphi_3\right)\epsilon^{-i\varphi_3} = dx_1 - idx_2, \tag{A.1.5}$$

we get

$$\begin{aligned} dx_1^2 + dx_2^2 &= (dx_1 + idx_2)(dx_1 - idx_2) \\ &= \left(-\frac{\eta}{\sqrt{1-\eta^2}}d\eta + i\sqrt{1-\eta^2}d\varphi_3\right)\left(-\frac{\eta}{\sqrt{1-\eta^2}}d\eta - i\sqrt{1-\eta^2}d\varphi_3\right) \\ &= \frac{\eta^2}{1-\eta^2}d\eta^2 + (1-\eta^2)d\varphi_3^2 \end{aligned} \tag{A.1.6}$$

Hence we obtain

$$ds^2 = \frac{1}{1-\eta^2}d\eta^2 + (1-\eta^2)d\varphi_3^2, \tag{A.1.7}$$

which is the same as eq.(5.1.16).

A.2 AdS space

Anti de Sitter space is a space of Lorentzian signature $(-++...+)$, but of constant *negative* curvature.

The anti in Anti de Sitter is because de Sitter space is defined as the space of Lorentzian signature and of constant *positive* curvature, thus a Lorentzian signature analog of the sphere (the sphere is the space of Euclidean signature and constant positive curvature).

In d dimensions, de Sitter space is defined by a sphere-like embedding in $d+1$ dimensions

$$\begin{aligned} ds^2 &= -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 + dx_{d+1}^2 \\ -x_0^2 + \sum_{i=1}^{d-1} x_i^2 + x_{d+1}^2 &= R^2 \end{aligned} \tag{A.2.1}$$

thus as mentioned, this is the Lorentzian version of the sphere, and it is clearly invariant under the group SO(1,d) (the d dimensional sphere would be invariant under SO(d+1) rotating the $d+1$ embedding coordinates).

Similarly, in d dimensions, Anti de Sitter space is defined by a Lobachevski-like embedding in d+1 dimensions

$$\begin{aligned} ds^2 &= -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_{d+1}^2 \\ -x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_{d+1}^2 &= -R^2 \end{aligned} \tag{A.2.2}$$

and is therefore the Lorentzian version of Lobachevski space. It is invariant under the group SO(2,d-1) that rotates the coordinates $x_\mu = (x_0, x_{d+1}, x_1, ..., x_{d-1})$ by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

The metric of this space can be written in different forms, corresponding to different coordinate systems. In Poincaré coordinates, it is

$$ds^2 = \frac{R^2}{x_0^2}(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dx_0^2) \tag{A.2.3}$$

where $-\infty < t, x_i < +\infty$, but $0 < x_0 < +\infty$. Up to a conformal factor therefore, this is just like (flat) 3d Minkowski space. However, one now discovers that one does not cover all of the space! In the finite coordinates τ, θ, one finds that one can now analytically continue past the diagonal boundaries (there is no obstruction to doing so).

In these Poincaré coordinates, we can understand Anti de Sitter space as a $d-1$ dimensional Minkowski space in $(t, x_1, ...x_{d-2})$ coordinates, with a "warp factor" (gravitational potential) that depends only on the additional coordinate x_0.

A coordinate system that does cover the whole space is called the global coordinates, and it gives the metric

$$ds_d^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\vec{\Omega}_{d-2}^2) \tag{A.2.4}$$

where $d\vec{\Omega}_{d-2}^2$ is the metric on the unit $d-2$ dimensional sphere. This metric is written in a suggestive form, since the metric on the d-dimensional sphere can be written in a similar way,

$$ds_d^2 = R^2(\cos^2\rho\, dw^2 + d\rho^2 + \sin^2\rho\, d\vec{\Omega}_{d-2}^2) \tag{A.2.5}$$

The change of coordinates $\tan\theta = \sinh\rho$ gives the metric

$$ds_d^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\vec{\Omega}_{d-2}^2) \tag{A.2.6}$$

where $0 \leq \theta \leq \pi/2$ in all dimensions except 2, (where $-\pi/2 \leq \theta \leq \pi/2$), and τ is arbitrary, and from it we infer the Penrose diagram of global AdS_2 space (Anti de Sitter space in 2 dimensions) which is an infinite strip between $\theta = -\pi/2$ and $\theta = +\pi/2$. The "Poincaré patch" covered by the Poincaré coordinates, is a triangle region of it.

Finally, let me mention that Anti de Sitter space is a solution of the Einstein equation with a constant energy-momentum tensor, known as a *cosmological constant*, thus $T_{\mu\nu} = \Lambda g_{\mu\nu}$, coming from a constant term in the action, $-\int d^4x\sqrt{-g}\Lambda$, so the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\Lambda g_{\mu\nu} \tag{A.2.7}$$

Appendix B

notation

B.1 Gamma matrices

Our choice of the 32×32 Gamma matrices is given by

$$\Gamma^{\underline{a}} = \begin{pmatrix} 0 & (\gamma^{\underline{a}})^{\underline{\alpha\beta}} \\ (\gamma^{\underline{a}})_{\underline{\alpha\beta}} & 0 \end{pmatrix}, \qquad \{\Gamma^{\underline{a}}, \Gamma^{\underline{b}}\} = 2\eta^{\underline{a}\underline{b}} 1_{32}, \tag{B.1.1}$$

with

$$\left((\gamma^{\underline{a}})^{\underline{\alpha\beta}}\right) = (1_{16}, \sigma^i), \qquad \left((\gamma^{\underline{a}})_{\underline{\alpha\beta}}\right) = (-1_{16}, \sigma^i), \tag{B.1.2}$$

where σ^i are real, symmetric 16×16 matrices which form the $SO(9)$ Clifford algebra:

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} 1_{16}, \qquad i, j = 1, 2, \ldots, 9. \tag{B.1.3}$$

The Gamma matrices in this "Majorana-Weyl" representation are real:

$$(\Gamma^{\underline{a}})^* = \Gamma^{\underline{a}}, \qquad \underline{a} = 0, 1, \ldots, 9. \tag{B.1.4}$$

One choice of the matrices σ^i is the following [72]:

$$\begin{aligned}
\sigma^1 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2, \\
\sigma^2 &= \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes 1_2, \\
\sigma^3 &= \tau_2 \otimes \tau_2 \otimes \tau_3 \otimes 1_2, \\
\sigma^4 &= \tau_2 \otimes 1_2 \otimes \tau_2 \otimes \tau_1, \\
\sigma^5 &= \tau_2 \otimes 1_2 \otimes \tau_2 \otimes \tau_3, \\
\sigma^6 &= \tau_2 \otimes \tau_1 \otimes 1_2 \otimes \tau_2, \\
\sigma^7 &= \tau_2 \otimes \tau_3 \otimes 1_2 \otimes \tau_2, \\
\sigma^8 &= \tau_1 \otimes 1_2 \otimes 1_2 \otimes 1_2, \\
\sigma^9 &= \tau_3 \otimes 1_2 \otimes 1_2 \otimes 1_2.
\end{aligned} \tag{B.1.5}$$

Here τ_k $(k = 1, 2, 3)$ are the Pauli matrices.

The charge conjugation matrix C is chosen as

$$C = \begin{pmatrix} 0 & 1_{16} \\ -1_{16} & 0 \end{pmatrix}. \tag{B.1.6}$$

The chirality matrix in $D = 9 + 1$ is defined by

$$\overline{\Gamma} := \Gamma^0 \Gamma^1 \cdots \Gamma^9 = \begin{pmatrix} 1_{16} & 0 \\ 0 & -1_{16} \end{pmatrix}. \tag{B.1.7}$$

Any 32-component spinor can be written as

$$\Theta = \begin{pmatrix} \theta^{\underline{\alpha}} \\ \chi_{\underline{\alpha}} \end{pmatrix}. \tag{B.1.8}$$

A Weyl spinor with positive chirality, $\overline{\Gamma}\Theta = +\Theta$, is given by

$$\Theta = \begin{pmatrix} \theta^{\underline{\alpha}} \\ 0 \end{pmatrix}, \tag{B.1.9}$$

and a Weyl spinor with negative chirality, $\overline{\Gamma}\Theta = -\Theta$, is given by

$$\Theta = \begin{pmatrix} 0 \\ \chi_{\underline{\alpha}} \end{pmatrix}. \tag{B.1.10}$$

In the 16-component notation, a spinor $\theta^{\underline{\alpha}}$ $(\chi_{\underline{\alpha}})$ with upper (lower) index $\underline{\alpha}$ represents a Weyl spinor with positive (negative) chirality.

In this Majorana-Weyl representation, the matrix ϱ is given by

$$C\Gamma^{012345} = \begin{pmatrix} \varrho_{\underline{\alpha}\underline{\beta}} & 0 \\ 0 & \varrho^{\underline{\alpha}\underline{\beta}} \end{pmatrix}, \tag{B.1.11}$$

$$(\varrho_{\underline{\alpha}\underline{\beta}}) = 1_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_3, \qquad (\varrho^{\underline{\alpha}\underline{\beta}}) = 1_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_3. \tag{B.1.12}$$

An antisymmetric product of Γ's is denoted by

$$\Gamma^{\underline{a}\underline{b}} = \Gamma^{[\underline{a}}\Gamma^{\underline{b}]} = \begin{pmatrix} (\gamma^{\underline{a}\underline{b}})^{\underline{\alpha}}{}_{\underline{\beta}} & 0 \\ 0 & (\gamma^{\underline{a}\underline{b}})_{\underline{\alpha}}{}^{\underline{\beta}} \end{pmatrix}. \tag{B.1.13}$$

B.2 Indices

We use the following conventions for the various indices in $AdS_5 \times S^5$.

(i) The world-sheet indices, $i, j = 0, 1$:

$$\xi^0 = \tau, \qquad \xi^1 = \sigma, \qquad \epsilon^{01} = 1.$$

(ii) The curved bosonic indices, $\underline{m}, \underline{n} = 0, 1, \ldots, 9$:
$\underline{m} = (\mathfrak{a}, m) = (\mathfrak{a}, a, s)$: $\mathfrak{a} = \pm$, $m = 1, 2, \ldots, 8$, $a = 1, 2, 3, 4$, $s = 1, 2, 3, 4$.

$$X^{\underline{m}}: \quad X^0 = t, \qquad X^9 = \varphi, \qquad X^{\pm} = \frac{1}{\sqrt{2}}(t \pm \varphi), \qquad X^a = z^a, \qquad X^{4+s} = y^s.$$

Also, $a', b' = 5, 6, 7, 8$: $X^{a'} = y^{a'-4}$.

(iii) The local Lorentz indices, $\underline{a}, \underline{b} = 0, 1, \ldots, 9$:

$$\eta_{\underline{a}\underline{b}} = \mathrm{diag}(-1, +1, \ldots, +1).$$

(iv) The indices for Weyl spinors, $\underline{\alpha}, \underline{\beta}, \bar{\underline{\alpha}}, \bar{\underline{\beta}} = 1, 2, \ldots, 16$.

The bar over $\underline{\alpha}$ and $\underline{\beta}$ has no physical meaning. It is occasionally used to distinguish a complex Weyl spinor from its conjugate:

$$\theta^{\underline{\alpha}} = \frac{1}{\sqrt{2}}(\theta^{1\underline{\alpha}} + \mathrm{i}\theta^{2\underline{\alpha}}), \qquad \bar{\theta}^{\bar{\underline{\alpha}}} = \frac{1}{\sqrt{2}}(\theta^{1\underline{\alpha}} - \mathrm{i}\theta^{2\underline{\alpha}}).$$

Here $\theta^{I\underline{\alpha}}$ $(I = 1, 2)$ are Majorana-Weyl spinors: $(\theta^{I\underline{\alpha}})^* = \theta^{I\underline{\alpha}}$.

The decomposition of Weyl spinors is given by

$$\theta^{\underline{\alpha}} = \begin{pmatrix} \theta^{+\alpha} \\ \theta^{-\dot{\alpha}} \end{pmatrix}, \qquad \bar{\theta}^{\underline{\bar{\alpha}}} = \begin{pmatrix} \bar{\theta}^{+\bar{\alpha}} \\ \bar{\theta}^{-\dot{\bar{\alpha}}} \end{pmatrix},$$

$\underline{\alpha} = (\alpha, \dot{\alpha})$, $\alpha = 1, 2, \ldots, 8$, $\dot{\alpha} = \dot{1}, \dot{2}, \ldots, \dot{8}$.

(v) The indices for combined spinors, $\hat{\alpha}, \hat{\beta} = 1, 2, \ldots, 16$:

$$(\Psi^{\hat{\alpha}}) = \begin{pmatrix} \theta^{+\alpha} \\ \bar{\theta}^{+\bar{\alpha}} \end{pmatrix}. \tag{B.2.1}$$

Appendix C

Details on the induced vielbein

The $psu(2,2|4)$ generators are given by

$$\widehat{P}_{\underline{a}} = (\widehat{P}_{\hat{a}}, \widehat{P}_{\hat{a}'}), \qquad \widehat{J}_{\hat{a}\hat{b}} = -\widehat{J}_{\hat{b}\hat{a}}, \qquad \widehat{J}_{\hat{a}'\hat{b}'} = -\widehat{J}_{\hat{b}'\hat{a}'}, \qquad \widehat{Q}_{\underline{\alpha}}, \qquad \widehat{\overline{Q}}_{\underline{\bar{\alpha}}}, \tag{C.0.1}$$

where $\underline{a} = 0,1,2,\ldots,9$, $\hat{a},\hat{b} = 0,1,2,3,4$, $\hat{a}',\hat{b}' = 5,6,7,8,9$, $\underline{\alpha} = 1,2,\ldots,16$, $\underline{\bar{\alpha}} = \bar{1},\bar{2},\ldots,\bar{16}$. The bosonic generators are chosen to be anti-Hermitian and $(\widehat{Q}_{\underline{\alpha}})^\dagger = \widehat{\overline{Q}}_{\underline{\bar{\alpha}}}$. The non-zero commutation relations are given by

$$[\widehat{P}_{\hat{a}}, \widehat{P}_{\hat{b}}] = \widehat{J}_{\hat{a}\hat{b}}, \qquad [\widehat{P}_{\hat{a}'}, \widehat{P}_{\hat{b}'}] = -\widehat{J}_{\hat{a}'\hat{b}'}, \tag{C.0.2}$$

$$[\widehat{P}_{\hat{a}}, \widehat{J}_{\hat{b}\hat{c}}] = \eta_{\hat{a}\hat{b}}\widehat{P}_{\hat{c}} - \eta_{\hat{a}\hat{c}}\widehat{P}_{\hat{b}}, \qquad [\widehat{P}_{\hat{a}'}, \widehat{J}_{\hat{b}'\hat{c}'}] = \delta_{\hat{a}'\hat{b}'}\widehat{P}_{\hat{c}'} - \delta_{\hat{a}'\hat{c}'}\widehat{P}_{\hat{b}'}, \tag{C.0.3}$$

$$[\widehat{J}_{\hat{a}\hat{b}}, \widehat{J}_{\hat{c}\hat{d}}] = \eta_{\hat{b}\hat{c}}\widehat{J}_{\hat{a}\hat{d}} + 3 \text{ terms}, \qquad [\widehat{J}_{\hat{a}'\hat{b}'}, \widehat{J}_{\hat{c}'\hat{d}'}] = \delta_{\hat{b}'\hat{c}'}\widehat{J}_{\hat{a}'\hat{d}'} + 3 \text{ terms}, \tag{C.0.4}$$

$$[\widehat{Q}_{\underline{\alpha}}, \widehat{P}_{\underline{a}}] = \frac{\mathrm{i}}{2}(\gamma_{\underline{a}}\varrho)_{\underline{\alpha}}{}^{\underline{\beta}}\widehat{Q}_{\underline{\beta}}, \qquad [\widehat{\overline{Q}}_{\underline{\bar{\alpha}}}, \widehat{P}_{\underline{a}}] = -\frac{\mathrm{i}}{2}(\gamma_{\underline{a}}\varrho)_{\underline{\bar{\alpha}}}{}^{\underline{\bar{\beta}}}\widehat{\overline{Q}}_{\underline{\bar{\beta}}}, \tag{C.0.5}$$

$$[\widehat{Q}_{\underline{\alpha}}, \widehat{J}_{\hat{a}\hat{b}}] = \frac{1}{2}(\gamma_{\hat{a}\hat{b}})_{\underline{\alpha}}{}^{\underline{\beta}}\widehat{Q}_{\underline{\beta}}, \qquad [\widehat{Q}_{\underline{\alpha}}, \widehat{J}_{\hat{a}'\hat{b}'}] = \frac{1}{2}(\gamma_{\hat{a}'\hat{b}'})_{\underline{\alpha}}{}^{\underline{\beta}}\widehat{Q}_{\underline{\beta}}, \tag{C.0.6}$$

$$[\widehat{\overline{Q}}_{\underline{\bar{\alpha}}}, \widehat{J}_{\hat{a}\hat{b}}] = \frac{1}{2}(\gamma_{\hat{a}\hat{b}})_{\underline{\bar{\alpha}}}{}^{\underline{\bar{\beta}}}\widehat{\overline{Q}}_{\underline{\bar{\beta}}}, \qquad [\widehat{\overline{Q}}_{\underline{\bar{\alpha}}}, \widehat{J}_{\hat{a}'\hat{b}'}] = \frac{1}{2}(\gamma_{\hat{a}'\hat{b}'})_{\underline{\bar{\alpha}}}{}^{\underline{\bar{\beta}}}\widehat{\overline{Q}}_{\underline{\bar{\beta}}}, \tag{C.0.7}$$

$$\{\widehat{Q}_{\underline{\alpha}}, \widehat{\overline{Q}}_{\underline{\bar{\beta}}}\} = -2\mathrm{i}(\gamma^{\underline{a}})_{\underline{\alpha}\underline{\bar{\beta}}}\widehat{P}_{\underline{a}} + (\gamma^{\hat{a}\hat{b}}\varrho)_{\underline{\alpha}\underline{\bar{\beta}}}\widehat{J}_{\hat{a}\hat{b}} - (\gamma^{\hat{a}'\hat{b}'}\varrho)_{\underline{\alpha}\underline{\bar{\beta}}}\widehat{J}_{\hat{a}'\hat{b}'}. \tag{C.0.8}$$

Here $\eta_{\hat{a}\hat{b}} = \mathrm{diag}(-,+,+,+,+)$. We define

$$\widehat{P}_\pm = \frac{1}{\sqrt{2}}(\widehat{P}_0 \pm \widehat{P}_9), \qquad \gamma_\pm = \frac{1}{2}(\gamma_0 \pm \gamma_9). \tag{C.0.9}$$

$$(\gamma_+)_{\underline{\alpha\beta}} = \begin{pmatrix} (\gamma_+)_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_8 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (\gamma_-)_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & 0 \\ 0 & (\gamma_-)_{\dot{\alpha}\dot{\beta}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1_8 \end{pmatrix}. \tag{C.0.10}$$

In our notation,

$$(\gamma^a)_{\underline{\alpha}\underline{\beta}} = \begin{pmatrix} 0 & (\gamma^a)_{\alpha\dot{\beta}} \\ (\gamma^a)_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \qquad (\gamma^{4+s})_{\underline{\alpha}\underline{\beta}} = \begin{pmatrix} 0 & (\gamma^{4+s})_{\alpha\dot{\beta}} \\ (\gamma^{4+s})_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \tag{C.0.11}$$

for $a = 1,2,3,4$ and $s = 1,2,3,4$.

If we decompose the fermionic generators into $\widehat{Q}_{\underline{\alpha}} = (\widehat{Q}^+_\alpha, \widehat{Q}^-_{\dot{\alpha}})$, $\widehat{\overline{Q}}_{\underline{\bar{\alpha}}} = (\widehat{\overline{Q}}^+_{\bar{\alpha}}, \widehat{\overline{Q}}^-_{\dot{\bar{\alpha}}})$, some of commutation relations can be rewritten as follows:

$$[\widehat{Q}^+_\alpha, \widehat{P}_+] = \frac{\mathrm{i}}{\sqrt{2}}(\gamma_+\varrho)_\alpha{}^\beta \widehat{Q}^+_\beta, \qquad [\widehat{Q}^-_{\dot{\alpha}}, \widehat{P}_+] = 0, \tag{C.0.12}$$

$$[\widehat{Q}^+_\alpha, \widehat{P}_-] = 0, \qquad [\widehat{Q}^-_{\dot{\alpha}}, \widehat{P}_-] = \frac{\mathrm{i}}{\sqrt{2}}(\gamma_-\varrho)_{\dot{\alpha}}{}^{\dot{\beta}} \widehat{Q}^-_{\dot{\beta}}, \tag{C.0.13}$$

$$\{\widehat{Q}^+_\alpha, \widehat{\overline{Q}}^+_{\bar{\beta}}\} = 2\sqrt{2}\mathrm{i}\,(\gamma_+)_{\alpha\bar{\beta}}\,\widehat{P}_- + (\gamma^{ab}\varrho)_{\alpha\bar{\beta}}\,\widehat{J}_{ab} - (\gamma^{a'b'}\varrho)_{\alpha\bar{\beta}}\,\widehat{J}_{a'b'}. \tag{C.0.14}$$

Here $a,b = 1,2,3,4$, $a',b' = 5,6,7,8$.

Using the coset representative (4.3.7) with

$$g_z = \exp\left(\mathcal{X}^a \widehat{P}_a\right), \qquad \mathcal{X}^a = \frac{z^a}{z}\log\left(\frac{1+(1/2)z}{1-(1/2)z}\right), \tag{C.0.15}$$

$$g_y = \exp\left(\mathcal{X}^{4+s} \widehat{P}_{4+s}\right), \qquad \mathcal{X}^{4+s} = -\mathrm{i}\frac{y^s}{y}\log\left(\frac{1+(\mathrm{i}/2)y}{1-(\mathrm{i}/2)y}\right), \tag{C.0.16}$$

we can calculate the vielbeins for the reduced type IIB superspace as follows:

$$G^{-1}\mathrm{d}G = E^{\underline{a}}\widehat{P}_{\underline{a}} + E^{\underline{\alpha}}\widehat{Q}_{\underline{\alpha}} + \overline{E}^{\underline{\bar{\alpha}}}\widehat{\overline{Q}}_{\underline{\bar{\alpha}}} + (\text{spin connection part}). \tag{C.0.17}$$

Let us define a 16×16 matrix $\mathcal{M}^2$ by

$$\mathcal{M}^2 = \begin{pmatrix} (\mathcal{M}^2)^\alpha{}_\beta & (\mathcal{M}^2)^\alpha{}_{\bar{\beta}} \\ (\mathcal{M}^2)^{\bar{\alpha}}{}_\beta & (\mathcal{M}^2)^{\bar{\alpha}}{}_{\bar{\beta}} \end{pmatrix}, \tag{C.0.18}$$

where the matrix elements are defined by

$$\mathrm{ad}^2(\theta^+\widehat{Q}^+ + \bar{\theta}^+\widehat{\overline{Q}}^+)(\widehat{Q}^+_\alpha) = \widehat{Q}^+_\beta(\mathcal{M}^2)^\beta{}_\alpha + \widehat{\overline{Q}}^+_{\bar{\beta}}(\mathcal{M}^2)^{\bar{\beta}}{}_\alpha, \tag{C.0.19}$$

$$\mathrm{ad}^2(\theta^+\widehat{Q}^+ + \bar{\theta}^+\widehat{\overline{Q}}^+)(\widehat{\overline{Q}}^+_{\bar{\alpha}}) = \widehat{Q}^+_\beta(\mathcal{M}^2)^\beta{}_{\bar{\alpha}} + \widehat{\overline{Q}}^+_{\bar{\beta}}(\mathcal{M}^2)^{\bar{\beta}}{}_{\bar{\alpha}}. \tag{C.0.20}$$

Explicit form of the matrix elements are given by

$$\begin{aligned}
(\mathcal{M}^2)^\alpha{}_\beta &= \frac{1}{2}(\theta^+\gamma_{ab})^\alpha(\bar{\theta}^+\gamma^{ab}\varrho)_\beta - \frac{1}{2}(\theta^+\gamma_{a'b'})^\alpha(\bar{\theta}^+\gamma^{a'b'}\varrho)_\beta, \\
(\mathcal{M}^2)^\alpha{}_{\bar{\beta}} &= -\frac{1}{2}(\theta^+\gamma_{ab})^\alpha(\theta^+\gamma^{ab}\varrho)_{\bar{\beta}} + \frac{1}{2}(\theta^+\gamma_{a'b'})^\alpha(\theta^+\gamma^{a'b'}\varrho)_{\bar{\beta}}, \\
(\mathcal{M}^2)^{\bar{\alpha}}{}_\beta &= \frac{1}{2}(\bar{\theta}^+\gamma_{ab})^{\bar{\alpha}}(\bar{\theta}^+\gamma^{ab}\varrho)_\beta - \frac{1}{2}(\bar{\theta}^+\gamma_{a'b'})^{\bar{\alpha}}(\bar{\theta}^+\gamma^{a'b'}\varrho)_\beta, \\
(\mathcal{M}^2)^{\bar{\alpha}}{}_{\bar{\beta}} &= -\frac{1}{2}(\bar{\theta}^+\gamma_{ab})^{\bar{\alpha}}(\theta^+\gamma^{ab}\varrho)_{\bar{\beta}} + \frac{1}{2}(\bar{\theta}^+\gamma_{a'b'})^{\bar{\alpha}}(\theta^+\gamma^{a'b'}\varrho)_{\bar{\beta}}.
\end{aligned} \tag{C.0.21}$$

Let

$$\frac{\cosh\mathcal{M}-1_{16}}{\mathcal{M}^2}=\begin{pmatrix}(K_{11})^{\alpha}{}_{\beta} & (K_{12})^{\alpha}{}_{\bar\beta}\\ (K_{21})^{\bar\alpha}{}_{\beta} & (K_{22})^{\bar\alpha}{}_{\bar\beta}\end{pmatrix},\qquad \frac{\sinh\mathcal{M}}{\mathcal{M}}=\begin{pmatrix}(L_{11})^{\alpha}{}_{\beta} & (L_{12})^{\alpha}{}_{\bar\beta}\\ (L_{21})^{\bar\alpha}{}_{\beta} & (L_{22})^{\bar\alpha}{}_{\bar\beta}\end{pmatrix}. \tag{C.0.22}$$

The induced vielbeins are calculated as follows:

$$\begin{aligned}
E_i^{\pm} &= e^{\pm}{}_{+}\partial_i X^{+} + e^{\pm}{}_{-}\mathcal{D}_i X^{-},\\
E_i^{a} &= \frac{1}{1-(z^2/4)}\mathcal{D}_i z^a,\\
E_i^{4+s} &= \frac{1}{1+(y^2/4)}\mathcal{D}_i y^s,\\
E_i^{+\alpha} &= U^{\alpha}{}_{\beta}\left((L_{11}\mathcal{D}_i\theta^{+})^{\beta}+(L_{12}\mathcal{D}_i\bar\theta^{+})^{\beta}\right),\\
E_i^{-\dot\alpha} &= V^{\dot\alpha}{}_{\beta}\left((L_{11}\mathcal{D}_i\theta^{+})^{\beta}+(L_{12}\mathcal{D}_i\bar\theta^{+})^{\beta}\right),\\
\overline{E}_i^{+\bar\alpha} &= \overline{U}^{\bar\alpha}{}_{\bar\beta}\left((L_{21}\mathcal{D}_i\theta^{+})^{\bar\beta}+(L_{22}\mathcal{D}_i\bar\theta^{+})^{\bar\beta}\right),\\
\overline{E}_i^{-\dot{\bar\alpha}} &= \overline{V}^{\dot{\bar\alpha}}{}_{\bar\beta}\left((L_{21}\mathcal{D}_i\theta^{+})^{\bar\beta}+(L_{22}\mathcal{D}_i\bar\theta^{+})^{\bar\beta}\right),
\end{aligned} \tag{C.0.23}$$

where

$$\begin{aligned}
e^{\pm}{}_{+} &= \frac{1}{2}\left[\left(\frac{1+(z^2/4)}{1-(z^2/4)}\right)\pm\left(\frac{1-(y^2/4)}{1+(y^2/4)}\right)\right],\\
e^{\pm}{}_{-} &= \frac{1}{2}\left[\left(\frac{1+(z^2/4)}{1-(z^2/4)}\right)\mp\left(\frac{1-(y^2/4)}{1+(y^2/4)}\right)\right],
\end{aligned} \tag{C.0.24}$$

$$U^{\alpha}{}_{\beta}=\frac{(\delta^{\alpha}{}_{\beta}+(1/4)z^a y^s(\gamma_a\gamma_{4+s})^{\alpha}{}_{\beta})}{(1-(z^2/4))^{1/2}(1+(y^2/4))^{1/2}},\qquad \overline{U}^{\bar\alpha}{}_{\bar\beta}=\frac{(\delta^{\bar\alpha}{}_{\bar\beta}+(1/4)z^a y^s(\gamma_a\gamma_{4+s})^{\bar\alpha}{}_{\bar\beta})}{(1-(z^2/4))^{1/2}(1+(y^2/4))^{1/2}}, \tag{C.0.25}$$

$$V^{\dot\alpha}{}_{\beta}=\frac{-\mathrm{i}z^a(\gamma_a\varrho)^{\dot\alpha}{}_{\beta}+\mathrm{i}y^s(\gamma_{4+s}\varrho)^{\dot\alpha}{}_{\beta}}{2(1-(z^2/4))^{1/2}(1+(y^2/4))^{1/2}},\qquad \overline{V}^{\dot{\bar\alpha}}{}_{\bar\beta}=\frac{\mathrm{i}z^a(\gamma_a\varrho)^{\dot{\bar\alpha}}{}_{\bar\beta}-\mathrm{i}y^s(\gamma_{4+s}\varrho)^{\dot{\bar\alpha}}{}_{\bar\beta}}{2(1-(z^2/4))^{1/2}(1+(y^2/4))^{1/2}}, \tag{C.0.26}$$

$$
\begin{aligned}
\mathcal{D}_i X^- &= \partial_i X^- + 2\sqrt{2}\mathrm{i}\left[(\bar\theta^+\gamma_+ K_{11})_\alpha + (\theta^+\gamma_+ K_{21})_\alpha\right]\mathcal{D}_i\theta^{+\alpha} \\
&\quad + 2\sqrt{2}\mathrm{i}\left[(\bar\theta^+\gamma_+ K_{12})_{\bar\alpha} + (\theta^+\gamma_+ K_{22})_{\bar\alpha}\right]\mathcal{D}_i\bar\theta^{+\bar\alpha}, \\
\mathcal{D}_i z^a &= \partial_i z^a - 2z_b\left[(\bar\theta^+\gamma^{ab}\varrho K_{11})_\alpha - (\theta^+\gamma^{ab}\varrho K_{21})_\alpha\right]\mathcal{D}_i\theta^{+\alpha} \\
&\quad - 2z_b\left[(\bar\theta^+\gamma^{ab}\varrho K_{12})_{\bar\alpha} - (\theta^+\gamma^{ab}\varrho K_{22})_{\bar\alpha}\right]\mathcal{D}_i\bar\theta^{+\bar\alpha}, \\
\mathcal{D}_i y^s &= \partial_i y^s + 2y_{s'}\left[(\bar\theta^+\gamma^{4+s,4+s'}\varrho K_{11})_\alpha - (\theta^+\gamma^{4+s,4+s'}\varrho K_{21})_\alpha\right]\mathcal{D}_i\theta^{+\alpha} \\
&\quad + 2y_{s'}\left[(\bar\theta^+\gamma^{4+s,4+s'}\varrho K_{12})_{\bar\alpha} + (\theta^+\gamma^{4+s,4+s'}\varrho K_{22})_{\bar\alpha}\right]\mathcal{D}_i\bar\theta^{+\bar\alpha}, \\
\mathcal{D}_i\theta^{+\alpha} &= \partial_i\theta^{+\alpha} - \frac{\mathrm{i}}{\sqrt{2}}(\theta^+\gamma_+\varrho)^\alpha\partial_i X^+, \\
\mathcal{D}_i\bar\theta^{+\bar\alpha} &= \partial_i\bar\theta^{+\bar\alpha} + \frac{\mathrm{i}}{\sqrt{2}}(\bar\theta^+\gamma_+\varrho)^{\bar\alpha}\partial_i X^+.
\end{aligned}
\tag{C.0.27}
$$

By comparing with (4.3.9), we can read off $\Lambda^-{}_{\hat\alpha}$, $\Lambda^m{}_{n\hat\alpha}$, $\Lambda^{\hat\alpha}{}_{\hat\beta}$. For example,

$$\Lambda^-{}_\alpha = 2\sqrt{2}\,\mathrm{i}\left[(\bar\theta^+\gamma_+ K_{11})_\alpha + (\theta^+\gamma_+ K_{21})_\alpha\right], \tag{C.0.28}$$

$$\Lambda^-{}_{\bar\alpha} = 2\sqrt{2}\,\mathrm{i}\left[(\bar\theta^+\gamma_+ K_{12})_{\bar\alpha} + (\theta^+\gamma_+ K_{22})_{\bar\alpha}\right]. \tag{C.0.29}$$

The fields $B_{\hat\alpha\hat\beta}$ in the Wess-Zumino term are read off from

$$
\begin{aligned}
&\frac{1}{2}\epsilon^{ij}B_{\hat\alpha\hat\beta}(Z)\mathcal{D}_i\Psi^{\hat\alpha}\mathcal{D}_j\Psi^{\hat\beta} \\
&= \epsilon^{ij}(E_i^{+\alpha}\varrho_{\alpha\beta}E_j^{+\beta} + E_i^{-\dot\alpha}\varrho_{\dot\alpha\dot\beta}E_j^{-\dot\beta} - \overline{E}_i^{+\bar\alpha}\varrho_{\bar\alpha\bar\beta}\overline{E}_j^{+\bar\beta} - \overline{E}_i^{-\dot{\bar\alpha}}\varrho_{\dot{\bar\alpha}\dot{\bar\beta}}\overline{E}_j^{-\dot{\bar\beta}}).
\end{aligned}
\tag{C.0.30}
$$

Our convention for the Levi-Civita symbol is $\epsilon^{01} = 1$.

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