The warping degree of a link diagram

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ABSTRACT

For an oriented link diagram D, the warping degree $d(D)$ is the smallest number of crossing changes which are needed to obtain a monotone diagram from D. We show that $d(D) + d(-D) + sr(D)$ is less than or equal to the crossing number of D, where $-D$ denotes the inverse of D and $sr(D)$ denotes the number of components which have at least one self-crossing. Moreover, we give a necessary and sufficient condition for the equality. We also consider the minimal $d(D) + d(-D) + sr(D)$ for all diagrams D. For the warping degree and warp-linking degree, we show some relations to the linking number, unknotting number, and the splitting number. We also discuss the complete splitting number of a lassoed link.

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CONTENTS

1. Introduction

An *r-component link* L is an image of a smooth embedding of the disjoint union of r circles into $\mathbb{R}^3$. In particular, a *knot* K is a link of one component. A *diagram* D of a link L is a generic immersion of L to a plane enhanced by information on overpasses and underpasses at double points. A *crossing point* p of D is such a double point of D with information on overpass and underpass. The warping degree and a monotone diagram is defined by Kawauchi for an oriented diagram of a knot, a link [10] or a spatial graph [11]. The warping degree represents such a complexity of a diagram, and depends on the orientation of the diagram. For an oriented link diagram D, we say that D is monotone if we meet every crossing point as an over-crossing first when we travel along all components of the oriented diagram with an order by starting from each base point. This notion is earlier used by Hoste [5] and by Lickorish-Millett [15] in computing polynomial invariants of knots and links. The warping degree $d(D)$ of an oriented link diagram D is the smallest number of crossing changes which are needed to obtain a monotone diagram from D in the usual way. We give the precise definitions of the warping degree and a monotone diagram in Section 2. A link diagram is *alternating* if the crossings alternate over and under as we go along the diagram. Let $-D$ be the diagram D with orientations reversed for all components, and we call $-D$ the inverse of D. Let $c(D)$ be the crossing number of D. We have the following theorem which is for a knot diagram:

Theorem 1.1. *Let D be an oriented knot diagram which has at least one crossing point. Then we have*

$$d(D) + d(-D) + 1 \leq c(D).$$

Further, the equality holds if and only if D is an alternating diagram.

We give the proof in Section 3. Let D be a diagram of an r-component link $(r \geq 1)$. Let D^i be a diagram on a knot component L^i of L, and we call D^i a component of D. A link diagram D is *equilibrial* if the number of non-self over-crossing points of D^i in $D^i \cup D^j$ is equal to the number of non-self under-crossing points of D^i in $D^i \cup D^j$ for every two-component sublink diagram $D^i \cup D^j$ of D. We assume that a knot diagram is equilibrial. In Figure 1, the diagram D is equilibrial and D' is not equilibrial although D and D' represent the same link. We generalize Theorem 1.1 to a link

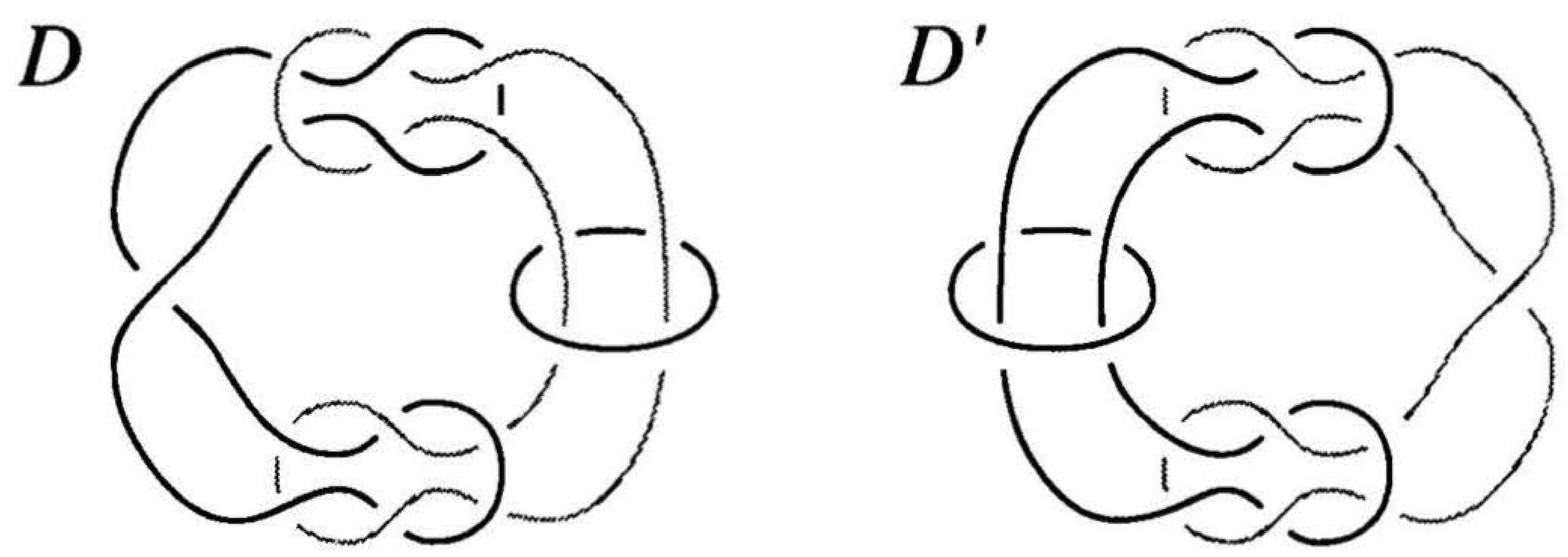


FIGURE 1

diagram:

Theorem 1.2. *Let D be an oriented link diagram, and $sr(D)$ the number of components D^i such that D^i has at least one self-crossing. Then we have*

$$d(D) + d(-D) + sr(D) \leq c(D).$$

Further, the equality holds if and only if D is equilibrial and every component D^i of D is alternating.

For example, the link diagram D in Figure 2 has $d(D) + d(-D) + sr(D) = 3 + 3 + 2 = 8 = c(D)$.

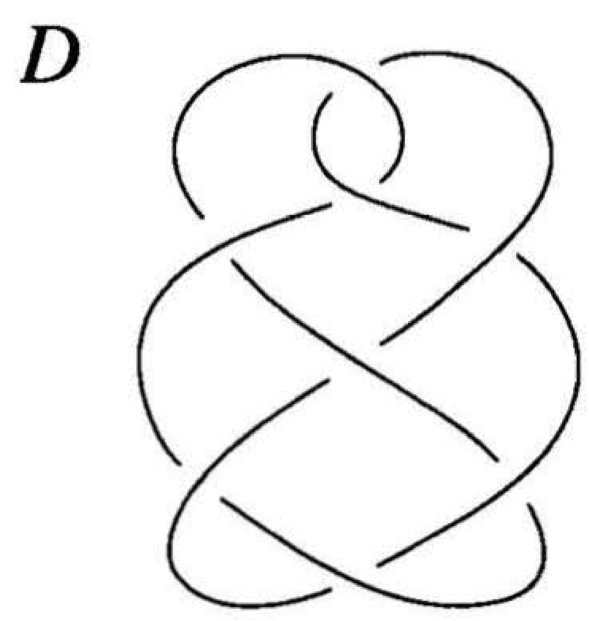


FIGURE 2

Let D be a diagram of a link. Let $u(D)$ be the unlinking number of D. As a lower bound for the value $d(D)+d(-D)+sr(D)$, we have the following inequality:

Theorem 1.3. *We have*

$$2u(D)+sr(D)\leq d(D)+d(-D)+sr(D).$$

The rest of this paper is organized as follows. In Section 2, we define the warping degree $d(D)$ of an oriented link diagram D. In Section 3, we prove Theorem 1.1. In Section 4, we define the warp-linking degree $ld(D)$, and consider the value $d(D)+d(-D)$ to prove Theorem 1.2. In Section 5, we show relations of the warp-linking degree and the linking number. In Section 6, we apply the warping degree to a link itself. In Section 7, we study relations to unlinking number and crossing number. In Section 8, we define the splitting number and consider relations between the warping degree and the splitting number. In Appendix A, we show methods for calculating the warping degrees and the warp-linking degrees. In Appendix B, we discuss the complete splitting number of a lassoed link.

2. Warping degree of an oriented link diagram

Let L be an oriented r-component link, and D a diagram of L. We take a sequence $\mathbf{a}$ of base points a_i $(i = 1, 2, \ldots, r)$, where every component has just one base point except at crossing points. Then $D_{\mathbf{a}}$, the pair of D and $\mathbf{a}$, is represented by $D_{\mathbf{a}} = D^1_{a_1} \cup D^2_{a_2} \cup \cdots \cup D^r_{a_r}$ with the order of $\mathbf{a}$. A self-crossing point p of $D^i_{a_i}$ is a *warping crossing point of* $D^i_{a_i}$ if we meet the point first at the under-crossing when we go along the oriented diagram $D^i_{a_i}$ by starting from a_i $(i = 1, 2, \ldots, r)$. A crossing point p of $D^i_{a_i}$ and $D^j_{a_j}$ is a *warping crossing point between* $D^i_{a_i}$ *and* $D^j_{a_j}$ if p is the under-crossing of $D^i_{a_i}$ $(1 \leq i < j \leq r)$. A crossing point p of $D_{\mathbf{a}}$ is a *warping crossing point of* $D_{\mathbf{a}}$ if p is a warping crossing point of $D^i_{a_i}$ or a warping crossing point between $D^i_{a_i}$ and $D^j_{a_j}$ [10].

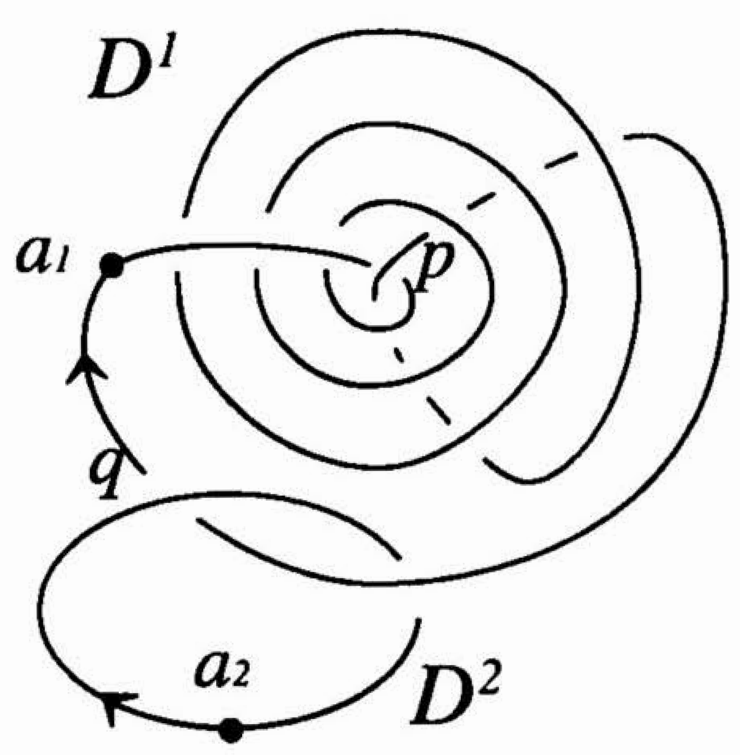


FIGURE 3

For example in Figure 3, p is a warping crossing point of $D^1_{a_1}$, and q is a warping crossing point between $D^1_{a_1}$ and $D^2_{a_2}$. We define the warping degree for an oriented link diagram [10]. The *warping degree* of $D_{\mathbf{a}}$, denoted by $d(D_{\mathbf{a}})$, is the number of warping crossing points of $D_{\mathbf{a}}$. The *warping degree* of D, denoted by $d(D)$, is the minimal warping degree $d(D_{\mathbf{a}})$ for all base point sequences $\mathbf{a}$ of D. Ozawa showed that a non-trivial link which has a diagram D with $d(D) = 1$ is a split union of a twist knot or the Hopf link and r trivial knots $(r \geq 0)$ [20]. Fung also showed that a non-trivial knot which has a diagram D with $d(D) = 1$ is a twist knot [27].

For an oriented link diagram with its base point sequence $D_{\mathbf{a}} = D^1_{a_1} \cup D^2_{a_2} \cup \cdots \cup D^r_{a_r}$, we denote by $d(D^i_{a_i})$ the number of warping crossing points of $D^i_{a_i}$. We denote by $d(D^i_{a_i}, D^j_{a_j})$ the number of warping crossing points between $D^i_{a_i}$ and $D^j_{a_j}$. By definition, we have that

$$d(D_{\mathbf{a}}) = \sum_{i=1}^{r} d(D^i_{a_i}) + \sum_{i<j} d(D^i_{a_i}, D^j_{a_j}).$$

Thus, the set of the warping crossing points of $D_{\mathbf{a}}$ is divided into two types in the sense that the warping crossing point is self-crossing or not. The pair $D_{\mathbf{a}}$ is *monotone* if $d(D_{\mathbf{a}}) = 0$. For example, $D_{\mathbf{a}}$ depicted in Figure 4 is monotone.

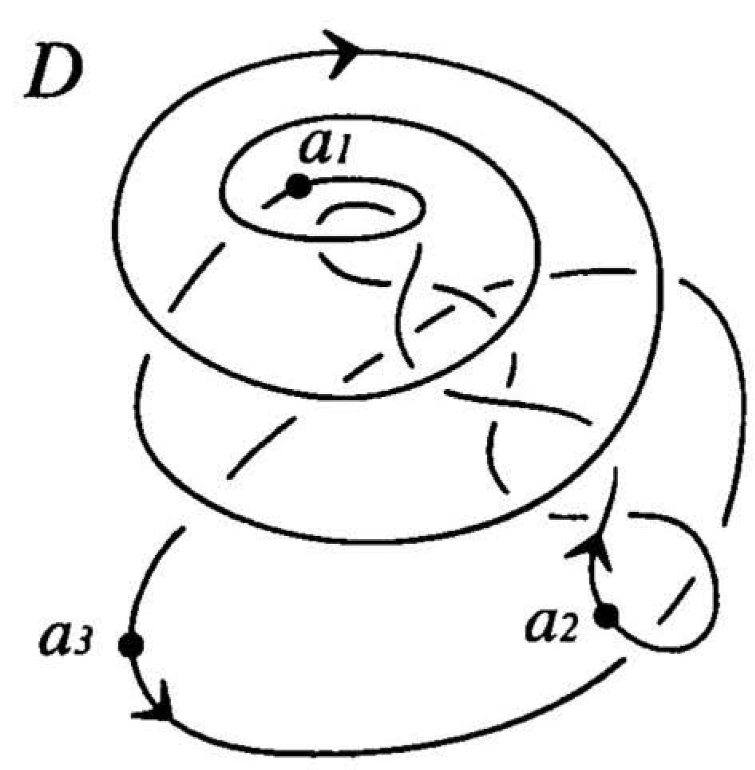


FIGURE 4

Note that a monotone diagram is a diagram of a trivial link. Hence we have $u(D) \leq d(D)$, where $u(D)$ is the unlinking number of D ([19], [28]).

3. Warping degree of a knot diagram

In this section, we discuss the warping degree of a knot diagram and prove Theorem 1.1. We first show some properties of the warping degree of a knot diagram. We have the following lemma:

Lemma 3.1. *Let D_b be a knot diagram with a base point b. Then we have*

$$d(D_b) + d(-D_b) = c(D).$$

Before proving Lemma 3.1, we introduce a method of judging locally whether a crossing point of an oriented knot diagram with a base point is a warping crossing point or not. For an oriented knot diagram D and a base point b, we notice that D_b is divided into $c(D) + 1$ arcs by cutting the base point and under-crossings. Then we label them along the orientation from the arc which has the base point as initial point as shown in Figure 5 . Every crossing point consists of two or three arcs. We label each crossing point via the indices of the arcs in the following definition.

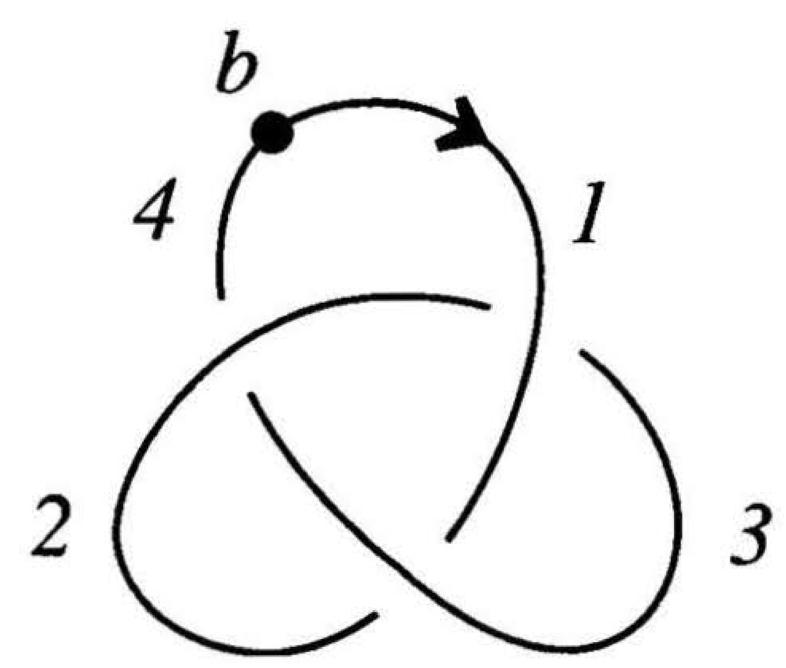


FIGURE 5

Definition 3.2. Let p be a crossing point of D_b which consists of an over-arc with the index α and the other two arcs with the index β, γ. We define the *cutting number* of p in D_b, denoted by $cut_{D_b}(p)$, by the following formula:

$$cut_{D_b}(p) = 2\alpha - \beta - \gamma.$$

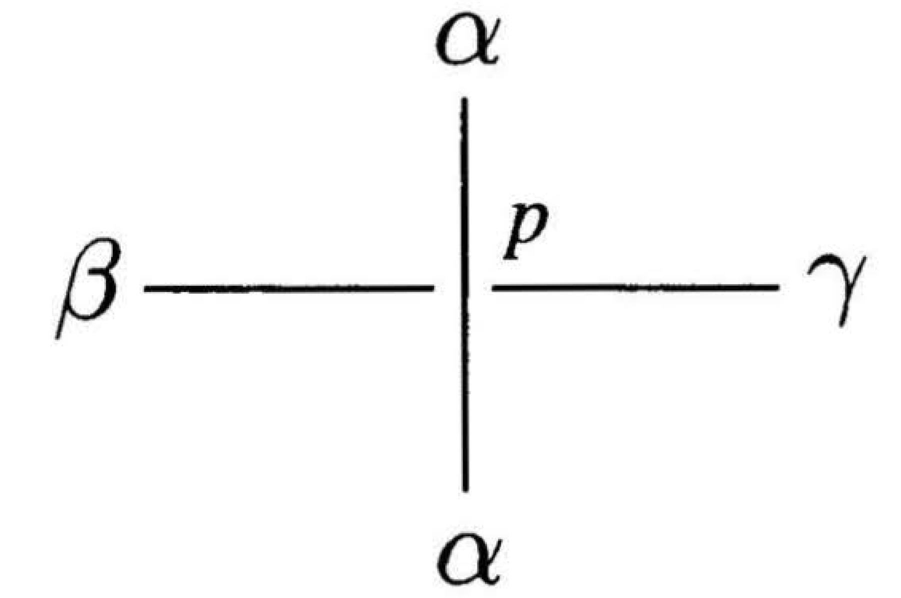


FIGURE 6

Note that the cutting number is always odd. Suppose β is smaller than γ, namely $\gamma = \beta + 1$. Then we have $cut_{D_b}(p) = 2(\alpha - \beta) - 1$ by substituting $\beta+1$ for γ. Hence $cut_{D_b}(p)$ is odd. By the definition of the warping crossing point, we have the following lemma:

Lemma 3.3. *A crossing point p is a warping crossing point of D_b if and only if $cut_{D_b}(p) > 0$, and p is a non-warping crossing point of D_b if and only if $cut_{D_b}(p) < 0$.*

We prove Lemma 3.1.

Proof of Lemma 3.1. Let $(k_1, k_2, \ldots, k_n)$ be the ordered set of arcs of D_b, where k_i has the index i. Let $(l_1, l_2, \ldots, l_n)$ be the ordered set of arcs of $-D_b$ as above. Then we notice that k_i and l_{n+1-i} are the same arc except the orientations for $i = 1, 2, \ldots, n$. Let p be a crossing point of D_b in Figure 6. Then we have the following equality:

$$\begin{aligned} cut_{-D_b}(p) &= 2(n+1-\alpha) - (n+1-\beta) - (n+1-\gamma) \\ &= -(2\alpha - \beta - \gamma) \\ &= -cut_{D_b}(p). \end{aligned}$$

Therefore p is a non-warping crossing point of $-D_b$ if and only if p is a warping crossing point of D_b. Hence we obtain that

$$\begin{aligned} c(D) &= d(D_b) + \#\{\text{non-warping crossing points of } D_b\} \\ &= d(D_b) + \#\{\text{warping crossing points of } -D_b\} \\ &= d(D_b) + d(-D_b). \end{aligned}$$

This completes the proof. □

We apply Lemma 3.1 to the mirror image in the following example:

Example 3.4. Let D_b be an oriented knot diagram with a base point b, and D_b^* the mirror image of D_b. Then we observe that a crossing point p is a non-warping crossing point of D_b^* if and only if p is a warping crossing point of D_b. By Lemma 3.1, p is a warping crossing point of D_b^* if and only if p is a warping crossing point of $-D_b$. Therefore we have $d(D^*) = d(-D)$.

For two base points which are put across a crossing point, we have the following lemma:

Lemma 3.5. *For the base points a_1, a_2 (resp. b_1, b_2) which are put across an over-crossing p (resp. under-crossing q) in Figure 7, we have $d(D_{a_2}) = d(D_{a_1}) + 1$ (resp. $d(D_{b_2}) = d(D_{b_1}) - 1$).*

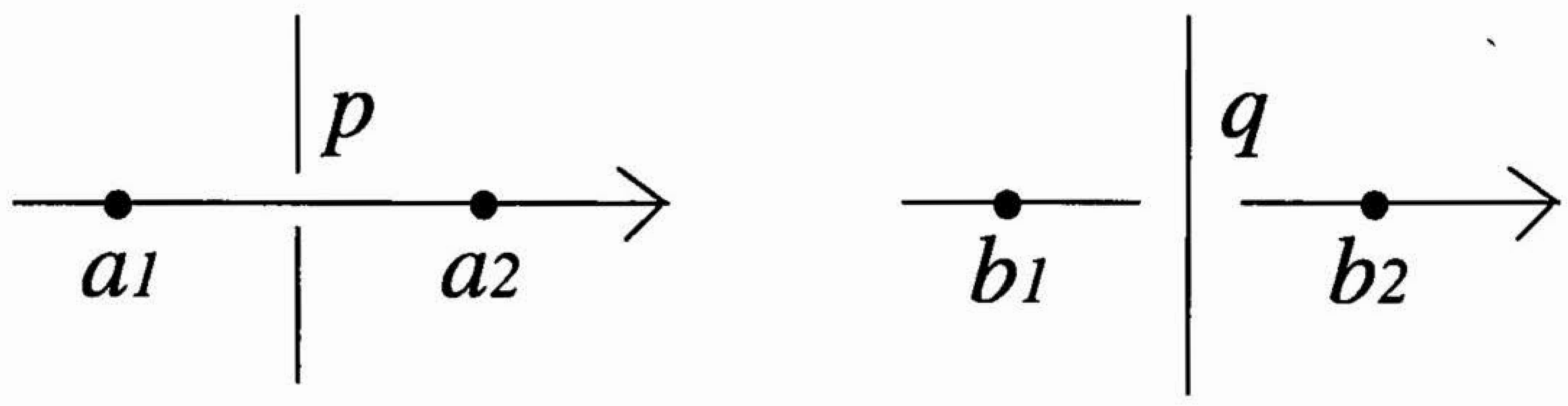


FIGURE 7

Proof. Except at p, D_{a_1} and D_{a_2} have warping crossing points at same crossing points, and p is not a warping crossing point of D_{a_1} and p is a warping crossing point of D_{a_2}. □

By Lemma 3.5, we obtain the following lemma:

Lemma 3.6. *Let D be an oriented alternating knot diagram. Let b be a base point of D which is just before an over-crossing as shown in Figure 8, then we have $d(D_b) = d(D)$.*

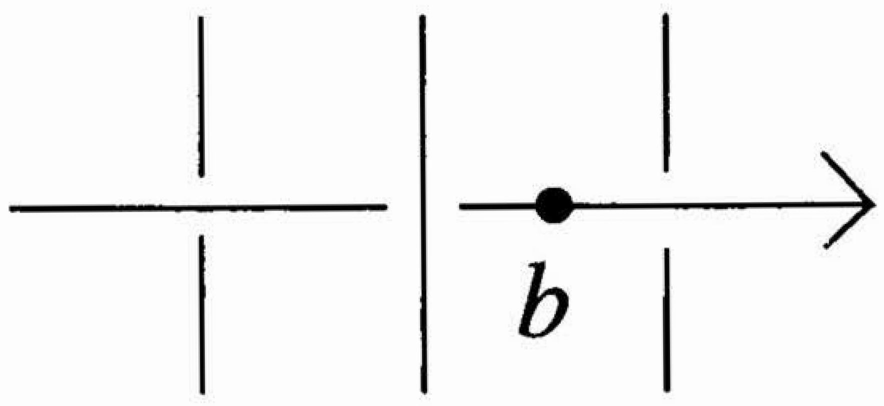


FIGURE 8

We prove Theorem 1.1.

Proof of Theorem 1.1. For an oriented knot diagram D with $c(D) \geq 1$, the following inequality holds:

$$\max_a d(D_a) - \min_a d(D_a) \geq 1. \tag{1}$$

From Lemma 3.5, the equality holds if D is an alternating diagram. On the other hand, if the equality holds, D is an alternating diagram, namely there do not exist any two over-crossings or two under-crossings which are next to each other as shown in Figure 9 .

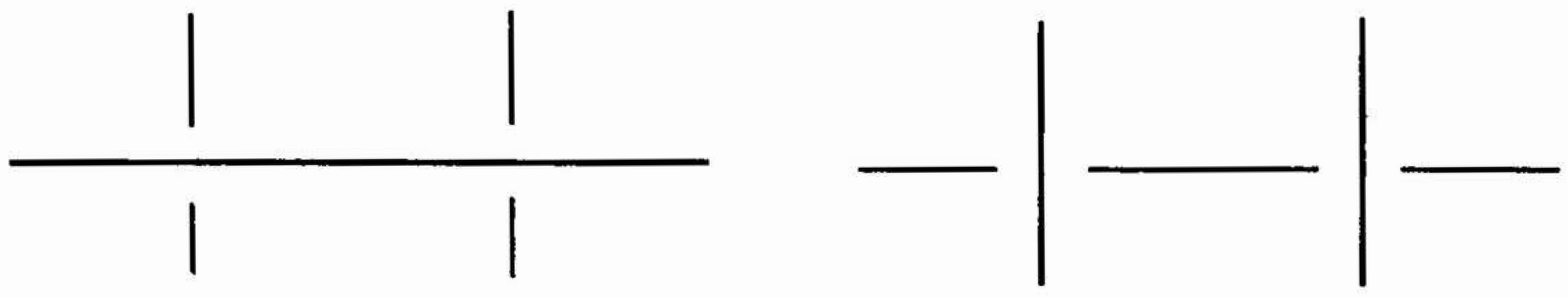


FIGURE 9

Let a and b be base points which satisfy $d(D_a) = d(D)$ and $d(-D_b) = d(-D)$. Then we notice that

$$\max_a d(D_a) = d(D_b),$$

because a' satisfies

$$d(-D_b) = \min_a d(-D_a)$$

and a warping crossing point of $-D_a$ is a non-warping crossing point of D_a. Hence the inequality (1) is equivalent to the following inequality:

$$d(D_b) - d(D_a) \geq 1,$$

and this is equivalent to

$$d(D_a) + 1 \leq d(D_b).$$

By adding $d(-D_b)$ to each side, we have

$$d(D_a) + d(-D_b) + 1 \leq d(D_b) + d(-D_b).$$

By Lemma 3.1 and the conditions of a and b, we obtain the following inequality:

$$d(D) + d(-D) + 1 \leq c(D),$$

where the equality holds if and only if D is an alternating diagram. □

Here is an example of Theorem 1.1.

Example 3.7. This table lists all standard knot diagrams based on Rolfsen's knot table with crossing number 9 or less [21]. In this table, $D(K)$ denotes the standard diagram of K with the orientation which has the smaller warping degree, and a knot marked with † is non-alternating.

K	$d(D(K))$	$d(-D(K))$	
3_1	1	1	
4_1	1	2	
5_i	2	2	$i = 1, 2$
6_i	2	3	$i = 1, 2, 3$
7_i	3	3	$i = 1, 2, \ldots, 6$
7_7	2	4	
8_i	3	4	$i = 1, 2, \ldots, 17$
8_{18}	2	5	
8_{19} †	3	3	
8_{20} †	2	3	
8_{21} †	2	2	
9_i	4	4	$i = 1, 2, \ldots, 13, 16, 18, 20, 21, 23, 25, 27, 28,$ $29, 30, 33, 35, 36, 38, 39, 40$
9_i	3	5	$i = 14, 15, 17, 19, 22, 24, 26, 31, 32, 34, 37, 41$
9_i †	3	3	$i = 42, 44, 45, 46$
9_i †	3	4	$i = 43, 49$
9_{47} †	2	5	
9_{48} †	2	3	

In the following lemma, we determine the warping degree of the standard diagram of a torus knot.

Lemma 3.8. *Let $T(p,q)$ be (p,q)-torus knot ($0 < p < q$, p and q are coprime) and $D(p,q)$ the standard diagram of $T(p,q)$ with the orientation as shown in Figure 10. Then we have the following:*

(1): $d(D(p,q)) = d(-D(p,q)) = (p-1)(q-1)/2$,

(2): $c(D(p,q)) - d(D(p,q)) - d(-D(p,q)) = p - 1$.

Proof. In $D(p,q)$, there are $(p-1)$ over-crossings and $(p-1)$ under-crossings alternately. By Lemma 3.5, a base point which is just before $(p-1)$ over-crossings realizes the warping degree of the diagram. For example, base points a and b in Figure 10 satisfy

$$d(D(p,q)_a) = d(D(p,q)),$$
$$d(-D(p,q)_b) = d(-D(p,q)),$$

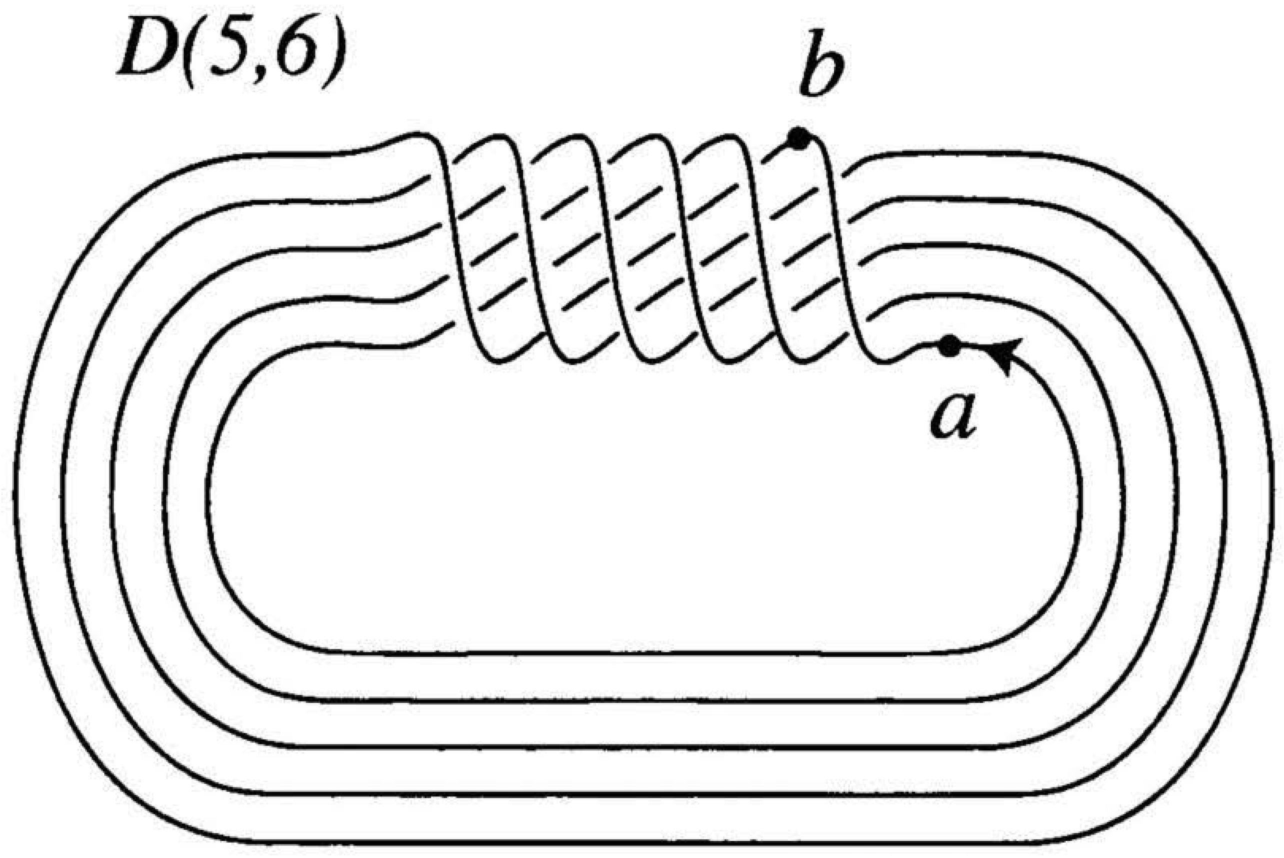


FIGURE 10

respectively. Considering the upside-down image of $D(p,q)$, we have

$$d(D(p,q)) = d(-D(p,q)),$$

that is,

$$d(D(p,q)_a) = d(-D(p,q)_b). \tag{2}$$

With respect to the crossing number of the diagram, we have

$$d(D(p,q)_a) + d(-D(p,q)_a) = c(D) = (p-1)q \tag{3}$$

by Lemma 3.1. And by Lemma 3.5, we have the following relation

$$d(-D(p,q)_b) = d(-D(p,q)_a) - (p-1). \tag{4}$$

By the formula (2) and (4), we have

$$d(D(p,q)_a) = d(-D(p,q)_a) - (p-1). \tag{5}$$

And by the formula (3) and (5), we have

$$2d(D(p,q)_a) = (p-1)q - (p-1).$$

Hence we obtain the warping degree

$$d(D(p,q)) = \frac{(p-1)(q-1)}{2}.$$

Hence we have

$$
\begin{aligned}
&c(D(p,q)) - d(D(p,q)) - d(-D(p,q)) \\
&= (p-1)q - \frac{(p-1)(q-1)}{2} \times 2 = p - 1.
\end{aligned}
$$

This means that $c(D) - d(D) - d(-D)$ depends only on p in this case. $\square$

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We first define the warp-linking degree, which is like a restricted warping degree and which has relations to the crossing number and the linking number (see also Section 5). The number of non-self warping crossing points does not depend on the orientation. We define the *warp-linking degree* of $D_{\mathbf{a}}$, denoted by $ld(D_{\mathbf{a}})$, by the following formula:

$$ld(D_{\mathbf{a}}) = \sum_{i<j} d(D^i_{a_i}, D^j_{a_j}) = d(D_{\mathbf{a}}) - \sum_{i=1}^{r} d(D^i_{a_i}),$$

where $D^i_{a_i}, D^j_{a_j}$ are components of $D_{\mathbf{a}}$. The *warp-linking degree* of D, denoted by $ld(D)$, is the minimal $ld(D_{\mathbf{a}})$ for all base point sequences $\mathbf{a}$. It does not depend on any choices of orientations of components. For example, the diagram D in Figure 11 has $ld(D) = 2$. A pair $D_{\mathbf{a}}$ is *stacked*

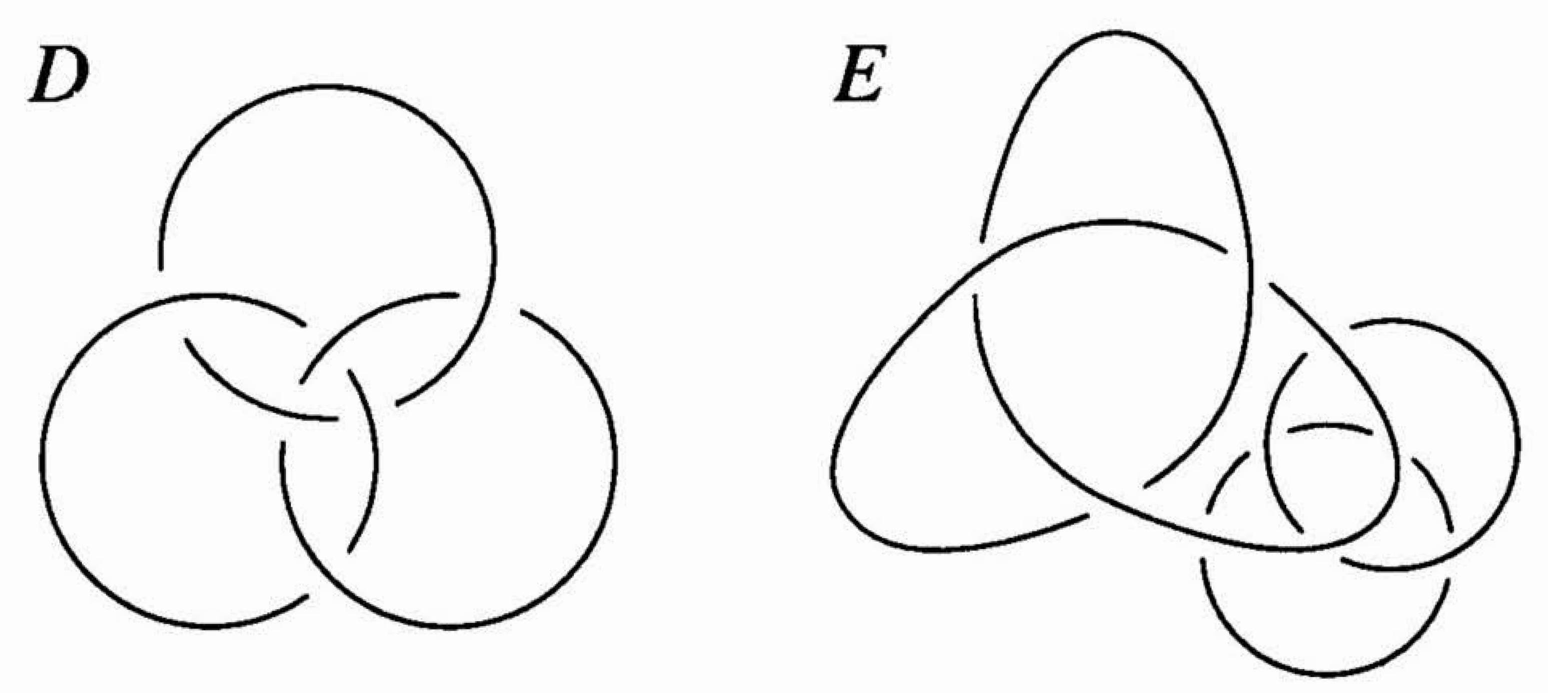


FIGURE 11

if $ld(D_{\mathbf{a}}) = 0$. A diagram D is *stacked* if $ld(D) = 0$. For example, the diagram E in Figure 11 is a stacked diagram. We remark that a similar notion is mentioned in [5]. Note that a monotone diagram is a stacked diagram. A link L is *completely splittable* if L has a diagram D without non-self crossings. We give the precise definition of complete splittable in Appendix B. We notice that a completely splittable link has some stacked diagrams.

The *non-self crossing number* of D, denoted by $lc(D)$, is the number of non-self crossing points of D. Remark that $lc(D)$ is always even. For an unordered diagram D, we assume that D^i and $D^i \cup D^j$ denote subdiagrams of D with an order. We have the following relation of warp-linking degree and non-self crossing number.

Lemma 4.1. *We have*

$$ld(D) \leq \frac{lc(D)}{2}.$$

Further, the equality holds if and only if D is an equilibrial diagram.

Proof. Let $\mathbf{a}$ be a base point sequence of D, and $\tilde{\mathbf{a}}$ the base point sequence $\mathbf{a}$ with the order reversed. We call $\tilde{\mathbf{a}}$ the reverse of $\mathbf{a}$. Since we have that $ld(D_{\mathbf{a}}) + ld(D_{\tilde{\mathbf{a}}}) = lc(D)$, we have the inequality $ld(D) \leq lc(D)/2$. Let D be an equilibrial diagram. Then we have $ld(D_{\mathbf{a}}) = lc(D)/2$ for every base point sequence $\mathbf{a}$. Hence we have $ld(D) = lc(D)/2$. On the other hand, we consider the case the equality $2ld(D) = lc(D)$ holds. For an arbitrary base point sequence $\mathbf{a}$ of D and its reverse $\tilde{\mathbf{a}}$, we have

$$ld(D_{\mathbf{a}}) \geq ld(D) = lc(D) - ld(D) \geq lc(D) - ld(D_{\mathbf{a}}) = ld(D_{\tilde{\mathbf{a}}}) \geq ld(D).$$

Then we have $lc(D) - ld(D_{\mathbf{a}}) = ld(D)$. Hence we have $ld(D_{\mathbf{a}}) = ld(D)$ for every base point sequence $\mathbf{a}$. Let $\mathbf{a}' = (a_1, a_2, \ldots, a_{k-1}, a_{k+1}, a_k, a_{k+2}, \ldots, a_r)$ be the base point sequence which is obtained from $\mathbf{a} = (a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_r)$ by exchanging a_k and a_{k+1} $(k = 1, 2, \ldots, r-1)$. Then, the number of over-crossings of D^k is equal to the number of under-crossings of D^k in the subdiagram $D^k \cup D^{k+1}$ of $D_{\mathbf{a}}$ because we have $ld(D_{\mathbf{a}}) = ld(D_{\mathbf{a}'})$. This completes the proof. □

We next consider the value $d(D) + d(-D)$ for an oriented link diagram D and the inverse $-D$. We have the following proposition:

Proposition 4.2. *Let D be an oriented link diagram. The value $d(D) + d(-D)$ does not depend on the orientation of D.*

Proof. Let D' be D with the same order and another orientation. Since we have $d(D^{i'}) = d(D^i)$ or $d(D^{i'}) = d(-D^i)$, we have $d(D^{i'}) + d(-D^{i'}) = d(D^i) + d(-D^i)$ for each D^i and $D^{i'}$. Then we have

$$\begin{aligned}
d(D') + d(-D') &= \sum_{i=1}^{r} d(D^{i'}) + ld(D') + \sum_{i=1}^{r} d(-D^{i'}) + ld(-D') \\
&= \sum_{i=1}^{r} \{d(D^{i'}) + d(-D^{i'})\} + 2ld(D') \\
&= \sum_{i=1}^{r} \{d(D^{i}) + d(-D^{i})\} + 2ld(D) \\
&= \sum_{i=1}^{r} d(D^{i}) + ld(D) + \sum_{i=1}^{r} d(-D^{i}) + ld(-D) \\
&= d(D) + d(-D).
\end{aligned}$$

□

A link diagram is a *self-crossing diagram* if every component of D has at least one self-crossing. In other words, a diagram D of an r-component link L is a self-crossing diagram if $sr(D) = r$. We have the following lemma:

Lemma 4.3. *Let D be a self-crossing diagram of an r-component link. Then we have*

$$d(D) + d(-D) + r \leq c(D).$$

Further, the equality holds if and only if D is equilibrial and every component D^i of D is alternating.

Proof. We have

$$\begin{aligned}
d(D)+d(-D)+r &= \sum_{i=1}^{r} d(D^i) + ld(D) + \sum_{i=1}^{r} d(-D^i) + ld(-D) + r \\
&= \sum_{i=1}^{r} \{d(D^i) + d(-D^i) + 1\} + 2ld(D) \\
&\leq \sum_{i=1}^{r} c(D^i) + 2ld(D) \\
&\leq \sum_{i=1}^{r} c(D^i) + lc(D) \\
&= c(D),
\end{aligned}$$

where the first inequality is obtained from Theorem 1.1, and the second inequality is obtained from Lemma 4.1. Hence we have the inequality. The equality holds if and only if D is equilibrial and every component D^i of D is alternating which is obtained by Theorem 1.1 and Lemma 4.1.

□

We give an example of Lemma 4.3.

Example 4.4. In Figure 12, there are three diagrams with 12 crossings. The diagram D is a diagram such that any component is alternating and has 3 over-non-self crossings and 3 under-non-self crossings. Then we have $d(D) + d(-D) + r = 12 = c(D)$. The diagram D' is a diagram which has a non-alternating component diagram. Then we have $d(D') + d(-D') + r = 10 < c(D')$. The diagram D'' is a diagram such that a component has 2 over-non-self crossings and 4 under-non-self crossings. Then we have $d(D'') + d(-D'') + r = 10 < c(D'')$.

Lemma 4.3 is only for self-crossing link diagrams. We prove Theorem 1.2 which is for every link diagram.

Proof of Theorem 1.2. For every component D^i such that D^i has no self-crossings, we apply a Reidemeister move of type I as shown in Figure 13. Then we obtain the diagram $D^{i'}$ from D^i, and $D^{i'}$ satisfies $d(D^{i'}) =$

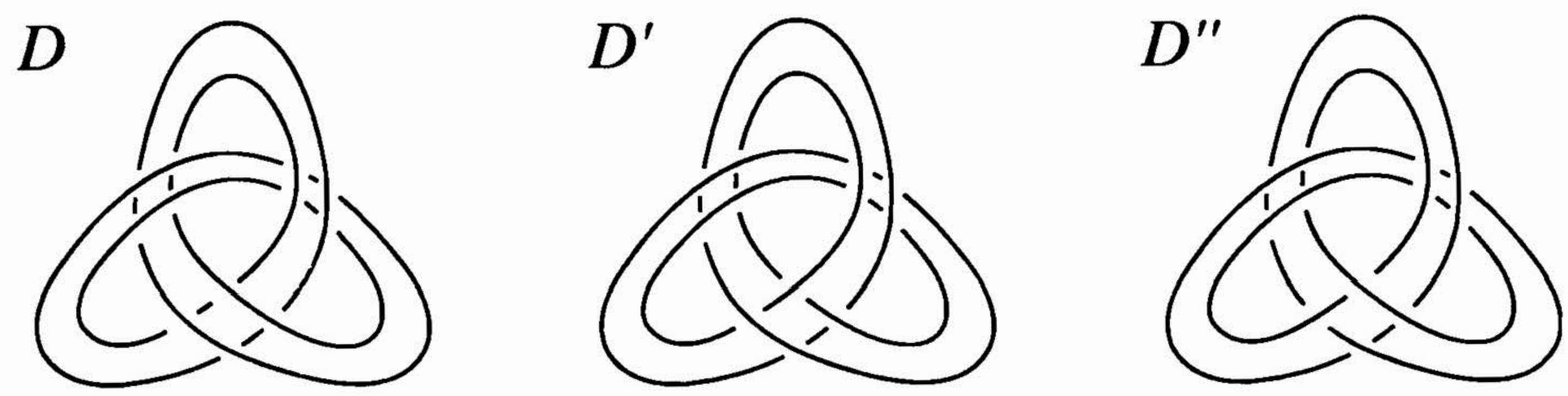


FIGURE 12

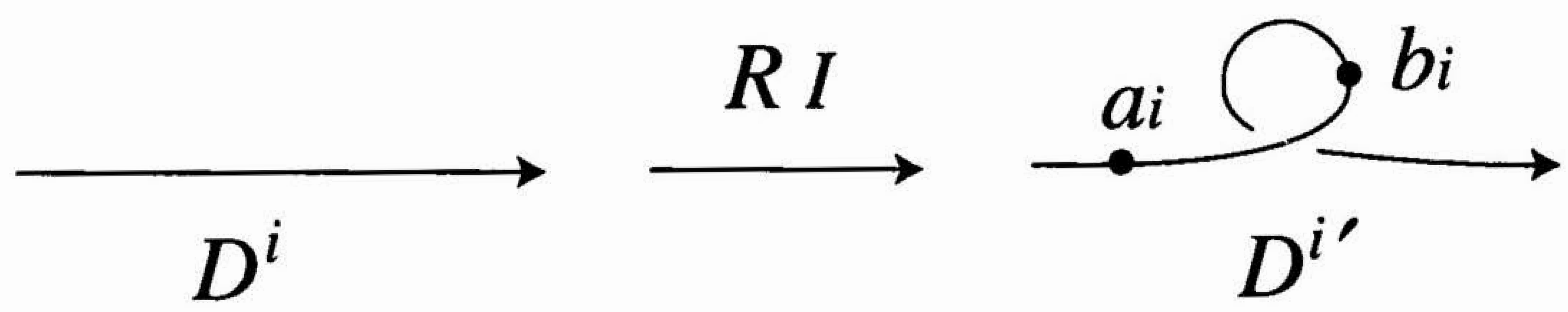


FIGURE 13

$d(-D^{i'}) = 0 = d(D^i) = d(-D^i)$ and $c(D^{i'}) = 1 = c(D^i) + 1$. For example the base points a_i, b_i in Figure 13 satisfy $d(D^i_{a_i}) = d(D^i) = 0$, $d(-D^i_{b_i}) = d(-D^i) = 0$. We remark that every D^i and $D^{i'}$ are alternating. We denote by D' the diagram obtained from D by this procedure. Since every component has at least one self-crossing, we apply Lemma 4.3 to D'. Then we have

$$d(D') + d(-D') + r \leq c(D').$$

And we obtain

$$d(D) + d(-D) + r \leq c(D) + (r - sr(D)).$$

Hence we have

$$d(D) + d(-D) + sr(D) \leq c(D).$$

The equality holds if and only if D is equilibrial and every component D^i of D is alternating. $\square$

5. Warp-linking degree and linking number

In this section, we consider the relation of the warp-linking degree and the linking number. For a crossing point p of an oriented diagram, $\varepsilon(p)$ denotes the sign of p, namely $\varepsilon(p) = +1$ if p is a positive crossing, and $\varepsilon(p) = -1$ if p is a negative crossing as shown in Figure 14. For an oriented

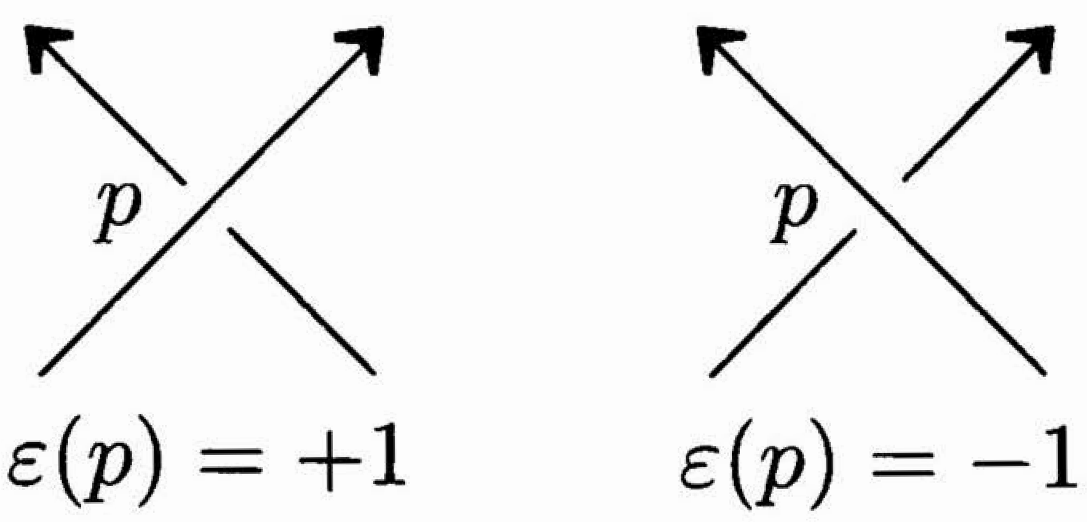


Figure 14

subdiagram $D^i \cup D^j$ $(i \neq j)$, the *linking number* of D^i with D^j is defined to be

$$\mathrm{Link}(D^i, D^j) = \frac{1}{2} \sum_{p \in D^i \cap D^j} \varepsilon(p).$$

The linking number of D^i with D^j is independent of the diagram (cf. [3], [10]). We have a relation of the warp-linking degree and the linking number of a link diagram in the following proposition:

Proposition 5.1. *For a link diagram D, we have the following (i) and (i i).*

(i): *We have*

$$\sum_{i<j} |\mathrm{Link}(D^i, D^j)| \leq ld(D).$$

Further, the equality holds if and only if non-self under-crossings of D^i in $D^i \cup D^j$ are all positive or all negative with an orientation for every subdiagram $D^i \cup D^j$ $(i < j)$ with an order which realizes the warp-linking degree.

(ii): *We have*

(6) $$\sum_{i<j} |\mathrm{Link}(D^i, D^j)| \equiv ld(D) \pmod 2.$$

Proof. **(i):** For a subdiagram $D^i \cup D^j$ $(i < j)$ with $d(D^i, D^j) = m$, we show that

$$|\mathrm{Link}(D^i, D^j)| \leq d(D^i, D^j).$$

Let $p_1, p_2, \ldots, p_m$ be the warping crossing points between D^i and D^j, and $\varepsilon(p_1)$,$\varepsilon(p_2)$,$\ldots$,$\varepsilon(p_m)$ the signs of them. Since a stacked diagram is a diagram of a completely splittable link, we have

(7) $$\mathrm{Link}(D^i, D^j) - (\varepsilon(p_1) + \varepsilon(p_2) + \cdots + \varepsilon(p_m)) = 0$$

by applying crossing changes at $p_1, p_2, \ldots, p_m$ for $D^i \cup D^j$. Then we have

$$|\mathrm{Link}(D^i, D^j)| = |\varepsilon(p_1) + \varepsilon(p_2) + \cdots + \varepsilon(p_m)| \leq m = d(D^i, D^j).$$

Hence we obtain

$$\sum_{i<j} |\mathrm{Link}(D^i, D^j)| \leq ld(D).$$

The equality holds if and only if non-self under-crossings of D^i in $D^i \cup D^j$ are all positive or all negative with an orientation for every subdiagram $D^i \cup D^j$ $(i < j)$ with an order which realizes the warp-linking degree.

(ii): By the above equality (7), we observe that $\mathrm{Link}(D^i, D^j) = \varepsilon(p_1) + \varepsilon(p_2) + \cdots + \varepsilon(p_m) = \varepsilon(q_1) + \varepsilon(q_2) + \cdots + \varepsilon(q_n)$, where p_k (resp. q_k) is a non-self under-crossing (resp. over-crossing) of D^i in $D^i \cup D^j$, $ld(D^i \cup D^j) = m$ and $lc(D^i \cup D^j) = m + n$. A similar fact is also mentioned in [21]. We have

$$\begin{aligned}\mathrm{Link}(D^i, D^j) &= \varepsilon(p_1) + \varepsilon(p_2) + \cdots + \varepsilon(p_m)\\ &\equiv m \pmod 2\\ &= d(D^i, D^j).\end{aligned}$$

Hence we have the modular equality

$$\sum_{i<j} |\text{Link}(D^i, D^j)| \equiv ld(D) \pmod 2.$$

□

Example 5.2. In Figure 15, D has $(0,2,3)$, E has $(0,2,2)$, and F has $(4,4,4)$, where (l,m,n) of D denotes that $\sum_{i<j} |\text{Link}(D^i, D^j)| = l$, $ld(D) = m$, and $lc(D)/2 = n$.

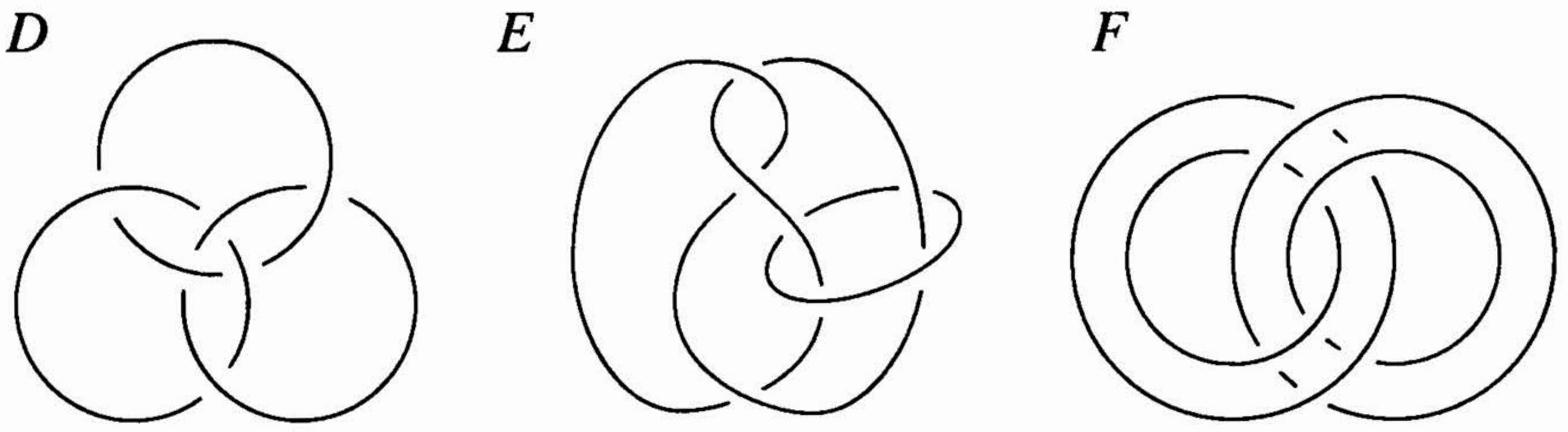


FIGURE 15

The *total linking number* of an oriented link L is defined to be $\sum_{i<j} \text{Link}(D^i, D^j)$ with a diagram and an order. We have the following corollary:

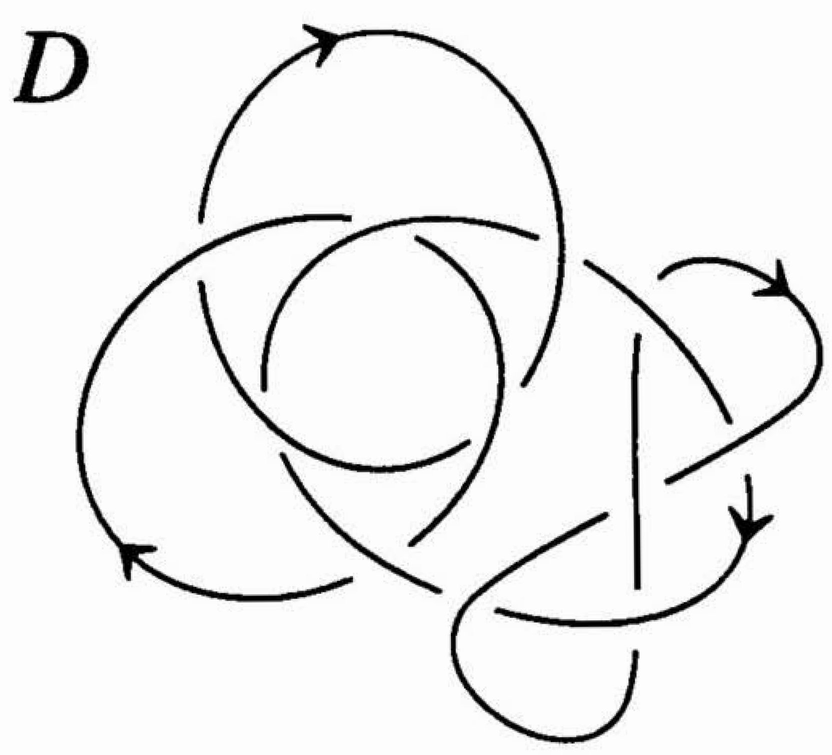


FIGURE 16

Corollary 5.3. *We have*

$$\sum_{i<j} \mathrm{Link}(D^i, D^j) = \sum_{k=1}^{r} \{\varepsilon(p_k) | p_k : \textit{a non-self warping crossing point of } D_{\mathbf{a}}\},$$

where $\mathbf{a}$ *is a base point sequence of* D.

Corollary 5.3 is useful in calculating the total linking number of a diagram. For example in Figure 16, the diagram D with 4 components and 11 crossing points has $ld(D) = 4$. We have that the total linking number of D is 0 by summing the signs of only 4 crossing points.

6. To a link invariant

In this section, we consider the minimal $d(D)+d(-D)$, $d(D)+d(-D)+sr(D)$ and $ld(D)$ for minimal crossing diagrams D for an oriented link L. We define $e(L)$ as follows:

$$e(L) = \min\{d(D) + d(-D) | D : \text{ a diagram of } L \text{ with } c(D) = c(L)\},$$

where $c(L)$ denotes the crossing number of L. In the case where L is a non-trivial knot K, we have the following theorem:

Theorem 6.1. *Let K be a non-trivial knot. We have the following (1) and (2).*

(1): *We have*

$$e(K) + 1 \leq c(K).$$

Further, the equality holds if and only if K is a prime alternating knot.

(2): *For any positive integer n, there exists a prime knot K such that*

$$c(K) - e(K) = n.$$

Proof. First, we show the equality of (1). By Theorem 1.1, we have the equality

$$d(D) + d(-D) + 1 = c(D) \tag{8}$$

if and only if D is an alternating diagram. If K is a prime alternating knot, then minimal crossing diagrams of K are alternating [17]. Hence we have the equality by considering the minimum of the equality (8). On the other hand, if K is a non-prime alternating knot, then there is a minimal crossing non-alternating diagram so that $e(K) + 1 < c(K)$ [16]. Next, we look to $c(T(p,q)) - e(T(p,q))$ for the (p,q)-torus knot $T(p,q)$ $(0 < p < q)$ to prove (2). Schubert mentioned in [22] (cf.[18]) that

$$c(T(p,q)) = (p-1)q.$$

Ozawa showed in [20] that the ascending number of $T(p,q)$, which is equal to the minimal warping degree for all diagrams of $T(p,q)$ and all orientations, is $(p-1)(q-1)/2$. Then we have

$$e(T(p,q)) = \frac{(p-1)(q-1)}{2} + \frac{(p-1)(q-1)}{2} = (p-1)(q-1)$$

because $d(D(p,q)) = d(-D(p,q))$. Hence we have

$$c(T(p,q)) - e(T(p,q)) = (p-1)q - (p-1)(q-1) = p-1.$$

$\square$

We next define $c^*(L)$ and $e^*(L)$ as follows:

$$c^*(L) = \min\{c(D) | D: \text{ a self-crossing diagram of } L\},$$

$$e^*(L) = \min\{d(D)+d(-D) | D: \text{ a self-crossing diagram of } L \text{ with } c(D) = c^*(L)\}.$$

As a generalization of Theorem 6.1, we have the following theorem:

Theorem 6.2. *For an r-component link L, we have*

$$e^*(L) + r \leq c^*(L).$$

Further, the equality holds if and only if every self-crossing diagram D of L with $c(D) = c^*(L)$ *is equilibrial and every component D^i of D is alternating.*

Proof. Let D be a self-crossing diagram of L with $c(D) = c^*(D)$. We assume that D satisfies the equality $d(D) + d(-D) = e^*(L)$. Then we have

$$\begin{aligned}
e^*(L) + r &= d(D) + d(-D) + r \\
&= \sum_{i=1}^{r} d(D^i) + ld(D) + \sum_{i=1}^{r} d(-D^i) + ld(-D) + r \\
&= \sum_{i=1}^{r} \{d(D^i) + d(-D^i) + 1\} + 2ld(D) \\
&\leq \sum_{i=1}^{r} c(D^i) + 2ld(D) \\
&\leq \sum_{i=1}^{r} c(D^i) + lc(D) \\
&= c(D) = c^*(L),
\end{aligned}$$

where the first inequality is obtained by Theorem 1.1, and the second inequality is obtained by Lemma 4.1. If D has a non-alternating component

D^i, or D has a diagram $D^i \cup D^j$ such that the number of over-crossings of D^i is not equal to the number of under-crossings of D^i, then we have $e^*(L) + r < c^*(L)$. On the other hand, the equality holds if D is equilibrial and every component D^i of D is alternating. $\square$

We have the following example:

Example 6.3. For non-trivial prime alternating knots $L^1, L^2, \ldots, L^r$ $(r \geq 2)$, we have a non-splittable link L by performing n_i-full twists for every L^i and L^{i+1} $(i = 1, 2, \ldots, r)$ with $L^{r+1} = L^1$ as shown in Figure 17, where we assume that n_1 and n_r have the same sign.

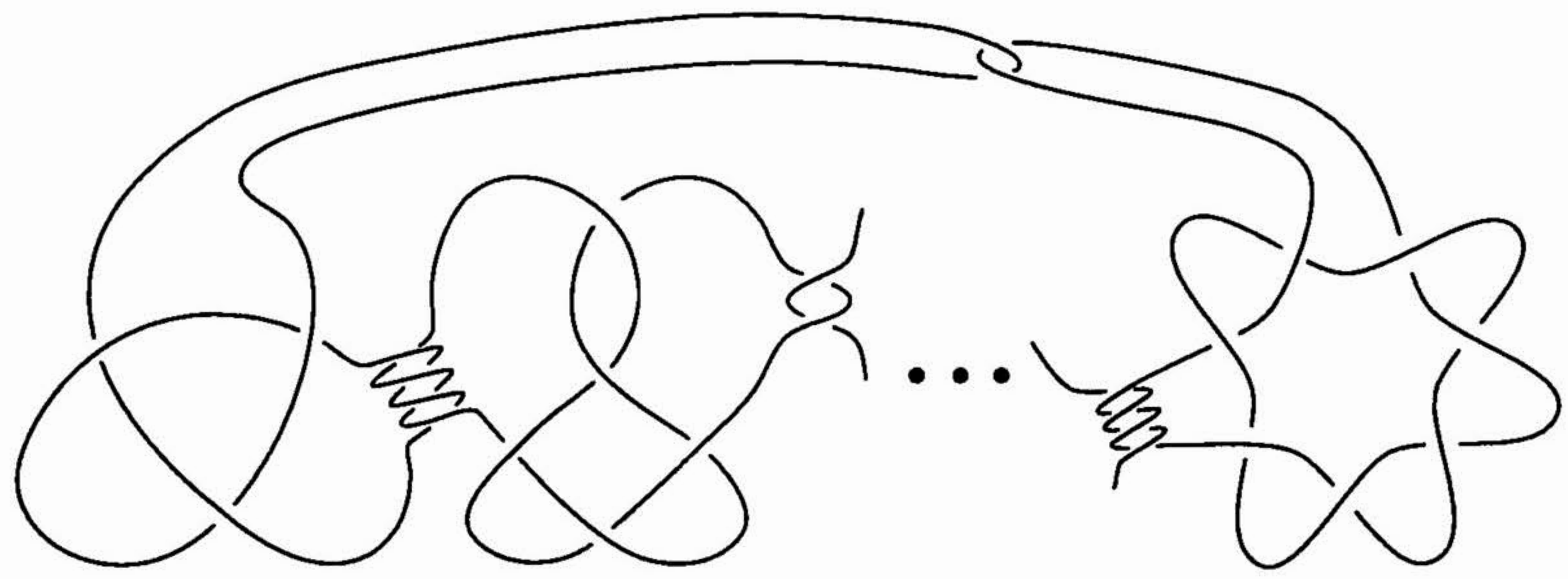


FIGURE 17

Note that we do not change the type of knot components L^i. Let D be a diagram of L with $c(D) = c(L)$. Then we notice that D is a self-crossing diagram with $c(D) = c^*(L)$. We also notice that D is equilibrial and every component D^i of D is alternating because $lc(D^i \cup D^j) = 2|n_i|$ and $\text{Link}(D^i, D^j) = n_i$, and $lc(D^1 \cup D^r) = 2|n_1 + n_r|$ and $\text{Link}(D^1, D^r) = n_1 + n_r$ in the case where $r = 2$. Hence we have $e^*(L) + r = c^*(L)$ in this case.

We have the following corollary:

Corollary 6.4. *Let L be an r-component link whose all components are non-trivial. Then we have*

$$e(L) + r \leq c(L).$$

Further, the equality holds if and only if every diagram D of L with $c(D) = c(L)$ is equilibrial and every component D^i of D is alternating.

Proof. Since every diagram D of L is a self-crossing diagram, we have $e(L) = e^*(L)$ and $c(L) = c^*(L)$. □

We also consider the minimal $d(D)+d(-D)+sr(D)$ and the minimal $sr(D)$ for diagrams D of L in the following formulae:

$$f(L) = \min\{d(D) + d(-D) + sr(D) | D : \text{ a diagram of } L\},$$

$$sr(L) = \min\{sr(D) | D : \text{ a diagram of } L\}.$$

Note that the value $f(L)$ and $sr(L)$ also do not depend on the orientation of L. Jin and Lee mentioned in [6] that every link has a diagram which restricts to a minimal crossing diagram for each component. Then we have the following proposition:

Proposition 6.5. *The value $sr(L)$ is equal to the number of non-trivial knot components of L.*

The following corollary is directly obtained from Theorem 1.2.

Corollary 6.6. *We have*

$$f(L) \leq c(L).$$

Proof. For a diagram D with $c(D) = c(L)$, we have

$$f(L) \leq d(D) + d(-D) + sr(D) \leq c(D) = c(L),$$

where the second inequality is obtained by Theorem 1.2. □

We have the following question:

Question 6.7. When does the equality $f(L) = c(L)$ hold?

Example 6.8. In Figure 18, there are two link diagrams D and E. We assume that D (resp. E) is a diagram of a link L (resp. M). We have $f(L) = c(L) = 5$ because we have $d(D) + d(-D) + sr(D) = 2 + 2 + 1$ and we know $d(D^i) \geq u(3_1) = 1$, $ld(D) \geq 1$, and $sr(D) \geq sr(L) = 1$, where D^i is any diagram of 3_1. On the other hand, we have that $f(M) < c(M)$ because $f(M) \leq d(E) + d(-E) + sr(E) = 3 + 3 + 1 = 7 < 10 = c(M)$.

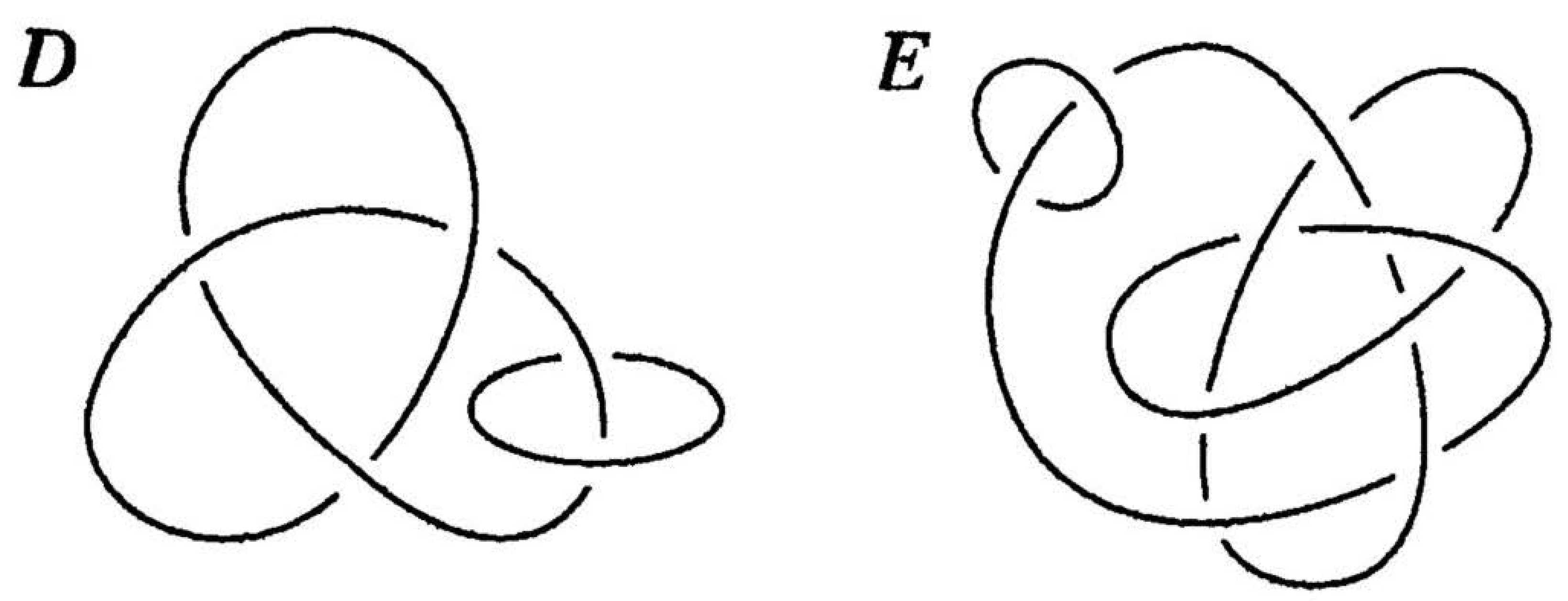


FIGURE 18

The *non-self crossing number* $lc(L)$ of a link L is the minimal non-self crossing number $lc(D)$ for all diagrams D of L. With respect to the warp-linking degree, we define $le(L)$ as follows:

$$le(L) = \min\{ld(D) | D : \text{a diagram of } L \text{ with } lc(D) = lc(L)\}.$$

We remark that $\text{lsplit}(L) \leq le(L)$, where $\text{lsplit}(L)$ is the linking complete splitting number of L which is defined in Section 8. A link L is *equilibrial* if all diagrams D of L with $lc(D) = lc(L)$ are equilibrial. We have the following theorem:

Theorem 6.9. *We have*

$$le(L) \leq \frac{lc(L)}{2}.$$

Further, the equality holds if and only if L is equilibrial.

Proof. If L is equilibrial, i.e., every D with $lc(D) = lc(L)$ is equilibrial, then every D satisfies that $ld(D) = lc(D)/2$ by Lemma 4.1. Then we have $le(L) = ld(D) = lc(D)/2 = lc(L)/2$. On the other hand, if the equality $le(L) = lc(L)/2$ holds, then every diagram D with $lc(D) = lc(L)$

satisfies $le(L) = lc(D)/2 = lc(L)/2$. Since $le(L) \leq ld(D) \leq lc(D)/2$, we have $ld(D) = lc(D)/2$ which means that D is equilibrial. Hence L is equilibrial. □

In the following example, we discuss $le(L)$ of the Whitehead link L.

Example 6.10. Let L be the Whitehead link. The three diagrams in Figure 19 represent the Whitehead link respectively. Since $L = L^1 \cup L^2$ is non-trivial and has $\text{Link}(L^1, L^2) = 0$ and $lc(L) \leq 4$, L has $lc(L) = 4$. Let $D = D^1 \cup D^2$ be a minimal non-self crossing diagram of L. We note that D is not a stacked diagram because L is not splittable. Therefore D^1 has even but non-zero number of non-self over-crossing points, that is, D^1 has two non-self over-crossing points and two non-self under-crossing points. Hence D is equilibrial, and L is equilibrial. By Theorem 6.9, we have $le(L) = lc(L)/2 = 2$.

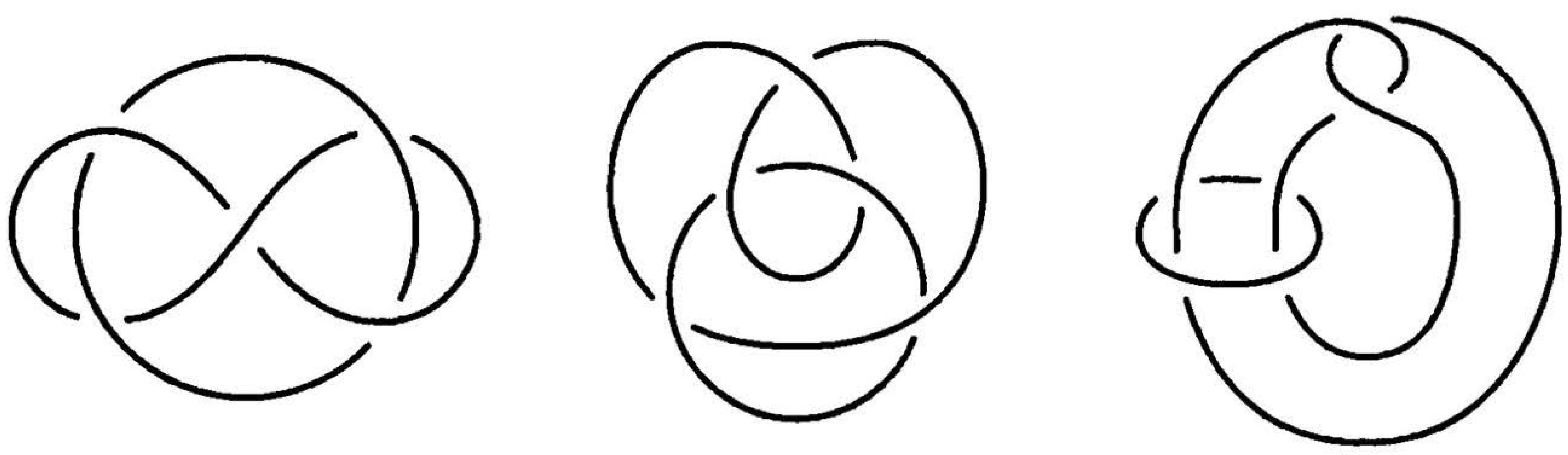


FIGURE 19

7. Relations of warping degree, unlinking number and crossing Number

In this section, we study several relations of the warping degree, the unknotting number or unlinking number, and the crossing number. Let $|D|$ be D with orientation forgotten. We define the minimal warping degree of D for all orientations as follows:

$$d(|D|) := \min\{d(D) | D : |D| \text{ with an orientation}\}.$$

Note that the minimal $d(|D|)$ for all diagrams D of L is equal to the ascending number $a(L)$ [20]:

$$a(L) = \min\{d(|D|) | D : \text{ a diagram of } L\}.$$

Let E be a knot diagram, and D a diagram of an r-component link. We review the relation of the unknotting number $u(E)$ (resp. the unlinking number $u(D)$) and the crossing number $c(E)$ (resp. $c(D)$) of E (resp. D). The following inequalities are well-known [19]:

$$u(E) \leq \frac{c(E) - 1}{2}, \tag{9}$$

$$u(D) \leq \frac{c(D)}{2}. \tag{10}$$

Moreover, Taniyama mentioned the following conditions [28]:

The necessary condition for the equality of (9) is that E is a reduced alternating diagram of some $(2, p)$-torus knot, or E is a diagram with $c(E) = 1$. The necessary condition for the equality of (10) is that every D^i is a simple closed curve on $\mathbb{S}^2$ and every subdiagram $D^i \cup D^j$ is an alternating diagram.

Hanaki and Kanadome characterized the link diagrams D which satisfy $u(D) = (c(D) - 1)/2$ as follows [4]:

Let $D = D^1 \cup D^2 \cup \cdots \cup D^r$ be a diagram of an r-component link. Then we have

$$u(D) = \frac{c(D) - 1}{2}$$

if and only if exactly one of $D^1, D^2, \ldots, D^r$ is a reduced alternating diagram of a $(2,p)$-torus knot, the other components are simple closed curves on $\mathbb{S}^2$, and the non-self crossings of the subdiagram $D^i \cup D^j$ are all positive, all negative, or empty for each $i \neq j$. In addition, they showed that any minimal crossing diagram D of a link L with $u(L) = (c(L)-1)/2$ satisfies $u(D) = (c(D)-1)/2$.

Abe, Hanaki and Higa study the knot diagrams D which satisfy

$$u(D) = \frac{c(D)-2}{2}.$$

Let D be a knot diagram with $u(D) = (c(D)-2)/2$. They showed in [1] that for any crossing point p of D, one of the components of D_p is a reduced alternating diagram of a $(2,p)$-torus knot and the other component of D_p has no self-crossings, where D_p is the diagram obtained from D by smoothing at p. In addition, they showed that any minimal crossing diagram D of a knot K with $u(K) = (c(K)-2)/2$ satisfies the above condition.

By adding to (9), we have the following corollary:

Corollary 7.1. *For a knot diagram E, we have*

$$u(E) \leq d(|E|) \leq \frac{c(E)-1}{2}.$$

Further, if we have

$$u(E) = d(|E|) = \frac{c(E)-1}{2},$$

then E is a reduced alternating diagram of some $(2,p)$-torus knot, or E is a diagram with $c(E) = 1$.

By adding to (10), we have the following corollary.

Corollary 7.2. **(i):** *For an r-component link diagram D, we have*

$$u(D) \leq d(|D|) \leq \frac{c(D)}{2}.$$

(ii): *We have*

$$u(D) \leq d(|D|) = \frac{c(D)}{2}$$

if and only if every D^i is a simple closed curve on $\mathbb{S}^2$ and the number of over-crossings of D^i is equal to the number of under-crossings of D^i in every subdiagram $D^i \cup D^j$ for each $i \neq j$.

(iii): *If we have*

$$u(D) = d(|D|) = \frac{c(D)}{2},$$

then every D^i is a simple closed curve on $\mathbb{S}^2$ and for each pair i, j, the subdiagram $D^i \cup D^j$ is an alternating diagram.

Proof. **(i):** The equality $u(D) \leq d(|D|)$ holds because $u(D) \leq d(D)$ holds for every oriented diagram. We show that $d(|D|) \leq c(D)/2$. Let D be an oriented diagram which satisfies

$$d(D) = \sum_{i=1}^{r} d(D^i) + ld(D) = d(|D|).$$

Then D also satisfies

(11) $$d(D^i) \leq \frac{c(D^i)}{2}$$

for every component D^i because of the orientation of D. By Lemma 4.1, we have

(12) $$ld(D) \leq \frac{lc(D)}{2}.$$

Then we have

$$\sum_{i=1}^{r} d(D^i) + ld(D) \leq \sum_{i=1}^{r} \frac{c(D^i)}{2} + \frac{lc(D)}{2}$$

by (11) and (12). Hence we obtain the inequality

$$d(|D|) \leq \frac{c(D)}{2}.$$

(ii): Suppose that the equality $d(|D|) = c(D)/2$ holds. Then the equalities

$$d(D^i) = \frac{c(D^i)}{2} \tag{13}$$

and

$$ld(D) = \frac{lc(D)}{2} \tag{14}$$

hold by (11) and (12), where D has an orientation such that $d(D) = d(|D|)$. The equality (13) is equivalent to that $c(D^i) = 0$ for every D^i. We prove this by an indirect proof. We assume that $c(D^i) > 0$ for a component D^i. In this case, we have the inequality

$$d(D^i) + d(-D^i) + 1 \leq c(D^i) \tag{15}$$

by Theorem 1.1 since D^i has a self-crossing. We also have

$$d(D^i) = d(-D^i) = \frac{c(D^i)}{2} \tag{16}$$

because $d(D^i) \leq d(-D^i)$ and (13). By substituting (16) for (15), we have

$$c(D^i) + 1 \leq c(D^i).$$

This implies that the assumption $c(D^i) > 0$ is contradiction. Therefore every D^i is a simple closed curve. The inequality (14) is equivalent to that the number of over-crossings of D^i is equal to the number of under-crossings of D^i in every subdiagram $D^i \cup D^j$ for each $i \neq j$ by Lemma 4.1. On the other hand, suppose that every D^i is a simple closed curve, and the number of over-crossings of D^i is equal to the number of under-crossings of D^i in every subdiagram $D^i \cup D^j$ for each $i \neq j$, then we have

$$d(|D|) = ld(D) = \frac{lc(D)}{2} = \frac{c(D)}{2}.$$

(iii): This holds by Corollary 7.2(i) and above Taniyama's condition.

□

Let K be a knot, and L an r-component link. Let $u(K)$ be the unknotting number of K, and $u(L)$ be the unlinking number of L. The following inequalities are also well-known [19]:

$$u(K) \leq \frac{c(K)-1}{2}, \tag{17}$$

$$u(L) \leq \frac{c(L)}{2}. \tag{18}$$

The following conditions are mentioned by Taniyama [28]:

The necessary condition for the equality of (17) is that K is a $(2,p)$-torus knot (p:odd,$\neq \pm 1$). The necessary condition for the equality of (18) is that L has a diagram D such that every D^i is a simple closed curve on $\mathbb{S}^2$ and every subdiagram $D^i \cup D^j$ is an alternating diagram.

By adding to (17) and Theorem 6.1, we have the following corollary:

Corollary 7.3. **(i):** *We have*

$$u(K) \leq \frac{e(K)}{2} \leq \frac{c(K)-1}{2}.$$

(ii): *We have*

$$u(K) \leq \frac{e(K)}{2} = \frac{c(K)-1}{2}$$

if and only if K is a prime alternating knot.

(iii): *If we have*

$$u(K) = \frac{e(K)}{2} = \frac{c(K)-1}{2},$$

then K is a $(2,p)$-torus knot (p:odd,$\neq \pm 1$).

By adding to (18), we have the following corollary:

Corollary 7.4. *For a diagram of an unoriented r-component link, we have*

$$u(L) \leq \frac{e(L)}{2} \leq \frac{c(L)}{2}.$$

Further, if the equality $u(L) = e(L)/2 = c(L)/2$ holds, then L has a diagram $D = D^1 \cup D^2 \cup \cdots \cup D^r$ such that every D^i is a simple closed curve on $\mathbb{S}^2$ and for each pair i, j, the subdiagram $D^i \cup D^j$ is an alternating diagram.

Proof. We prove the inequality $u(L) \leq e(L)/2$. Let D be a minimal crossing diagram of L which satisfies $e(L) = d(D) + d(-D)$. Then we obtain

$$e(L) = d(D) + d(-D) \geq 2u(D) \geq 2u(L).$$

The condition for the equality is due to above Taniyama's condition. $\square$

8. Relation of warping degree and splitting Number

In this section, we define the splitting number and consider relations of the warping degree and the complete splitting number. The *splitting number* (resp. *complete splitting number*) of D, denoted by $\mathrm{Split}(D)$ (resp. $\mathrm{split}(D)$), is the smallest number of crossing changes which are needed to obtain a diagram of a splittable (resp. completely splittable) link from D. The splitting number of a link which is the minimal $\mathrm{Split}(D)$ for all diagrams D is defined by Adams [2]. The *linking splitting number* (resp. *linking complete splitting number*) of D, denoted by $\mathrm{lSplit}(D)$ (resp. $\mathrm{lsplit}(D)$), is the smallest number of crossing changes at non-self crossings which are needed to obtain a diagram of a splittable (resp. completely splittable) link from D. In Appendix B, we also discuss the complete splitting number of a link. We have the following propositions:

Proposition 8.1. **(i):** *We have*

$$\mathrm{split}(D) \leq d(|D|).$$

(ii): *We have*

$$\mathrm{split}(D) \leq \mathrm{lsplit}(D) \leq ld(D) \leq \frac{lc(D)}{2} \leq \frac{c(D)}{2}.$$

We give examples of Proposition 8.1.

Example 8.2. The diagram D in Figure 20 has $\mathrm{split}(D) = 2 < d(|D|) = 3$. The diagram E in Figure 20 has $\mathrm{split}(E) = d(|E|) = 3$.

Example 8.3. The diagram D in Figure 21 has $\mathrm{split}(D) = 1 < \mathit{lsplit}(D) = 2$. The diagram E in Figure 21 has $\mathrm{split}(E) = \mathit{lsplit}(E) = 2$.

Example 8.4. The diagram D in Figure 22 has $\mathrm{lsplit}(D) = 3 < ld(D) = 5$. The diagram E in Figure 22 has $\mathrm{lsplit}(E) = ld(E) = 5$.

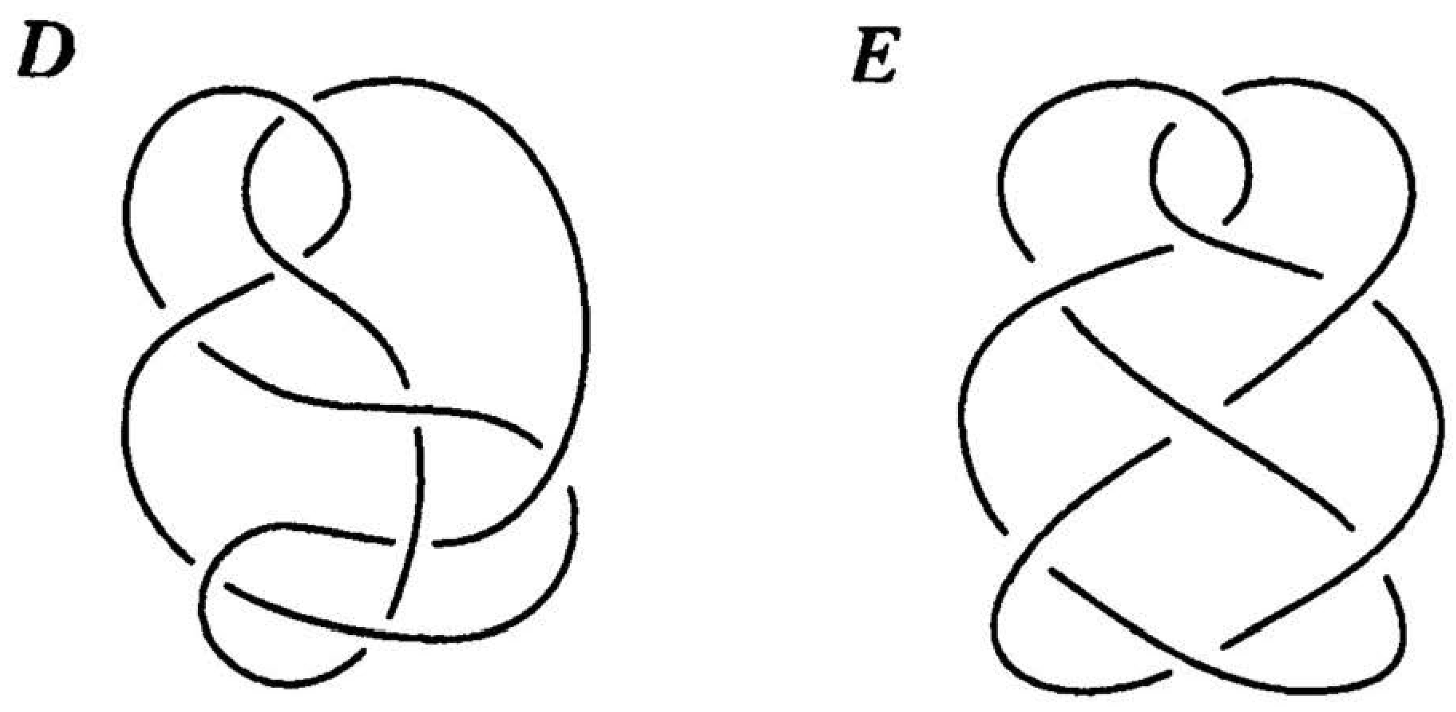


Figure 20

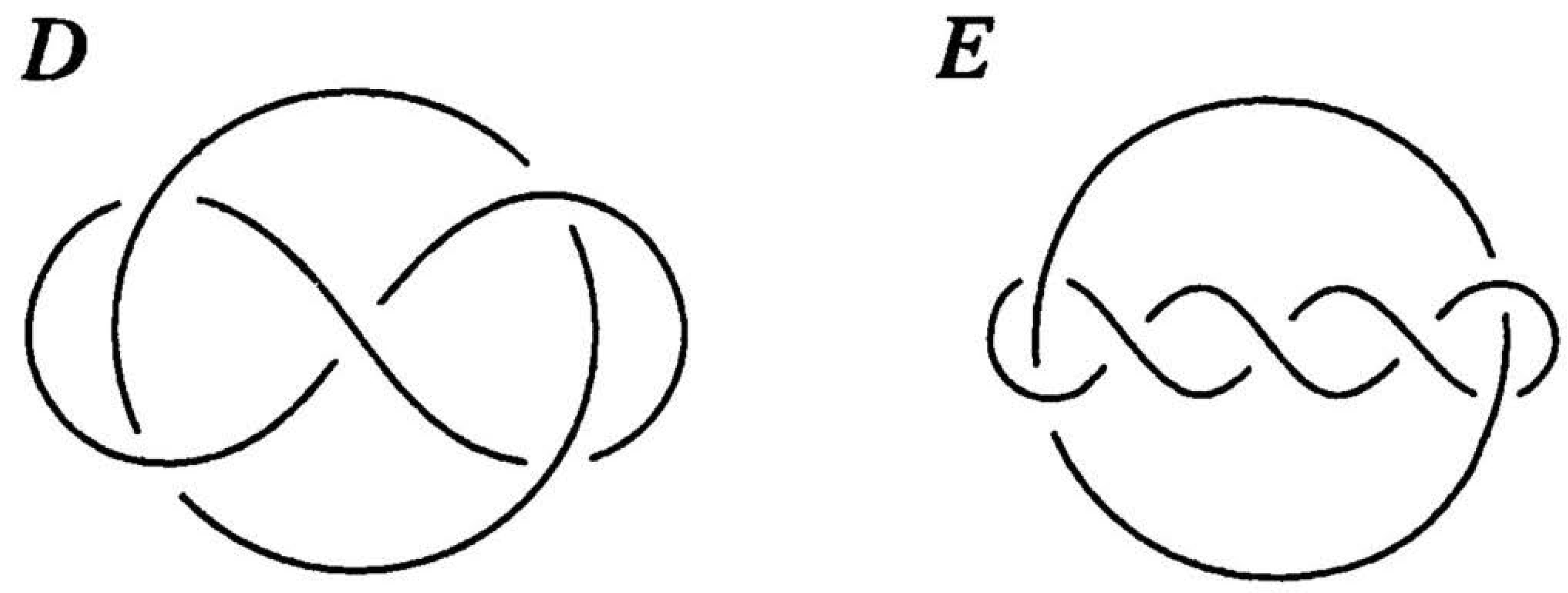


Figure 21

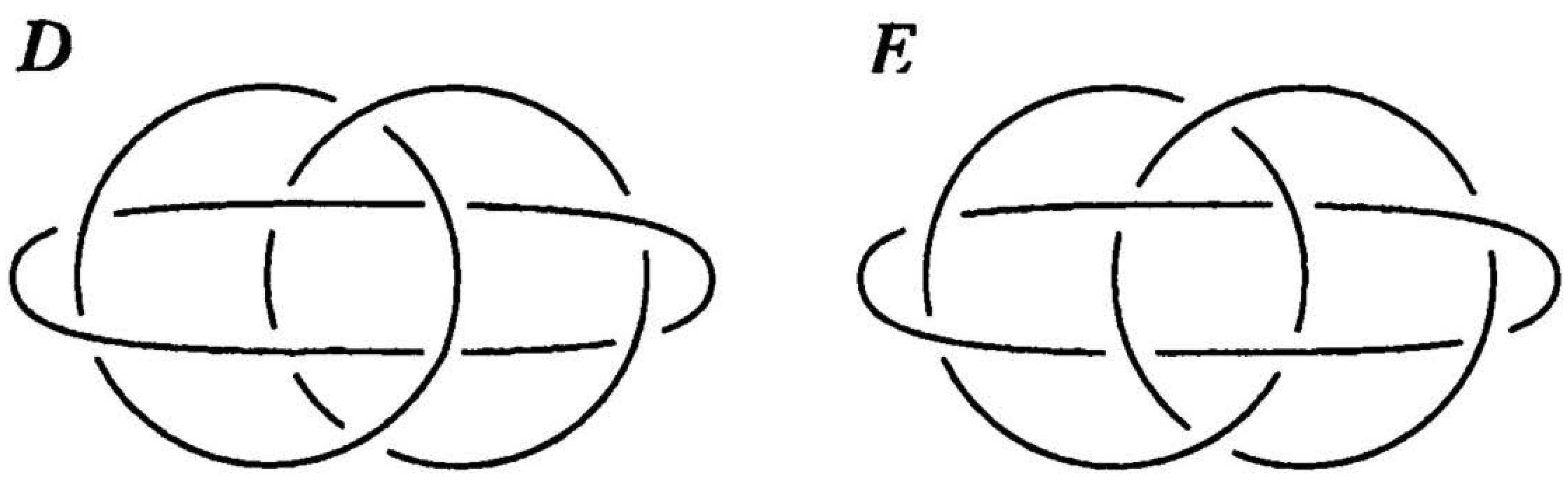


Figure 22

We raise the following question:

Question 8.5. When does the equality

$$\mathrm{split}(D) = d(|D|),$$
$$\mathrm{split}(D) = \mathrm{lsplit}(D)$$

or

$$lsplit(D) = ld(D)$$

hold?

APPENDIX A

In this appendix, we show methods for calculating the warping degrees and warp-linking degrees by using matrices. First, we give a method for calculating the warping degree $d(D)$ of an oriented knot diagram D. Let a be a base point of D. We can obtain the warping degree $d(D_a)$ of D_a by counting the warping crossing points easily. Let $[D_a]$ be the sequence of some "o" and "u", which is obtained as follows. When we go along the oriented diagram D from a, we write down "o" (resp. "u") if we reach a crossing point as an over-crossing (resp. under-crossing) in numerical order. We next perform a normalization to $[D_a]$ by deleting the subsequence "ou" repeatedly, to obtain the normalized sequence $\lfloor D_a \rfloor$. Then we have

$$d(D) = d(D_a) - \frac{1}{2}\sharp\lfloor D_a \rfloor,$$

where $\sharp\lfloor D_a \rfloor$ denotes the number of entries in $\lfloor D_a \rfloor$. Thus, we obtain the warping degree $d(D)$ of D. In the following example, we find the warping degree of a knot diagram by using the above algorithm.

Example 8.6. For the oriented knot diagram D and the base point a in Figure 23, we have $d(D_a) = 4$ and $[D_a] = [oouuouuouuouoouoou]$. By normalizing $[D_a]$, we obtain $\lfloor D_a \rfloor = [uuoo]$. Hence we find the warping degree of D as follows:

$$d(D) = 4 - \frac{1}{2} \times 4 = 2.$$

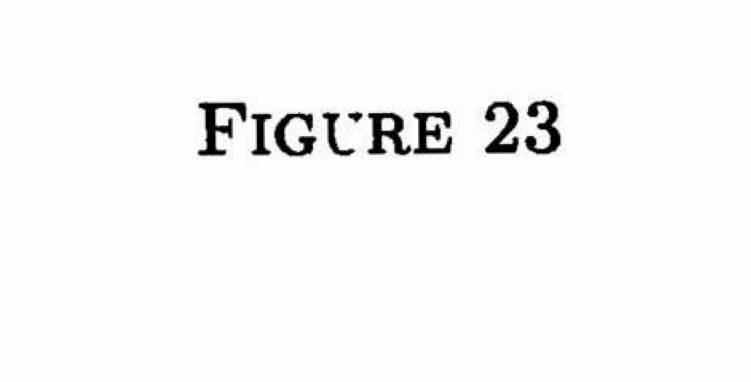


FIGURE 23

For some types of knot diagram, this algorithm is useful in formulating the warping degree or looking into its properties. We show a property of an oriented diagram of a pretzel knot of odd type in the following example:

Example 8.7. Let $D = P(\varepsilon_1 n_1, \varepsilon_2 n_2, \dots, \varepsilon_m n_m)$ be an oriented pretzel knot diagram of odd type ($\varepsilon_i \in +1, -1, n_i, m$: odd$>0$), where the orientation is given as shown in Figure 24. We take base points a, b in Figure 24. Then we have

$$d(D_a) = d(-D_b) = \frac{c(D)}{2} + \sum_i \frac{(-1)^{i+1}\varepsilon_i}{2}$$

and

$$\sharp\lfloor D_a \rfloor = \sharp\lfloor -D_b \rfloor.$$

Hence we have $d(D) = d(-D)$ in this case. In particular, if D is alternating i.e. $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = \pm 1$, then we have

$$d(D) = \frac{c(D)}{2} - \frac{1}{2}.$$

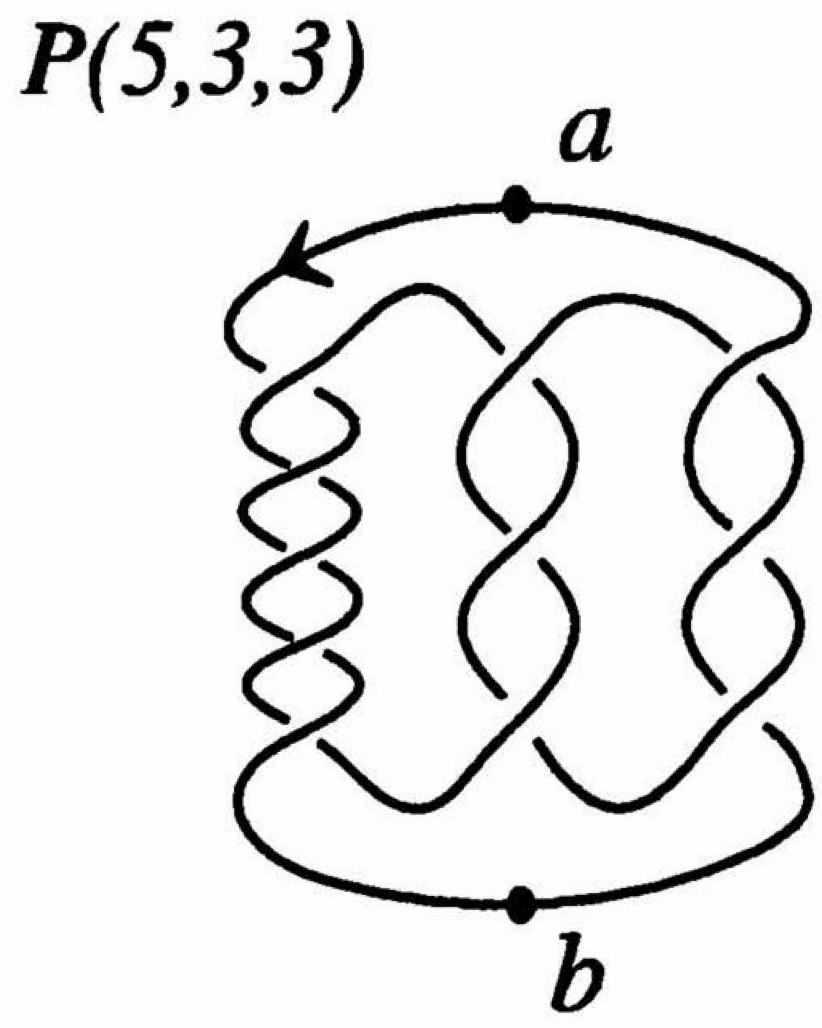


FIGURE 24

We next consider how to calculate the warp-linking degree $ld(D)$ by using matrices. For a link diagram D and a base point sequence $\mathbf{a}$ of D, we define an r-square matrix $M(D_{\mathbf{a}}) = (m_{ij})$ by the following rule:

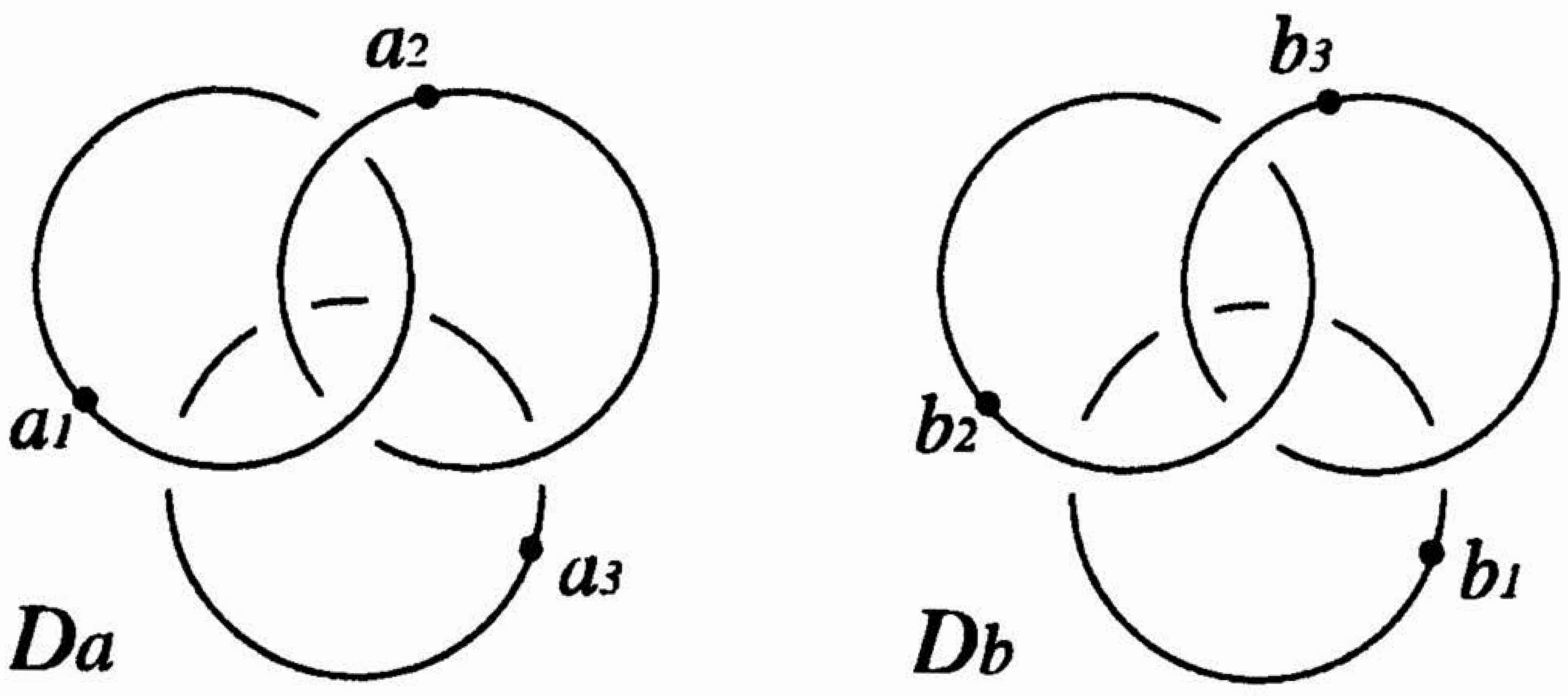


FIGURE 25

- For $i \neq j$, m_{ij} is the number of crossings of D^i and D^j which are under-crossings of D^i.
- For $i = j$, $m_{ij} = d(D^i)$.

We show an example.

Example 8.8. For $D_{\mathbf{a}}$ and $D_{\mathbf{b}}$ in Figure 25, we have

$$M(D_{\mathbf{a}}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}, \quad M(D_{\mathbf{b}}) = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We note that $ld(D_{\mathbf{a}})$ is obtained by summing the upper triangular entries of $M(D_{\mathbf{a}})$, that is

$$ld(D_{\mathbf{a}}) = \sum_{i<j} m_{ij},$$

and we notice that

$$d(D_{\mathbf{a}}) = \sum_{i \leq j} m_{ij},$$

where m_{ij} is an entry of $M(D_{\mathbf{a}})$ $(i, j = 1, 2, \ldots, r)$. For the base point sequence $\mathbf{a}' = (a_1, a_2, \ldots, a_{k+1}, a_k, \ldots, a_r)$ which is obtained from a base point sequence $\mathbf{a}$ by exchanging a_k and a_{k+1} $(k = 1, 2, \ldots, r-1)$, the matrix $M(D_{\mathbf{a}'})$ is obtained as follows:

$$M(D_{\mathbf{a}'}) = P_k M(D_{\mathbf{a}}) P_k^{-1},$$

where

$$P_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}; m_{ij} = \begin{cases} 1 \text{ for } (i,j) = (k,k+1), (k+1,k) \\ \quad \text{and}(i,j) = (i,i)(i \neq k, k+1), \\ 0 \text{ otherwise.} \end{cases}$$

With respect to the warp-linking degree, we have

$$ld(D_{\mathbf{a}'}) = ld(D_{\mathbf{a}}) - m_{kk+1} + m_{k+1k},$$

where m_{kk+1}, m_{k+1k} are entries of $M(D_{\mathbf{a}})$. To enumerate the permutation of the order of $\mathbf{a} = (a_1, a_2, \ldots, a_r)$, we consider a matrix $Q = P^{r-1}P^{r-2}\ldots P^2P^1$, where P^n denotes $P_nP_{n+1}\cdots P_{k_n}$ $(n \leq k_n \leq r-1)$ or the identity matrix E_r. Since Q depends on the choices of k_n $(n = 1, 2, \ldots, r-1)$, we also denote Q by $Q_{\mathbf{k}}$, where $\mathbf{k} = (k_1, k_2, \ldots, k_{r-1})$ $(n \leq k_n \leq r)$ and we regard $P^n = E_r$ in the case $k_n = r$. Hence we obtain the following formula:

$$ld(D) = \min_{\mathbf{k}}\{\sum_{i<j} m_{ij} | m_{ij} : \text{ an entry of } Q_{\mathbf{k}}M(D_{\mathbf{a}})Q_{\mathbf{k}}^{-1}\}.$$

Thus, we obtain the warping degree of an oriented link diagram by summing the warping degrees $d(D^i)$ $(i = 1, 2, \ldots, r)$ and the warp-linking degree $ld(D)$.

APPENDIX B

In this appendix, we discuss the complete splitting number of a lassoed link.

8.1. Introduction. The splittability of a link is one of the basic concepts in knot theory. For example, the splittability interacts with polynomial invariants: the Alexander polynomial and the Conway polynomial take zero for a splittable link. Jones polynomial and skein polynomial have specific formulae with respect to the split sum. Moreover, the splittabilities of links or spatial graphs are studied and applied to other subjects: chemistry, biology, psychology, etc. For example, Kawauchi proposed a model of prion proteins as a spatial graph [8], and Yoshida studied its splittability which concerns with the study of prion diseases: mad cow disease, scrapie, Creutzfeldt-Jakob disease, etc. [13]. Another example is about a model of human mind which is also proposed by Kawauchi [7], [8]; by considering one's mind as a knot and by considering a mind relation of n persons as an n-component link, the models "mind knots" and "mind links" are studied. The splittability of a link corresponds to the "self-releasability" of a mind link.

For a two-component link, Adams defined the splitting number which represents how distant the link is from a splittable link [2]. In this paper, we define for an n-component link L ($n = 2, 3, 4, \dots$) the complete splitting number $\mathrm{split}(L)$ which represents how distant the link is from a completely splittable link. The *unlinking number* $u(L)$ of a link L is the minimal number of crossing changes in any diagram of L which are needed to obtain the trivial link L. Since a trivial link is completely splittable, we have $\mathrm{split}(L) \leq u(L)$. Lassoing is a crossing-changing and loop-adding local move as shown in Figure 26 (we give the precise definitions of completely splittable, complete splitting number, and a lassoing in Subsection 8.2). For any r-component link $L = L^1 \cup L^2 \cup \dots \cup L^r$ ($r = 1, 2, 3, \dots$) with the Conway polynomial $\nabla(L) \neq 0$, there are ℓ-iterated lassoings from L to an algebraically completely splittable link L^* with $\nabla(L^*) \neq 0$ where $\ell = \sum_{i<j} |Link(L_i, L_j)|$ (we define an algebraically completely splittable link in Subsection 8.2). For any s-component link $K = K^1 \cup K^2 \cup \dots \cup K^s$ ($s \geq 1$) with $\nabla(K) \neq 0$, there are $(\ell + u)$-iterated lassoings from K to an algebraically completely splittable link L with trivial components such

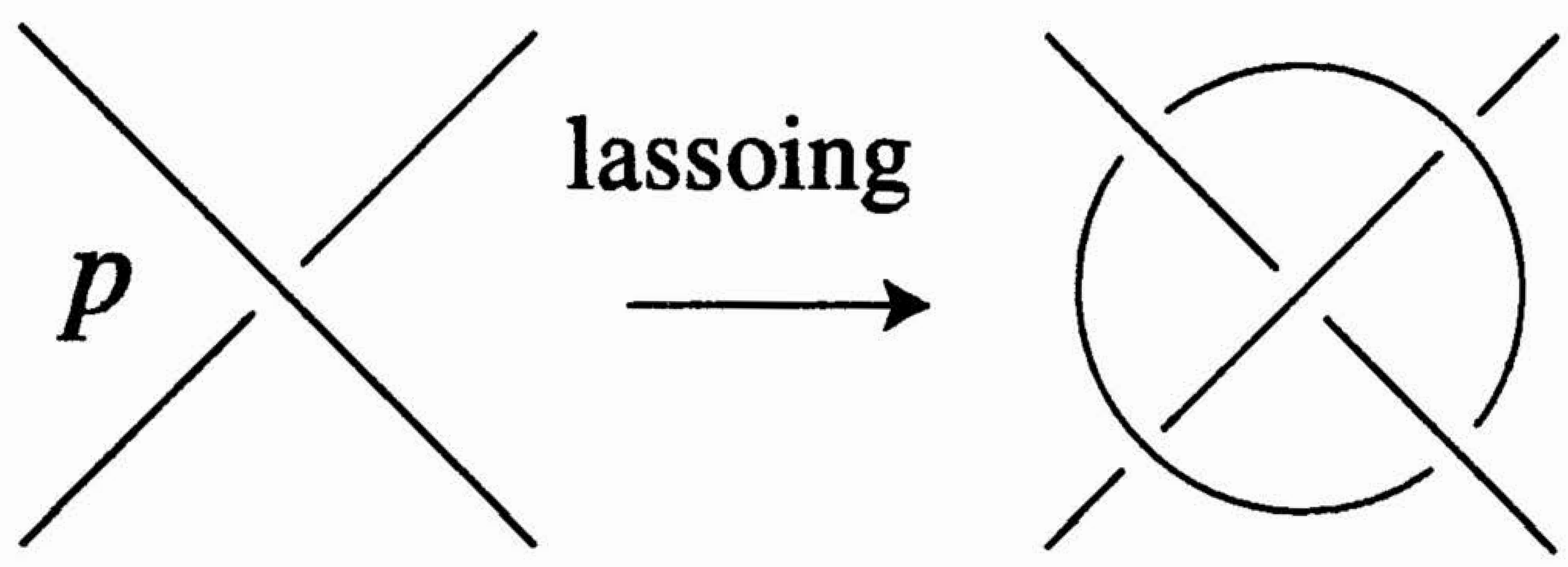


FIGURE 26

that $\nabla(L) \neq 0$ where $\ell = \sum_{i<j} |Link(K^i, K^j)|$ and $u = \sum_{i=1}^{s} u(K^i)$. In this appendix, we show the following theorem:

Theorem 8.9. *Any link L obtained from any s-component link $K = K^1 \cup K^2 \cup \ldots \cup K^s$ $(s = 1, 2, 3, \ldots)$ with $\nabla(K) \neq 0$ by r-iterated lassoings $(r = 0, 1, 2, \ldots)$ satisfies*

$$r + split(K) \geq split(L) \geq r + s - 1.$$

We have the following corollaries:

Corollary 8.10. *For any s-component link $K = K^1 \cup K^2 \cup \ldots \cup K^s$ $(s = 1, 2, 3, \ldots)$ with* $\mathrm{split}(K) = s - 1$*, and any integer $r \geq \ell + u$ where $\ell = \sum_{i<j} |Link(K^i, K^j)|$ and $u = \sum_{i=1}^{s} u(K^i)$, there are r-iterated lassoings from K to an algebraically completely splittable link L with trivial components such that* $\mathrm{split}(L) = r + s - 1$.

Corollary 8.11. *Let K be a knot. Let L be a link which is obtained from K by r-iterated lassoings $(r = 1, 2, 3, \ldots)$. Then L has* $\mathrm{split}(L) = r$.

We define a *component-lassoing* to be the lassoing at a self-crossing point of a diagram. We have the following corollary:

Corollary 8.12. *Every link L obtained from a knot K by r-iterated component-lassoings* ($r = 1, 2, 3, \dots$) *is an* $(r+1)-$*component algebraically completely splittable link with* $\mathrm{split}(L) = r$.

7_6^2

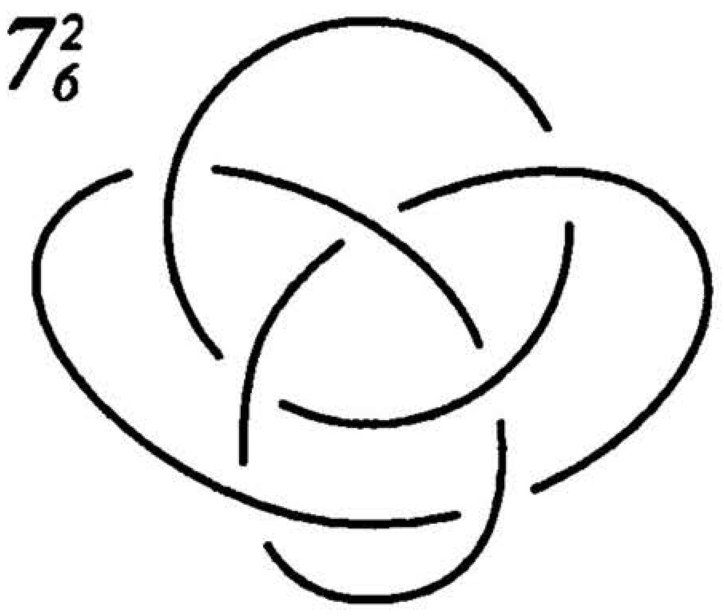


FIGURE 27

For example, the link 7_6^2 depicted in Figure 27 which is a link obtained from a trefoil knot by a single component-lassoing, has the linking number zero and $\mathrm{split}(7_6^2) = 1$. We also remark that $u(7_6^2) = 2$ (cf. [14]). Adams also showed in [2] that there is a two-component link, each component of which is trivial, but such that its splitting number is less than its unlinking number, like the link 7_6^2. We show in Subsection 8.5 that for any integer $r > 0$ and any knot K with Nakanishi's index $e(K) > 2r$, any link L obtained from K by r-iterated lassoings is a link such that $\mathrm{split}(L) < u(L)$, i.e., L is non-trivial by any r crossing changes.

8.2. **Complete splitting number.** Let $L = L_1 \cup L_2 \cup \cdots \cup L_r$ be a link consisting of sublinks L_i ($i = 1, 2, \dots, r$). A link L is *splittable into* $L_1, L_2, \dots, L_r$ if there exist mutually disjoint 3-balls B_i ($i = 1, 2, \dots, r$) in S^3 such that $L_i \subset B_i$. For example, the link M in Figure 28 is splittable into M_1 and M_2 whereas the link N is not splittable into N_1 and N_2. A link L is *splittable* if L is splittable into subdiagrams L_1 and L_2, where

$L = L_1 \cup L_2$, $L_1, L_2 \neq \phi$. For example, the link M in Figure 28 is a splittable link.

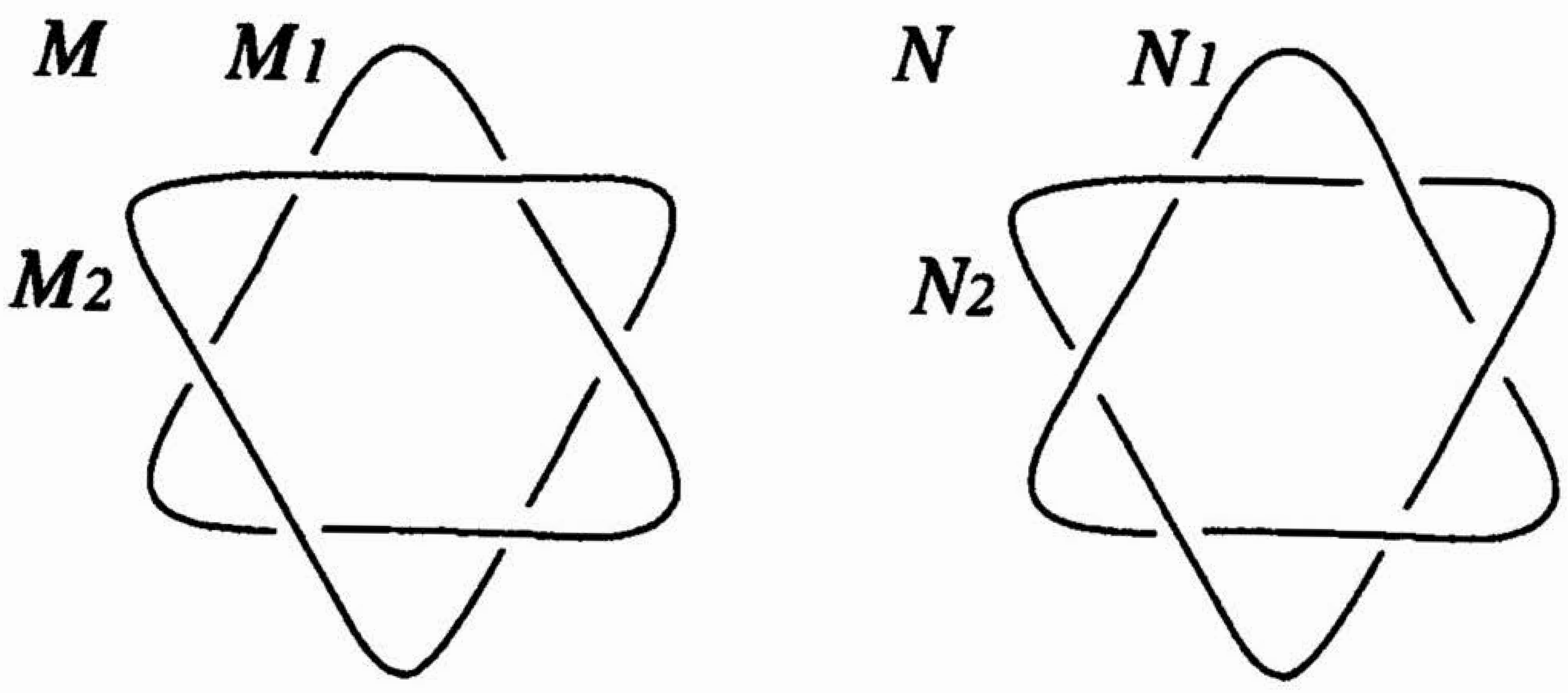


FIGURE 28

A link L is *completely splittable* if L is splittable into all the knot components of L. In particular, a knot is assumed as a non-splittable link but a completely splittable link. A link L is *algebraically completely splittable* if every two knot components K^i and K^j of L have the linking number $\mathrm{Link}(K^i, K^j) = 0$. For example, the link E in the left hand of Figure 29 is not completely splittable but algebraically it is completely splittable.

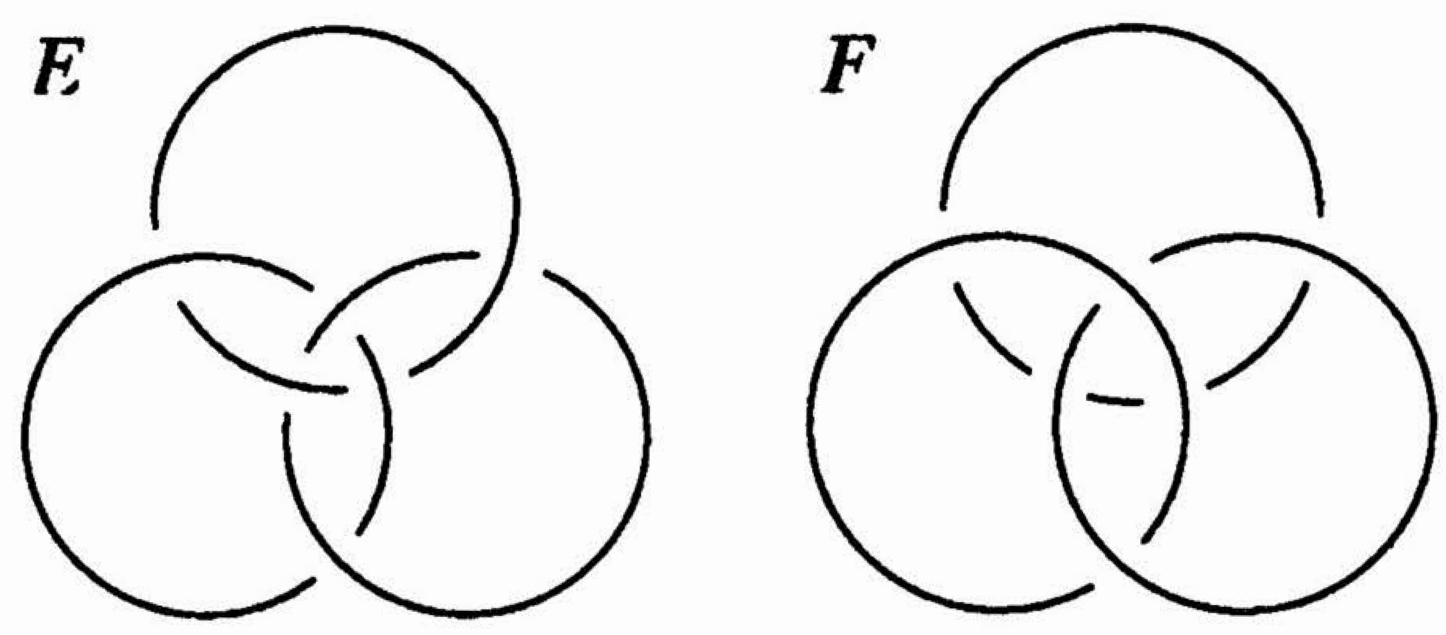


FIGURE 29

The *complete splitting number* split(D) of a link diagram D is the minimal number of crossing changes which are needed to obtain a diagram of a completely splittable link from D. For example, the link diagram F in the right hand in Figure 29 has split$(F) = 1$. As a relation to the warp-linking degree $ld(D)$ of D, we have split$(D) \leq ld(D)$. The *complete splitting number* split(L) of a link L is the minimal number of crossing changes in any diagram of the link which are needed to obtain a completely splittable link.

Let p be a crossing point of a link diagram D. We put a lasso around p, i.e., we apply a crossing change at p, and add a loop alternately around the crossing as shown in Figure 26. Then, we obtain another link diagram D'. The diagram D' is said to be obtained from D by *lassoing* at p. Let L' be the link which has the diagram D'. The link L' is said to be obtained from L by a *lassoing*. For example, we obtain the Borromean ring from the Hopf link by a lassoing (see Figure 30).

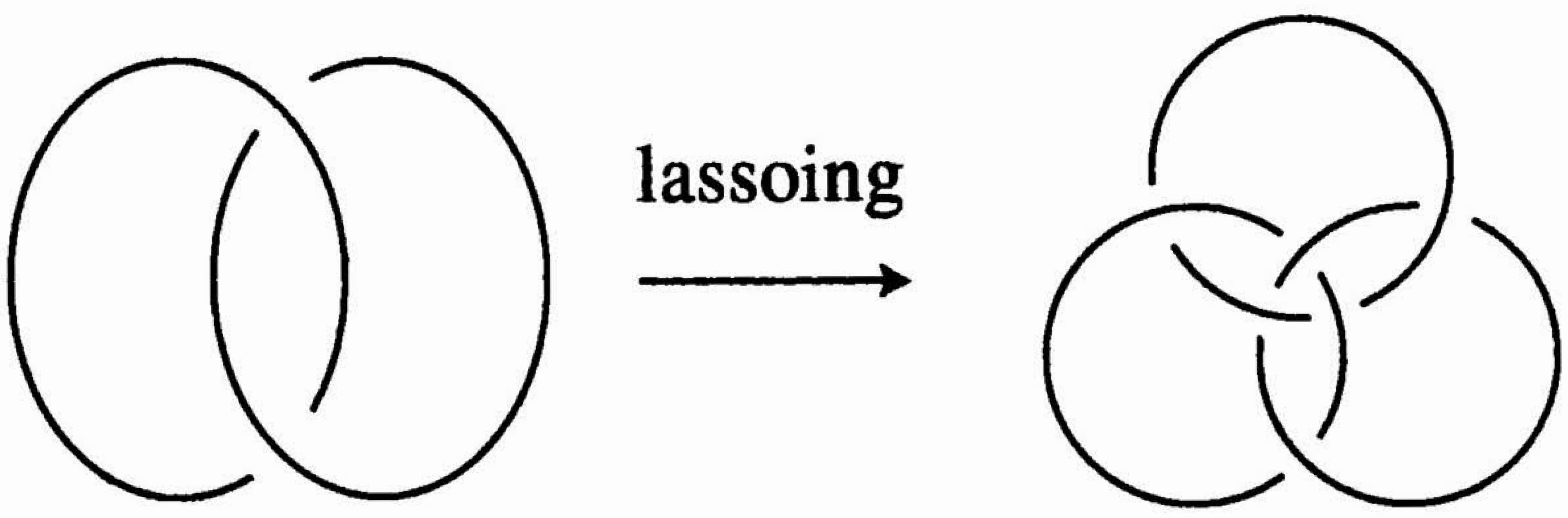


FIGURE 30

A link L' is said to be obtained from L by *r-iterated lassoings* if L' is obtained from L by lassoings r times iteratively. For example, the link L in Figure 31 is a link obtained from a trivial knot by two-iterated lassoings. Since a lassoing depends on the choice of a crossing point and the choice of a diagram of the link, we may have many types of link by a lassoing.

8.3. **Conway polynomial and Alexander polynomial.** In this subsection, we study the Conway polynomials and the Alexander polynomials of lassoed links. Let $\nabla(L; z)$ be the Conway polynomial of a link L with an orientation. We have the following lemma:

Lemma 8.13. *We have*

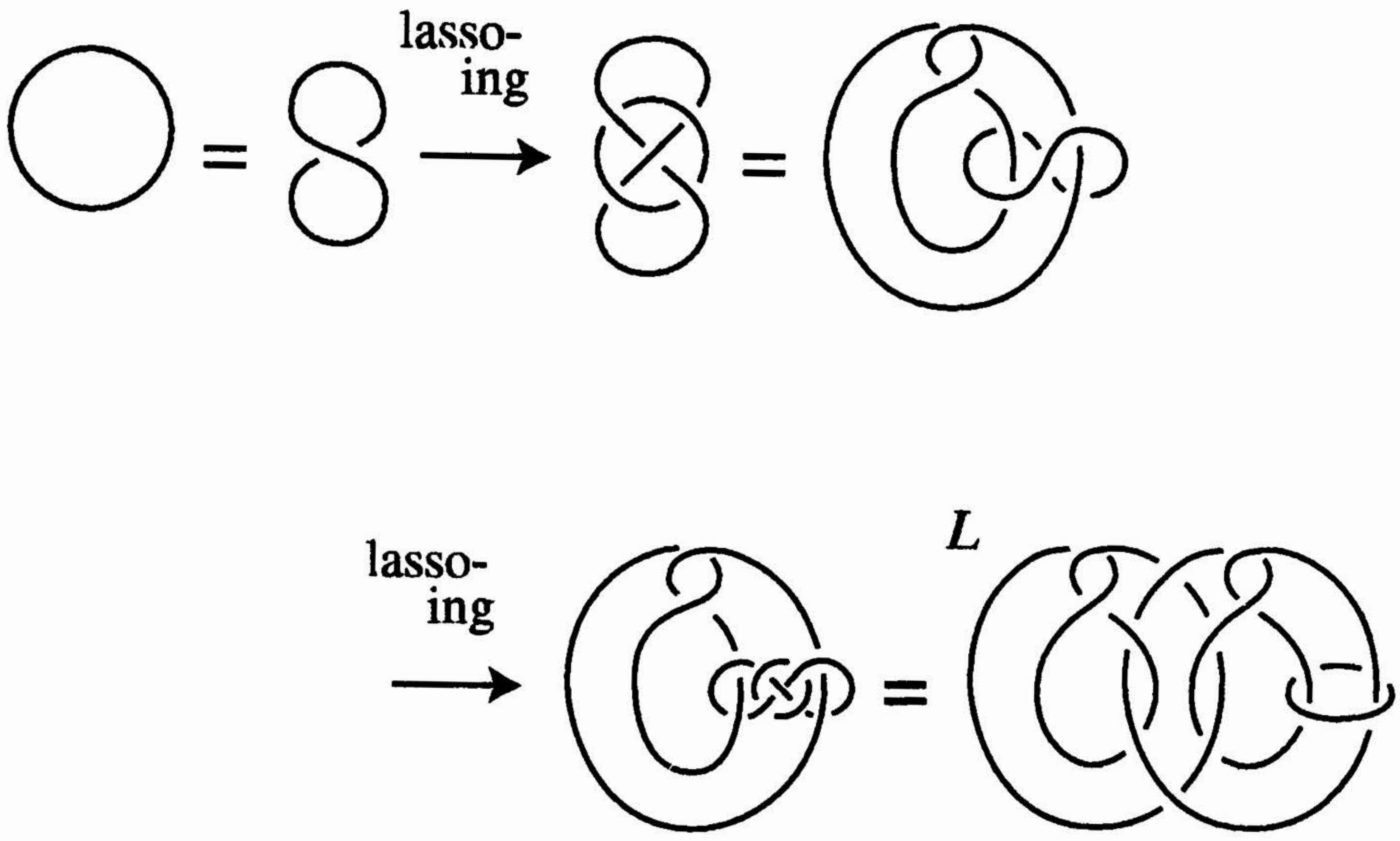
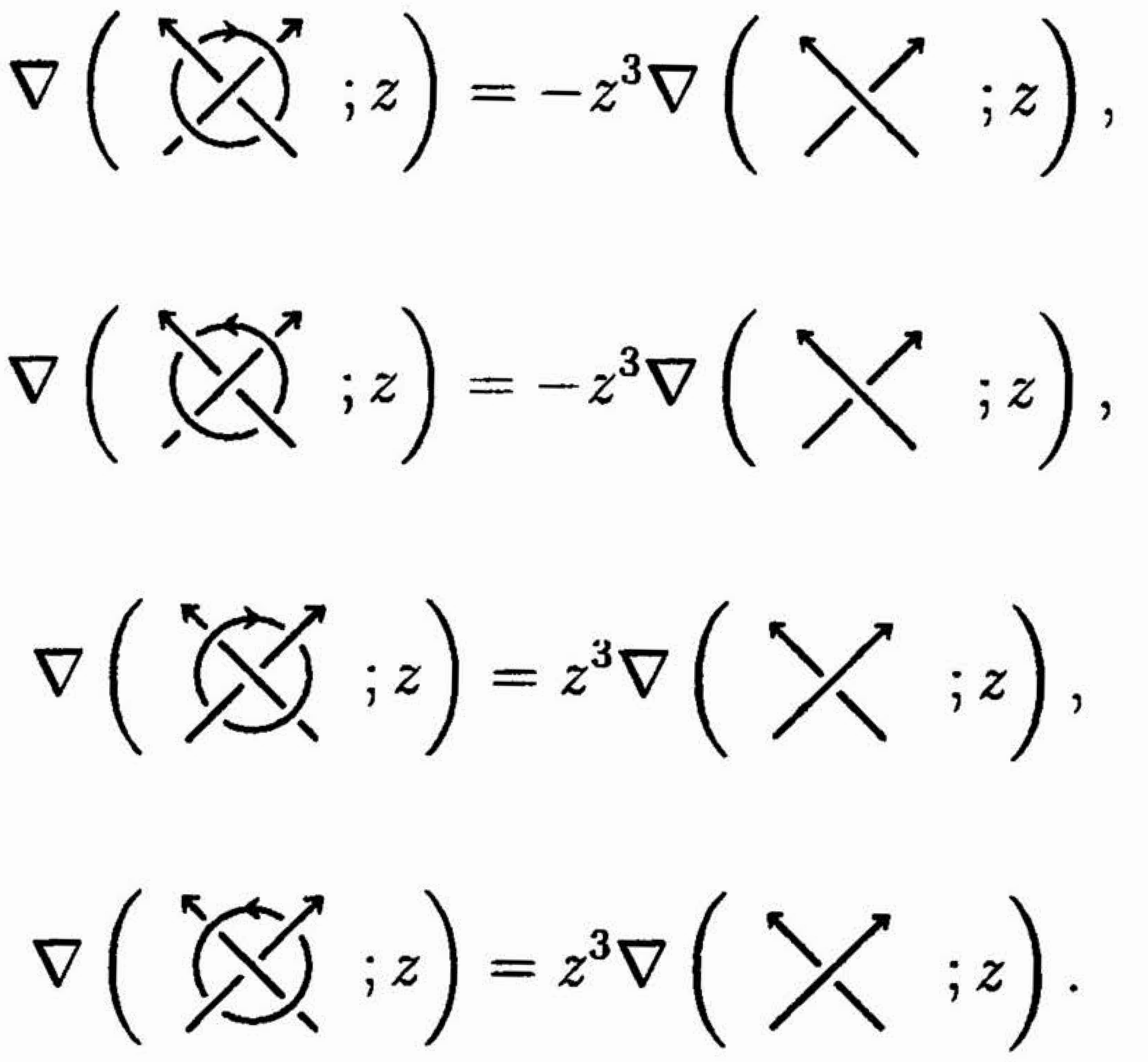


FIGURE 31



Proof. We obtain the first equality by the skein relations in Figure 32. The other equalities are similarly obtained. □

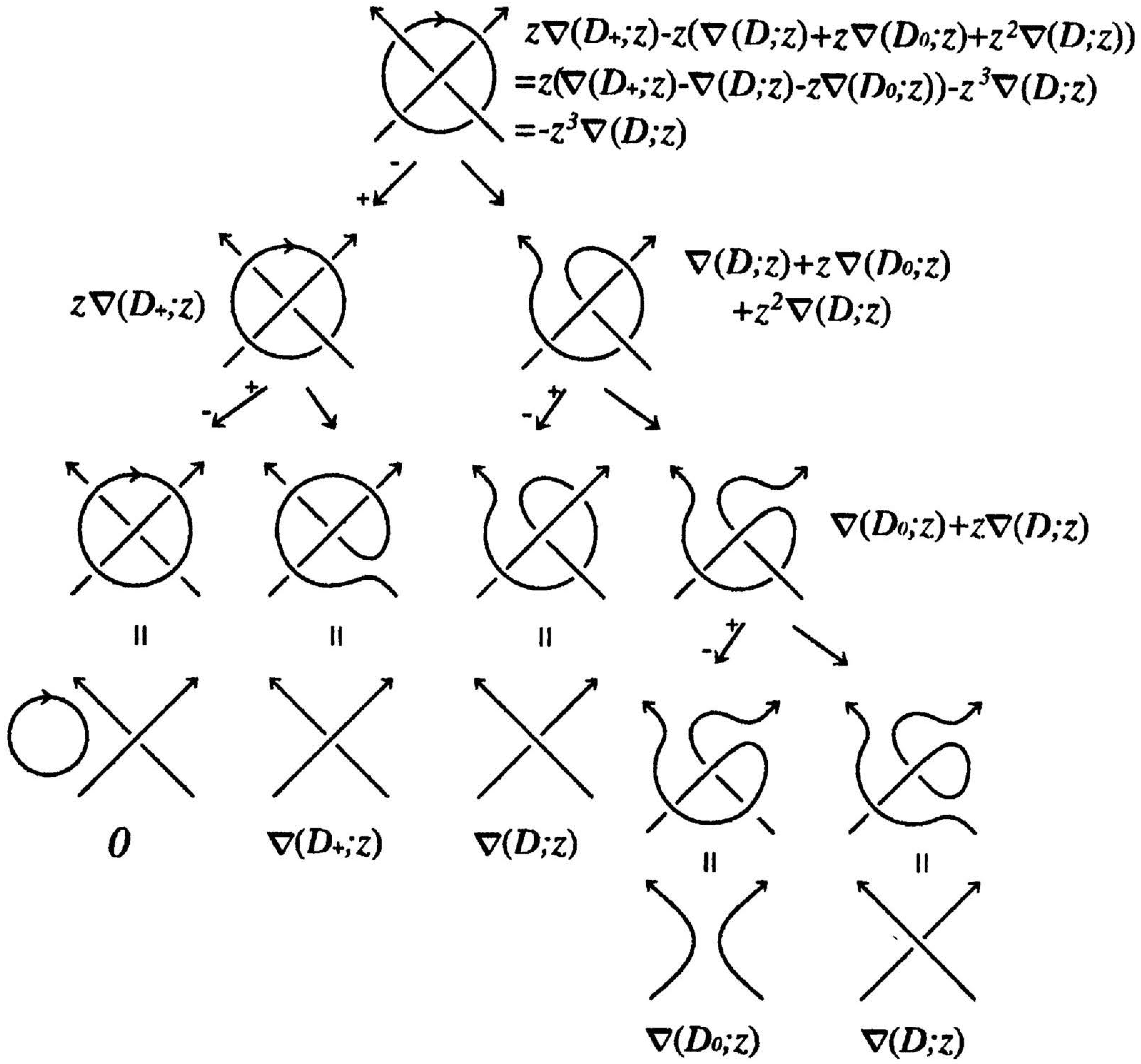


FIGURE 32

Example 8.14. The link diagram D in Figure 33 is obtained from a diagram of a trefoil knot by 2-iterated lassoings. Then we have $\nabla(L) = z^3 \times z^3 \times \nabla(3_1) = z^6(1 + z^2)$, where L is a link represented by D, and 3_1 is a trefoil knot.

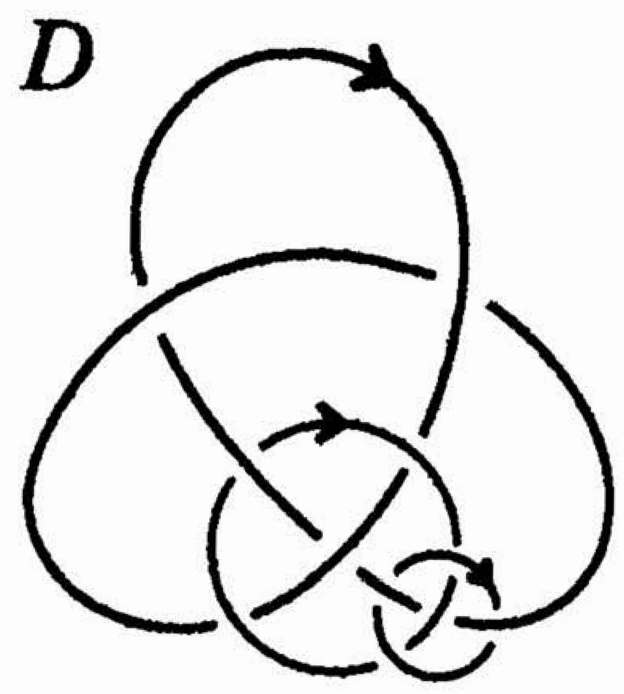


FIGURE 33

We remark that for a link L' with $\nabla(L') = 0$, there are no lassoings from L' to L with $\nabla(L) \neq 0$. We have the following corollary:

Corollary 8.15. *Let L be a link obtained from a link L' with $\nabla(L') \neq 0$, in particular from any knot K, by r-iterated lassoings ($r = 1, 2, 3, \dots$). Then we have $\nabla(L) \neq 0$.*

Let Λ be the integral Laurent polynomial ring, i.e., $\Lambda = \mathbb{Z}[t, t^{-1}]$. With respect to the one-variable Alexander polynomial, we have the following corollary by Lemma 8.13 by substituting $t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ for z:

Corollary 8.16. *Let L' be a link which is obtained from a link L by a lassoing. Then we have*

$$\Delta_{L'}(t) \doteq (t-1)^3 \Delta_L(t),$$

where $\Delta_L(t)$ is the one-variable Alexander polynomial of L, and $\doteq$ means equal up to multiplications of the units of Λ.

We show an example.

Example 8.17. We have

$$\Delta(5^2_1) \doteq \Delta(7^2_8) \doteq \Delta(8^2_{15}) \doteq (t-1)^3 \Delta(0),$$
$$\Delta(7^2_6) \doteq \Delta(9^2_{55}) \doteq \Delta(9^2_{56}) \doteq (t-1)^3 \Delta(3_1),$$

$$\Delta(8^2_{13}) \doteq (t-1)^3\Delta(4_1),$$
$$\Delta(9^2_{31}) \doteq (t-1)^3\Delta(5_1),$$
$$\Delta(9^2_{32}) \doteq \Delta(9^2_{33}) \doteq (t-1)^3\Delta(5_2),$$

where $\Delta(L) = \Delta_L(t)$. All the two-component links with the crossing number nine or less which are obtained from knots by lassoings have been listed above.

Up to multiplications of $t-1$, the one-variable Alexander polynomial of any link is the Alexander polynomial of an algebraically completely splittable link consisting of trivial components:

Corollary 8.18. *Let $(t-1)^m f(t)$ be the Alexander polynomial of a link, where m is a non-negative integer, $f(t) \in \Lambda$, and $f(1) \neq 0$. Then, there exists a non-negative integer n such that the Laurent polynomial $(t-1)^{m+3n} f(t)$ is the Alexander polynomial of an algebraically completely splittable link consisting of trivial components.*

Proof. We can change a crossing by a lassoing. □

8.4. **Proof of Theorem 8.9.** In this subsection, we prove Theorem 8.9. Before the proof, we define some notions which are due to [9] to prove Theorem 8.9. For the integral Laurent polynomial ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, a *multiplicative set* of Λ is a subset $S \subset \Lambda - \{0\}$ which satisfies the following three conditions: the units $\pm t^i$ ($i \in \mathbb{Z}$) are in S, the product gg' of any elements g and g' of S is in S, and every prime factor of any element $g \in S$ is in S. For the quotient field $Q(\Lambda)$ of Λ and a multiplicative set S of Λ, $\Lambda_S = \{f/g \in Q(\Lambda) | f \in \Lambda, g \in S\}$ is a subring of $Q(\Lambda)$. For a finitely generated Λ-module H, let H_S be the Λ_S-module $H \otimes_\Lambda \Lambda_S$, and $e_S(H)$ the minimal number of Λ_S-generators of H_S. We take $e_S(H) = 0$ when $H = 0$. We call $e_S(H)$ the Λ_S-*rank* of H. Let L be an oriented link in S^3, and $E(L) = cl(S^3 - L)$ the compact exterior of L. Let $\tilde{E}(L) \to E(L)$ be the infinite cyclic covering which is induced from the epimorphism $\gamma_L : \pi_1(E(L)) \to \mathbb{Z}$ sending each oriented meridian of L to $1 \in \mathbb{Z}$. Then we can regard $H_1(\tilde{E}(L))$ as a finitely Λ-module. We denote $e_S(H_1(\tilde{E}(L)))$ by

$e_S(L)$. Let L, L' be links which have the same number of components. By Theorem 2.3 in [9], we immediately have

$$d^X(L, L') \geq |e_s(L) - e_s(L')|, \tag{19}$$

where $d^X(L, L')$ denotes the X-distance between L and L'. We prove Theorem 8.9.

Proof of Theorem 8.9. Let L be a link which is obtained from a link $K = K^1 \cup K^2 \cup \cdots \cup K^s$ with $\nabla(K) \neq 0$ by r-iterated lassoings ($r = 1, 2, 3, \ldots$). Let L' be a completely splittable link which is obtained from L by m crossing changes, where $m = \text{split}(L) = d^X(L, L')$. We set $S = \Lambda - \{0\}$. Since L' is completely splittable and the number of components of L' is $r + s$, we have

$$e_s(L') = r + s - 1. \tag{20}$$

The Alexander polynomial of L is non-zero because the Conway polynomial of L is non-zero by Corollary 8.15. Hence we have

$$e_s(L) = 0. \tag{21}$$

By substituting the equalities (20), (21) and $d^X(L, L') = \text{split}(L)$ into the inequality (19), we have

$$\text{split}(L) \geq r + s - 1.$$

From the r-iterated lassoings, we have

$$r + \text{split}(K) \geq \text{split}(L).$$

Hence we have the inequality

$$r + \text{split}(K) \geq \text{split}(L) \geq r + s - 1.$$

□

As the contraposition of Theorem 8.9, we have the following corollary:

Corollary 8.19. *Let $K = K^1 \cup K^2 \cup \cdots \cup K^s$ be an s-component link. If K has* $\text{split}(K) < s - 1$, *then* $\nabla(K) = 0$.

8.5. **Non-triviality.** In this subsection, we discuss the non-trivialities of completely splittable links which are obtained from L in Corollary 8.11 by r crossing changes ($r = 1, 2, \dots$). For a link L obtained from a knot K by r-iterated lassoings, we have the following theorem:

Theorem 8.20. *If a link L is obtained from a knot K with $e(K) > 2r$ by r-iterated lassoings ($r = 1, 2, 3, \dots$), then we have* $\operatorname{split}(L) = r$ *and* $u(L) > r$.

We remark that in Theorem 8.20 the link L is an algebraically completely splittable link if the r-iterated lassoings are all component-lassoings. Before proving Theorem 8.20, we have the following Lemma:

Lemma 8.21. *Let $L_0 = K^1 + K^2 + \cdots + K^r$ be a completely splittable link with r components. Then we have*

$$u(L_0) = \sum_{i=1}^{r} u(K^i).$$

Proof. We have $u(L_0) = u + u_1 + u_2 + \cdots + u_r$, where u is the number of non-self crossing changes and u_i is the number of crossing changes on K^i which are needed to obtain the trivial link from L_0. Then we have

$$u(L_0) = u + u_1 + \cdots + u_r \geq u_1 + \cdots + u_r \geq \sum_{i=1}^{r} u(K^i).$$

Since L_0 is completely splittable, we have

$$u(L_0) \leq \sum_{i=1}^{r} u(K^i).$$

Therefore the equality holds. □

We show Theorem 8.20

Proof of Theorem 8.20. Let $L_0 = K^1 + K^2 + \cdots + K^{r+1}$ be a completely splittable link which is obtained from L by r crossing changes. For the integral Laurent polynomial ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, we take the multiplicative set S of Λ so that S is the set of units of Λ. Then $e_S(L)$ is equivalent to Nakanishi's index $e(L)$ [9]. Since we can consider $L_0 = K^1 + K^2 + \cdots + K^{r+1}$ to be a connected sum $O^{r+1} \# K^1 \# K^2 \# \ldots \# K^{r+1}$, we have

$$\begin{aligned} H_1(\tilde{E}(L_0)) &\cong H_1(\tilde{E}(O^{r+1})) \oplus H_1(\tilde{E}(K^1 \# K^2 \# \ldots \# K^{r+1})) \\ &\cong \Lambda^r \oplus H_1(\tilde{E}(K^1 \# K^2 \# \ldots \# K^{r+1})). \end{aligned}$$

And by [12], we have $e(L_0) = r + e(K^1 \# K^2 \# \ldots \# K^{r+1})$. By substituting this into the inequality (19), we have

$$d^X(L, L_0) \geq |e(L) - e(L_0)| \geq e(L) - r - e(K_1 \# K_2 \# \ldots \# K_{r+1}).$$

Recall that $d^X(L, L_0) = \mathrm{split}(L) = r$. Then we have

$$r \geq e(L) - r - e(K^1 \# K^2 \# \ldots \# K^{r+1}). \tag{22}$$

Next, we consider another completely splittable link $K + O^r$ which is obtained from L by the r anti-lassoings (see Figure 34).

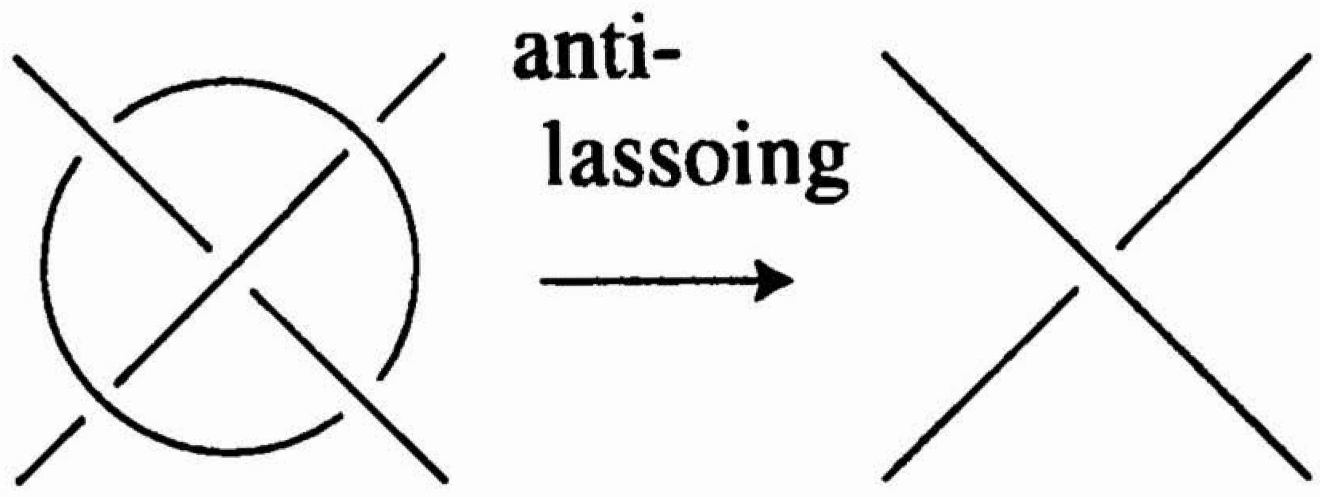


FIGURE 34

Since $K + O^r = O^{r+1} \# K$, we have

$$H_1(\tilde{E}(L_0)) \cong \Lambda^r \oplus H_1(\tilde{E}(K)).$$

And by [12], we have $e(K + O^r) = r + e(K)$. Hence we have

$$r \geq r + e(K) - e(L) \tag{23}$$

by [9]. By summing the inequalities (22) and (23), we have

$$2r \geq e(K) - e(K^1 \# K^2 \# \ldots \# K^{r+1}).$$

From Lemma 8.21, we have

$$u(L_0) = \sum_{i=1}^{r+1} u(K^i) \geq u(K^1\#K^2\#\ldots\#K^{r+1}) \geq e(K^1\#K^2\#\ldots\#K^{r+1})$$
$$\geq e(K) - 2r.$$

Hence L_0 is non-trivial if $e(K) > 2r$. □

For a knot which has Nakanishi's index large enough, we can construct a link such that the unlinking number is greater than the complete splitting number. Here is an example.

Example 8.22. Since the knot K in Figure 35 which is the connected sum of $2r + 1$ trefoil knots has Nakanishi's index $e(K) = 2r + 1$, any link L obtained from K by r-iterated lassoings has the unlinking number more than r whereas $\text{split}(L) = r$.

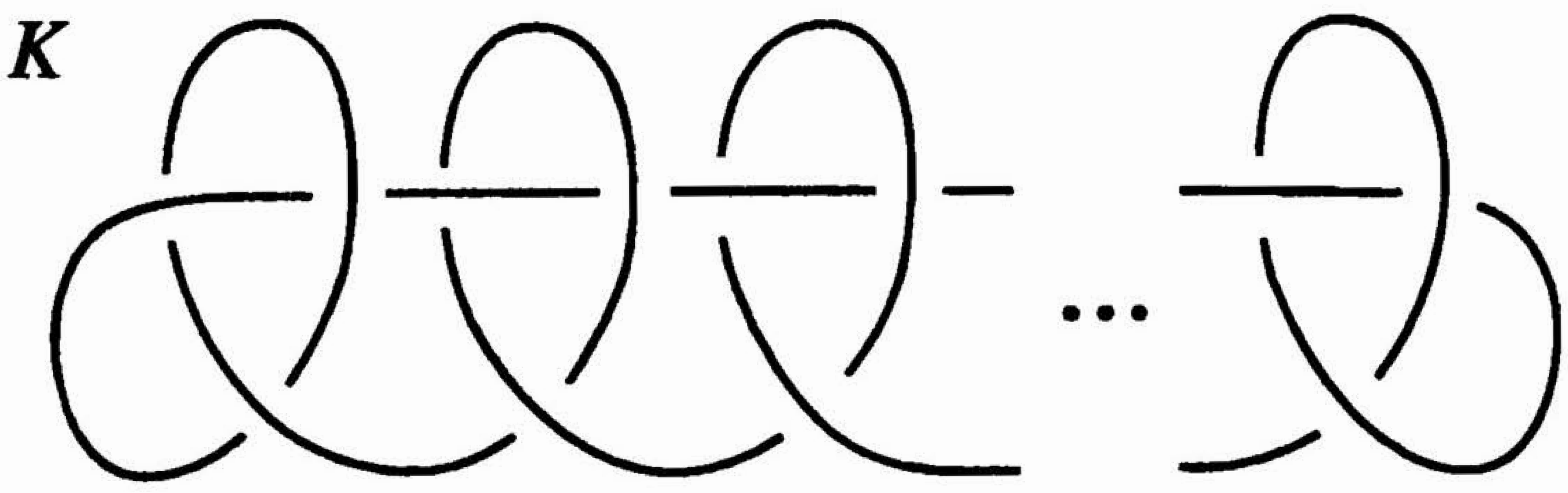


FIGURE 35

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