Topology of toric manifolds and graphs

（トーリック多様体のトポロジーとグラフ）

理学研究科
数物系専攻

平成27年度

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Chapter 1

Introduction

A *toric variety* is a normal algebraic variety of complex dimension n with an algebraic action of a complex torus having an open dense orbit. The family of toric varieties one-to-one corresponds to that of fans which are objects in combinatorics. Via this correspondence, we can describe geometrical properties of toric varieties in terms of the corresponding fans. A toric variety may not be compact and nonsingular but we are mainly concerned with compact nonsingular toric varieties and we call them *toric manifolds*.

A toric manifold X is not necessarily projective but if it is projective, then it admits a moment map and the moment map image of X is a nonsingular polytope P called a *Delzant polytope*. The normal fan to P agrees with the fan corresponding to X and Delzant's theorem tells us that the family of projective toric manifolds one-to-one corresponds to Delzant polytopes up to some equivalence.

One can associate a Delzant polytope to a simple graph (see section 3.2) and a Delzant polytope associated to a simple graph is called a *graph associahedron*. Important polytopes such as permutohedron, cyclohedon and associahedron (or Stasheff polytope) are graph associahedra. Since a graph associahedron is a Delzant polytope, it associates a projective toric manifold. Consequently, a simple graph associates a (projective) toric manifold.

In this doctoral thesis, we consider the following four topics.

1. Unique decomposition problem for toric manifolds.

2. Characterization of spin toric manifolds associated to simple graphs.

3. Facet vectors of toric manifolds associated to simple graphs and root systems.

4. Cohomology representations of toric manifolds associated to simple graphs.

We shall explain these four topics in more detail.

We discuss topic (1) in Chapter 2. We say that a toric manifold is *algebraically indecomposable* if it does not decompose into the product of two toric manifolds of positive dimension *as varieties*. Using the bijective correspondence between toric varieties and fans, one can see that the direct decomposition of a toric manifold into algebraically indecomposable toric manifolds as algebraic varieties is unique up to order of the factors (Theorem 2.1.2).

An algebraically indecomposable toric manifold happens to decompose into the product of two toric manifolds of positive dimension as *smooth manifolds*.

We say that a toric manifold is *differentially indecomposable* if it does not decompose into the product of two toric manifolds of positive dimension *as smooth manifolds*. Our concern is the following problem.

Unique decomposition problem for toric manifolds ([14]). *Is the direct decomposition of a toric manifold into the product of differentially indecomposable toric manifolds unique up to order of the factors?*

As far as the author knows, nothing was known for the above problem. We prove that the problem is affirmative if the complex dimension of every factor in the product is less than or equal to two (Theorem 2.2.1). Note that a toric manifold of complex dimension one is diffeomorphic to $\mathbb{C}P^1$ and that of complex dimension two is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \sharp q \overline{\mathbb{C}P^2}$ ($q \geq 0$).

Simply connected closed smooth 4-manifolds with smooth actions of $(S^1)^2$ are of the form

$$S^4 \sharp p \mathbb{C}P^2 \sharp q \overline{\mathbb{C}P^2} \sharp r (\mathbb{C}P^1 \times \mathbb{C}P^1) \quad (p + q + r \geq 0) \tag{1.0.1}$$

(see [18]). Our method used to prove Theorem 2.2.1 can be applied to products of copies of $\mathbb{C}P^1$ and manifolds in (1.0.1) and yields a more general result (Theorem 2.3.4) than Theorem 2.2.1.

One can consider the unique decomposition problem for *real* toric manifolds, where a real toric manifold is the set of real points in a toric manifold. It has been shown in [4] that the unique decomposition problem is affirmative for real Bott manifolds which are a special class of *real* toric manifolds. Real Bott manifolds are compact flat manifolds and it is shown in [3] that there are non-diffeomorphic compact flat manifolds whose products with S^1 are diffeomorphic. This means that the unique decomposition property does not hold for general compact flat manifolds while it does for the special class of compact flat manifolds consisting of real Bott manifolds. We prove that the unique decomposition problem is affirmative for real toric manifolds if the real dimension of every factor in the product is less than or equal to two (Theorem 2.4.1).

We discuss topic (2) in Chapter 3. As is well-known, a smooth manifold admits a spin structure if and only if its first and second Stiefel-Whitney classes vanish. Using this criterion, we give a necessary and sufficient condition for a toric manifold M to admit a spin structure in terms of the corresponding fan (Proposition 3.1.1). It turns out that this is equivalent to the real part of M being orientable ([16]).

As mentioned above, a simple graph associates a toric manifold. Using Proposition 3.1.1, we characterize simple graphs whose associated toric manifolds admit spin structures (Theorem 3.2.8).

One can also associate a toric manifold to a pseudograph which may have multiedges and loops ([2]). We will see that Theorem 3.2.8 can be generalized to pseudographs (Theorem 3.3.13).

We discuss topic (3) in Chapter 4. Let G be a connected simple graph and P_G be its graph associahedron. We call a primitive (outward) normal vector to a facet of P_G a *facet vector* and denote by $F(G)$ the set of facet vectors of P_G. One can observe that when G is a complete graph, $F(G)$ agrees with the primitive edge vectors of the fan formed by the Weyl chambers of a root system of type A, in other words, $F(G)$ is *dual* to a root system of type A when G is a complete graph. Motivated by this observation, we ask whether $F(G)$ itself forms a root system for a simple graph G. It turns out that $F(G)$ forms a root system if and only if G is a cycle graph (Theorem 4.0.2).

We discuss topic (4) in Chapter 5. The automorphism group $\mathrm{Aut}(G)$ of a simple graph G induces a cohomology representation of the toric manifold associated to the graph G. When G is a complete graph, $\mathrm{Aut}(G)$ is a symmetric group and the toric manifold associated to the complete graph G is what is called a permutohedral variety whose fan is formed by Weyl chambers of a root system of type A as mentioned above. Procesi ([19]) initiated the study of the cohomology representations for permutohedral varieties (i.e. when G is a complete graph) and obtained a recursive formula to find the representations. Then more work has been done by Stanley ([20]), Stembridge ([21]) and Dolgachev-Luntz ([8]) in this case, and Henderson ([10]) gave a closed formula to find the cohomology representations of *real* toric manifolds associated to complete graphs.

Using the argument of Procesi, we investigate the cohomology representations when G is a cycle graph or a graph obtained by removing one edge from a complete graph. The automorphism group of a cycle graph is a dihedral group and irreducible representations of a dihedral group are well-known. We describe the cohomology representations when G is a cycle graph with 3, 4, or 5 nodes. When G is a graph obtained by removing one edge from a complete graph, $\mathrm{Aut}(G)$ is the product of a symmetric group and a group of order 2. In this case, we obtain a recursive formula similar to that obtained by Procesi for complete graphs (Theorem 5.2.2).

Acknowledgement. I would like to express my special appreciation and thanks to my advisor Professor Mikiya Masuda. He has been a tremendous mentor for me. I would like to thank him for encouraging my research and for allowing me to grow as a mathematician. His advice on research as well as on my career have been priceless.

Special thanks to my family. Words cannot express how grateful I am to my mother. I would also like to thank Professor Megumi Harada and all of my friends who supported me; Hiraku Abe, Yukiko Fukukawa, Syumi Kinjyo, Hideya Kuwata, Megumi Hashizume, Tatsuya Horiguchi, Mika Nishimoto, Mika Sakata, Yusuke Suyama, Yuriko Umemoto, and Haozhi Zeng. This work was partially supported by Grant-in-Aid for JSPS Fellows 27 · 0184.

Chapter 2

Uniqueness of the direct decomposition of toric manifolds

In Chapter 2, we study the following unique decomposition problem.

Unique decomposition problem for toric manifolds ([14]). *Is the direct decomposition of a toric manifold into the product of differentially indecomposable toric manifolds unique up to order of the factors?*

2.1 Direct decomposition of toric manifolds as algebraic varieties

We briefly review toric geometry and refer the reader to [9] and [17] for details.

A *toric variety* is a normal algebraic variety of complex dimension n with an algebraic action of a complex torus $(\mathbb{C}^*)^n$ having an open dense orbit. The fundamental theorem in toric geometry says that the category of toric varieties of (complex) dimension n is isomorphic to the category of fans of (real) dimension n. Here, a *fan* Δ of dimension n is a collection of rational strongly convex polyhedral cones in $\mathbb{R}^n$ satisfying the following conditions:

· Each face of a cone in Δ is also a cone in Δ.

· The intersection of two cones in Δ is a face of each.

A rational strongly convex polyhedral cone in $\mathbb{R}^n$ is a cone with apex at the origin, generated by a finite number of vectors; "rational" means that it is generated by vectors in the lattice $\mathbb{Z}^n$, and "strong" convexity that it contains no line through the origin. The union of cones in the fan Δ coincides with $\mathbb{R}^n$ if and only if the corresponding toric variety is compact, and the generators of each cone in Δ are a part of a basis of $\mathbb{Z}^n$ if and only if the corresponding toric variety is nonsingular. In Chapter 2, we will treat only compact nonsingular toric varieties and call them *toric manifolds.*

The fundamental theorem in toric geometry implies that two toric manifolds M and N of complex dimension n are weakly equivariantly isomorphic as algebraic varieties if and only if the corresponding fans are isomorphic, i.e., there is an automorphism of $\mathbb{Z}^n$ sending cones to cones in the corresponding fans. Here a map $f\colon M \to N$ is said to be weakly equivariant if there is an automorphism ρ of $(\mathbb{C}^*)^n$ such that $f(gx) = \rho(g)f(x)$ for any $g \in (\mathbb{C}^*)^n$ and $x \in M$.

Proposition 2.1.1. *Two toric manifolds are isomorphic as algebraic varieties* *if and only if they are weakly equivariantly isomorphic as algebraic varieties. Therefore, two toric manifolds are isomorphic as algebraic varieties if and only if their corresponding fans are isomorphic.*

Proof. This proposition is well-known but since there seems no literature, we shall sketch the proof.

It suffices to prove the "only if" part in the former statement because the "if" part is trivial and the latter statement follows from the former statement and the fundamental theorem in toric geometry as remarked above. Let $\mathrm{Aut}(M)$ be the group of automorphisms of a toric manifold M. This is a (finite dimensional) algebraic group, and the torus $T_M = (\mathbb{C}^*)^n$ acting on M is a subgroup of $\mathrm{Aut}(M)$, in fact, it is a maximal torus in $\mathrm{Aut}(M)$. Now, let f be an isomorphism (as algebraic varieties) from M to another toric manifold N. Then f induces a group isomorphism $\hat{f}\colon \mathrm{Aut}(N) \to \mathrm{Aut}(M)$ mapping $g \in \mathrm{Aut}(N)$ to $f^{-1} \circ g \circ f \in \mathrm{Aut}(M)$. Since $\hat{f}(T_N)$ is a maximal torus in $\mathrm{Aut}(M)$ and all maximal tori in an algebraic group are conjugate to each other, there exists $h \in \mathrm{Aut}(M)$ satisfying $\hat{f}(T_N) = hT_M h^{-1}$. Then $f \circ h$ is a weakly equivariant isomorphism from M to N. □

We say that a toric manifold is *algebraically indecomposable* if it does not decompose into the product of two toric manifolds of positive dimension as algebraic varieties. Again, the fundamental theorem in toric geometry implies that a toric manifold is algebraically indecomposable if and only if the corresponding fan is *indecomposable*, i.e., it does not decompose into the product of two fans of positive dimension.

Theorem 2.1.2. *The direct decomposition of a toric manifold into algebraically indecomposable toric manifolds as algebraic varieties is unique up to order of the factors. Namely, if M_i $(1 \leq i \leq k)$ and M'_j $(1 \leq j \leq \ell)$ are algebraically indecomposable toric manifolds and $\prod_{i=1}^{k} M_i$ and $\prod_{j=1}^{\ell} M'_j$ are isomorphic as algebraic varieties, then $k = \ell$ and there exists an element σ in the symmetric group S_k on k letters such that M_i is isomorphic to $M'_{\sigma(i)}$ as algebraic varieties for all $1 \leq i \leq k$.*

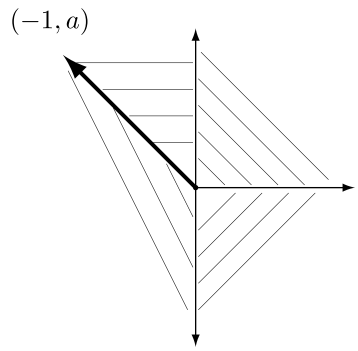
Proof. Denote the fan of M_i by Δ_i and that of M'_j by Δ'_j, and let ψ be an isomorphism from $\prod_{i=1}^{k} \Delta_i$ to $\prod_{j=1}^{\ell} \Delta'_j$. Let p_j be the projection from $\prod_{j=1}^{\ell} \Delta'_j$ onto Δ'_j. Since an edge in Δ_i maps to an edge in $\prod_{j=1}^{\ell} \Delta'_j$ by ψ, the image $\psi(\Delta_i)$ coincides with the product $\prod_{j=1}^{\ell} p_j(\psi(\Delta_i))$. This together with the indecomposability of Δ_i implies that $p_j(\psi(\Delta_i))$ consists of only the origin except for one j, namely $\psi(\Delta_i)$ is contained in some Δ'_j. Applying the same argument to ψ^{-1}, one concludes that $\psi(\Delta_i) = \Delta'_j$. This together with Proposition 2.1.1 proves the theorem. □

The following corollary follows from Theorem 2.1.2.

Corollary 2.1.3 (cancellation)**.** *Let M, M' and M'' be toric manifolds. If the direct products $M \times M''$ and $M' \times M''$ are isomorphic as varieties, then so are M and M'.*

2.2 Direct decomposition of toric manifolds as smooth manifolds

In this section, we will consider the direct decomposition of toric manifolds as smooth manifolds. We say that a toric manifold M is *differentially indecomposable* if M does not decompose into two toric manifolds of positive dimension as smooth manifolds. We note that the algebraic indecomposability does not imply the differential indecomposability for toric manifolds. For example, the Hirzebruch surface F_a $(a \in \mathbb{Z})$ corresponding to the fan described below is algebraically indecomposable unless $a = 0$ but diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ as smooth manifolds if a is even.



$a \in \mathbb{Z}$

Toric manifolds of complex dimension one are diffeomorphic to $\mathbb{C}P^1$, and those of complex dimension two are diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}$ $(q \in \mathbb{Z}_{\geq 0})$. The purpose of this section is to prove the following theorem.

Theorem 2.2.1. *Let M_i $(1 \leq i \leq k)$ and M'_j $(1 \leq j \leq \ell)$ be differentially* *indecomposable toric manifolds of complex dimension less than or equal to two. If $\prod_{i=1}^{k} M_i$ and $\prod_{j=1}^{\ell} M'_j$ are diffeomorphic, then $k = \ell$ and there exists an element σ in the symmetric group S_k on k letters such that M_i and $M'_{\sigma(i)}$ are diffeomorphic for all $1 \leq i \leq k$. Therefore, the unique decomposition problem mentioned in the Introduction is affirmative for products of differentially indecomposable toric manifolds of complex dimension less than or equal to two.*

For the proof of this theorem, we consider

$$A(X;R) = \{u \in H^2(X;R)\backslash\{0\} \mid u^2 = 0\} \tag{2.2.1}$$

for a topological space X and a commutative ring R.

Lemma 2.2.2. *Let R be $\mathbb{Z}$ or a field, and let X_i $(1 \leq i \leq k)$ be a connected topological space such that $H^q(X_i;R)$ is finitely generated for any q and $H^1(X_i;R) = H^3(X_i;R) = 0$. Moreover, when $R = \mathbb{Z}$, we suppose that $H^q(X_i;\mathbb{Z})$ $(q \leq 4)$ is a free module. (Toric manifolds satisfy these conditions.) Then, there exists a natural identification*

$$A(\prod_{i=1}^{k} X_i;R) \cong \coprod_{i=1}^{k} A(X_i;R).$$

Proof. By the Künneth formula, the cohomology group $H^2(\prod_{i=1}^{k} X_i;R)$ is isomorphic to $\bigoplus_{i=1}^{k} H^2(X_i;R)$. So an element u in $H^2(\prod_{i=1}^{k} X_i;R)$ can be written as $u = u_1 + \cdots + u_k$ $(u_i \in H^2(X_i;R))$. Again, by the Künneth formula,

$$\begin{aligned} H^4(\prod_{i=1}^{k} X_i;R) \cong & \Big(\bigoplus_{i=1}^{k} H^4(X_i;R)\Big) \\ & \oplus \Big(\bigoplus_{1\leq i<j\leq k} H^2(X_i;R) \otimes H^2(X_j;R)\Big) \end{aligned}$$

and via this isomorphism

$$u^2 = \sum_{i=1}^{k} u_i^2 + 2 \sum_{1 \le i < j \le k} u_i \otimes u_j.$$

So if $u^2 = 0$, then $u_i = 0$ except one i. Therefore, the lemma holds. $\square$

Differentially indecomposable toric manifolds of complex dimension less than or equal to two are diffeomorphic to $\mathbb{C}P^1$ or $\mathbb{C}P^2 \sharp q \overline{\mathbb{C}P^2}$ ($q \in \mathbb{Z}_{\ge 0}$). Their cohomology rings are as follows:

$$\begin{aligned} & H^*(\mathbb{C}P^1; R) \cong R[x]/(x^2 = 0) \\ & H^*(\mathbb{C}P^2 \sharp q \overline{\mathbb{C}P^2}; R) \\ & \qquad \cong R[x, y_1, \dots, y_q] / \begin{pmatrix} x^2 = -y_i^2, & xy_i = 0 \ (\forall i), \\ y_i y_j = 0 \ (i \ne j) & \end{pmatrix} \end{aligned} \tag{2.2.2}$$

Lemma 2.2.3. (1) *$A(\mathbb{C}P^1; R) \cong \{a \in R \backslash \{0\}\}$. In particular, the set $A(\mathbb{C}P^1; \mathbb{R})$ consists of two one dimensional connected components, and $A(\mathbb{C}P^1; \mathbb{Z}/2)$ consists of one element.*

(2) *$A(\mathbb{C}P^2 \sharp q \overline{\mathbb{C}P^2}; R) \cong \{(a, b_1, \dots, b_q) \in R^{q+1} \backslash \{0\} \mid a^2 = b_1^2 + \cdots + b_q^2\}$. In particular, $A(\mathbb{C}P^2; \mathbb{R})$ and $A(\mathbb{C}P^2; \mathbb{Z}/2)$ are empty, $A(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}; \mathbb{R})$ consists of four one dimensional connected components, and $A(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}; \mathbb{Z}/2)$ consists of one element. When $q \ge 2$, $A(\mathbb{C}P^2 \sharp q \overline{\mathbb{C}P^2}; \mathbb{R})$ consists of two q dimensional connected components.*

Proof. (1) This easily follows from the former isomorphism in (2.2.2).

(2) Using the latter isomorphism in (2.2.2), one can write an element u in

$H^2(\mathbb{C}P^2\sharp q\overline{\mathbb{C}P^2};R)$ as

$$u = ax + b_1y_1 + \cdots + b_qy_q \quad (a, b_1, \ldots, b_q \in R),$$

so we have $u^2 = (a^2 - b_1^2 - \cdots - b_q^2)x^2$, which implies (2). □

Proof of Theorem 2.2.1. Let m (resp, m_q) be the number of M_i's diffeomorphic to $\mathbb{C}P^1$ (resp, $\mathbb{C}P^2\sharp q\overline{\mathbb{C}P^2}$). Similarly, let m' (resp, m'_q) be the number of M'_j's diffeomorphic to $\mathbb{C}P^1$ (resp, $\mathbb{C}P^2\sharp q\overline{\mathbb{C}P^2}$). Then

$$\begin{aligned} M &:= \prod_{i=1}^{k} M_i = (\mathbb{C}P^1)^m \times \prod_{q\geq 0}(\mathbb{C}P^2\sharp q\overline{\mathbb{C}P^2})^{m_q} \\ M' &:= \prod_{j=1}^{\ell} M'_j = (\mathbb{C}P^1)^{m'} \times \prod_{q\geq 0}(\mathbb{C}P^2\sharp q\overline{\mathbb{C}P^2})^{m'_q}. \end{aligned} \tag{2.2.3}$$

By assumption, $H^*(M;\mathbb{Z})$ and $H^*(M';\mathbb{Z})$ are isomorphic as graded rings, and an isomorphism between them induces an isomorphism between $H^*(M;R)$ and $H^*(M';R)$ for any commutative ring R and a bijection between $A(M;R)$ and $A(M';R)$. When $R = \mathbb{R}$, we compare the number of connected components of dimension t in $A(M;\mathbb{R})$ and $A(M';\mathbb{R})$. Since the bijection between $A(M;\mathbb{R})$ and $A(M';\mathbb{R})$ is a homeomorphism, we obtain

$$2m + 4m_1 = 2m' + 4m'_1, \qquad 2m_t = 2m'_t \quad (t \geq 2) \tag{2.2.4}$$

from Lemmas 2.2.2 and 2.2.3. Moreover, comparing the number of elements in $A(M;\mathbb{Z}/2)$ and $A(M';\mathbb{Z}/2)$, we obtain

$$m + m_1 = m' + m'_1 \tag{2.2.5}$$

from the fact $m_t = m'_t$ $(t \geq 2)$ in (2.2.4), Lemmas 2.2.2 and 2.2.3. The identities (2.2.4) and (2.2.5) imply $m = m'$ and $m_t = m'_t$ $(t \geq 1)$. These together with the equality of the dimensions of M and M' (which are respectively $m+2\sum_{t\geq 0} m_t$ and $m'+2\sum_{t\geq 0} m'_t$ by (2.2.3)) imply $m_0 = m'_0$. Therefore the theorem is proved. □

The following corollary follows from Theorem 2.2.1.

Corollary 2.2.4 (cancellation)**.** *Let M, M' and M'' be products of toric manifolds of complex dimension less than or equal to two. If $M\times M''$ and $M'\times M''$ are diffeomorphic, then so are M and M'.*

2.3 Simply connected compact 4-manifolds with $(S^1)^2$-actions

In this section, we show that the idea developed to prove Theorem 2.2.1 works for products of $\mathbb{C}P^1$ and simply connected compact smooth 4-manifolds with smooth actions of compact torus $(S^1)^2$. By Orlik-Raymond ([18]), these 4-manifolds are diffeomorphic to

$$S^4 \sharp p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2} \sharp r(\mathbb{C}P^1 \times \mathbb{C}P^1) \quad (p+q+r \geq 0). \tag{2.3.1}$$

Proposition 2.3.1. *A manifold in* (2.3.1) *is diffeomorphic to one of the following:*

$$S^4, \quad p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2} \ (p \geq q \geq 0,\ p+q \geq 1),$$
$$r(\mathbb{C}P^1 \times \mathbb{C}P^1) \ (r \geq 1).$$

Moreover these manifolds are not diffeomorphic to each other.

Proof. This proposition must be known but since there seems no literature, we shall give a proof.

Claim $\mathbb{C}P^2 \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ *and* $\overline{\mathbb{C}P^2} \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ *are diffeomorphic to* $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$.

The fan corresponding to the blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and that of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ are isomorphic, so $\overline{\mathbb{C}P^2} \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ and $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ are isomorphic as algebraic varieties, in particular, $\overline{\mathbb{C}P^2} \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ is diffeomorphic to $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$.

Moreover $\mathbb{C}P^2 \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ and $\overline{\mathbb{C}P^2} \sharp (\overline{\mathbb{C}P^1 \times \mathbb{C}P^1})$ are diffeomorphic, and since there is an orientation preserving diffeomorphism from $\overline{\mathbb{C}P^1 \times \mathbb{C}P^1}$ to

$\mathbb{C}P^1 \times \mathbb{C}P^1$ (i.e., an orientation reversing diffeomorphism from $\mathbb{C}P^1 \times \mathbb{C}P^1$ to itself), $\overline{\mathbb{C}P^2} \sharp (\overline{\mathbb{C}P^1 \times \mathbb{C}P^1})$ is diffeomorphic to $\overline{\mathbb{C}P^2} \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$. So $\mathbb{C}P^2 \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ and $\overline{\mathbb{C}P^2} \sharp (\mathbb{C}P^1 \times \mathbb{C}P^1)$ are diffeomorphic. Therefore the claim is proved.

From the Claim above and the fact that $p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}$ and $q\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2}$ are diffeomorphic, we see that a manifold in (2.3.1) is diffeomorphic to one of the manifolds in Proposition 2.3.1.

We shall prove that the manifolds in Proposition 2.3.1 are not diffeomorphic to each other. The manifolds $p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}$ are not spin manifolds (i.e. their second Stiefel-Whitney classes do not vanish) while $r(\mathbb{C}P^1 \times \mathbb{C}P^1)$ are spin manifolds. Therefore, they are not homotopy equivalent, in particular, not diffeomorphic. Euler characteristic χ and the absolute value of signature σ are homotopy invariants, and

$$\chi(p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}) = p + q + 2, \qquad \sigma(p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}) = p - q$$

$$\chi(r(\mathbb{C}P^1 \times \mathbb{C}P^1)) = 2r + 2, \qquad \sigma(r(\mathbb{C}P^1 \times \mathbb{C}P^1)) = 0$$

$$\chi(S^4) = 2$$

so the manifolds in Proposition 2.3.1 are not homotopy equivalent to each other, in particular, they are not diffeomorphic to each other. □

We find $A(M; R)$ in (2.2.1) for the manifolds M in Proposition 2.3.1 and any commutative ring R. Since

$$\begin{aligned} &H^*(p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}; R) \\ &\qquad \cong R[x_1, \dots, x_p, y_1, \dots, y_q] \Big/ \begin{pmatrix} x_i^2 = -y_j^2, & x_i y_j = 0 (\forall i, j), \\ x_i x_j = 0, & y_i y_j = 0 (\forall i \neq j) \end{pmatrix}, \end{aligned}$$

$$H^*(r(\mathbb{C}P^1 \times \mathbb{C}P^1); R)$$

$$\cong R[z_1, \ldots, z_r, w_1, \ldots, w_r] \Big/ \begin{pmatrix} z_i w_i = z_j w_j, \\ z_i z_j = w_i w_j = 0 \ (\forall i, j), \\ z_i w_j = 0 \ (\forall i \neq j) \end{pmatrix},$$

$$H^*(S^4; R) \cong R[x]/(x^2 = 0),$$

we see that

$$A(p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}; R)$$

$$\cong \{(a_1, \ldots, a_p, b_1, \ldots, b_q) \in R^{p+q} \backslash \{0\} \mid a_1^2 + \cdots + a_p^2 = b_1^2 + \cdots + b_q^2\}, \tag{2.3.2}$$

$$A(r(\mathbb{C}P^1 \times \mathbb{C}P^1); R)$$

$$\cong \{(c_1, \ldots, c_r, d_1, \ldots, d_r) \in R^{2r} \backslash \{0\} \mid c_1 d_1 + \cdots + c_r d_r = 0\}, \tag{2.3.3}$$

$$A(S^4; R) = \emptyset.$$

Lemma 2.3.2. (1) $A(p\mathbb{C}P^2; \mathbb{R})$ *is empty.*

(2) *When* $p \geq q \geq 1$, $A(p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}; \mathbb{R})$ *is homeomorphic to* $S^{p-1} \times S^{q-1} \times \mathbb{R}$.

(3) $A(r(\mathbb{C}P^1 \times \mathbb{C}P^1); \mathbb{R})$ *is homeomorphic to* $S^{r-1} \times S^{r-1} \times \mathbb{R}$.

Proof. (1) This easily follows from (2.3.2).

(2) For each positive real number c, the set

$$\{(a_1, \ldots, a_p, b_1, \ldots, b_q) \in \mathbb{R}^{p+q} \backslash \{0\} \mid a_1^2 + \cdots + a_p^2 = b_1^2 + \cdots + b_q^2 = c\}$$

is homeomorphic to the product of spheres $S^{p-1} \times S^{q-1}$. So, the space $A(p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}; \mathbb{R})$ is homeomorphic to $S^{p-1} \times S^{q-1} \times \mathbb{R}_{>0}$ by (2.3.2) and hence to $S^{p-1} \times S^{q-1} \times \mathbb{R}$.

(3) For each i, we change the variables in (2.3.3) as follows:

$$c_i = a_i + b_i,\ d_i = a_i - b_i.$$

Then one sees that the space $A(r(\mathbb{C}P^1 \times \mathbb{C}P^1); \mathbb{R})$ is homeomorphic to the space $A(r\mathbb{C}P^2 \sharp r\overline{\mathbb{C}P^2}; \mathbb{R})$. □

Lemma 2.3.3. *For a finite set A, we denote the cardinality of A by $|A|$. Then*

(1) $|A(p\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2}; \mathbb{Z}/2)| = 2^{2p-1} - 1,$

(2) $|A(r(\mathbb{C}P^1 \times \mathbb{C}P^1); \mathbb{Z}/2)| = 2^{2r-1} + 2^{r-1} - 1.$

Proof. (1) By (2.3.2), we count the number of elements $(a_1, \dots, a_p, b_1, \dots, b_p) \in (\mathbb{Z}/2)^{2p} \backslash \{0\}$ satisfying

$$a_1^2 + \cdots + a_p^2 = b_1^2 + \cdots + b_p^2.$$

This equation is equivalent to the existence of even number of "1" in $a_1, \dots, a_p, b_1, \dots, b_p$. Therefore,

$$|A(p\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2}; \mathbb{Z}/2)| + 1 = \begin{pmatrix} 2p \\ 0 \end{pmatrix} + \begin{pmatrix} 2p \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} 2p \\ 2p \end{pmatrix} = 2^{2p-1}.$$

(2) By (2.3.3), it is enough to show the following:

$$\begin{aligned} &|\{(c_1, \dots, c_r, d_1, \dots, d_r) \in (\mathbb{Z}/2)^{2r} \mid c_1 d_1 + \cdots + c_r d_r = 0\}| \\ &\qquad = 2^{2r-1} + 2^{r-1}. \end{aligned} \tag{2.3.4}$$

We show this by induction. When $r = 1$, we can check (2.3.4) easily. Suppose that (2.3.4) holds when $r = k$, and we consider the case $r = k + 1$. When $c_{k+1} d_{k+1} = 0$ (i.e., (c_{k+1}, d_{k+1}) is $(0, 0), (1, 0)$ or $(0, 1)$), the number of elements

$(c_1, \ldots, c_k, d_1, \ldots, d_k)$ in $(\mathbb{Z}/2)^{2k}$ satisfying $c_1d_1 + \cdots + c_kd_k = 0$ is $2^{2k-1} + 2^{k-1}$ by assumption of induction. When $c_{k+1}d_{k+1} = 1$ (i.e., $(c_{k+1}, d_{k+1}) = (1, 1)$), the number of elements $(c_1, \ldots, c_k, d_1, \ldots, d_k)$ in $(\mathbb{Z}/2)^{2k}$ satisfying $c_1d_1 + \cdots + c_kd_k = 1$ is $2^{2k} - (2^{2k-1} + 2^{k-1})$. So

$$
\begin{aligned}
&|\{(c_1, \ldots, c_{k+1}, d_1, \ldots, d_{k+1}) \in (\mathbb{Z}/2)^{2(k+1)} \mid c_1d_1 + \cdots + c_{k+1}d_{k+1} = 0\}| \\
&= 3(2^{2k-1} + 2^{k-1}) + 2^{2k} - (2^{2k-1} + 2^{k-1}) = 2^{2k+1} + 2^k.
\end{aligned}
$$

Therefore (2.3.4) also holds when $r = k + 1$. □

Note that the manifolds in Proposition 2.3.1 except $\mathbb{C}P^1 \times \mathbb{C}P^1$ do not decompose into the product of two manifolds of positive dimension. The following theorem generalizes Theorem 2.2.1.

Theorem 2.3.4. *Let M_i $(1 \leq i \leq k)$ and M'_j $(1 \leq j \leq \ell)$ be $\mathbb{C}P^1$ or the manifolds in Proposition 2.3.1 except $\mathbb{C}P^1 \times \mathbb{C}P^1$. If $\prod_{i=1}^k M_i$ and $\prod_{j=1}^\ell M'_j$ are diffeomorphic, then $k = \ell$ and there exists an element σ in the symmetric group S_k on k letters such that M_i and $M'_{\sigma(i)}$ are diffeomorphic for all $1 \leq i \leq k$.*

Proof. Let m (resp, $m_{p,q}$, n_r or n) be the number of M_i's diffeomorphic to $\mathbb{C}P^1$ (resp, $p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}$ $(p \geq q \geq 0,\ p + q \geq 1)$, $r(\mathbb{C}P^1 \times \mathbb{C}P^1)$ $(r \geq 2)$ or S^4). Similarly, let m' (resp, $m'_{p,q}$, n'_r or n') be the number of M'_j's diffeomorphic to $\mathbb{C}P^1$ (resp, $p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2}$ $(p \geq q \geq 0,\ p + q \geq 1)$, $r(\mathbb{C}P^1 \times \mathbb{C}P^1)$ $(r \geq 2)$ or S^4).

Therefore,

$$
\begin{aligned}
M &:= \prod_{i=1}^{k} M_i \cong (\mathbb{C}P^1)^m \times \prod_{p\geq q} (p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2})^{m_{p,q}} \\
&\qquad \times \prod_{r\geq 2} (r(\mathbb{C}P^1 \times \mathbb{C}P^1))^{n_r} \times (S^4)^n \\
M' &:= \prod_{j=1}^{\ell} M'_j \cong (\mathbb{C}P^1)^{m'} \times \prod_{p\geq q} (p\mathbb{C}P^2 \sharp q\overline{\mathbb{C}P^2})^{m'_{p,q}} \\
&\qquad \times \prod_{r\geq 2} (r(\mathbb{C}P^1 \times \mathbb{C}P^1))^{n'_r} \times (S^4)^{n'}
\end{aligned} \tag{2.3.5}
$$

By assumption, $H^*(M;\mathbb{Z})$ and $H^*(M';\mathbb{Z})$ are isomorphic as graded rings, and an isomorphism φ between them induces an isomorphism between $H^*(M;R)$ and $H^*(M';R)$ for any commutative ring R and induces a bijection between $A(M;R)$ and $A(M';R)$. When $R = \mathbb{R}$, the bijection is a homeomorphism. Comparing the homeomorphism type and the number of connected components of $A(M;\mathbb{R})$ and $A(M';\mathbb{R})$ using Lemmas 2.2.2 and 2.3.2, we obtain

$$
\begin{aligned}
&2m + 4m_{1,1} = 2m' + 4m'_{1,1}, \quad m_{p,q} = m'_{p,q} \ (p > q \geq 1), \\
&m_{p,p} + n_p = m'_{p,p} + n'_p \ (p \geq 2).
\end{aligned} \tag{2.3.6}
$$

The linear subspace spanned by all one dimensional connected components in $A(M;\mathbb{R})$ (resp, $A(M';\mathbb{R})$) is $H^2((\mathbb{C}P^1)^m \times (\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2})^{m_{1,1}};\mathbb{R})$ (resp, $H^2((\mathbb{C}P^1)^{m'} \times(\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2})^{m'_{1,1}};\mathbb{R})$). Therefore, the isomorphism φ induces an isomorphism between $H^2((\mathbb{C}P^1)^m \times (\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2})^{m_{1,1}};\mathbb{Z})$ and $H^2((\mathbb{C}P^1)^{m'} \times (\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2})^{m'_{1,1}};\mathbb{Z})$. In particular, φ induces an isomorphism between the cohomology rings with $\mathbb{Z}/2$ coefficients. It follows from Lemma 2.2.2 that

$$
\begin{aligned}
&m|A(\mathbb{C}P^1;\mathbb{Z}/2)| + m_{1,1}|A(\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2};\mathbb{Z}/2)| \\
&= m'|A(\mathbb{C}P^1;\mathbb{Z}/2)| + m'_{1,1}|A(\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2};\mathbb{Z}/2)|
\end{aligned}
$$

and hence we have $m + m_{1,1} = m' + m'_{1,1}$ by Lemma 2.2.3. This together with the first identity in (2.3.6) implies that

$$m = m', \quad m_{1,1} = m'_{1,1}. \tag{2.3.7}$$

The linear subspace spanned by all connected components homeomorphic to $S^{p-1} \times S^{p-1} \times \mathbb{R}$ ($p \geq 2$) in $A(M;\mathbb{R})$ (resp, $A(M';\mathbb{R})$) is $H^2((p\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2})^{m_{p,p}} \times (p(\mathbb{C}P^1 \times \mathbb{C}P^1))^{n_p};\mathbb{R})$ (resp, $H^2((p\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2})^{m'_{p,p}} \times (p(\mathbb{C}P^1 \times \mathbb{C}P^1))^{n'_p};\mathbb{R})$). Therefore, it follows from Lemma 2.2.2 that

$$\begin{aligned} &m_{p,p}|A(p\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2};\mathbb{Z}/2)| + n_p|A(p(\mathbb{C}P^1 \times \mathbb{C}P^1);\mathbb{Z}/2)| \\ =&m'_{p,p}|A(p\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2};\mathbb{Z}/2)| + n'_p|A(p(\mathbb{C}P^1 \times \mathbb{C}P^1);\mathbb{Z}/2)| \end{aligned}$$

and hence we have

$$\begin{aligned} &m_{p,p}(2^{2p-1} - 1) + n_p(2^{2p-1} + 2^{p-1} - 1) \\ &\quad = m'_{p,p}(2^{2p-1} - 1) + n'_p(2^{2p-1} + 2^{p-1} - 1) \end{aligned} \tag{2.3.8}$$

by Lemma 2.3.3. So by (2.3.6), (2.3.7), and (2.3.8), we have

$$m = m', \quad m_{p,q} = m'_{p,q} \ (p \geq q \geq 1), \quad n_p = n'_p \ (p \geq 2). \tag{2.3.9}$$

It remains to prove $n = n'$ and $m_{p,0} = m'_{p,0}$ ($p \geq 1$). Since $H^*(M;\mathbb{Z})$ and $H^*(M';\mathbb{Z})$ are isomorphic by assumption, the Poincaré polynomials of M and M' must coincide. So, the Poincaré polynomials of $(S^4)^n \times \prod_{p\geq 1}(p\ \mathbb{C}P^2)^{m_{p,0}}$ and $(S^4)^{n'} \times \prod_{p\geq 1}(p\mathbb{C}P^2)^{m'_{p,0}}$ must coincide by (2.3.5) and (2.3.9). It follows that

$$(1 + x^2)^n \times \prod_{p\geq 1}(1 + px + x^2)^{m_{p,0}} = (1 + x^2)^{n'} \times \prod_{p\geq 1}(1 + px + x^2)^{m'_{p,0}}$$

where x is a variable. This implies that $n = n'$ and $m_{p,0} = m'_{p,0}$. □

Similarly to Corollary 2.2.4, the following corollary follows from Theorem 2.3.4.

Corollary 2.3.5 (cancellation)**.** *Let M, M' and M'' be products of copies of $\mathbb{C}P^1$ and manifolds in Proposition 2.3.1. If $M \times M''$ and $M' \times M''$ are diffeomorphic, then so are M and M'.*

A *topological toric manifold* introduced by Ishida, Fukukawa, and Masuda ([12]) is a compact smooth manifold of real dimension $2n$ with a smooth action of complex torus $(\mathbb{C}^*)^n$ that is locally equivariantly diffeomorphic to a smooth faithful representation space of $(\mathbb{C}^*)^n$. A toric manifold regarded as a smooth manifold is a topological toric manifold. A topological toric manifold of real dimension two is diffeomorphic to $\mathbb{C}P^1$ and the manifolds in Proposition 2.3.1 except S^4 are topological toric manifolds. Therefore, it follows from Theorem 2.3.4 that Theorem 2.2.1 holds for topological toric manifolds, so we may ask the unique decomposition problem for topological toric manifolds and no counterexample is known even to this extended problems.

2.4 Direct decomposition of real toric manifolds

In this section, we will consider the real case of section 2.2. A real part of a toric manifold is a real manifold and is called a *real toric manifold*. Similarly to section 2.2, we deal with real toric manifolds of dimension less than or equal to 2. A real toric manifold of dimension less than or equal to 2 is diffeomorphic to one of the following manifolds;

$$\mathbb{R}P^1, \quad q\mathbb{R}P^2 \ (q \geq 0).$$

Theorem 2.4.1. *Let M_i $(1 \leq i \leq k)$ and M'_j $(1 \leq j \leq \ell)$ be real toric manifolds of dimension less than or equal to two. If $\prod_{i=1}^{k} M_i$ and $\prod_{j=1}^{\ell} M'_j$ are diffeomorphic, then $k = \ell$ and there exists an element σ in the symmetric group S_k on k letters such that M_i and $M'_{\sigma(i)}$ are diffeomorphic for all $1 \leq i \leq k$.*

Proof. Let M be diffeomorphic to $(\mathbb{R}P^1)^\alpha \times \prod_{q\geq 1}(q\mathbb{R}P^2)^{\beta_q}$, and M' be diffeomorphic to $(\mathbb{R}P^1)^{\alpha'} \times \prod_{q\geq 1}(q\mathbb{R}P^2)^{\beta'_q}$. A cohomology ring of each factor of M, M' is the following;

$$H^*(\mathbb{R}P^1; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^2 = 0).$$

$$H^*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^3 = 0).$$

$$H^*(q\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \dots, x_q]/\left(x_1^2 = \cdots = x_q^2, \ \ x_i x_j = 0 \ (i \neq j)\right).$$

The Poincaré polynomial of each factor of M, M' is the following;

$$P(\mathbb{R}P^1) = 1 + x.$$

$$P(\mathbb{R}P^2) = 1 + x + x^2.$$

$$P(q\mathbb{R}P^2) = 1 + qx + x^2.$$

Since $H^*(M;\mathbb{Z}_2)$ is isomorphic to $H^*(M';\mathbb{Z}_2)$, the Poincaré polynomial of M is the same as one of M'. So,

$$(1+x)^{\alpha}\prod_{q\geq 1}(1+qx+x^2)^{\beta_q} = (1+x)^{\alpha'}\prod_{q\geq 1}(1+qx+x^2)^{\beta'_q}.$$

Therefore,

$$(1+x)^{\alpha+2\beta_2}\prod_{q\geq 1, q\neq 2}(1+qx+x^2)^{\beta_q} = (1+x)^{\alpha'+2\beta'_2}\prod_{q\geq 1, q\neq 2}(1+qx+x^2)^{\beta'_q}.$$

Since equations $1+x=0,\ 1+qx+x^2=0$ $(q \geq 1, q \neq 2)$ have different solutions each other,

$$\alpha + 2\beta_2 = \alpha' + 2\beta'_2, \quad \beta_q = \beta'_q \ \ (q \geq 1, q \neq 2). \tag{2.4.1}$$

We consider $A(M;\mathbb{Z}_2) := \{u \in H^1(M;\mathbb{Z}_2) \mid u^2 = 0\}$ and $n(M) := \dim A(M)$. $A(M;\mathbb{Z}_2)$ is a vector space over $\mathbb{Z}_2$, and $n(M)$ is an invariant of cohomology rings. The invariant of each factor of M, M' is the followings;

$$n(\mathbb{R}P^1) = 1, \quad n(\mathbb{R}P^2) = 0, \quad n(q\mathbb{R}P^2) = q - 1.$$

In fact, we can take a basis $x_1 + x_2, x_2 + x_3, \ldots, x_{q-1} + x_q$ of $A(q\mathbb{R}P^2)$. So,

$$n(M) = \alpha + \beta_2 + 2\beta_3 + \cdots + (q-1)\beta_q,$$

$$n(M') = \alpha' + \beta'_2 + 2\beta'_3 + \cdots + (q-1)\beta'_q.$$

Since $n(M) = n(M')$, by (2.4.1) the following holds;

$$\alpha + \beta_2 = \alpha' + \beta'_2.$$

By (2.4.1), $\alpha = \alpha', \beta_2 = \beta'_2$. □

The following corollary follows from Theorem 2.4.1.

Corollary 2.4.2 (cancellation). *Let M, M' and M'' be products of real toric manifolds of dimension less than or equal to two. If $M \times M''$ and $M' \times M''$ are diffeomorphic, then so are M and M'.*

Chapter 3

Spin toric manifolds associated to graphs

3.1 Spin toric manifolds and the orientability of real toric manifolds

In this section, we give a necessary and sufficient condition for a projective toric manifold to admit a spin structure and for a real toric manifold to be orientable. Let P be a Delzant polytope of dimension n in $\mathbb{R}^n$ with m facets, λ be a function mapping each facet of P to its facet vector (i.e. a normal primitive vector to the facet), and λ' be the modulo 2 reduction of λ. A toric manifold constructed from P is written by $M(P)$, and its real part (i.e. its real toric manifold) is written by $M_{\mathbb{R}}(P)$.

Proposition 3.1.1. *The followings are equivalent.*

(1) *The toric manifold $M(P)$ admits a spin structure.*

(2) *The real toric manifold $M_{\mathbb{R}}(P)$ is orientable.*

(3) *There is a homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0, 1\}$ such that $\epsilon(\lambda'(\mathbf{F})) = \{1\}$, where $\mathbf{F}$ is the set of facets of the Delzant polytope P.*

Proof. We prove the equivalence between (1) and (3). We can prove the equivalence between (2) and (3) similarly, so we omit the proof. The equivalence between (2) and (3) was proved by [16], however the following proof is different from their proof.

A manifold M admits a spin structure if and only if its first Stiefel-Whitney class $w_1(M)$ and second Stiefel-Whitney class $w_2(M)$ vanish. Since the cohomology group $H^1(M(P))$ of the projective toric manifold $M(P)$ is trivial, its first Stiefel-Whitney class $w_1(M(P))$ vanishes. So, it is enough to prove the equivalence between (3) and the vanishing of $w_2(M(P))$.

Let T^n be a compact torus $(S^1)^n$, $M = M(P)$, and $\pi : ET^n \times_{T^n} M \to BT^n$ be the Borel construction of M. Since the Serre spectral sequence of π degenerates at the E_2-level, we have the following exact sequence.

$$0 \longrightarrow H^2(BT^n; \mathbb{Z}_2) \xrightarrow{\pi^*} H^2_{T^n}(M; \mathbb{Z}_2) \xrightarrow{\rho^*} H^2(M; \mathbb{Z}_2) \longrightarrow 0, \qquad (3.1.1)$$

where ρ^* is the surjection induced from an inclusion of the fiber $\rho : M \to ET^n \times_{T^n} M$.

Let $F_1, \dots, F_m$ be the facets of P and $\tau_1, \dots, \tau_m$ be elements in $H^2_{T^n}(M; \mathbb{Z}_2)$ which are Poincaré dual to the characteristic submanifolds of M corresponding to $F_1, \dots, F_m$. Then, $\pi^*(u)$ is written as a linear combination of $\tau_1, \dots, \tau_m$ as follows (see [15] for example):

$$\pi^*(u) = \sum_{i=1}^{m} v_i(u)\tau_i.$$

Here, v_i can be regarded as an element in $\mathrm{Hom}(H^2(BT^n); \mathbb{Z}_2) = H_2(BT^n\ ; \mathbb{Z}_2)$. So, $\pi^*(u)$ is written as follows:

$$\pi^*(u) = \sum_{i=1}^{m} \langle u, v_i \rangle \tau_i,$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between cohomology and homology. Let λ' be a homomorphism $\mathbf{F}$ to $H_2(BT^n; \mathbb{Z}_2)$ which maps F_i to v_i. Then,

$$\pi^*(u) = \langle u, \lambda'(F_1) \rangle \tau_1 + \cdots + \langle u, \lambda'(F_m) \rangle \tau_m.$$

It is known that the equivariant Stiefel-Whitney class $w^{T^n}(M)$ is of the form

$$w^{T^n}(M) = \prod_{i=1}^{m} (1 + \tau_i),$$

so we have $w_2^{T^n}(M) = \sum_{i=1}^{m} \tau_i$ ([7]). The second Stiefel-Whitney class $w_2(M)$ is the image of $w_2^{T^n}(M)$ by ρ^* in (3.1.1). Since (3.1.1) is an exact sequence, the equation $w_2(M) = 0$ is equivalent to the existence of an element u in $H^2(BT^n; \mathbb{Z}_2)$ such that $\pi^*(u) = w_2^{T^n}(M)$. So we have

$$\sum_{i=1}^{m} \langle u, \lambda'(F_i) \rangle \tau_i = \sum_{i=1}^{m} \tau_i.$$

Therefore, $w_2(M)$ vanishes if and only if $\langle u, \lambda'(F_i) \rangle$ is 1 for each $i = 1, \dots, m$, which implies the equivalence between (1) and (3). □

Remark 3.1.2. The same proof as above shows that Proposition 3.1.1 holds for a toric manifold whose realization of the underlying simplicial complex of the corresponding fan is a disk ([1, 13]), for a quasitoric manifold ([7]) and for a topological toric manifold ([12]).

A truncation of a Delzant polytope P along a face corresponds to blowing-up along the submanifold of $M(P)$ corresponding to the face. To be precise, let F be a codimension k face which is an intersection of k facets $F_1, \ldots, F_k$ of a Delzant polytope P, and $\lambda(F_i)$ be the facet vector of the facet F_i for each i. A face truncation at F is to cut P along the face F in such a way that the facet vector of the new facet is $\lambda(F_1) + \cdots + \lambda(F_k)$ (Figure 3.1). The projective toric manifold corresponding to the truncated Delzant polytope is formed by blowing-up $M(P)$ along the submanifold corresponding to the face F.

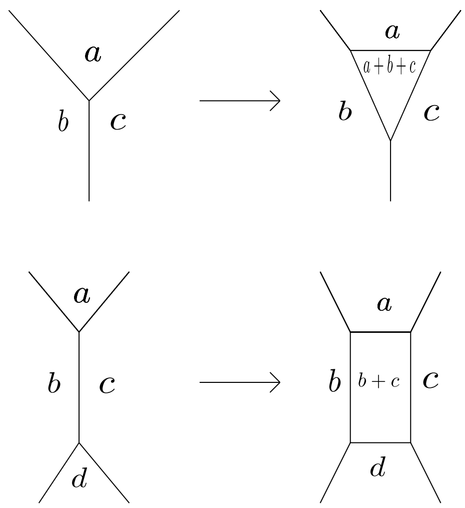


Figure 3.1: face truncations and new facet vectors corresponding to blowing-up

3.2 Spin toric manifolds associated to simple graphs and building sets

We set $[n+1] := \{1, \ldots, n+1\}$. In this section, we assume that a graph G is finite and simple, and review the construction of a toric manifold $M(G)$ (resp. $M(B)$) from a finite simple graph G (resp. a building set B on $[n+1]$), and characterize a graph G (resp. a building set B) whose associated toric manifold $M(G)$ (resp. $M(B)$) admits a spin structure. There are two kinds of constructions of a Delzant polytope from G (resp. B). One is to realize a Delzant polytope in $\mathbb{R}^{n+1}$ by Minkowski sum, and the other is to truncate faces of a simplex in $\mathbb{R}^n$. In this section, we use the second construction.

Let G be a simple graph with $n+1$ nodes, and its node set $V(G)$ be $[n+1]$. We set

$$B(G) := \{I \subset V(G) \mid G|I \text{ is connected}\},$$

where $G|I$ is a maximal subgraph of G with the node set I (i.e. the induced subgraph). The empty set $\emptyset$ is not in $B(G)$. We call $B(G)$ a *graphical building set* of G. A graphical building set $B(G)$ is a building set on $V(G)$, so we review the construction of a toric manifold from a building set.

Definition 3.2.1. A *building set* B on $[n+1]$ is a collection of nonempty subsets of $[n+1]$ such that

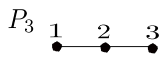
(1) B contains all singletons $\{i\}$ for every i,

(2) if $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.

If $[n+1] \in B$, then B is called a *connected* building set.

Example 3.2.2. We consider the following path graph P_3.

Then,



$$B(P_3) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\},$$

which we simply express as follows:

$$B(P_3) = \{1, 2, 3, 12, 23, 123\}.$$

When a building set B is connected, we construct a Delzant polytope P_B in $\mathbb{R}^n$ as follows ([6]). We take an n-simplex in $\mathbb{R}^n$ such that its facet vectors are $e_1, \ldots, e_n$, and $-e_1 - \cdots - e_n$, where $e_1, \ldots, e_n$ are the standard basis of $\mathbb{R}^n$. Each facet vector e_i $(1 \leq i \leq n)$ corresponds to an element i in B, and the facet vector $-e_1 - \cdots - e_n$ corresponds to an element $n+1$ in B, where i in B means the singleton $\{i\}$ in B. We truncate the n-simplex along faces in increasing order of dimension. Let F_i denote the facet corresponding to an element i in B. For every element $I = i_1 \ldots i_k$ in $B \setminus [n+1]$ we truncate the simplex along a face $F_{i_1} \cap \cdots \cap F_{i_k}$ in such a way that the facet vector of the new facet, denoted F_I, is the sum of the facet vectors of the facets $F_{i_1}, \ldots, F_{i_k}$. Then the resulting polytope, denoted P_B, is a Delzant polytope, and called a *nestohedron.* The set $B \setminus [n+1]$ one-to-one corresponds to the set of facets of P_B. Let $M(B)(M_{\mathbb{R}}(B))$ denote a (real) toric manifold corresponding to P_B. A nestohedron constructed from a graphical building set $B(G)$ is called a *graph associahedron*, and the associated (real) toric manifold is denoted by $M(G)(M_{\mathbb{R}}(G))$. When a building set B is disconnected, the corresponding nestohedron is defined as the product of nestohedra associated to connected building sets in B. The corresponding (real) toric manifold is also defined as the product of (real) toric manifolds associated to connected building sets in B.

Remark 3.2.3. The size of an n-simplex is not important because the size does not affect the topology of the associated toric manifolds. The important data are a simple polytope and its facet vectors.

Example 3.2.4.

(1) When a graph G is a point, the associated (real) toric manifold is also a point. We understand that a point is orientable and admits a spin structure.

(2) When G is a connected graph with 2 nodes, the corresponding graph associahedron P_G in $\mathbb{R}$ is an 1-simplex (Figure 3.2), and the associated (real) toric manifold is diffeomorphic to $\mathbb{C}P^1$ ($\mathbb{R}P^1$). $\mathbb{C}P^1$ admits a spin structure and $\mathbb{R}P^1$ is orientable.

(3) When G is a connected graph with 3 nodes, G is a path graph P_3 or cycle graph C_3. If G is the path graph P_3, then its graphical building set $B(P_3)$ is $\{1, 2, 3, 12, 23, 123\}$, and the corresponding graph associahedron P_{P_3} is a pentagon (Figure 3.2). So, the associated toric manifold is diffeomorphic to $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ and does not admit a spin structure. If G is the cycle graph C_3,

then its graphical building set $B(C_3)$ is $\{1,2,3,12,23,31,123\}$, and the corresponding graph associahedron P_{C_3} in $\mathbb{R}^2$ is a hexagon (Figure 3.2). So, the associated toric manifold is diffeomorphic to $\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}$ and also does not admit a spin structure.

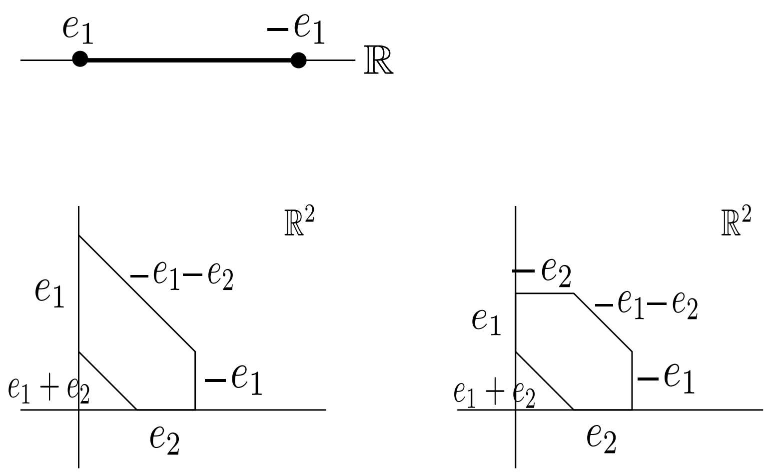


Figure 3.2: graph associahedra and facet vectors in (2) and (3)

Example 3.2.5.

(1) A building set on [1] is only $\{1\}$, so the corresponding nestohedron $P_{\{1\}}$ is a point, and the associated (real) toric manifold is a point.

(2) Building sets on [2] are $\{1,2\}$ and $\{1,2,12\}$. If B is $\{1,2\}$, then its nestohedron P_B is a point, so the associated (real) toric manifold is a point. If B is $\{1,2,12\}$, then its nestohedron P_B is an 1-simplex, so the associated (real) toric manifold is diffeomorphic to $\mathbb{C}P^1$ ($\mathbb{R}P^1$).

(3) Building sets on [3] are essentially the following.

$$\{1,2,3\},\ \{1,2,3,12\},\ \{1,2,3,12,23,123\},$$

$$\{1,2,3,12,23,31,123\}, \{1,2,3,123\},\ \{1,2,3,12,123\}.$$

Each nestohedron P_B is a point, 1-simplex, pentagon, hexagon, 2-simplex, and square. The last two are not constructed from any graph, and the corresponding Delzant polytopes are as in Figure 3.3. The toric manifolds $M(B)$ (resp, real toric manifolds $M_{\mathbb{R}}(B)$) associated to the building sets are respectively diffeomorphic to a point, $\mathbb{C}P^1, \mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$, $\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}$, $\mathbb{C}P^2$, and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ (resp, a point, $\mathbb{R}P^1, 3\mathbb{R}P^2, 4\mathbb{R}P^2, \mathbb{R}P^2$, and $2\mathbb{R}P^2$).

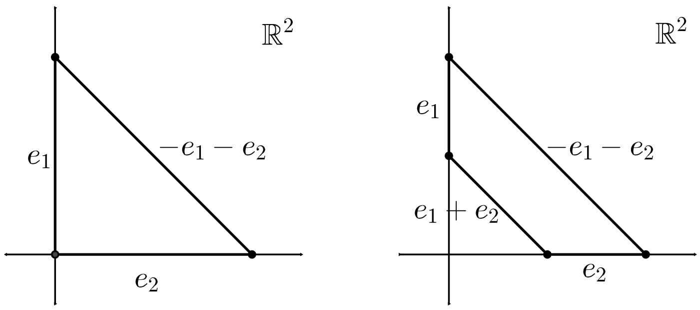


Figure 3.3: nestohedra corresponding to {1,2,3,123} and {1,2,3,12,123}

Lemma 3.2.6. *Let B be a connected building set on $[n+1]$. Then the following are equivalent.*

(1) *The toric manifold $M(B)$ admits a spin structure.*

(2) *The real toric manifold $M_{\mathbb{R}}(B)$ is orientable.*

(3) *$n+1$ is even and any element in $B \backslash \{[n+1]\}$ has odd order.*

Proof. Let $\mathbf{F}$ be the set of facets of the nestohedron P_B, λ be a function mapping each facet of P_B to its facet vector, and λ' be the modulo 2 reduction of λ. By Proposition 3.1.1, it is enough to show the equivalence between (3) and the existence of a homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0,1\}$ satisfying $\epsilon(\lambda'(\mathbf{F})) = \{1\}$.

The nestohedron P_B has $e_1, \ldots, e_n$, and $e_1 + \cdots + e_n$ as facet vectors modulo 2, where the facets associated to these facet vectors correspond to the singletons in B, that is, $\lambda'(F_1) = e_1, \ldots, \lambda'(F_n) = e_n, \lambda'(F_{n+1}) = e_1 + \cdots + e_n$. Suppose that there is a homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0,1\}$ such that $\epsilon(\lambda'(\mathbf{F})) = \{1\}$. Then n is odd. We assume that there is an element I with an even order in $B \backslash \{[n+1]\}$, and let F_I be the facet of P_B corresponding to I. Then, since $\epsilon(\lambda'(F_1)) = \cdots = \epsilon(\lambda'(F_{n+1})) = 1$, we have $\epsilon(\lambda'(F_I)) = 0$. This is a contradiction.

If (3) holds, then we can take the homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0,1\}$ mapping each e_i to 1. $\square$

Lemma 3.2.7. *Suppose that a smooth manifold M is diffeomorphic to the product of smooth manifolds $M_1, \ldots, M_k$. Then the followings hold.*

(1) *M is orientable if and only if each factor M_i is orientable.*

(2) *M admits a spin structure if and only if each factor M_i admits a spin structure.*

Proof. We use the following formula. Let ξ, η be vector bundles over base spaces B_1, B_2. Then the l-th Stiefel-Whitney class of the product bundle $\xi \times \eta$ over $B_1 \times B_2$ is

$$w_l(\xi \times \eta) = \sum_{i=0}^{l} w_i(\xi) \times w_{l-i}(\eta). \tag{3.2.1}$$

In particular

$$w_1(M) = w_1(M_1) + \cdots + w_1(M_k).$$

Therefore, $w_1(M) = 0$ if and only if $w_1(M_1) = \cdots = w_1(M_k) = 0$ since there is no relation among $w_1(M_1), \ldots, w_1(M_k)$. This means (1).

If M admits a spin structure, then $w_1(M_1) = \cdots = w_1(M_k) = 0$ because of the orientability of each M_i. So, it follows from (3.2.1) that

$$w_2(M) = w_2(M_1) + \cdots + w_2(M_k).$$

Therefore $w_2(M) = 0$ if and only if there is no relation among $w_2(M_1), \ldots, w_2(M_k)$, so $w_2(M_1) = \cdots = w_2(M_k) = 0$. This means (2). □

The following theorem follows from Lemmas 3.2.6 and 3.2.7.

Theorem 3.2.8. *Let B be an union of connected building sets $B_1, \ldots, B_k$ on subsets $S_1, \ldots, S_k$ in $[n+1]$. Then the following are equivalent.*

(1) *The toric manifold $M(B)$ admits a spin structure.*

(2) *The real toric manifold $M_{\mathbb{R}}(B)$ is orientable.*

(3) *Each building set B_i satisfies either of the following.*

(I) $|S_i| = 1$.

(II) $|S_i|$ *is even and any element in $B_i \backslash \{S_i\}$ has an odd order.*

Corollary 3.2.9. *Let G be a finite simple graph.*

(1) The toric manifold $M(G)$ admits a spin structure if and only if $M(G)$ is diffeomorphic to $(\mathbb{C}P^1)^k$.

(2) The real toric manifold $M_{\mathbb{R}}(G)$ is orientable if and only if $M_{\mathbb{R}}(G)$ is diffeomorphic to $(\mathbb{R}P^1)^k$.

Moreover, the corresponding graph is the disjoint union of k connected graphs with 2 nodes and finitely many points.

Proof. We assume that a graph G has k connected component $G_1, \ldots, G_k$. Then we can take the graphical building set of G as B in Theorem 3.2.8, the graphical building set of G_i as B_i, and the node set of G_i as S_i. (3)(I) in Theorem 3.2.8 means that G_i is a point, and (3)(II) means that G_i is a connected graph with 2 nodes. In fact, G_i has even nodes because $|S_i|$ is even, and if G_i has more than or equal to 4 nodes, then G_i has a connected proper subgraph with 2 nodes, which gives an even order element in $B_i \setminus \{S_i\}$. □

Remark 3.2.10. A toric manifold M has trivial 1-st cohomology group ([9]), so that M admits only one spin structure if M admits a spin structure.

3.3 Spin toric manifolds associated to pseudographs

In this section, we construct a toric manifold $M(G)$ from a pseudograph G (i.e. a graph may have multiedges and loops) ([2]), and characterize a pseudograph G whose associated toric manifold $M(G)$ admits a spin structure. We assume that a pseudograph G is finite. A toric manifold $M(G)$ is not compact when G is a pseudograph with at least one loop. So, we call a nonsingular toric variety a *toric manifold* in this section.

Definition 3.3.1. Let G be a pseudograph.

(1) A *tube* G_t of G is a proper connected subgraph of G such that if a pair of nodes of G_t is connected by an edge of G, then G_t contains at least one edge connecting the pair.

(2) Two tubes are *compatible*, if one is included in the other, or they are disjoint and cannot be connected by an edge of G.

(3) A *tubing* of G is the set of pairwise compatible tubes and the union of such tubes is not G.

Example 3.3.2. (a) and (b) in Figure 3.4 are tubings. However, (c) in Figure 3.4 is not a tubing because two tubes are not compatible. (d) in Figure 3.4 is also not a tubing because the union of the tubes is the whole graph.

Definition 3.3.3. Let G be a pseudograph.

(1) Suppose that a pair of nodes is connected by at least two edges. Then the

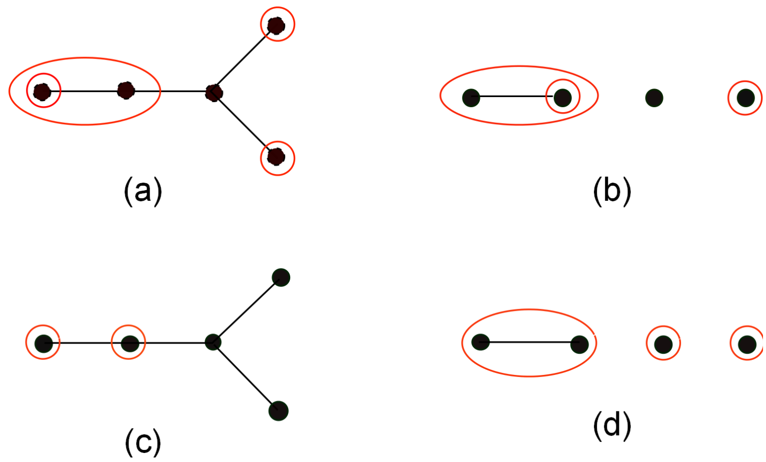


Figure 3.4: tubings and non-tubings

set of all edges connecting the pair of nodes is called a *bundle*.

(2) The *underlying simple graph* G_s of G is the graph obtained by deleting all loops and replacing each bundle to an edge.

Example 3.3.4. The underlying simple graph of the left pseudograph in Figure 3.5 is the right simple graph. Here, B_1 and B_2 are bundles.

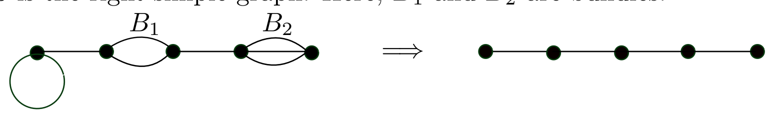


Figure 3.5: underlying simple graph

For each tube G_t of a pseudograph G, we define a set S as follows.

(1) All nodes of G_t are in S.

(2) All edges of G_t except for edges not contained in bundles and all loops of G_t are in S.

(3) All edges in bundles of G not containing edges of G_t are in S.

(4) All loops not incident to any node of G_t are in S.

We call S a *label* of G_t.

Definition 3.3.5. A tube G_t is called *full,* if it is a subgraph that consists of some of the nodes of the original graph and all of the edges that connect them in the original graph (i.e. an induced subgraph of G).

Example 3.3.6. Figure 3.6 shows examples of full tubes of a graph and their associated labeling. Here, $3abcd$ means the set $\{3, a, b, c, d\}$.

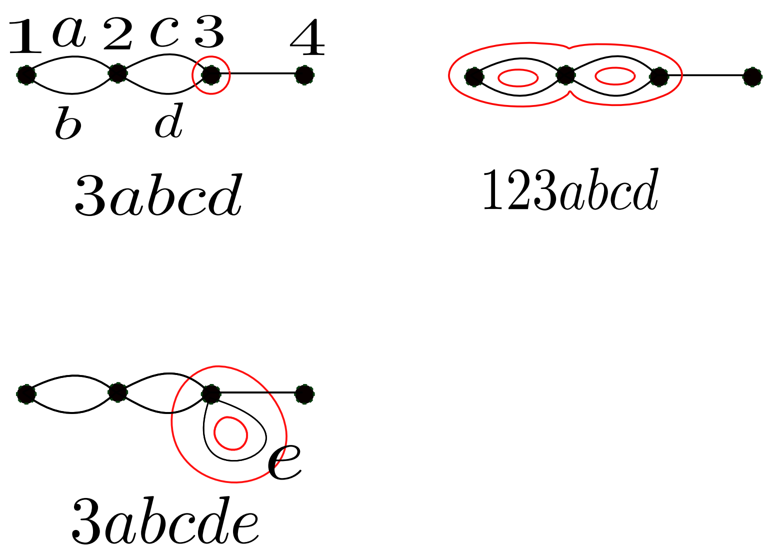


Figure 3.6: full tubes and corresponding labels

Let G be a pseudograph with $n+1$ nodes and l loops, $B_1, \ldots, B_k$ be bundles of G with $b_1+1, \ldots, b_k+1$ edges, Δ^s be an s-simplex, and ρ be a ray. We define

$$\Sigma_G := \Delta^n \times \prod_{i=1}^{k} \Delta^{b_i} \times \rho^l,$$

and label every face in Σ_G as follows.

(1) Each facet of Δ^n corresponds to a node of G. Each face of Δ^n corresponds to a proper subset of the node set of G and is the intersection of the facets associated to nodes in that subset.

(2) Each vertex of Δ^{b_i} corresponds to an edge of the bundle B_i. Each face of Δ^{b_i} corresponds to a subset of an edge set of B_i defined by the vertices spanning the face.

(3) Each ρ corresponds to a loop of G.

Each face of Σ_G is labeled by the product of each factor naturally.

Remark 3.3.7. Let G_t be a tube of G. Suppose that the label of G_t contains k nodes of G and does not contain l edges in bundles and m loops. Then the face of Σ_G corresponding to G_t is of codimension $k+l+m$ by the way of labeling faces of Σ_G.

Facets of Σ_G are

$$\begin{aligned} &(\text{facets of } \Delta^n) \times \prod_{i=1}^{k} \Delta^{b_i} \times \rho^l, \\ &\Delta^n \times (\text{facets of } \Delta^{b_j}) \times \prod_{i=1, i\neq j}^{k} \Delta^{b_i} \times \rho^l \quad (j = 1, \dots, k), \text{ and} \\ &\Delta^n \times \prod_{i=1}^{k} \Delta^{b_i} \times (\text{facets of } \rho^l). \end{aligned}$$

The number of facets in each line above is $n+1, \displaystyle\sum_{j=1}^{k}(b_j+1)$, and l respectively. We embed Σ_G in an Euclidean space such that a facet vector of each facet is respectively

$$e_1, \dots, e_n, -e_1 - \cdots - e_n,$$

$$e_{n+1}, \dots, e_{n+b_1}, -e_{n+1} - \cdots - e_{n+b_1},$$

$$e_{n+b_1+1}, \dots, e_{n+b_1+b_2}, -e_{n+b_1+1} - \cdots - e_{n+b_1+b_2},$$

$$\vdots$$

$$e_{n+b_1+\cdots+b_{k-1}+1}, \dots, e_{n+b_1+\cdots+b_k}, -e_{n+b_1+\cdots+b_{k-1}+1} - \cdots - e_{n+b_1+\cdots+b_k},$$

$$e_{n+b_1+\cdots+b_k+1}, \dots, e_{n+b_1+\cdots+b_k+l}.$$

Here, $\{e_i\}_i$ is the standard basis in the Euclidean space of the dimension of Σ_G.

Example 3.3.8. We consider the pseudograph G drawn below. We embed Σ_G in $\mathbb{R}^3$ in such a way that each facet vector is

$$1ab \to e_1, \quad 2ab \to e_2, \quad 3ab \to -e_1 - e_2, \quad 123a \to e_3, \quad 123b \to -e_3.$$

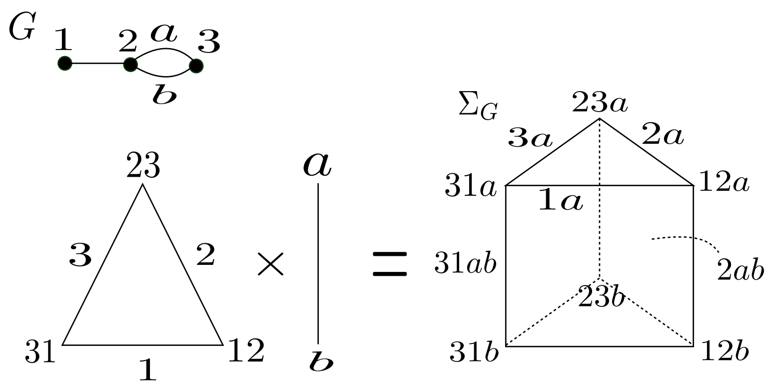


Figure 3.7: Σ_G and labels of faces

Then, we construct a pseudograph associahedron KG by truncating Σ_G along some faces. At first, one truncates Σ_G along faces with labels corresponding to full tubes as follows. If a face F of Σ_G with a label corresponding to a full tube is denoted by $F_1 \cap \cdots \cap F_k$, where each F_i is a facet of Σ_G, then truncate Σ_G along the face F in such a way that the facet vector of the new facet is the sum of the facet vectors of $F_1, \ldots, F_k$. We repeat this truncation from low dimensional faces to high dimensional faces. The label corresponding to a full tube is (nodes of this full tube)(every edge in bundles and every loop in G). Therefore, if we truncate Σ_G along all faces with labels corresponding to full tubes, then Σ_G turns to

$$P_{G_s} \times \prod_{i=1}^{k} \Delta^{b_i} \times \rho^l, \tag{3.3.1}$$

where P_{G_s} is the graph associahedron corresponding to the underlying simple graph G_s of G. Next, one truncates (3.3.1) along faces with labels corresponding to non-full tubes in the same way as full tubes.

Proposition 3.3.9. ([2]) *Let G be a pseudograph, and KG be the pseudograph associahedron constructed from G. If G does not have any loop, then KG is a Delzant polytope and if G has a loop, then KG is a simple polyhedral cone. Its face poset is isomorphic to the set of tubings of G, ordered under the reverse subset containment. In particular, there is a one-to-one correspondence between facets of KG and tubes of G.*

We denote the (real) toric manifold corresponding to KG by $M(G)$ ($M_{\mathbb{R}}(G)$).

Example 3.3.10. We shall observe the pseudograph associahedron KG for the pseudograph G in Example 3.3.8. Figure 3.8 indicates all tubes of G and the corresponding labels. The first line indicates full tubes, and the second line indicates non-full tubes. Truncating Σ_G along faces with labels corresponding

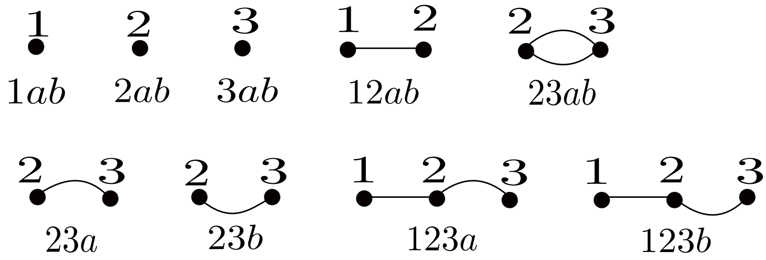


Figure 3.8: tubes and corresponding labels

to the full tubes, Σ_G turns into the left in Figure 3.9. This is the product of 1-simplex and the graph associahedron constructed from the underlying simple graph of G. Moreover, truncating the left in Figure 3.9 along faces with labels corresponding to non-full tubes, the left turns into the right in Figure 3.9. This is the pseudograph associahedron KG associated to G. Each facet vector is as

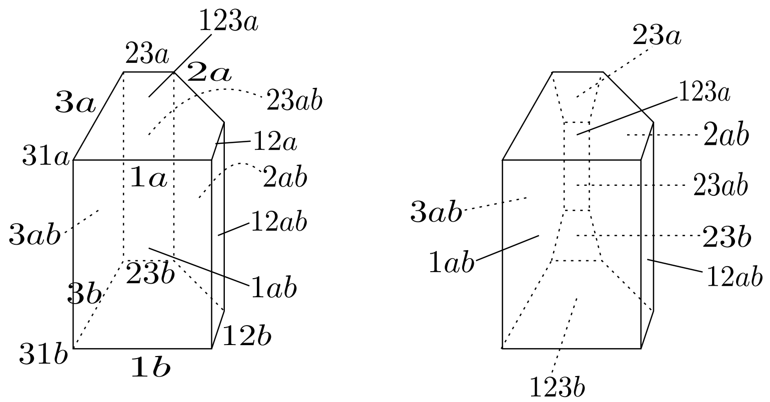


Figure 3.9: pseudograph associahedron

follows:

$$1ab \to e_1, \qquad 2ab \to e_2, \qquad 3ab \to -e_1 - e_2,$$

$$12ab \to e_1 + e_2, \qquad 23ab \to -e_1, \qquad 23a \to -e_1 + e_3,$$

$$23b \to -e_1 - e_3, \qquad 123a \to e_3, \qquad 123b \to -e_3$$

Example 3.3.11. When G is the disjoint union of $n+1$ nodes, the pseudograph associahedron KG is as follows. The polytope Σ_G is an n-simplex, and the nodes of G correspond to the $n+1$ facets of the n-simplex. Every tube of G is 1 node and full. Suppose that the tube G_i of G is the node i of G, then the label of G_i is i. So, KG is an n-simplex since KG is a polytope obtained by truncating the n-simplex along $n+1$ facets. Therefore, the associated toric manifold $M(G)$ is diffeomorphic to $\mathbb{C}P^n$.

Remark 3.3.12. The graph associahedron P_G of G above is a point. If a simple graph G is not connected, then the associated pseudograph associahedron KG is different from the graph associahedron P_G.

Theorem 3.3.13. *Let G be a pseudograph.*
(1) The toric manifold $M(G)$ admits a spin structure if and only if $M(G)$ is diffeomorphic to one of $\mathbb{C}P^{k-1}(k : 1 \text{ or even}), \mathbb{C}P^1, \mathbb{C}P^1 \times \mathbb{C}P^1$, and $\mathbb{C}$.
(2) The real toric manifold $M_{\mathbb{R}}(G)$ is orientable if and only if $M_{\mathbb{R}}(G)$ is diffeomorphic to one of $\mathbb{R}P^{k-1}(k : 1 \text{ or even}), \mathbb{R}P^1, \mathbb{R}P^1 \times \mathbb{R}P^1$, and $\mathbb{R}$.
Moreover, the associated pseudograph is respectively the disjoint union of k

nodes, a connected simple graph with 2 nodes, a connected pseudograph with 2 nodes and 2 multiedges, and 1 node with 1 loop.

Remark 3.3.14. If G is a pseudograph with loops, then the realization of the underlying simplicial complex which is dual to the boundary complex of KG is a disk. Because truncating Σ_G along faces preserves the homeomorphic type of a realization of the underlying simplicial complex. So, by Remark 3.1.2, Proposition 3.1.1 can be applied even if G has loops.

Proof. If $M(G)$ is diffeomorphic to one of $\mathbb{C}P^{k-1}$ (k : 1 or even), $\mathbb{C}P^1$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $\mathbb{C}$, then $M(G)$ admits a spin structure.

The toric manifold $M(G)$ does not admit any spin structure unless the following two conditions are satisfied:

The cardinality of the node set $V(G)$is 1 or even. (3.3.2)

The number of multiedges in any bundle is even. (3.3.3)

Because if the cardinality $n+1$ of the node set $V(G)$ is more than one, then KG has facet vectors $e_1, \ldots, e_n, -e_1 - \cdots - e_n$, so (3) in Proposition 3.1.1 implies (3.3.2) if $M(G)$ admits a spin structure. A similar argument implies (3.3.3). If Σ_G is truncated along a codimension 2 face, then (3) in Proposition 3.1.1 is not satisfied. Therefore, it is enough to consider G which satisfies (3.3.2) and (3.3.3) and whose associated pseudograph associahedron KG is constructed without truncating Σ_G along any codimension 2 faces.

Suppose that G contains a proper full tube shown in Figure 3.10. The label of this full tube is ij(all edges in all bundles and all loops), so this tube

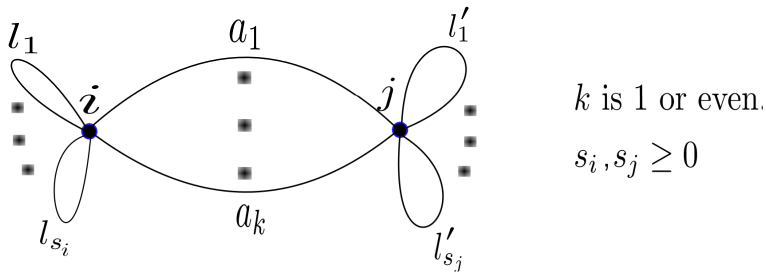


Figure 3.10: proper full tube

corresponds to a codimension 2 face of Σ_G by Remark 3.3.7. Therefore, G does not contain the proper full tube in Figure 3.10 if $M(G)$ admits a spin structure.

(1) Assume that G is a connected pseudograph in Figure 3.10 with the node set $\{1, 2\}$ and has more than or equal to 2 loops (Figure 3.11). Labels of two full tubes are $1a_1 \ldots a_k l_1 \ldots l_{s_1} l'_1 \ldots l'_{s_2}$ and $2a_1 \ldots a_k l_1 \ldots l_{s_1}\ l'_1 \ldots l'_{s_2}$, and corresponding faces of Σ_G are two facets. Since truncating Σ_G along facets does not change Σ_G, a non-full tube obtained by removing 2 loops from G corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

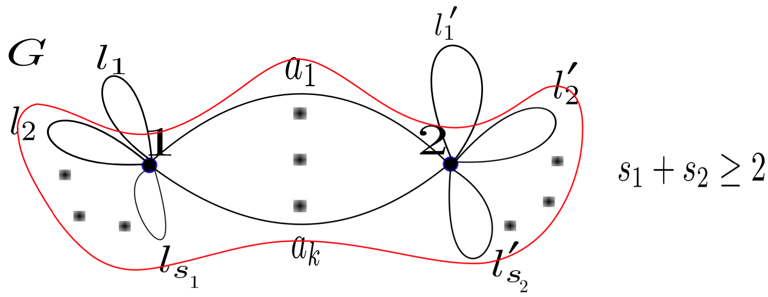


Figure 3.11: pseudograph and non-full tube in (1)

(2) Assume that G is a pseudograph with the node set $\{1, 2\}$, edges $a_1, \ldots, a_k$

(k is 1 or even) and a loop l incident to the node 1 (Figure 3.12). Labels of two full tubes of G are $1a_1 \dots a_k l$ and $2a_1 \dots a_k l$, and corresponding faces of Σ_G are two facets. Similarly to (1), a non-full tube which is the node 1 corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

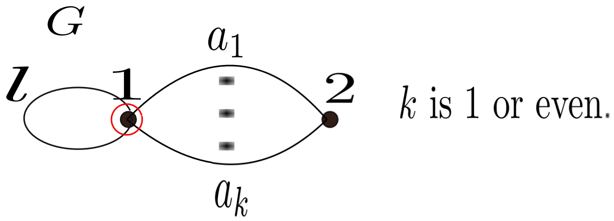


Figure 3.12: pseudograph and non-full tube in (2)

(3) Assume that G is a pseudograph with the node set $\{1, 2\}$ and multiedges $a_1, \dots, a_k$ ($k \geq 4$, even) (Figure 3.13). Labels of full tubes are $1a_1 \dots a_k$ and $2a_1 \dots a_k$, and corresponding faces of Σ_G are two facets. So, a non-full tube obtained by removing 2 edges from G corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

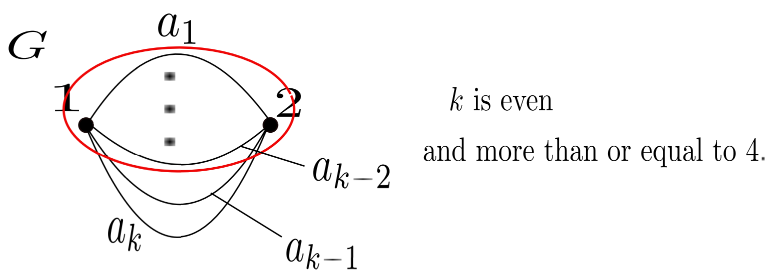


Figure 3.13: pseudograph and non-full tube in (3)

(4) If G is a pseudograph with the node set $\{1, 2\}$ and has 1 or 2 multiedges

but does not have loops (Figure 3.14), then the associated toric manifolds $\mathbb{C}P^1$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ admit spin structures.

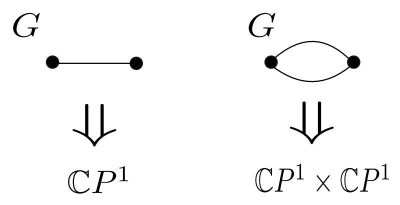


Figure 3.14: (4)

(5) Assume that G is a pseudograph with 1 node and s loops ($s \geq 2$) (Figure 3.15). There is no full tube, so a non-full tube obtained by removing 2 loops from G corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

(6) If G is a pseudograph with 1 node and 1 loop, then the associated toric manifold $\mathbb{C}$ admits a spin structure. If G is 1 node, then the associated toric manifold is a point and admits a spin structure (Figure 3.16).

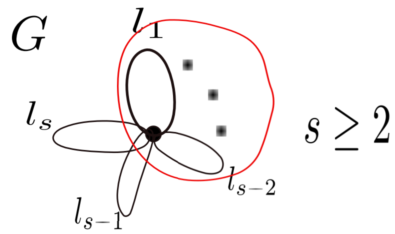


Figure 3.15: pseudograph and non-full tube in (5)

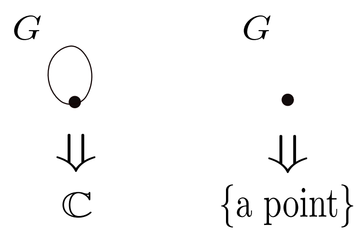


Figure 3.16: (6)

The above observation shows that if G is connected, then the associated toric manifold admits a spin structure if and only if G is 1 node, 1 node with 1 loop, a path graph with 2 nodes, or a pseudograph with 2 nodes and 2 multiedges.

Suppose that G is not connected. Then each connected component of G has only 1 node since G does not contain a proper full tube in Figure 3.10. If a connected component of G has s loops ($s \geq 1$), then a tube obtained by removing 1 loop from the connected component corresponds to a codimension 2 face of Σ_G. So, if G is not connected, then each connected component of G is 1 node if $M(G)$ admits a spin structure. □

Chapter 4

Relation between a root system and Delzant polytope constructed from a connected simple graph

Let G be a connected simple graph with $n+1$ nodes ($n \geq 1$) and $V(G)$ be the node set of G. In section 3.2 we explained how to construct a graph associahedron P_G and how to take facet vectors of P_G. We denote by $F(G)$ the set of facet vectors of P_G.

As mentioned in the Introduction, $F(G)$ is *dual* to a root system of type A when G is a complete graph. We shall explain what this means. If G is a complete graph K_{n+1} with $n+1$ nodes, then the graphical building set $B(K_{n+1})$

(see section 3.2) consists of all subsets of $[n+1]$ so that the graph associahedron $P_{K_{n+1}}$ is a permutohedron obtained by cutting all faces of the n-simplex with facet vectors $e_1, \dots, e_n, -(e_1 + \cdots + e_n)$, where $e_1, \dots, e_n$ denote the standard base of $\mathbb{R}^n$ as before. It follows that

$$F(K_{n+1}) = \big\{ \pm e_I \mid \emptyset \neq I \subset [n] \big\} \quad \text{where } e_I = \sum_{i \in I} e_i. \tag{4.0.1}$$

On the other hand, consider the standard root system $\Delta(A_n)$ of type A_n given by

$$\Delta(A_n) := \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\} \tag{4.0.2}$$

which lies on the hyperplane H of $\mathbb{R}^{n+1}$ defined by $e_1 + \cdots + e_{n+1} = 0$. Take $e_1 - e_2, e_2 - e_3, \dots, e_n - e_{n+1}$ as a base of $\Delta(A_n)$ as usual. Then their dual base with respect to the standard inner product is what is called the fundamental dominant weights given by

$$\lambda_i = (e_1 + \cdots + e_i) - \frac{i}{n+1}(e_1 + \cdots + e_{n+1}) \quad (i = 1, 2, \dots, n) \tag{4.0.3}$$

which also lie on the hyperplane H. The Weyl group action permutes $e_1, \dots, e_{n+1}$ so that it preserves H. We identify H with the quotient vector space H^* of $\mathbb{R}^{n+1}$ by the line spanned by $e_1 + \cdots + e_{n+1}$ using the inner product, namely put the condition $e_1 + \cdots + e_{n+1} = 0$. Then the set of elements obtained from the orbits of $\lambda_1, \dots, \lambda_n$ by the Weyl group action is

$$\big\{ \sum_{j \in J} e_J \mid \emptyset \neq J \subset [n+1] \big\} \quad \text{in } H^*.$$

This set agrees with $F(K_{n+1})$ in (4.0.1) because $e_{n+1} = -(e_1 + \cdots + e_n)$. In this sense $F(K_{n+1})$ is *dual* to $\Delta(A_n)$.

We note that $F(K_{n+1})$ itself forms a root system (of type A_n) when $n = 1$ or 2. However the following holds.

Lemma 4.0.1. *If $n \geq 3$, then $F(K_{n+1})$ does not form a root system.*

Proof. Suppose that $F(K_{n+1})$ forms a root system for $n \geq 3$. Then $F(K_{n+1})$ is of rank n by (4.0.1). Let $\{\alpha_1, \ldots.\alpha_n\}$ be a base of the root system $F(K_{n+1})$. Then $\sum_{j \in J} \alpha_j$ are in $F(K_{n+1})$ for any nonempty subset J of $[n]$ (see [11, the first corollary in p.50 and the latter Lemma A in p.52]). This shows that the number of positive roots in $F(K_{n+1})$ is at least $2^n - 1$ and hence $|F(K_{n+1})| \geq 2^{n+1} - 2$ while $|F(K_{n+1})| = 2^{n+1} - 2$ by (4.0.1), where $|\quad|$ denotes cardinality. This means that any element in $F(K_{n+1})$ is of the form $\pm \sum_{j \in J} \alpha_j$.

By (4.0.1) $\alpha_j = \pm e_{I_j}$ for some subset I_j of $[n]$. We may assume that α_j is e_{I_j} for $1 \leq j \leq k$ and $-e_{I_j}$ for $k+1 \leq j \leq n$ for some k without loss of generality. We note that

$$\begin{aligned} &e_I + e_{I'} \text{ (resp. } e_I - e_{I'}) \in F(K_{n+1}) \\ &\Leftrightarrow I \cap I' = \emptyset \text{ (resp. } I \subset I' \text{ or } I \supset I'). \end{aligned} \tag{4.0.4}$$

When $k = n$ or 0, $|I_j| = 1$ because otherwise one cannot express every $\pm e_i$ as the form $\pm \sum_{j=1}^{n} \alpha_j = \pm \sum_{j=1}^{n} e_{I_j}$. Therefore, the base is of the form $\{e_1, \ldots, e_n\}$ or $\{-e_1, \ldots, -e_n\}$ when $k = n$ or 0.

When $1 \leq k \leq n-1$, a similar observation shows that $|I_j| = 1$ or $|I_j| = 2$ and I_j contains a unique $I_{j'}$ with $|I_{j'}| = 1$, where $1 \leq j \leq k < j' \leq n$ or $1 \leq j' \leq k < j \leq n$ in the latter case by (4.0.4). In fact, if $\alpha_1 = e_1 + e_2 + e_3, \alpha_2 = e_4 + e_5, \ldots,$ then there exist $j_1, j_2,$ and j_3 such that $\alpha_{j_1} = -e_2 - e_3, \alpha_{j_2} = -e_1 - e_3,$ and $\alpha_{j_3} = -e_1 - e_2$. Because $e_1, e_2,$ and e_3 are in $F(K_{n+1})$. However, $\alpha_{j_1}, \alpha_{j_2},$

and α_{j_3} are linearly dependent. In this case, one can see that $\pm\sum_{j\in J}^{n}\alpha_j = \pm\sum_{j\in J}^{n} e_{I_j}$ does not belong to $F(K_{n+1})$ for some subset J of $[n]$, namely the case where $1 \leq k \leq n-1$ does not occur. For instance, if $\alpha_1 = e_1+e_2$, $\alpha_2 = -e_2$, $\alpha_3 = e_3$, then $\alpha_2 + \alpha_3 = -e_2 + e_3$ does not belong to $F(K_4)$.

The argument above shows that the number of bases of the root system $F(K_{n+1})$ is two. However, the number of bases agrees with the order of the Weyl group of the root system and it is more than two when the rank of the root system is more than two. This is a contradiction and proves the lemma. □

Motivated by the observation above, we ask "Characterize a connected finite simple graph G such that $F(G)$ forms a root system" and the following theorem answers the question.

Theorem 4.0.2. *Let G be a connected finite simple graph with more than two nodes. Then the set $F(G)$ of facet vectors of the graph associahedron associated to G forms a root system if and only if G is a cycle graph. Moreover, the root system associated to the cycle graph with $n+1$ nodes is of type A_n.*

The rest of this section is devoted to the proof of Theorem 4.0.2. We begin with the following lemma.

Lemma 4.0.3. *Let C_{n+1} be the cycle graph with $n+1$ nodes. Then $F(C_{n+1})$ forms a root system of type A_n.*

Proof. An element I in the graphical building set $B(C_{n+1})$ different from the entire set $[n+1]$ is one of the following:

1. $\{i, i+1, \ldots, j\}$ where $1 \leq i \leq j \leq n$,

2. $\{i, i+1, \ldots, n+1\}$ where $2 \leq i \leq n+1$,

3. $\{i, i+1, \ldots, n+1, 1, \ldots, j\}$ where $1 \leq j < i \leq n+1$ and $i-j \geq 2$.

Therefore the facet vector of the facet corresponding to I is respectively given by

$$\sum_{k=i}^{j} e_k, \qquad -\sum_{k=1}^{i-1} e_k, \qquad -\sum_{k=j+1}^{i-1} e_k \tag{4.0.5}$$

according to the cases (1), (2), (3) above. It follows that

$$F(C_{n+1}) = \Big\{ \pm \sum_{k=i}^{j} e_k \mid 1 \leq i < j \leq n \Big\}. \tag{4.0.6}$$

This set forms a root system of type A_n. Indeed, an isomorphism from $\mathbb{Z}^n$ to the sublattice

$$\big\{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\big\} \subset \mathbb{Z}^{n+1}$$

sending e_i to $e_i - e_{i+1}$ for $i = 1, 2, \ldots, n$ maps $F(C_{n+1})$ to the standard root system $\Delta(A_n)$ of type A_n in (4.0.2). □

The following lemma is a key observation to prove that there is no other connected simple graph G such that $F(G)$ forms a root system.

Lemma 4.0.4. *Let G be a connected simple graph. Suppose that $F(G)$ is* centrally symmetric*, which means that $\alpha \in F(G)$ if and only if $-\alpha \in F(G)$ (note that $F(G)$ is centrally symmetric if $F(G)$ forms a root system). Then the following holds.*

$$I \in B(G) \implies V(G) \setminus I \in B(G). \tag{4.0.7}$$

Proof. By definition $B(G)$ contains elements $\{1\}, \ldots, \{n\}, \{n+1\}$ and the facet vector of the facet of the graph associahedron P_G corresponding to $\{i\}$ is e_i

when $1 \leq i \leq n$ and $-(e_1 + \cdots + e_n)$ when $i = n+1$. Let I be an element in $B(G)$ and α_I be the facet vector of the facet of P_G corresponding to I.

If $n+1$ is not in I, then $\alpha_I = \sum_{i \in I} e_i$. Therefore

$$-\alpha_I = -\sum_{i \in I} e_i = -(e_1 + \cdots + e_n) + \sum_{j \in [n] \setminus I} e_j.$$

This means that the element in $B(G)$ corresponding to the facet vector $-\alpha_I$ is $([n] \setminus I) \cup \{n+1\} = V(G) \setminus I$ and hence $V(G) \setminus I$ is in $B(G)$.

If $n+1$ is in I, then $\alpha_I = \sum_{i \in I \setminus \{n+1\}} e_i - (e_1 + \cdots + e_n)$. Therefore

$$-\alpha_I = (e_1 + \cdots + e_n) - \sum_{i \in I \setminus \{n+1\}} e_i = \sum_{j \in [n+1] \setminus I} e_j.$$

This means that the element in $B(G)$ corresponding to the facet vector $-\alpha_I$ is also $V(G) \setminus I$ and hence $V(G) \setminus I$ is in $B(G)$. □

Using Lemma 4.0.4, we prove the following.

Lemma 4.0.5. *Let G be a connected finite simple graph. Then $B(G)$ satisfies (4.0.7) if and only if G is a cycle or complete graph.*

Proof. If G is a cycle or complete graph, then $F(G)$ is centrally symmetric by (4.0.1) or (4.0.6) and hence $B(G)$ satisfies (4.0.7) by Lemma 4.0.4. So the "if" part is proven.

We shall prove the "only if" part. Suppose that $B(G)$ satisfies (4.0.7). If $B(G)$ does not contain $\{i, j\}$, then $B(G)$ does not contain $V(G) \setminus \{i, j\}$ by (4.0.7), so the induced subgraph $G|(V(G) \setminus \{i, j\})$ is not connected. Since $B(G)$ contains $\{i\}$ and $\{j\}$, $B(G)$ contains $V(G) \setminus \{i\}$ and $V(G) \setminus \{j\}$. So,

$$G|(V(G) \setminus \{i\}),\ G|(V(G) \setminus \{j\}) : \text{connected subgraph.} \tag{4.0.8}$$

We suppose that the number of connected components of $G|(V(G)\setminus\{i,j\})$ is k ($k \geq 3$) (Figure 4.1). We denote by $G_1, \ldots, G_k$ the k connected components of $G|(V(G)\setminus\{i,j\})$. By (4.0.8), the nodes i and j are respectively joined to every connected component by at least one edge. Since $G|(V(G_1)\cup\{i,j\})$ is connected, $G|(V(G_2)\cup\ldots V(G_k))$ is also connected by (4.0.7). However, since $k \geq 3$ and $G|(V(G_2)\cup\ldots V(G_k))$ is the disjoint union of $G_2, \ldots, G_k$, this contradicts our assumption that $G_2, \ldots, G_k$ are connected components.

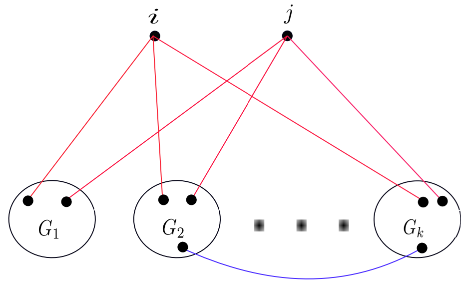


Figure 4.1: the case that connected components are more than or equal to 3

By the above argument, we may assume that the number of connected components of $G|(V(G)\setminus\{i,j\})$ is two. We denote by G_1, G_2 the connected components of $G|(V(G)\setminus\{i,j\})$. Similarly to the above, the nodes i and j are joined to both G_1 and G_2.

Suppose that G_1, G_2 are both path graphs and the node i is joined to one end node of G_1, G_2 respectively and the node j is joined to the other end node of G_1, G_2 (Figure 4.2). Then G is a cycle graph.

We consider the other case, that is,

1. either G_1 or G_2 is not a path graph, or

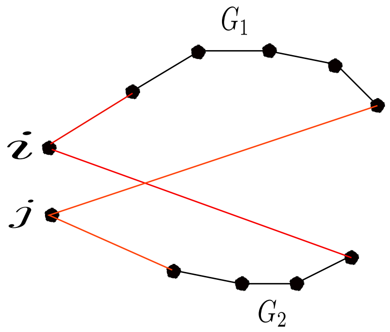


Figure 4.2: the case of cycle graph

2. both G_1 and G_2 are path graphs but the nodes i and j are not joined to the end points of G_1 and G_2 as in Figure 4.2,

(see Figure 4.3, left). Then there exist nodes $i_1, j_1 \in V(G_1)$ and $i_2, j_2 \in V(G_2)$ such that

1. i_1 and i_2 are joined to i,
2. j_1 and j_2 are joined to j, and
3. either the shortest path P_1 from i_1 to j_1 in G_1 is not the entire G_1 or the shortest path P_2 from i_2 to j_2 in G_2 is not the entire G_2.

Without loss of generality we may assume that $P_1 \neq G_1$. Since the induced subgraph $G|(V(P_1) \cup \{i, j, i_2, j_2\})$ is connected, so is $G|(V(G) \setminus (V(P_1) \cup \{i, j, i_2, j_2\}))$ by (4.0.7). This means that there is at least one edge joining G_1 and G_2 (Figure 4.3, right), and hence $G|(V(G) \setminus \{i, j\})$ is connected. This contradicts our assumption that $G|(V(G) \setminus \{i, j\})$ consists of two connected components. Therefore $\{i, j\}$ is in $B(G)$. Since i and j are arbitrary, G is a

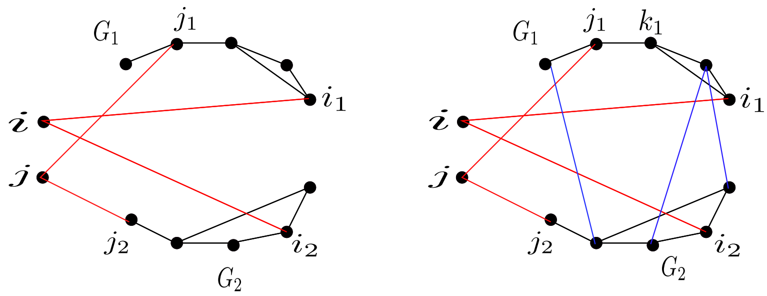


Figure 4.3: the other case

complete graph. □

Now Theorem 4.0.2 follows from Lemmas 4.0.1, 4.0.3, 4.0.4 and 4.0.5.

Chapter 5

Cohomology representations of toric manifolds associated to some simple graphs

In this Chapter, we consider the following problem.

Problem 5.0.1. *Let G be a connected finite simple graph, and $X(G)$ be the toric manifold associated to G. Describe the $Aut(G)$-representation on the cohomology ring $H^*(X(G);\mathbb{C})$ induced by the $Aut(G)$ action on G.*

5.1 Representations of the case of cycle graphs

We study Problem 5.0.1 when G is a cycle graph with 3, 4, or 5 nodes.

In section 3.2 we explained how to construct a graph associahedron P_G and how to take facet vectors of P_G. For I and $J (I \neq J)$ in $B(G)$, if the induced subgraphs $G|I$ and $G|J$ satisfy one of the followings, then facets F_I and F_J of P_G intersect.

(1) One properly contains the other.

(2) They are disjoint and cannot be connected by a single edge of G.

For $I_1, \ldots, I_k$ in $B(G)$, if any pair of the corresponding induced subgraphs satisfies one of (1) and (2), then the set $\{G_{I_1}, \ldots, G_{I_k}\}$ is called *tubing* of G, and the intersection of facets $F_{I_1}, \ldots F_{I_k}$ is a codimension k face of P_G. Let Δ_G be the fan corresponding to P_G, and the tubing $\{G_{I_1}, \ldots, G_{I_k}\}$ of G corresponds to a cone of k dimension in Δ_G.

We prove that $Aut(G)$ induces the cohomology representation of $X(G)$. It is enough to prove that $Aut(G)$ induces an action on Δ_G.

Lemma 5.1.1. *Let G be a connected simple graph with $n+1$ nodes. If g is an automorphism of G, then g induces an automorphism $\tilde{g}$ of Δ_G.*

Proof. We suppose that $g(i_1) = 1, \ldots, g(i_n) = n, g(i_{n+1}) = n+1$ ($i_j \neq i_k$ for $j \neq k$), where the set $\{i_1, \ldots, i_{n+1}\} = \{1, \ldots, n+1\}$ is the node set of G. Edge vectors of Δ_G corresponding to elements $1, \ldots, n$ in $B(G)$ are $e_1, \ldots, e_n$, and an edge vector of Δ_G corresponding to $n+1$ in $B(G)$ is $e_{n+1} := -e_1 - \cdots - e_n$ by the way of taking facet vectors of P_G. One defines an automorphism $\tilde{g}$ of $\mathbb{Z}^n$ by $\tilde{g}(e_{i_1}) = e_1, \ldots, \tilde{g}(e_{i_n}) = e_n$. Since $e_{i_{n+1}} = -e_{i_1} - \cdots - e_{i_n}$, $\tilde{g}(e_{i_{n+1}}) = e_{n+1}$.

Since g is an automorphism of G, g induces a bijection g' from $B(G)$ to $B(G)$, that is, if $g(i_{j_1}) = j_1, \ldots, g(i_{j_k}) = j_k$ in G, then $g'(i_{j_1} \ldots i_{j_k}) = j_1 \ldots j_k$

in $B(G)$. Since $\tilde{g}$ is an automorphism of $\mathbb{Z}^n$, $\tilde{g}(e_{i_{j_1}} + \cdots + e_{i_{j_k}}) = e_{j_1} + \cdots + e_{j_k}$. So, $\tilde{g}$ maps the edge vector of Δ_G corresponding to $i_{j_1} \dots i_{j_k}$ in $B(G)$ to the edge vector of Δ_G corresponding to $j_1 \dots j_k$ in $B(G)$.

Since g is an automorphism of G, g induces a bijection from the set {tubing of G} to {tubing of G}. So, $\tilde{g}$ maps an l dimensional cone of Δ_G to an l dimensional cone of Δ_G. □

Let C_{n+1} be a cycle graph with $n+1$ nodes. The automorphism group of C_{n+1} is the dihedral group D_{n+1} :

$$D_{n+1} = \langle \sigma, \tau \mid \sigma^{n+1} = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle \subset \mathfrak{S}_{n+1},$$

where e is the identity element of D_{n+1}, σ is a rotation, τ is a reflection, and $\mathfrak{S}_{n+1}$ is the symmetric group on $n+1$ letters;

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n & n+1 \\ 2 & 3 & \dots & n+1 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & \dots & n & n+1 \\ n+1 & n & \dots & 2 & 1 \end{pmatrix}.$$

Figure 5.1 is the case $n = 4$.

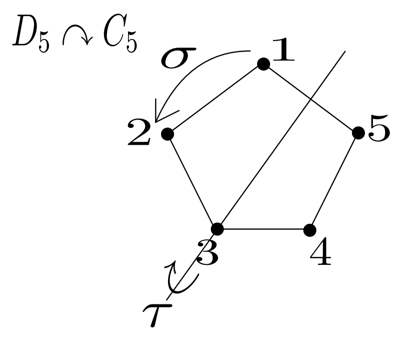


Figure 5.1: dihedral group D_5

We take representatives of conjugacy classes of D_n as follows: $e, \sigma, \sigma^2, \ldots, \sigma^{\frac{n}{2}}$, $\tau, \sigma\tau$ when n is even, and $e, \sigma, \sigma^2, \ldots, \sigma^{\frac{n-1}{2}}, \tau$ when n is odd.

Irreducible representations of D_n are as follows. If n is even, then there are four 1-dimensional representations $\rho_1, \rho_2, \rho_3, \rho_4$ of D_n and $\frac{n}{2} - 1$ 2-dimensional representations $\rho'_1, \ldots, \rho'_{\frac{n}{2}-1}$ of D_n. Representations $\rho_1, \rho_2, \rho_3, \rho_4$ are defined as follows:

$$\rho_1(\sigma) = id, \quad \rho_1(\tau) = id, \qquad \rho_2(\sigma) = id, \quad \rho_2(\tau) = -id,$$

$$\rho_3(\sigma) = -id, \quad \rho_3(\tau) = id, \qquad \rho_4(\sigma) = -id, \quad \rho_4(\tau) = -id.$$

The representation ρ'_i for each i is defined as follows:

$$\rho'_i(\sigma) = \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}, \quad \rho'_i(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If n is odd, there are two 1-dimensional representations ρ_1, ρ_2 of D_n and $\frac{n-1}{2}$ 2-dimensional representations $\rho'_1, \ldots, \rho'_{\frac{n-1}{2}}$ of D_n. Representations ρ_i, ρ'_i are the same as above.

We study the toric manifold $X(C_{n+1})$ corresponding to the cycle graph C_{n+1} and the D_{n+1}-representation on the cohomology ring $H^*(X(C_{n+1}))$. Let I be a subset in $[n+1]$, and π_I be a subvariety in $\mathbb{C}P^n$ defined as follows:

$$\pi_I := \{[z_1, \ldots, z_{n+1}] \in \mathbb{C}P^n \mid z_i = 0 \ \ (i \in I)\}.$$

If I is of order $n-k$, then π_I is a subvariety of dimension k. We define a subvariety X^k as follows for $k = 0, 1, \ldots, n-2$;

$$X^k := \bigcup_{\substack{|I|=n-k, \\ I \in B(C_{n+1})}} \pi_I.$$

The subvariety X^0 is in $\mathbb{C}P^n$ and over the vertices of n-simplex Δ^n. For I with cardinality n we define $\tilde{\pi}_I := \pi_I$ and

$$\tilde{X}^0 := \coprod_{\substack{|I|=n,\\ I\in B(C_{n+1})}} \tilde{\pi}_I.$$

We denote by Y_0 a variety obtained by blowing-up $\mathbb{C}P^n$ along $\tilde{X}^0$ and by P_0 a simple polytope under Y_0. Then P_0 is a polytope obtained by truncating all vertices of Δ^n. The subvariety X^1 is in $\mathbb{C}P^n$ and over edges of Δ^n corresponding to all I in $B(C_{n+1})$ with cardinality $n-1$. We denote by $\tilde{\pi}_I$ and $\tilde{X}^1$ strict transforms of π_I ($|I| = n-1$) and X^1 by the previous blow-up (i.e. $\tilde{\pi}_I$ and $\tilde{X}^1$ are closures of images of π_I and X^1 by the previous blow-up), by Y_1 a variety obtained by blowing-up Y_0 along $\tilde{X}^1$ and by P_1 a simple polytope under Y_1. The subvariety X^2 is in $\mathbb{C}P^n$ and over 2 dimensional faces of Δ^n corresponding to all I in $B(C_{n+1})$ with cardinality $n-2$. We denote by $\tilde{\pi}_I$ and $\tilde{X}^2$ strict transforms of π_I ($|I| = n-2$) and X^2 by the previous two blow-ups, by Y_2 a variety obtained by blowing-up Y_1 along $\tilde{X}^2$ and by P_2 a simple polytope under Y_2. By repeating this construction, we obtain the toric manifold $X(C_{n+1})$ as Y_{n-2}. The simple polytope under Y_{n-2} is called a *cyclohedron*;

$$\begin{array}{ccccccccc}
\mathbb{C}P^n & \xleftarrow[\text{along } \tilde{X}^0]{\text{blow-up}} & Y_0 & \xleftarrow[\text{along } \tilde{X}^1]{\text{blow-up}} & Y_1 \leftarrow \cdots \leftarrow & Y_{n-3} & \xleftarrow[\text{along } \tilde{X}^{n-2}]{\text{blow-up}} & Y_{n-2} \cong X(C_{n+1}) \\
\downarrow & & \downarrow & & \downarrow \quad \cdots & \downarrow & & \downarrow \\
\Delta^n & \xleftarrow[\text{truncation}]{\text{vertices}} & P_0 & \xleftarrow[\text{truncation}]{\text{edges}} & P_1 \leftarrow \cdots \leftarrow & P_{n-3} & \xleftarrow[\text{truncation}]{n-2 \text{ dim faces}} & P_{n-2}
\end{array}$$

Lemma 5.1.2. *If n is more than or equal to 5, then $\tilde{X}^3$ is a singular variety.*

Proof. The subvariety $\tilde{X}^3$ is a strict transform of X^3 in Y_2;

$$\tilde{X}^3 = \coprod_{\substack{|I|=n-3, \\ I \in B(C_{n+1})}} \pi_I.$$

For a subset I in $[n+1]$, we denote by F_I the intersection of facets F_i of Δ^n over all $i \in I$. If $I = \{1, 2, \ldots, n-3\}$, then F_I is a 3-simplex in Δ^n. The face F_I has five faces which are not truncated until F_I reaches a face of P_2, and these are faces corresponding to sets $I_1 = \{1, 2, \ldots, n-2, n\}$, $I_2 = \{1, 2, \ldots, n-3, n-1, n\}$, $I_3 = \{n+1, 1, 2, \ldots, n-3, n-1\}$, $I_4 = \{1, 2, \ldots, n-3, n-1\}$, and $I_5 = \{1, 2, \ldots, n-3, n\}$. Faces corresponding to I_1, I_2 and I_3 are edges and faces corresponding to I_4 and I_5 are of dimension 2. On the other hand, if $J = \{2, 3, \ldots, n-2\}$, then F_J is also a 3-simplex in Δ^n and F_J has five faces which are not truncated until F_J reaches a face of P_2, and these are faces corresponding to sets $J_1 = \{2, 3, \ldots, n-2, n, n+1\}$, $J_2 = \{2, 3, \ldots, n-1, n+1\}$, $J_3 = \{1, 2, \ldots, n-2, n\}$, $J_4 = \{2, 3, \ldots, n-2, n\}$, and $J_5 = \{2, 3, \ldots, n-2, n+1\}$. We denote by $\tilde{F}_I$ (resp, $\tilde{F}_J$) a face in P_2 where F_I (resp, F_J) reaches. Since $I_1 = J_3$, $\tilde{F}_I$ intersects with $\tilde{F}_J$ in P_2. This means that $\tilde{X}^3$ is a singular variety. $\square$

We shall investigate how the D_{n+1}-representation $H^*(X(C_{n+1}))$ decomposes into irreducible ones. In general, if we blow up a smooth subvariety A of codimension k in a smooth complete variety B, then the cohomology of the subvariety obtained by blowing up B is additively isomorphic to

$$H^*(B) \oplus (H^*(A) \otimes H^+(\mathbb{C}P^{k-1})),$$

where H^+ denotes the cohomology group of positive degree. Moreover, if a group

G acts on B preserving A, then the above isomorphism is compatible with the natural actions of G on $H^*(B)$, $H^*(A)$ and the trivial action on $H^+(\mathbb{C}P^{k-1})$.

The general fact mentioned above shows that if n is less than or equal to 4, then the following isomorphism is a D_{n+1}-isomorphism.

$$H^*(X(C_{n+1})) \cong H^*(\mathbb{C}P^n) \oplus \left(\bigoplus_{k=0}^{n-2} \left(H^*(\tilde{X}^k) \otimes H^+(\mathbb{C}P^{n-k-1}) \right) \right). \qquad (5.1.1)$$

Here, the second and fourth representations are trivial. To describe the D_{n+1}-representation $H^*(X(C_{n+1}))$, it is enough to describe the D_{n+1}-representation $H^*(\tilde{X}^k)$ for each k.

Definition 5.1.3. We define $R(X(G);t)$ to be a polynomial in t whose coefficient of t^i is the $Aut(G)$-representation $H^{2i}(X(G))$.

We shall study $R(X(C_3);t)$ (Figure 5.2). The graphical building set $B(C_3)$ of the cycle graph C_3 is the set $\{1, 2, 3, 12, 23, 31, 123\}$, so

$$X^0 = \pi_{12} \cup \pi_{23} \cup \pi_{31} = \tilde{X}^0 = \tilde{\pi}_{12} \sqcup \tilde{\pi}_{23} \sqcup \tilde{\pi}_{31}.$$

The subvariety $\tilde{X}^0$ is smooth, so the following isomorphism is a D_3-isomorphism.

$$\begin{array}{ccccccc} H^*(X(C_3)) & \cong & H^*(\mathbb{C}P^2) & \oplus & \Big(H^*(\tilde{X}^0) & \otimes & H^+(\mathbb{C}P^1) \Big). \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ D_3 & & D_3 & & D_3 & & D_3 \end{array}$$

Here, the second and fourth representations are trivial. So, to determine $R(X(C_3);t)$, it is enough to describe the D_3-representation $H^*(\tilde{X}^0)$.

The dihedral group D_3 is as follows;

$$D_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle.$$

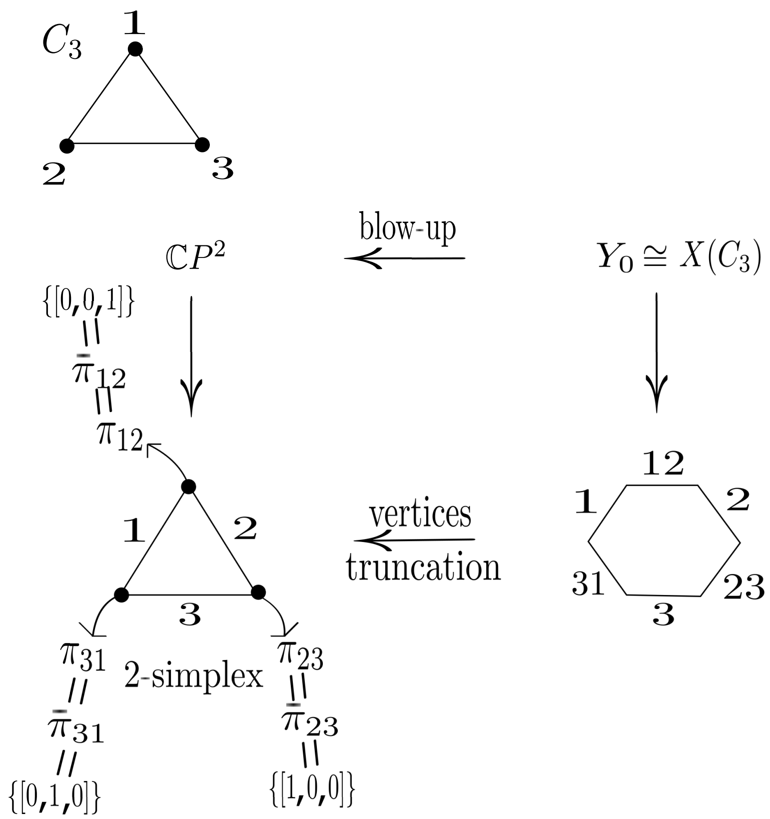


Figure 5.2: the case of cycle graph with 3 nodes

Representatives of conjugacy classes of D_3 are e, σ and τ. The irreducible representations of D_3 are ρ_1, ρ_2 and ρ_1', and the character table is as follows.

	e	σ	τ
ρ_1	1	1	1
ρ_2	1	1	-1
ρ_1'	2	-1	0

Since $\tilde{X}^0$ is the disjoint union of $\tilde{\pi}_{12}, \tilde{\pi}_{23}$, and $\tilde{\pi}_{31}$,

$$H^0(\tilde{X}^0) \cong H^0(\tilde{\pi}_{12}) \oplus H^0(\tilde{\pi}_{23}) \oplus H^0(\tilde{\pi}_{31}).$$

The D_3-representation $H^0(\tilde{X}^0)$ is induced by the action of D_3 on C_3, so

$$\text{tr } e = 3, \quad \text{tr } \sigma = 0, \quad \text{tr } \tau = 1.$$

By the character table above, the D_3-representation $H^0(\tilde{X}^0)$ is the sum of ρ_1 and ρ_1'.

The Poincaré polynomial $P(X(C_3), t)$ of $X(C_3)$ is

$$P(X(C_3), t) = 1 + 4t + t^2,$$

and

$$R(X(C_3); t) = \rho_1 + (2\rho_1 + \rho_1')t + \rho_1 t^2 = \rho_1(1+t)^2 + \rho_1' t.$$

We shall study $R(X(C_4); t)$. The graphical building set $B(C_4)$ of the cycle graph C_4 is the set $\{1, 2, 3, 4, 12, 23, 34, 41, 123, 234, 341, 412, 1234\}$. Subvarieties

$X^0, \tilde{X}^0$, and X^1 are as follows.

$$X^0 = \pi_{123} \cup \pi_{234} \cup \pi_{341} \cup \pi_{412} = \tilde{X}^0 = \tilde{\pi}_{123} \sqcup \tilde{\pi}_{234} \sqcup \tilde{\pi}_{341} \sqcup \tilde{\pi}_{412}.$$

$$X^1 = \pi_{12} \cup \pi_{23} \cup \pi_{34} \cup \pi_{41}.$$

The variety Y_0 is that obtained by blowing-up $\mathbb{C}P^3$ along $\tilde{X}^0$, and $\tilde{X}^1$ is a strict transform of X^1 by this blowing-up ;

$$\tilde{X}^1 = \tilde{\pi}_{12} \sqcup \tilde{\pi}_{23} \sqcup \tilde{\pi}_{34} \sqcup \tilde{\pi}_{41}.$$

The variety Y_1 is that obtained by blowing-up Y_0 along $\tilde{X}^1$ and is $X(C_4)$;

$$\mathbb{C}P^3 \xleftarrow[\text{along } \tilde{X}^0]{\text{blow-up}} Y_0 \xleftarrow[\text{along } \tilde{X}^1]{\text{blow-up}} Y_1 \cong X(C_4).$$

Subvarieties $\tilde{X}^0$ and $\tilde{X}^1$ are smooth, so the following isomorphism is a D_4-isomorphism ;

$$H^*(X(C_4)) \cong H^*(\mathbb{C}P^3) \oplus \left(H^*(\tilde{X}^0) \otimes H^+(\mathbb{C}P^2)\right) \oplus \left(H^*(\tilde{X}^1) \otimes H^+(\mathbb{C}P^1)\right).$$

Here, the cohomology representations of complex projective spaces are trivial. So, to determine $R(X(C_4);t)$, it is enough to determine D_4-representations $H^*(\tilde{X}^0)$ and $H^*(\tilde{X}^1)$.

The dihedral group D_4 is as follows;

$$D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle.$$

Representatives of conjugacy classes of D_4 are $e, \sigma, \sigma^2, \tau$, and $\sigma^3\tau$. The irreducible representations of D_4 are ρ_1, ρ_2, ρ_3, ρ_4, and ρ_1', and the character table is as follows.

	e	σ	σ^2	τ	$\sigma^3\tau = \tau\sigma$
ρ_1	1	1	1	1	1
ρ_2	1	1	1	-1	-1
ρ_3	1	-1	1	1	-1
ρ_4	1	-1	1	-1	1
ρ_1'	2	0	-2	0	0

Since $\tilde{X}^0$ is the disjoint union of $\tilde{\pi}_{123}, \tilde{\pi}_{234}, \tilde{\pi}_{341}$, and $\tilde{\pi}_{412}$,

$$H^0(\tilde{X}^0) \cong H^0(\tilde{\pi}_{123}) \oplus H^0(\tilde{\pi}_{234}) \oplus H^0(\tilde{\pi}_{341}) \oplus H^0(\tilde{\pi}_{412}).$$

The D_4-representation $H^0(\tilde{X}^0)$ is induced by an action of D_4 on C_4, so

$$\text{tr } e = 4, \quad \text{tr } \sigma = 0, \quad \text{tr } (\sigma^2) = 0, \quad \text{tr } \tau = 0, \quad \text{tr } (\tau\sigma) = 2.$$

By the character table above, the D_4-representation $H^0(\tilde{X}^0)$ is the sum of ρ_1, ρ_4, and ρ_1'.

Since $\tilde{X}^1$ is the disjoint union of the $\tilde{\pi}_{12}, \tilde{\pi}_{23}, \tilde{\pi}_{34}$, and $\tilde{\pi}_{41}$,

$$H^*(\tilde{X}^1) \cong H^*(\tilde{\pi}_{12}) \oplus H^*(\tilde{\pi}_{23}) \oplus H^*(\tilde{\pi}_{34}) \oplus H^*(\tilde{\pi}_{41}).$$

Since $\tilde{\pi}_{12}, \tilde{\pi}_{23}, \tilde{\pi}_{34}$, and $\tilde{\pi}_{41}$ are diffeomorphic to $\mathbb{C}P^1$, the following holds for the D_4-representation $H^0(\tilde{X}^1)$;

$$\text{tr } e = 4, \quad \text{tr } \sigma = 0, \quad \text{tr } (\sigma^2) = 0, \quad \text{tr } \tau = 2, \quad \text{tr } (\tau\sigma) = 0.$$

By the character table above, the D_4-representation $H^0(\tilde{X}^1)$ is the sum of ρ_1, ρ_3, and ρ_1'. Since traces of $e, \sigma, \sigma^2, \tau$, and $\sigma^3\tau$ in D_4 of the D_4-representation

$H^2(\tilde{X}^1)$ are the same as traces of $e, \sigma, \sigma^2, \tau$, and $\sigma^3\tau$ in D_4 of the D_4-representation $H^0(\tilde{X}^1)$, the D_4-representation $H^2(\tilde{X}^1)$ is also the sum of ρ_1, ρ_3, and ρ_1'.

Thus $R(X(C_4);t)$ is as follows ;

$$R(X(C_4);t) = \rho_1 + (3\rho_1 + \rho_3 + \rho_4 + 2\rho_1')t + (3\rho_1 + \rho_3 + \rho_4 + 2\rho_1')t^2 + \rho_1 t^3$$
$$= \rho_1(1+t)^3 + (\rho_3 + \rho_4)t(1+t) + 2\rho_1' t(1+t).$$

We shall study $R(X(C_5);t)$. The graphical building set $B(C_5)$ of the cycle graph C_5 is the set

$$\{1, 2,3, 4, 5, 12, 23, 34, 45, 51, 123, 234, 345, 451, 512,$$
$$1234, 2345, 3451, 4512, 5123, 12345\}.$$

Subvarieties $X^0, \tilde{X}^0, X^1$, and X^2 are as follows ;

$$X^0 = \pi_{1234} \cup \pi_{2345} \cup \pi_{3451} \cup \pi_{4512} \cup \pi_{5123}$$
$$= \tilde{X}^0 = \tilde{\pi}_{1234} \sqcup \tilde{\pi}_{2345} \sqcup \tilde{\pi}_{3451} \sqcup \tilde{\pi}_{4512} \sqcup \tilde{\pi}_{5123}.$$
$$X^1 = \pi_{123} \cup \pi_{234} \cup \pi_{345} \cup \pi_{451} \cup \pi_{512}.$$
$$X^2 = \pi_{12} \cup \pi_{23} \cup \pi_{34} \cup \pi_{45} \cup \pi_{51}.$$

The variety Y_0 is that obtained by blowing-up $\mathbb{C}P^4$ along $\tilde{X}^0$, and $\tilde{X}^1$ is a strict transform of X^1 by this blowing-up ;

$$\tilde{X}^1 = \tilde{\pi}_{123} \sqcup \tilde{\pi}_{234} \sqcup \tilde{\pi}_{345} \sqcup \tilde{\pi}_{451} \sqcup \tilde{\pi}_{512}.$$

The variety Y_1 is that obtained by blowing-up Y_0 along $\tilde{X}^1$ and $\tilde{X}^2$ is a strict transform of X^2 by these two blowing-ups ;

$$\tilde{X}^2 = \tilde{\pi}_{12} \sqcup \tilde{\pi}_{23} \sqcup \tilde{\pi}_{34} \sqcup \tilde{\pi}_{45} \sqcup \tilde{\pi}_{51}.$$

The variety Y_2 is that obtained by blowing-up Y_1 along $\tilde{X}^2$ and is $X(C_5)$;

$$\mathbb{C}P^4 \xleftarrow[\text{along } \tilde{X}^0]{\text{blow-up}} Y_0 \xleftarrow[\text{along } \tilde{X}^1]{\text{blow-up}} Y_1 \xleftarrow[\text{along } \tilde{X}^2]{\text{blow-up}} Y_2 \cong X(C_5).$$

Since $\tilde{X}^0, \tilde{X}^1$, and $\tilde{X}^2$ are smooth, the following isomorphism is a D_5-isomorphism;

$$\begin{aligned} H^*(X(C_5)) \cong & H^*(\mathbb{C}P^4) \oplus \left(H^*(\tilde{X}^0) \otimes H^+(\mathbb{C}P^3)\right) \\ & \oplus \left(H^*(\tilde{X}^1) \otimes H^+(\mathbb{C}P^2)\right) \oplus \left(H^*(\tilde{X}^2) \otimes H^+(\mathbb{C}P^1)\right). \end{aligned}$$

Here, the representations on the cohomology groups of complex projective spaces are trivial. So, to determine $R(X(C_5); t)$, it is enough to determine the D_5-representations $H^*(\tilde{X}^0), H^*(\tilde{X}^1)$ and $H^*(\tilde{X}^2)$.

The dihedral group D_5 is as follows;

$$D_5 = \langle \sigma, \tau \mid \sigma^5 = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle.$$

Representatives of conjugacy classes of D_5 are e, σ, σ^2, and τ. The irreducible representations of D_5 are ρ_1, ρ_2, ρ_1', and ρ_2', and the character table is as follows.

	e	σ	σ^2	τ
ρ_1	1	1	1	1
ρ_2	1	1	1	-1
ρ_1'	2	$\omega + \omega^{-1}$	$\omega^2 + \omega^{-2}$	0
ρ_2'	2	$\omega^2 + \omega^{-2}$	$\omega^4 + \omega^{-4}$	0

Since $\tilde{X}^0$ is the disjoint union of $\tilde{\pi}_{1234}, \tilde{\pi}_{2345}, \tilde{\pi}_{3451}, \tilde{\pi}_{4512}$, and $\tilde{\pi}_{5123}$,

$$H^0(\tilde{X}^0) \cong H^0(\tilde{\pi}_{1234}) \oplus H^0(\tilde{\pi}_{2345}) \oplus H^0(\tilde{\pi}_{3451}) \oplus H^0(\tilde{\pi}_{4512}) \oplus H^0(\tilde{\pi}_{5123}).$$

The D_5-representation $H^0(\tilde{X}^0)$ is induced by an action of D_5 on C_5, so

$$\text{tr } e = 5, \quad \text{tr } \sigma = 0, \quad \text{tr } (\sigma^2) = 0, \quad \text{tr } \tau = 1.$$

By the character table above, the D_5-representation $H^0(\tilde{X}^0)$ is the sum of ρ_1, ρ_1' and ρ_2'.

Since $\tilde{X}^1$ is the disjoint union of $\tilde{\pi}_{123}, \tilde{\pi}_{234}, \tilde{\pi}_{345}, \tilde{\pi}_{451}$, and $\tilde{\pi}_{512}$,

$$H^*(\tilde{X}^1) \cong H^*(\tilde{\pi}_{123}) \oplus H^*(\tilde{\pi}_{234}) \oplus H^*(\tilde{\pi}_{345}) \oplus H^*(\tilde{\pi}_{451}) \oplus H^*(\tilde{\pi}_{512}).$$

Since $\tilde{\pi}_{123}, \tilde{\pi}_{234}, \tilde{\pi}_{345}, \tilde{\pi}_{451}$, and $\tilde{\pi}_{512}$ are diffeomorphic to $\mathbb{C}P^1$, the following holds for the D_5-representation $H^0(\tilde{X}^1)$.

$$\text{tr } e = 5, \quad \text{tr } \sigma = 0, \quad \text{tr } (\sigma^2) = 0, \quad \text{tr } \tau = 1.$$

By the character table above, the D_5-representation $H^0(\tilde{X}^1)$ is the sum of ρ_1, ρ_1', and ρ_2'. Since traces of e, σ, σ^2, and τ of the D_5-representation $H^2(\tilde{X}^1)$ are the same as traces of e, σ, σ^2, and τ in D_5 of the D_5-represent-ation $H^0(\tilde{X}^1)$, the D_5-representation $H^2(\tilde{X}^1)$ is also the sum of ρ_1, ρ_1', and ρ_2'.

Since $\tilde{X}^2$ is the disjoint union of $\tilde{\pi}_{12}, \tilde{\pi}_{23}, \tilde{\pi}_{34}, \tilde{\pi}_{45}$, and $\tilde{\pi}_{51}$,

$$H^*(\tilde{X}^2) \cong H^*(\tilde{\pi}_{12}) \oplus H^*(\tilde{\pi}_{23}) \oplus H^*(\tilde{\pi}_{34}) \oplus H^*(\tilde{\pi}_{45}) \oplus H^*(\tilde{\pi}_{51}).$$

Traces of e, σ, σ^2, and τ of the D_5-representations $H^0(\tilde{X}^2)$ and $H^4(\tilde{X}^2)$ are also as follows:

$$\text{tr } e = 5, \quad \text{tr } \sigma = 0, \quad \text{tr } (\sigma^2) = 0, \quad \text{tr } \tau = 1.$$

So, these representations are the sum of ρ_1, ρ_1', and ρ_2' respectively. We determine the D_5-representation $H^2(\tilde{X}^2)$.

$$H^2(\tilde{X}^2) \cong H^2(\tilde{\pi}_{12}) \oplus H^2(\tilde{\pi}_{23}) \oplus H^2(\tilde{\pi}_{34}) \oplus H^2(\tilde{\pi}_{45}) \oplus H^2(\tilde{\pi}_{51}),$$

and we find traces of e, σ, σ^2, and τ. Clearly tr σ = tr $(\sigma^2) = 0$. The element τ in D_5 preserves only $H^2(\tilde{\pi}_{51})$, so traces of τ on $H^2(\tilde{\pi}_{51})$ and $H^2(\tilde{X}^2)$ are the same. The intersection of F_1 and F_5 in 4-simplex Δ^4 is a 2-simplex, and three vertices of the 2-simplex correspond to 3451, 4512, and 5123 in $B(C_5)$ respectively (Figure 5.3). Recall that F_i is the facet of the n-simplex corresponding to i in $B(G)$ for a simple graph G with $n+1$ nodes. In the truncation of vertices

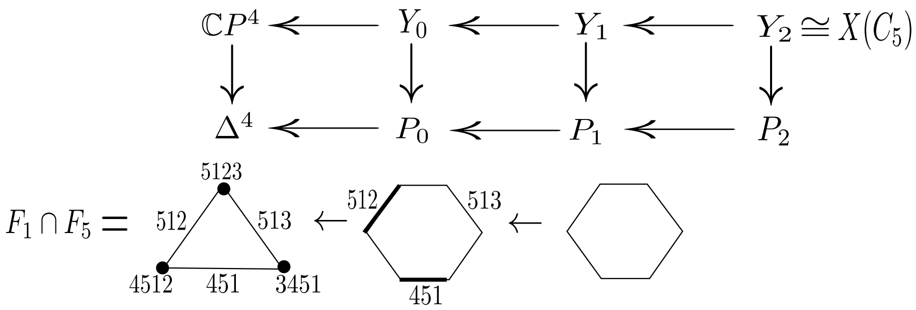


Figure 5.3: the intersection of F_1 and F_5

three vertices of the 2-simplex corresponding to 3451, 4512, and 5123 in $B(C_5)$ are truncated, and this 2-simplex turns into a hexagon in P_0. In the truncation of edges the two edges of the hexagon corresponding to 512, and 451 in $B(C_5)$ are truncated. The subvariety $\tilde{\pi}_{51}$ in Y_1 is over the hexagon in P_1 obtained by truncating the vertices and edges above. Clearly, $\tilde{\pi}_{51}$ is a variety obtained by blowing-up $\mathbb{C}P^2$ along three points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$, and the following isomorphism holds as modules;

$$H^*(\tilde{\pi}_{51}) \cong H^*(\mathbb{C}P^2) \oplus \left(H^*(3 \text{ points}) \otimes H^+(\mathbb{C}P^1) \right).$$

The element τ in D_5 acts on each cohomology group, and this isomorphism is τ-equivariant. The τ-action on the cohomology group of a complex projective space is trivial. So, to determine the τ-action on $H^2(\tilde{\pi}_{51})$, it is enough to determine the τ-action on $H^*(3 \text{ points})$. The trace of the τ-action on $H^2(\tilde{\pi}_{51})$ is 2 which can be found by seeing the τ-action of $F_1 \cap F_5$ in Δ^4. So, tr $\tau = 2$ for the τ-action on $H^2(\tilde{X}^2)$. Since the Betti number of degree two of $\tilde{\pi}_{51}$ is 4, tr $e = 20$ for the e-action on $H^2(\tilde{X}^2)$. By the character table above, the D_5-representation $H^2(\tilde{X}^2)$ is the sum of ρ_1, ρ_2, $4\rho_1'$, and $4\rho_2'$.

Thus we obtain

$$\begin{aligned} R(X(C_5);t) &= \rho_1 + (4\rho_1 + 3\rho_1' + 3\rho_2')t + (7\rho_1 + \rho_2 + 7\rho_1' + 7\rho_2')t^2 \\ &\quad + (4\rho_1 + 3\rho_1' + 3\rho_2')t^3 + \rho_1 t^4 \\ &= \rho_1\{(1+t)^4 + t^2\} + t^2\rho_2 + (3t + 7t^2 + 3t^3)(\rho_1' + \rho_2'). \end{aligned}$$

5.2 Representations of the case of graphs obtained by removing an edge from complete graphs

In this section, we consider Problem 5.0.1 when G is a graph obtained by removing an edge from a complete graph. Procesi solved Problem 5.0.1 when G is a complete graph ([19]). Let K_{n+1} be a complete graph with $n+1$ nodes, and G_{n+1} be a graph obtained by removing the edge $\{1, n+1\}$ from K_{n+1}. We denote the graphical building set of G_{n+1} (resp, K_{n+1}) by $B(G_{n+1})$ (resp, $B(K_{n+1})$). Then the following holds;

$$B(G_{n+1}) = B(K_{n+1}) \setminus \{\{1, n+1\}\}. \tag{5.2.1}$$

We define $\pi_I, \tilde{\pi}_I, X^k, \tilde{X}^k, Y_k, P_k$ $(k = 0, 1, \ldots, n-2)$ similarly to the case of cycle graphs. Then Y_{n-2} is isomorphic to $X(G_{n+1})$ and

$$\pi_I = \{[z_1, \ldots, z_{n+1}] \in \mathbb{C}P^n \mid z_i = 0 \ \ (i \in I)\},$$

$$\tilde{\pi}_I : \text{strict transform of } \pi_I,$$

$$X^k := \bigcup_{\substack{|I|=n-k,\\ I \in B(G_{n+1})}} \pi_I, \qquad \tilde{X}^k : \text{strict transform of } X^k,$$

$$\begin{array}{ccccccccc}
\mathbb{C}P^n & \xleftarrow[\text{along } \tilde{X}^0]{\text{blow-up}} & Y_0 & \xleftarrow[\text{along } \tilde{X}^1]{\text{blow-up}} & Y_1 \leftarrow \cdots \leftarrow & Y_{n-3} & \xleftarrow[\text{along } \tilde{X}^{n-2}]{\text{blow-up}} & Y_{n-2} \cong X(G_{n+1}) \\
\downarrow & & \downarrow & & \downarrow \quad \cdots & \downarrow & & \downarrow \\
\Delta^n & \xleftarrow[\text{truncation}]{\text{vertices}} & P_0 & \xleftarrow[\text{truncation}]{\text{edges}} & P_1 \leftarrow \cdots \leftarrow & P_{n-3} & \xleftarrow[\text{truncation}]{n-2 \text{ dim faces}} & P_{n-2}
\end{array}$$

Remark 5.2.1. In the case of complete graphs K_{n+1}, we can also define $\pi_I, \tilde{\pi}_I, X^k, \tilde{X}^k, Y_k, P_k$ $(k = 0, 1, \dots, n-2)$. By (5.2.1), $Y_0, \dots, Y_{n-3}$ in the case of G_{n+1} are the same as $Y_0, \dots, Y_{n-3}$ in the case of K_{n+1}, and the symmetric group $\mathfrak{S}_{n+1}$ acts on $Y_0, \dots, Y_{n-3}$. However, since $\tilde{X}^{n-2}$ in the cases of G_{n+1} is different from $\tilde{X}^{n-2}$ in the case of K_{n+1}, Y_{n-2} in the cases of G_{n+1} is also different from Y_{n-2} in the case of K_{n+1}.

The automorphism group of G_{n+1} is $H := \mathfrak{S}_{n-1} \times \mathfrak{S}_2$, which is a subgroup of $\mathfrak{S}_{n+1}$. The following holds as H-modules;

$$H^*(X(G_{n+1})) \cong H^*(Y_{n-3}) \oplus \left(H^*(\tilde{X}^{n-2}) \otimes H^+(\mathbb{C}P^1) \right).$$

Here, the H-representation $H^+(\mathbb{C}P^1)$ is trivial, and the other H-representations are induced by the H-action on G_{n+1}. So, to determine the H-representation $H^*(X(G_{n+1}))$, it is enough to determine the H-representations $H^*(Y_{n-3})$, and $H^*(\tilde{X}^{n-2})$.

We determine the H-representation $H^*(Y_{n-3})$. Since $\tilde{X}^k$ $(k = 0, 1, \dots, n-3)$ are smooth, the following holds as $\mathfrak{S}_{n+1}$-modules;

$$H^*(Y_{n-3}) \cong H^*(\mathbb{C}P^n) \oplus \left(\bigoplus_{k=0}^{n-3} \left(H^*(\tilde{X}^k) \otimes H^+(\mathbb{C}P^{n-k-1}) \right) \right).$$

Here, the $\mathfrak{S}_{n+1}$-representations $H^*(\mathbb{C}P^n)$ and $H^+(\mathbb{C}P^{n-k-1})$ are trivial, and the $\mathfrak{S}_{n+1}$-representation $H^*(\tilde{X}^k)$ is induced by the $\mathfrak{S}_{n+1}$-action on K_{n+1}. By ([19]) $\tilde{\pi}_I$ is isomorphic to $X(K_{k+1})$ for I with cardinality $n-k$, so the following holds;

$$R(Y_{n-3};t) = S_{(n+1)}(1+t+\cdots+t^n) + \sum_{k=0}^{n-3}\left\{R(\tilde{X}^k;t)(\sum_{i=1}^{n-k-1} t^i)\right\}$$

$$= S_{(n+1)}(1+t+\cdots+t^n)$$
$$+\sum_{k=0}^{n-3}\left\{\left(\mathrm{Ind}_{\mathfrak{S}_{k+1}\times\mathfrak{S}_{n-k}}^{\mathfrak{S}_{n+1}} R(X(K_{k+1});t)\boxtimes S_{(n-k)}\right)\left(\sum_{i=1}^{n-k-1} t^i\right)\right\},$$

where S_λ denotes the irreducible representation of $\mathfrak{S}_{n+1}$ corresponding to a Young diagram λ with $n+1$ boxes. We denote by λ_i the number of boxes in the i-th row of λ, then λ may be regarded as a partition $(\lambda_1, \lambda_2, \ldots, \lambda_j)$ of $n+1$. The H-representation $H^*(Y_{n-3})$ is the restriction of the $\mathfrak{S}_{n+1}$-representation $H^*(Y_{n-3})$. So, the following holds;

$$R(Y_{n-3};t) = S_{(n-1)}\boxtimes S_{(2)}(1+t+\cdots+t^n)$$
$$+\sum_{k=0}^{n-3}\left\{\mathrm{Res}_H^{\mathfrak{S}_{n+1}}\left(\mathrm{Ind}_{\mathfrak{S}_{k+1}\times\mathfrak{S}_{n-k}}^{\mathfrak{S}_{n+1}} R(X(K_{k+1});t)\boxtimes S_{(n-k)}\right)\left(\sum_{i=1}^{n-k-1} t^i\right)\right\}.$$

In general, we can describe induced and restricted representations of $\mathfrak{S}_{d+m} \supset \mathfrak{S}_d \times \mathfrak{S}_m$ with Littlewood-Richardson numbers $C^\nu_{\lambda,\mu}$;

$$\mathrm{Ind}_{\mathfrak{S}_d\times\mathfrak{S}_m}^{\mathfrak{S}_{d+m}} S_\lambda \boxtimes S_\mu = \sum_{\nu\vdash d+m} C^\nu_{\lambda,\mu} S_\nu,$$

$$\mathrm{Res}_{\mathfrak{S}_d\times\mathfrak{S}_m}^{\mathfrak{S}_{d+m}} S_\nu = \sum_{\lambda\vdash d,\mu\vdash m} C^\nu_{\lambda,\mu} S_\lambda \boxtimes S_\mu,$$

where $\lambda \vdash n$ means that λ is a partition of n (i.e. λ is a Young diagram with n boxes).

We determine the H-representation $H^*(\tilde{X}^{n-2})$.

$$\tilde{X}^{n-2} = \coprod_{\substack{I\in B(G_{n+1}),\\ |I|=2}} \tilde{\pi}_I,$$

and $\tilde{\pi}_I$ is isomorphic to $X(K_{n-1})$. If I does not contain 1 and $n+1$, then the maximal subgroup H_1 of H acting on $\tilde{\pi}_I$ is $\mathfrak{S}_2 \times \mathfrak{S}_{n-3} \times \mathfrak{S}_2$. Here, the first $\mathfrak{S}_2$ is the symmetric group on $I \subset [n+1]$ and trivially acts on $\tilde{\pi}_I$, the second $\mathfrak{S}_2$ is the symmetric group on $\{1, n+1\}$, and $\mathfrak{S}_{n-3}$ is the symmetric group on the set of other elements in $[n+1]$. The subgroup H_1 acts on $H^*(\tilde{\pi}_I)$, and H transitively acts on $\displaystyle\bigoplus_{\substack{I \in B(G_{n+1}),\\ |I|=2,\ 1,n+1 \notin I}} H^*(\tilde{\pi}_I)$. If I contains either 1 or $n+1$ (e.g. $I = \{1, l\}, l \neq n+1$), then the maximal subgroup H_2 of H acting on $\tilde{\pi}_I$ is $\mathfrak{S}_{n-2} \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_1$. Here, $\mathfrak{S}_{n-2}$ is the symmetric group on $[n+1] \setminus \{l, 1, n+1\}$. The subgroup H_2 acts on $H^*(\tilde{\pi}_I)$, and H transitively acts on $\displaystyle\bigoplus_{\substack{I \in B(G_{n+1}),\\ |I|=2,\ \text{otherwise}}} H^*(\tilde{\pi}_I)$. So, the H-representation $H^*(\tilde{X}^{n-2})$ is as follows;

$$\mathrm{Ind}_{H_1}^{H} H^*(\tilde{\pi}_I) + \mathrm{Ind}_{H_2}^{H} H^*(\tilde{\pi}_I).$$

If I does not contain 1 and $n+1$, then the H_1-representation $H^*(\tilde{\pi}_I)$ is the restriction of the $\mathfrak{S}_2 \times \mathfrak{S}_{n-1}$-representation $H^*(\tilde{\pi}_I)$. Here, $\mathfrak{S}_2$ in $\mathfrak{S}_2 \times \mathfrak{S}_{n-1}$ is the symmetric group on I and trivially acts on $H^*(\tilde{\pi}_I)$. So, the H_1-representation $H^*(\tilde{\pi}_I)$ is $\mathrm{Res}_{H_1}^{\mathfrak{S}_2 \times \mathfrak{S}_{n-1}} H^*(X(K_{n-1}))$. Therefore,

$$\mathrm{Ind}_{H_1}^{H} H^*(\tilde{\pi}_I) = \mathrm{Ind}_{H_1}^{H} \left(\mathrm{Res}_{H_1}^{\mathfrak{S}_2 \times \mathfrak{S}_{n-1}} H^*(X(K_{n-1})) \right).$$

On the other hand, if I contains either 1 or $n+1$ (e.g. $I = \{1, l\}, l \neq n+1$), then the H_2-representation $H^*(\tilde{\pi}_I)$ is the restriction of the $\mathfrak{S}_2 \times \mathfrak{S}_{n-1}$-representation $H^*(\tilde{\pi}_I)$. Here, $\mathfrak{S}_2$ in $\mathfrak{S}_2 \times \mathfrak{S}_{n-1}$ is the symmetric group on I and trivially acts on $H^*(\tilde{\pi}_I)$. So, the H_2-representation $H^*(\tilde{\pi}_I)$ is $\mathrm{Res}_{H_2}^{\mathfrak{S}_2 \times \mathfrak{S}_{n-1}} H^*(X$

(K_{n-1})). Therefore,

$$\mathrm{Ind}_{H_2}^{H} H^*(\tilde{\pi}_I) = \mathrm{Ind}_{H_2}^{H}\left(\mathrm{Res}_{H_2}^{\mathfrak{S}_2\times\mathfrak{S}_{n-1}} H^*(X(K_{n-1}))\right).$$

Summing up the above argument, we obtain the following.

Theorem 5.2.2. *Let G_{n+1} be a graph obtained by removing an edge from a complete graph K_{n+1} with $n+1$ nodes, and $X(G_{n+1})$ be the toric manifold associated to G_{n+1}. Let H be the automorphism group $\mathfrak{S}_{n-1}\times\mathfrak{S}_2$ of G_{n+1}. Then the H-representation $R(X(G_{n+1});t)$ on the cohomology ring of $X(G_{n+1})$ is as follows;*

$$\begin{aligned}
R(X(G_{n+1});t) &= S_{(n-1)} \boxtimes S_{(2)}(1+t+\cdots+t^n)\\
&+\sum_{k=0}^{n-3}\left\{\mathrm{Res}_{H}^{\mathfrak{S}_{n+1}}\left(\mathrm{Ind}_{\mathfrak{S}_{k+1}\times\mathfrak{S}_{n-k}}^{\mathfrak{S}_{n+1}} R(X(K_{k+1});t)\boxtimes S_{(n-k)}\right)\left(\sum_{i=1}^{n-k-1} t^i\right)\right\}\\
&+\left\{\mathrm{Ind}_{H_1}^{H}\left(\mathrm{Res}_{H_1}^{\mathfrak{S}_2\times\mathfrak{S}_{n-1}} S_{(2)}\boxtimes R(X(K_{n-1});t)\right)\right.\\
&\qquad\left.+\mathrm{Ind}_{H_2}^{H}\left(\mathrm{Res}_{H_2}^{\mathfrak{S}_2\times\mathfrak{S}_{n-1}} S_{(2)}\boxtimes R(X(K_{n-1});t)\right)\right\}t.
\end{aligned}$$

Below are explicit computations for $n = 2, 3, 4$ using Theorem 5.2.2.

$$R(X(G_3);t) = S_{(1)}\boxtimes S_{(2)}(1+t^2) + (2S_{(1)}\boxtimes S_{(2)} + S_{(1)}\boxtimes S_{(1,1)})t.$$

$$\begin{aligned}
R(X(G_4);t) =& S_{(2)}\boxtimes S_{(2)}(1+t^3)\\
&+(5S_{(2)}\boxtimes S_{(2)} + 2S_{(2)}\boxtimes S_{(1,1)} + 2S_{(1,1)}\boxtimes S_{(2)}\\
&+S_{(1,1)}\boxtimes S_{(1,1)})(t+t^2).
\end{aligned}$$

$$R(X(G_5);t) = S_{(3)} \boxtimes S_{(2)}(1+t^4)$$
$$+ (8S_{(3)} \boxtimes S_{(2)} + 3S_{(3)} \boxtimes S_{(1,1)} + 5S_{(2,1)} \boxtimes S_{(2)} + 2S_{(2,1)} \boxtimes S_{(1,1)})(t+t^3)$$
$$+ (15S_{(3)} \boxtimes S_{(2)} + 7S_{(3)} \boxtimes S_{(1,1)} + 12S_{(2,1)} \boxtimes S_{(2)}$$
$$+ 7S_{(2,1)} \boxtimes S_{(1,1)} + S_{(1,1,1)} \boxtimes S_{(2)} + S_{(1,1,1)} \boxtimes S_{(1,1)})t^2.$$

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