Equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties

(スプリンガー多様体と正則な冪零ヘッセンバーグ多様体の同変コホモロジー環)

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Abstract

Hessenberg varieties are subvarieties of the flag variety $Flags(\mathbb{C}^n)$. We are mainly concerned with the nilpotent type of Hessenberg varieties. Springer varieties and regular nilpotent Hessenberg varieties are the two extreme cases of nilpotent Hessenberg varieties. Springer varieties are associated with representations of the symmetric group on n letters. DeConcini-Procesi gave an explicit presentation of the cohomology rings of Springer varieties, and Tanisaki simplified their presentation. Regular nilpotent Hessenberg varieties are generalizations of the Peterson variety which is associated with the quantum cohomology of $Flags(\mathbb{C}^n)$. The (equivariant) cohomology rings of regular nilpotent Hessenberg varieties have been studied by Brion-Carrell, Insko, and Fukukawa-Harada-Masuda.

In this dissertation we give an explicit presentation of the equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties. Also, we give an explicit presentation of the equivariant cohomology rings of Peterson varieties in all Lie types.

Contents

Chapter 1

Introduction

The flag variety $Flags(\mathbb{C}^n)$ is the collection of nested sequences $V_\bullet := (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ of subspaces in $\mathbb{C}^n$ where V_i is i-dimensional vector space. Given a linear operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ and a nondecreasing function $h : [n] \to [n]$ such that $h(i) \geq i$ for each i where $[n] := \{1, 2, \ldots, n\}$, we consider the flags

$$\mathrm{Hess}(A, h) = \{V_\bullet \in Flags(\mathbb{C}^n) \mid AV_i \subseteq V_{h(i)} \text{ for all } 1 \leq i \leq n\}. \qquad (1.0.1)$$

The subvariety $\mathrm{Hess}(A, h)$ is called a **Hessenberg variety** (in type A) and the map h is called a **Hessenberg function**. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \ldots, h(n) = n)$. If $h = (n, n, \ldots, n)$ or A is the zero operator, $\mathrm{Hess}(A, h) = Flags(\mathbb{C}^n)$. Thus the flag variety $Flags(\mathbb{C}^n)$ is itself a special case of Hessenberg varieties. We also remark that if $g \in \mathrm{GL}(n, \mathbb{C})$, then $\mathrm{Hess}(A, h)$ and $\mathrm{Hess}(gAg^{-1}, h)$ can be identified via the usual action of $\mathrm{GL}(n, \mathbb{C})$ on $Flags(\mathbb{C}^n)$. Therefore, we may assume that A is in Jordan canonical form.

If $A = N$ is a nilpotent operator, we call $\mathrm{Hess}(N, h)$ a **nilpotent Hessenberg variety**. We are mainly concerned with the nilpotent type. If $h =$

$(1, 2, \ldots, n)$, we call $\mathrm{Hess}(N, h)$ a **Springer variety**. We denote the Springer variety $\mathrm{Hess}(N, h)$ by $\mathcal{S}_\lambda$ where $h = (1, 2, \ldots, n)$ and N is in Jordan canonical form with Jordan blocks of weakly decreasing sizes $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. If $N = \mathsf{N}$ is the regular nilpotent operator, that is, N is the nilpotent operator in Jordan canonical form with one Jordan block, we call $\mathrm{Hess}(\mathsf{N}, h)$ a **regular nilpotent Hessenberg variety**. These varieties are the two extreme cases of nilpotent Hessenberg varieties.

Springer constructed a representation of the symmetric group S_n on the cohomology $H^*(\mathcal{S}_\lambda)$ considered as a complex vector space, and this representation on the top degree part is the irreducible representation of type λ ([34], [35]). Here, we call the S_n-representation on the total cohomology $H^*(\mathcal{S}_\lambda)$ Springer's representation. DeConcini-Procesi [9] used Springer's representation to give a presentation of the cohomology ring $H^*(\mathcal{S}_\lambda)$ as a quotient of a polynomial ring by an ideal. Tanisaki [37] gave another set of generators of this ideal which simplifies the DeConcini-Procesi presentation; this ideal is now called Tanisaki's ideal.

On the other hand, regular nilpotent Hessenberg varieties are generalizations of the **Peterson variety**, denoted by Pet, which is the case when $h = (2, 3, 4, \ldots, n, n)$ for $\mathrm{Hess}(\mathsf{N}, h)$. The Peterson variety arises in the study of the quantum cohomology of the flag variety $Flags(\mathbb{C}^n)$ ([25], [32]). The (equivariant) cohomology rings of $\mathrm{Hess}(\mathsf{N}, h)$ have been studied in [6], [22], [11].

In this thesis we give an explicit presentation of the equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties. Recall that the following standard torus

$$T = \left\{ \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix} \middle| \; g_i \in \mathbb{C}^* \;\; (1 \leq i \leq n) \right\} \tag{1.0.2}$$

acts on the flag variety $Flags(\mathbb{C}^n)$ in a natural way. This T-action does not preserve the subvariety $\mathrm{Hess}(N,h)$ in general, but the following one-dimensional subtorus S of T preserves $\mathrm{Hess}(N,h)$ ([15, Lemma 5.1]):

$$S = \left\{ \begin{pmatrix} g & & & \\ & g^2 & & \\ & & \ddots & \\ & & & g^n \end{pmatrix} \middle| \; g \in \mathbb{C}^* \right\}. \tag{1.0.3}$$

We are mainly concerned with the S-equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties.

In Chapter 3 we give an explicit presentation of the S-equivariant cohomology rings of the Springer variety $\mathcal{S}_{(n-k,k)}$ for $\lambda = (n-k,k)$ with $\mathbb{Q}$-coefficients. We call $\mathcal{S}_{(n-k,k)}$ the **$(n{-}k,k)$ Springer variety**. The presentation of $H^*_S(\mathcal{S}_{(n-k,k)};\mathbb{Q})$ yields a presentation of the ordinary cohomology $H^*(\mathcal{S}_{(n-k,k)};\mathbb{Q})$ as a ring. However, the resulting presentation is slightly different from Tanisaki's presentation given in [37]. From the presentation in Chapter 3 we will see that the S-equivariant cohomology $H^*_S(\mathcal{S}_{(n-k,k)})$ does **not** extend Springer's representation of S_n. This problem can be rectified by considering instead the action of the following ℓ-dimensional subtorus T^ℓ of T , which does preserve $\mathcal{S}_\lambda$ for

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$:

$$T^\ell = \left\{ \begin{pmatrix} h_1 E_{\lambda_1} & & & \\ & h_2 E_{\lambda_2} & & \\ & & \ddots & \\ & & & h_\ell E_{\lambda_\ell} \end{pmatrix} \middle| \; h_i \in \mathbb{C}^* \ (1 \leq i \leq \ell) \right\} \tag{1.0.4}$$

where E_i is the identity matrix of size i.

In Chapter 4 we construct an S_n-representation on the T^ℓ-equivariant cohomology $H^*_{T^\ell}(\mathcal{S}_\lambda)$ which lifts Springer's representation. We use this S_n-representation to give an explicit presentation of the T^ℓ-equivariant cohomology rings of Springer varieties $\mathcal{S}_\lambda$ with $\mathbb{Z}$-coefficients. More precisely, we give a T^ℓ-equivariant version of Tanisaki's presentation given in [37]. Our method is the T^ℓ-equivariant analogue of [37]. This is joint work with Hiraku Abe ([2]).

In Chapter 5 we give an explicit presentation of the S-equivariant cohomology rings of all regular nilpotent Hessenberg varieties $\mathrm{Hess}(\mathsf{N}, h)$ with $\mathbb{Q}$-coefficients. As a corollary, we have an explicit presentation of the ordinary cohomology ring of $\mathrm{Hess}(\mathsf{N}, h)$:

$$H^*(\mathrm{Hess}(\mathsf{N}, h); \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/\check{I}_h \tag{1.0.5}$$

where $\check{I}_h$ is an ideal generated by polynomials $\sum_{k=1}^{j} \left(x_k \prod_{\ell=j+1}^{h(j)} (x_k - x_\ell) \right)$ for $1 \leq j \leq n$ and we take $\prod_{\ell=j+1}^{h(j)} (x_k - x_\ell) = 1$ if $h(j) = j$. Using the presentation of (1.0.5), we see that the cohomology ring $H^*(\mathrm{Hess}(\mathsf{N}, h))$ is a Poincare duality algebra. We remark that $\mathrm{Hess}(\mathsf{N}, h)$ has singular points in general ([25], [23]). This is joint work with Hiraku Abe, Megumi Harada and Mikiya Masuda ([1]).

In Chapter 6 we give an explicit presentation of the S-equivariant cohomology rings of the Peterson varieties in all Lie types with $\mathbb{Q}$-coefficients. More precisely, we are able to give a uniform description of the relevant ideal J, valid for all Lie types, using the Cartan matrix associated to the Lie algebra $\mathfrak{g}$ of G. In particular, our analysis shows that the ideal J is generated by quadratics. Also, our result generalizes the presentation of Fukukawa-Harada-Masuda given in [11], which was only for Lie type A. This is joint work with Megumi Harada and Mikiya Masuda ([14]).

This thesis is organized as follows. We briefly recall the localization theorem and some properties of Hessenberg varieties in Chapter 2. Using localization tequniques, we calculate the equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties. We give an explicit presentation of the S-equivariant cohomology rings of $(n-k,k)$ Springer varieties and the T^ℓ-equivariant cohomology rings of all Springer varieties in Chapter 3 and Chapter 4 respectively. We give an explicit presentation of the S-equivariant cohomology rings of all regular nilpotent Hessenberg varieties in Chapter 5. We give an explicit presentation of the S-equivariant cohomology rings of Peterson varieties in all Lie types in Chapter 6.

The material in Chapter 3 is contained in [17]. Chapter 4 is joint work with Hiraku Abe in [2]. Chapter 5 is joint work with Hiraku Abe, Megumi Harada, and Mikiya Masuda in [1]. Chapter 6 is joint work with Megumi Harada and Mikiya Masuda in [14].

Chapter 2

Preliminaries

2.1 Localization theorem

First, we recall the definition of torus equivariant cohomology ring. Let T be an n-dimensional torus. Then there exists a contractible space ET on which T acts freely. Let X be a topological space with T-action. Then the T-equivariant cohomology ring of X is defined by the cohomology ring of the Borel construction of X, i.e.

$$H_T^*(X) := H^*(ET \times_T X) \tag{2.1.1}$$

where $ET \times_T X$ denotes the orbit space of $ET \times X$ with respect to the diagonal T-action. Let $BT := ET/T$. Then the projection $ET \to BT$ is a principal T-bundle, so the Borel construction of X produces a fiber bundle over BT with fiber X. The bundle $X \to ET \times_T X \to BT$ is called the **Borel fibration**. The Borel fibration induces the following homomorphisms:

$$H^*(BT) \to H_T^*(X) \to H^*(X). \tag{2.1.2}$$

In particular, $H_T^*(X)$ has an $H^*(BT)$-module structure.

Let X^T be the set of T-fixed points of X. The inclusion map $X^T \subset X$ induces the homomorphism:

$$\iota : H_T^*(X) \to H_T^*(X^T). \tag{2.1.3}$$

The following theorem called the localization theorem plays a role in this thesis.

Theorem 2.1.1. ([19], [31], [8]) *Let X be a locally contractible compact Hausdorff space with T-action and R be a multiplicatively closed set $H^*(BT) \setminus \{0\}$. Then the localization of the homomorphism ι in* (2.1.3) *with respect to R is an isomorphism:*

$$R^{-1}\iota : R^{-1}H_T^*(X;\mathbb{Q}) \cong R^{-1}H_T^*(X^T;\mathbb{Q}). \tag{2.1.4}$$

In paricular, we see that if $H_T^*(X;\mathbb{Q})$ is a torsion-free $H^*(BT;\mathbb{Q})$-module, then $\iota : H_T^*(X;\mathbb{Q}) \to H_T^*(X^T;\mathbb{Q})$ in (2.1.3) is injective for a locally contractible compact Hausdorff space X with T-action. In fact, in the commutative diagram

$$\begin{array}{ccc} H_T^*(X;\mathbb{Q}) & \xrightarrow{\iota} & H_T^*(X^T;\mathbb{Q}) \\ \downarrow & & \downarrow \\ R^{-1}H_T^*(X;\mathbb{Q}) & \xrightarrow[\cong]{R^{-1}\iota} & R^{-1}H_T^*(X^T;\mathbb{Q}) \end{array}$$

the left vertical map is injective because $H_T^*(X;\mathbb{Q})$ is a torsion-free $H^*(BT;\mathbb{Q})$-module. Therefore, we have the injectivity of $\iota : H_T^*(X;\mathbb{Q}) \to H_T^*(X^T;\mathbb{Q})$. Furthermore, if $H_T^*(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module, we have the injectivity of $\iota : H_T^*(X;\mathbb{Z}) \to H_T^*(X^T;\mathbb{Z})$.

Let X be path-connected. We assume that the cohomology of X vanishes in all odd degrees and $H^*(X)$ is a free module. Using the Serre spectral sequence, we obtain the following isomorphism as $H^*(BT)$-modules:

$$H_T^*(X) \cong H^*(BT) \otimes H^*(X).$$

In particular, we see that $H_T^*(X)$ is a free $H^*(BT)$-module.

Therefore, we have the following corollary from Theorem 2.1.1.

Corollary 2.1.2. ([19], [31], [8]) *Let X be a locally contractible compact Hausdorff space with T-action. We assume that $H^i(X;\mathbb{Q}) = 0$ for odd i. Then the homomorphism induced from the inclusion map $X^T \subset X$*

$$\iota : H_T^*(X;\mathbb{Q}) \to H_T^*(X^T;\mathbb{Q})$$

is injective. Furthermore, if $H^*(X;\mathbb{Z})$ is a free module, then

$$\iota : H_T^*(X;\mathbb{Z}) \to H_T^*(X^T;\mathbb{Z})$$

is injective.

Example. As is well-known, the flag variety $Flags(\mathbb{C}^n)$ is a locally contractible compact Hausdorff space. The n-dimensional torus T in (1.0.2) naturally acts on $Flags(\mathbb{C}^n)$. Since $Flags(\mathbb{C}^n)$ admits a complex cellular decomposition called Schubert cell, the cohomology of $Flags(\mathbb{C}^n)$ vanishes in all odd degrees and $H^*(Flags(\mathbb{C}^n);\mathbb{Z})$ is a free module. Therefore, the homomorphism induced from the inclusion map $Flags(\mathbb{C}^n)^T \subset Flags(\mathbb{C}^n)$

$$\iota : H_T^*(Flags(\mathbb{C}^n);\mathbb{Z}) \to H_T^*(Flags(\mathbb{C}^n)^T;\mathbb{Z})$$

is injective from Corollary 2.1.2. The T-fixed point set $Flags(\mathbb{C}^n)^T$ is given by

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\}$$

where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$ and S_n is the permutation group on n letters $\{1, 2, \ldots, n\}$ (cf. [13, Lemma 2 in Section 10.1]). We may identify

$Flags(\mathbb{C}^n)^T$ with S_n through the correspondence:

$$(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle) \mapsto w \tag{2.1.5}$$

So we can embed the T-equivariant cohomology ring $H^*_T(Flags(\mathbb{C}^n);\mathbb{Z})$ into a direct sum of a polynomial ring:

$$\iota : H^*_T(Flags(\mathbb{C}^n);\mathbb{Z}) \to \bigoplus_{w\in S_n} \mathbb{Z}[t_1,\ldots,t_n] \tag{2.1.6}$$

because $H^*_T(pt;\mathbb{Z}) = H^*(BT;\mathbb{Z}) = \mathbb{Z}[t_1,\ldots,t_n]$ where $t_i \in H^2(BT)$ denotes the first Chern class of the line bundle $ET \times_T \mathbb{C}_i$ over BT. Here, $\mathbb{C}_i$ is the one dimensional representation of T through a map $T \to \mathbb{C}^*$ given by $diag(g_1,\ldots,g_n) \mapsto g_i$.

Let E_i be the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n) \times \mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\bar{x}_i \in H^2_T(Flags(\mathbb{C}^n))$. Then, using the injectivity of (2.1.6), one can see that the following relations hold:

$$e_i(\bar{x}_1,\ldots,\bar{x}_n) - e_i(t_1,\ldots,t_n) = 0 \quad \text{for all} \quad 1 \le i \le n \tag{2.1.7}$$

where e_i is the i-th elementary symmetric polynomial. In fact, it is known that the T-equivariant cohomology ring of $Flags(\mathbb{C}^n)$ is given by

$$H^*_T(Flags(\mathbb{C}^n);\mathbb{Z}) \cong \mathbb{Z}[x_1,\ldots,x_n,t_1,\ldots,t_n]/I \tag{2.1.8}$$

where I is an ideal generated by symmetric polynomials $e_i(x_1,\ldots,x_n)-e_i(t_1,\ldots,t_n)$ for $1 \le i \le n$.

For instance, we can check the relations in the case $n = 3$ using the injectivity of i in (2.1.6). We arrange all the T-fixed points in $Flags(\mathbb{C}^3)^T \cong S_3$ by an order:

$$123,\ 132,\ 213,\ 231,\ 312,\ 321$$

where all the above permutations are in one-line notation. The w-component of $\iota(\bar{x}_i)$ is given by $t_{w(i)}$, so we have

$$\iota(\bar{x}_1) = (t_1, t_1, t_2, t_2, t_3, t_3),$$
$$\iota(\bar{x}_2) = (t_2, t_3, t_1, t_3, t_1, t_2),$$
$$\iota(\bar{x}_3) = (t_3, t_2, t_3, t_1, t_2, t_1).$$

Also, the w-component of $\iota(t_i)$ is given by t_i, so we have

$$\iota(\bar{t}_1) = (t_1, t_1, t_1, t_1, t_1, t_1),$$
$$\iota(\bar{t}_2) = (t_2, t_2, t_2, t_2, t_2, t_2),$$
$$\iota(\bar{t}_3) = (t_3, t_3, t_3, t_3, t_3, t_3).$$

Therefore, $\iota\big(e_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) - e_i(t_1, t_2, t_3)\big) = 0$ holds for $i = 1, 2, 3$. From the injectivity of ι, we have the relations in (2.1.7) for $n = 3$. The relations in (2.1.7) for general n can be checked in a similar way.

Let X be a path-connected space with T-action. We assume that the cohomology of X vanishes in all odd degrees and $H^i(X)$ is a finitely generated free module for every i. Then the homomorphism $H^*(BT) \to H^*_T(X)$ in (2.1.2) is injective (cf. [27, Theorem 4.2 in CHAPTER III]), so we may think of t_i as an element of $H^*_T(X)$ where we identify $H^*(BT)$ with a polynomial ring in n variables $t_1, \cdots, t_n$. Also, the homomorphism $H^*_T(X) \to H^*(X)$ in (2.1.2) is surjective and its kernel is generated by $t_1, \cdots, t_n$ (cf. [27, Theorem 4.2 in CHAPTER III]). Therefore, we have the following isomorphism:

$$H^*(X) \cong H^*_T(X)/(t_1, \cdots, t_n). \tag{2.1.9}$$

Example. Taking $t_i = 0$ for $i = 1, \ldots, n$ in (2.1.8), we have well-known presentation by Borel:

$$H^*(Flags(\mathbb{C}^n);\mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/\check{I} \qquad (2.1.10)$$

where $\check{I}$ is an ideal generated by the elementary symmetric polynomials $e_i(x_1, \ldots, x_n)$ for $1 \leq i \leq n$.

2.2 Properties of Hessenberg varieties

We have already seen the localization tequnique in Section 2.1. To use the localization tequnique for a topological space X, it is enough to check the conditions that the cohomology of X vanishes in all odd degrees and $H^*(X;\mathbb{Z})$ is a free module. In this section we recall pavings, sometimes called cellular decompositions, and the construction of pavings by complex affines of Hessenberg varieties given by Tymoczko. As a result, we can use the localization tequnique for all Hessenberg varieties.

Definition. Let X be a variety. A **paving** of X is an ordered partition $X = \amalg_{i=0}^{\infty} X_i$ so that each finite union $\amalg_{i=0}^{j} X_i$ is Zariski-closed in X. If in addition each X_i is homeomorphic to affine space $\mathbb{R}^{d_i}$ then $\amalg_{i=0}^{\infty} X_i$ is called a paving by affines. We call the X_i **cells**.

Example. Let w be a permutation in the n-th symmetric group S_n and C_w be the Schubert cell in the flag variety $Flags(\mathbb{C}^n)$ which is homeomorphic to affine space $\mathbb{C}^{\ell(w)}$. Here, $\ell(w)$ is the number of inversions of w. Then $Flags(\mathbb{C}^n)$ is paved by affines $\amalg_{w \in S_n} C_w$.

Proposition 2.2.1. (cf. [13, Lemma 6 in Appendix B]) *Let X be a variety and $X = \amalg_i X_i$ be a paving by a finite number of affines X_i with each X_i homeomorphic to $\mathbb{C}^{d_i}$. Then the nonzero cohomology groups of X are giben by*

$$H^k(X;\mathbb{Z}) \cong \bigoplus_{i \text{ such that } 2d_i = k} \mathbb{Z}.$$

In particular, the cohomology of X vanishes in all odd degrees and $H^*(X;\mathbb{Z})$ is a free module.

Therefore, we have the following corollary from Corollary 2.1.2.

Corollary 2.2.2. *Let X be a variety and compact Hausdorff space with T-action. We assume that X has admits a paving by a finite number of complex affine spaces. Then the homomorphism induced from the inclusion map $X^T \subset X$*

$$\iota : H_T^*(X;\mathbb{Z}) \to H_T^*(X^T;\mathbb{Z})$$

is injective.

The following theorem implies that we can apply the localization tequnique to Hessenberg varieties.

Theorem 2.2.3. ([38]) *Fix a Hessenberg function $h : [n] \to [n]$ and a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$. The Hessenberg variety* $\mathrm{Hess}(A,h)$ *admits a paving by a finite number of complex affine spaces.*

Remark. In [38, Theorem 6.1], Tymoczko gave explicitly a paving by complex affines of $\mathrm{Hess}(A,h)$. In fact, let $\{C_w\}_{w\in S_n}$ be the Schubert cells in the flag variety $Flags(\mathbb{C}^n)$. If A is in highest form and in permuted Jordan form (cf. [38, Section 4]), then the intersections $C_w \cap \mathrm{Hess}(A,h)$ form a paving by complex affines of $\mathrm{Hess}(A,h)$.

Remark. A paving by complex affines of Springer variety $\mathcal{S}_\lambda$ is given by [33].

The n-dimensional torus T in (1.0.2) acts on the flag variety $Flags(\mathbb{C}^n)$ in a natural way. However, this T-action does not preserve the subvariety $\mathrm{Hess}(A,h)$ in general. This problem can be rectified by considering instead the action of the one-dimensional subtorus S of T in (1.0.3), which does preserve $\mathrm{Hess}(A,h)$ ([15, Lemma 5.1]).

From Corollary 2.2.2 and Theorem 2.2.3, we have the following theorem.

Theorem 2.2.4. *The homomorphism induced from the inclusion map* $\mathrm{Hess}(A,h)^S \subset \mathrm{Hess}(A,h)$

$$\iota : H_S^*(\mathrm{Hess}(A,h);\mathbb{Z}) \to H_S^*(\mathrm{Hess}(A,h)^S;\mathbb{Z})$$

is injective.

We are mainly concerned with nilpotent Hessenberg varieties $\mathrm{Hess}(N,h)$. Since the S-fixed points of the flag variety $Flags(\mathbb{C}^n)$ coincide with the T-fixed points of the flag variety $Flags(\mathbb{C}^n)$, we identify the S-fixed points of nilpotent Hessenberg variety $\mathrm{Hess}(N,h)$ with a subset of the permutation group S_n under the identification (2.1.5).

So we can embed the S-equivariant cohomology ring $H_S^*(\mathrm{Hess}(N,h);\mathbb{Z})$ into a direct sum of a polynomial ring:

$$\iota : H_T^*(Flags(\mathbb{C}^n);\mathbb{Z}) \to \bigoplus_{w \in \mathrm{Hess}(N,h)^S \subseteq S_n} \mathbb{Z}[t]. \tag{2.2.1}$$

Note that $H_S^*(pt;\mathbb{Z}) = H^*(BS;\mathbb{Z}) = \mathbb{Z}[t]$ where $t \in H^2(BS)$ denotes the first Chern class of the line bundle $ES \times_S \mathbb{C}$ over BS. Here, $\mathbb{C}$ is the one dimensional representation of S through a map $S \cong \mathbb{C}^*$ given by $diag(g, g^2, \ldots, g^n) \mapsto g$.

Putting the injective maps ι in (2.1.6) and (2.2.1) together, we have the following commutative diagram

$$\begin{array}{ccc} H_T^*(Flags(\mathbb{C}^n);\mathbb{Z}) & \xrightarrow{\iota} & \bigoplus_{w\in Flags(\mathbb{C}^n)^T=S_n} \mathbb{Z}[t_1,\dots,t_n] \\ \rho\big\downarrow & & \big\downarrow\pi \\ H_S^*(\mathrm{Hess}(N,h);\mathbb{Z}) & \xrightarrow{\iota} & \bigoplus_{w\in \mathrm{Hess}(N,h)^S\subseteq S_n} \mathbb{Z}[t] \end{array} \tag{2.2.2}$$

where the map ρ is induced from inclusion maps $\mathrm{Hess}(N,h) \subseteq Flags(\mathbb{C}^n)$ and $S \subseteq T$, and the map π is naturally induced from the map $\mathbb{Z}[t_1,\dots,t_n] \to \mathbb{Z}[t]$ given by $t_i \mapsto it$ for $i = 1,\dots,n$.

Using the commutative diagram (2.2.2), we will find relations between ring generators of $H_S^*(\mathrm{Hess}(N,h))$ when $\mathrm{Hess}(N,h)$ is a Springer variety or a regular nilpotent Hessenberg variety. Using the relations, we will give an explicit presentation of the equivariant cohomology rings of Springer varieties or regular nilpotent Hessenberg varieties. As a corollary, we have an explicit presentation of the ordinary cohomology rings of Springer varieties or regular nilpotent Hessenberg varieties from (2.1.9).

Remark. DeConcini-Procesi [9] gave an explicit presentation of the ordinary cohomology rings of Springer varieties, and Tanisaki [37] simplified their presentation. The localization tequnique is not used in their arguments.

Chapter 3

The S-equivariant cohomology rings of $(n-k,k)$ Springer varieties

In Chapter 3 we give an explicit presentation of the S-equivariant cohomology rings of $(n-k,k)$ Springer varieties. Chapter 3 is organized as follows. We briefly recall the necessary background in Section 3.1 and Section 3.2. Our main theorem, Theorem 3.3.3, is formulated in Section 3.3 and proved in Section 3.4. This is a work in [17].

3.1 Definition of $(n-k,k)$ Springer varieties

We begin by recalling the definition of the Springer varieties in type A in Chapter 1. Since we work exclusively with type A in this chapter, we henceforth omit it from our terminology.

Definition. Let N be a nilpotent matrix of size n in Jordan canonical form with Jordan blocks of weakly decreasing sizes $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The **Springer variety** $\mathcal{S}_\lambda$ associated to N is defined as

$$\mathcal{S}_\lambda = \{V_\bullet \in Flags(\mathbb{C}^n) \mid NV_i \subseteq V_i \text{ for all } 1 \le i \le n\}.$$

We denote by $\mathcal{S}_{(n-k,k)}$ the Springer variety corresponding to the partition $\lambda = (n-k,k)$ with $2k \le n$.

Lemma 3.1.1.

$$\mathcal{S}_\lambda = \{V_\bullet \in Flags(\mathbb{C}^n) \mid NV_i \subseteq V_{i-1} \text{ for all } 1 \le \text{i} \le \text{n}\}$$

Here, we take $V_0 := \{0\}$.

Proof. We prove that an element of $\mathcal{S}_\lambda$ satisfies the condition $NV_i \subseteq V_{i-1}$ for $i = 1, \ldots, n$. Let $V_\bullet \in \mathcal{S}_\lambda$ and $v_1, v_2, \ldots, v_i$ be generators for V_i. Since $Nv_1, Nv_2, \ldots, Nv_i$ generate NV_i, it is enough to prove that $Nv_i \in V_{i-1}$. Now $Nv_i \in V_i$, so we can write

$$Nv_i = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_i v_i \tag{3.1.1}$$

for some $c_1, \ldots, c_i \in \mathbb{C}$ and we have

$$N^2 v_i - c_i N v_i = N(Nv_i - c_i v_i) \in NV_{i-1} \subseteq V_{i-1}.$$

From (3.1.1), this implies

$$N^2 v_i - c_i^2 v_i \in V_{i-1}.$$

Inductively, we have

$$N^m v_i - c_i^m v_i \in V_{i-1} \tag{3.1.2}$$

for any positive integer m. Since N is nilpotent, (3.1.2) implies

$$c_i^m v_i \in V_{i-1}$$

for some m. If $c_i \neq 0$, then $v_i \in V_{i-1}$ and we have $V_i = V_{i-1}$. This is a contradiction. Therefore, we have $c_i = 0$. From (3.1.1), $Nv_i \in V_{i-1}$. □

3.2 S-fixed points of $(n-k,k)$ Springer varieties

We first recall that the T-fixed point set $Flags(\mathbb{C}^n)^T$ of $Flags(\mathbb{C}^n)$ is given by

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\}$$

where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$ and S_n is the permutation group on n letters $\{1, 2, \ldots, n\}$, so we identify $Flags(\mathbb{C}^n)^T$ with S_n as is standard. Also, since the S-fixed point set $Flags(\mathbb{C}^n)^S$ of $Flags(\mathbb{C}^n)$ agrees with $Flags(\mathbb{C}^n)^T$ where S is defind by (1.0.3), we have

$$\mathcal{S}_N^S = \mathcal{S}_N \cap Flags(\mathbb{C}^n)^S = \mathcal{S}_N \cap Flags(\mathbb{C}^n)^T \subset S_n.$$

We next describe the S-fixed points in $\mathcal{S}_{(n-k,k)}$. Let $w_{\ell_1,\ell_2,\ldots,\ell_k}$ be an element of S_n defined by

$$w_{\ell_1,\ell_2,\ldots,\ell_k}(i) = \begin{cases} n-k+j & \text{if } i = \ell_j, \\ i-j & \text{if } \ell_j < i < \ell_{j+1}, \end{cases} \tag{3.2.1}$$

where $\ell_0 := 0$, $\ell_{k+1} := n+1$. Note that $w^{-1}_{\ell_1,\ell_2,\ldots,\ell_k}(i) < w^{-1}_{\ell_1,\ell_2,\ldots,\ell_k}(i')$ if $1 \leq i < i' \leq n-k$ or $n-k+1 \leq i < i' \leq n$.

Example. Take $n=4$ and $k=2$. Using one-line notation, the set of permutations of the form described in (3.2.1) are as follows:

$$[3,4,1,2],[3,1,4,2],[3,1,2,4],[1,3,4,2],[1,3,2,4],[1,2,3,4].$$

Lemma 3.2.1. *The S-fixed points $\mathcal{S}^S_{(n-k,k)}$ of the Springer variety $\mathcal{S}_{(n-k,k)}$ is the set*

$$\{w_{\ell_1,\ell_2,\ldots,\ell_k}\in S_n \mid 1\le \ell_1<\ell_2<\cdots<\ell_k\le n\}.$$

Proof. Since $\mathcal{S}^S_{(n-k,k)}\subset Flags(\mathbb{C}^n)^T$, any element $V_\bullet$ of $\mathcal{S}^S_{(n-k,k)}$ is of the form

$$V_\bullet=(\langle e_{w(1)}\rangle\subset\langle e_{w(1)},e_{w(2)}\rangle\subset\cdots\subset\langle e_{w(1)},e_{w(2)},\ldots,e_{w(n)}\rangle)$$

for some $w\in S_n$. Since N is the nilpotent operator consisting of two Jordan blocks with weakly decreasing sizes $(n-k,k)$,

$$Ne_i=\begin{cases}0 & \text{if } i=1 \text{ or } n-k+1,\\ e_{i-1} & \text{otherwise.}\end{cases}$$

Therefore, if $V_\bullet$ belongs to $\mathcal{S}_{(n-k,k)}$, then $w(1)=1$ or $n-k+1$. If $w(1)=1$ then $w(2)=2$ or $n-k+1$. If $w(1)=n-k+1$ then $w(2)=1$ or $n-k+2$, and so on. This shows that $w=w_{\ell_1,\ell_2,\ldots,\ell_k}$ for some $1\le\ell_1<\ell_2<\cdots<\ell_k\le n$. Conversely, one can easily see that $w_{\ell_1,\ell_2,\ldots,\ell_k}\in\mathcal{S}^S_{(n-k,k)}$. □

3.3 The main theorem in Chapter 3

In this section, we formulate our main theorem which gives an explicit presentation of the S-equivariant cohomology ring of the $(n-k,k)$ Springer variety.

We consider the following commutative diagram:

$$\begin{array}{ccc} H_T^*(Flags(\mathbb{C}^n);\mathbb{Q}) & \xrightarrow{\iota_1} & H_T^*(Flags(\mathbb{C}^n)^T;\mathbb{Q}) = \bigoplus_{w\in S_n} \mathbb{Q}[t_1,\dots,t_n] \\ {\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ H_S^*(\mathcal{S}_\lambda;\mathbb{Q}) & \xrightarrow{\iota_2} & H_S^*(\mathcal{S}_\lambda^S;\mathbb{Q}) = \bigoplus_{w\in \mathcal{S}_\lambda^S\subset S_n} \mathbb{Q}[t] \end{array} \tag{3.3.1}$$

where all the maps are induced from inclusion maps and we consider the identification $H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1,\dots,t_n]$ and $H^*(BS;\mathbb{Q}) = \mathbb{Q}[t]$. The maps ι_1 and ι_2 in (3.3.1) are injective since the odd degree cohomology groups of $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ vanish. The map π_1 in (3.3.1) is known to be surjective (cf. [18]) and the map π_2 is obviously surjective.

Let E_i be the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n)\times\mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\bar{x}_i \in H_T^2(Flags(\mathbb{C}^n);\mathbb{Q})$ and the image $\pi_1(\bar{x}_i)$ of $\bar{x}_i$ for each i by τ_i.

Since π_1 is surjective, we have the following lemma.

Lemma 3.3.1. *The S-equivariant cohomology ring $H_S^*(\mathcal{S}_\lambda;\mathbb{Q})$ is generated by $\tau_1,\dots,\tau_n,t$ as a ring where τ_i is the image of $\bar{x}_i$ under the map π_1 in* (3.3.1)*.*

We next consider relations between $\tau_1,\dots,\tau_n$, and t. We have

$$\iota_2(\tau_i)|_w = w(i)t$$

because $\iota_1(\bar{x}_i)|_w = t_{w(i)}$, $\iota_1(t_i)|_w = t_i$, and $\pi_2(t_i) = it$, where $f|_w$ denotes the w-component of $f \in \bigoplus_{w\in S_n} \mathbb{Q}[t_1,\dots,t_n]$.

Lemma 3.3.2. *The elements $\tau_1,\dots,\tau_n,t$ satisfy the following relations:*

$$\sum_{1\le i\le n} \tau_i - \frac{n(n+1)}{2}t = 0, \tag{3.3.2}$$

$$(\tau_i+\tau_{i-1}-(n-k+i)t)(\tau_i-\tau_{i-1}-t)=0 \qquad (1\le i\le n), \qquad (3.3.3)$$

$$\prod_{0\le j\le k}(\tau_{i_j}-(i_j-j)t))=0 \qquad (1\le i_0<\cdots<i_k\le n) \qquad (3.3.4)$$

where $\tau_0=0$.

Proof. The relation (3.3.2) follows from a relation in $H_T^*(Flags(\mathbb{C}^n);\mathbb{Q})$. In fact,

$$\sum_{1\le i\le n}\tau_i-\frac{n(n+1)}{2}t=\pi_1((e_1(\bar{x}_1,\dots,\bar{x}_n)-e_1(t_1,\dots,t_n)))=0.$$

In the following, we denote $\iota_2(\tau_i)$ by the same notation τ_i for each i. To prove the relation (3.3.3), it is sufficient to prove either

$$(\tau_i+\tau_{i-1}-(n-k+i)t)|_{w_{\ell_1,\ell_2,\dots,\ell_k}}=0 \ \text{ or } \ (\tau_i-\tau_{i-1}-t)|_{w_{\ell_1,\ell_2,\dots,\ell_k}}=0 \qquad (3.3.5)$$

for any $w_{\ell_1,\ell_2,\dots,\ell_k}\in\mathcal{S}^S_{(n-k,k)}$ since the restriction map ι_2 in (3.3.1) is injective.

We first treat the case $i=1$. By the definition of $w_{\ell_1,\ell_2,\dots,\ell_k}$ in (3.2.1) the following holds:

$$\tau_1|_{w_{\ell_1,\ell_2,\dots,\ell_k}}=w_{\ell_1,\ell_2,\dots,\ell_k}(1)t=\begin{cases}(n-k+1)t & \text{if } \ell_1=1,\\ t & \text{if } \ell_1\neq 1.\end{cases}$$

This shows (3.3.5) for $i=1$ because $\tau_0=0$.

We now treat the case $1<i\le n$. Note that

$$(\tau_i-\tau_{i-1})|_{w_{\ell_1,\ell_2,\dots,\ell_k}}=(w_{\ell_1,\ell_2,\dots,\ell_k}(i)-w_{\ell_1,\ell_2,\dots,\ell_k}(i-1))t, \qquad (3.3.6)$$

$$(\tau_i+\tau_{i-1})|_{w_{\ell_1,\ell_2,\dots,\ell_k}}=(w_{\ell_1,\ell_2,\dots,\ell_k}(i)+w_{\ell_1,\ell_2,\dots,\ell_k}(i-1))t. \qquad (3.3.7)$$

We take four cases depending on whether $i-1$ and i appear in $\ell_1,\dots,\ell_k$ or not.

(i) If $\ell_j=i-1<i=\ell_{j+1}$ for some $1\le j\le k-1$, then by (3.2.1) and

(3.3.6),

$$(\tau_i - \tau_{i-1})|_{w_{\ell_1,\ell_2,\ldots,\ell_k}} = ((n-k+j+1) - (n-k+j))t = t.$$

(ii) If $\ell_j < i-1 < i < \ell_{j+1}$ for some $0 \le j \le k$, then by (3.2.1) and (3.3.6),

$$(\tau_i - \tau_{i-1})|_{w_{\ell_1,\ell_2,\ldots,\ell_k}} = ((i-j) - (i-j-1))t = t.$$

(iii) If $\ell_j = i-1 < i < \ell_{j+1}$ for some $1 \le j \le k$, then by (3.2.1) and (3.3.7),

$$(\tau_i + \tau_{i-1})|_{w_{\ell_1,\ell_2,\ldots,\ell_k}} = ((i-j) + (n-k+j))t = (n-k+i)t.$$

(iv) If $\ell_{j-1} < i-1 < i = \ell_j$ for some $1 \le j \le k$, then by (3.2.1) and (3.3.7),

$$(\tau_i + \tau_{i-1})|_{w_{\ell_1,\ell_2,\ldots,\ell_k}} = ((n-k+j) + (i-j))t = (n-k+i)t.$$

Therefore, (3.3.5) holds in all cases, proving the relations (3.3.3).

Finally we prove the relations (3.3.4). For any $w_{\ell_1,\ell_2,\ldots,\ell_k} \in \mathcal{S}^S_{(n-k,k)}$, there is a positive integer i_j such that $\ell_j < i_j < \ell_{j+1}$ for some $0 \le j \le k$. Thus, we have

$$w_{\ell_1,\ell_2,\ldots,\ell_k}(i_j) = i_j - j.$$

This means that

$$\prod_{0 \le j \le k} (\tau_{i_j} - (i_j - j)t)|_{w_{\ell_1,\ell_2,\ldots,\ell_k}} = 0.$$

Therefore, the relations (3.3.4) hold, and the proof is complete. □

It follows from Lemma 3.3.2 that we obtain a well-defined ring homomorphism

$$\varphi : \mathbb{Q}[x_1, \ldots, x_n, t]/I \to H^*_S(\mathcal{S}_{(n-k,k)}; \mathbb{Q}) \tag{3.3.8}$$

where I is the ideal of a polynomial ring $\mathbb{Q}[x_1,\dots,x_n,t]$ generated by the following three types of elements:

$$\sum_{1\le i\le n} x_i - \frac{n(n+1)}{2}t, \tag{3.3.9}$$

$$(x_i + x_{i-1} - (n-k+i)t)(x_i - x_{i-1} - t) \qquad (1\le i\le n), \tag{3.3.10}$$

$$\prod_{0\le j\le k} (x_{i_j} - (i_j - j)t) \qquad (1\le i_0 < \cdots < i_k \le n) \tag{3.3.11}$$

where $x_0 = 0$. Moreover, φ is surjective by Lemma 3.3.1.

The following is our main theorem and will be proved in the next section.

Theorem 3.3.3. *Let $\mathcal{S}_{(n-k,k)}$ be the $(n-k,k)$ Springer variety with $0\le k\le n/2$ and let the circle group S act on $\mathcal{S}_{(n-k,k)}$ as described in Section 3.2. Then the S-equivariant cohomology ring of $\mathcal{S}_{(n-k,k)}$ is given by*

$$H_S^*(\mathcal{S}_{(n-k,k)};\mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_n,t]/I$$

where $H_S^*(pt;\mathbb{Q}) = \mathbb{Q}[t]$ and I is the ideal of the polynomial ring $\mathbb{Q}[x_1,\dots,x_n,t]$ generated by the elements listed in (3.3.9), (3.3.10), *and* (3.3.11).

Since the ordinary cohomology ring of $\mathcal{S}_{(n-k,k)}$ can be obtained by taking $t=0$ in Theorem 3.3.3, we obtain the following corollary.

Corollary 3.3.4. *Let $\mathcal{S}_{(n-k,k)}$ be $(n-k,k)$ Springer variety with $0\le k\le n/2$. Then the ordinary cohomology ring of $\mathcal{S}_{(n-k,k)}$ is given by*

$$H^*(\mathcal{S}_{(n-k,k)};\mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_n]/J$$

where J is the ideal of the polynomial ring $\mathbb{Q}[x_1,\dots,x_n]$ generated by the fol-

lowing three types of elements:

$$\sum_{1\leq i\leq n} x_i,$$
$$x_i^2 \qquad (1\leq i\leq n),$$
$$\prod_{1\leq j\leq k+1} x_{i_j} \qquad (1\leq i_1<\cdots<i_{k+1}\leq n).$$

Remark. A ring presentation of the cohomology ring of the Springer variety $\mathcal{S}_\lambda$ is given in [37] for an arbitrary nilpotent operator N. Specifically, it is the quotient of a polynomial ring by an ideal called Tanisaki's ideal. When $\lambda_N = (n-k,k)$, Tanisaki's ideal is generated by the following three types of elements:

$$e_1(x_1,\ldots,x_n),$$
$$e_2(x_{i_1},\ldots,x_{i_{n-1}}) \qquad (1\leq i_1<\cdots<i_{n-1}\leq n),$$
$$e_{k+1}(x_{i_1},\ldots,x_{i_{k+1}}) \qquad (1\leq i_1<\cdots<i_{k+1}\leq n),$$

where e_i is the ith elementary symmetric polynomial. Note that the first and third elements above are the same as those in Corollary 3.3.4. In fact, one can easily check that Tanisaki's ideal above agrees with the ideal J in Corollary 3.3.4 although the generators are slightly different.

3.4 Proof of the main theorem in Chapter 3

This section is devoted to the proof of Theorem 3.3.3. More precisely, we will prove that the epimorphism φ in (3.3.8) is an isomorphism. For this, we first find generators of $\mathbb{Q}[x_1,\ldots,x_n,t]/I$ as a $\mathbb{Q}[t]$-module.

Recall that a **filling** of λ by the alphabet $\{1,\ldots,n\}$ is an injective placing of the integers $\{1,\ldots,n\}$ into the boxes of λ.

Definition. Let λ be a Young diagram with n boxes. A filling of λ is a **permissible filling** if for every horizontal adjacency $\boxed{a}\boxed{b}$ we have $a<b$. Also, a permissible filling is a **standard tableau** if for every vertical adjacency $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$ we have $a<b$.

Let T be a permissible filling of $(n-\ell,\ell)$ with $0\le \ell\le k$. Let $j_1,j_2,\dots,j_\ell$ be the numbers in the bottom row of T. We define $x_T := x_{j_1}x_{j_2}\dots x_{j_\ell}$ and $x_{T_0}:=1$ where T_0 is the standard tableau on (n).

Proposition 3.4.1. *The set* $\{x_T \mid T$ *standard tableau on* $(n-\ell,\ell)$ *with* $0\le \ell\le k\}$ *generates* $\mathbb{Q}[x_1,\dots,x_n,t]/I$ *as a* $\mathbb{Q}[t]$*-module.*

Proof. It is sufficient to prove that $x_{b_1}x_{b_2}\cdots x_{b_\ell}\quad(1\le b_1\le b_2\le\cdots\le b_\ell\le n)$ can be written in $\mathbb{Q}[x_1,\dots,x_n,t]/I$ as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau. We prove this by induction on ℓ. The base case $\ell=0$ is clear. Now we assume that $\ell\ge 1$ and the claim holds for $\ell-1$. The relations (3.3.10) imply that

$$x_i^2=(n-k+i+1)tx_i+t\sum_{1\le p\le i-1}x_p-\sum_{1\le p\le i}(n-k+p)t^2\quad(1\le i\le n)\qquad(3.4.1)$$

by an inductive argument on i, so we may assume $b_1<b_2<\cdots<b_\ell$.

To prove the claim for ℓ, we consider two cases: $1\le\ell\le k$ and $\ell\ge k+1$.

(Case i). Suppose $1\le \ell\le k$. We write $x_{b_1}x_{b_2}\cdots x_{b_\ell}=x_U$ where

$$U=\begin{array}{|c|c|c|c|c|c|}\hline a_1&\cdots&a_\ell&a_{\ell+1}&\cdots&a_{n-\ell}\\\hline b_1&\cdots&b_\ell\\\cline{1-3}\end{array}$$

is a permissible filling of $(n-\ell,\ell)$. Let j be the minimal positive integer in the set $\{r\mid a_r>b_r, 1\le r\le\ell\}$, i.e.,

$$a_i<b_i\qquad(1\le i<j),\qquad(3.4.2)$$

$$a_j>b_j.\qquad(3.4.3)$$

We consider the following equation which follows from the relation (3.3.9):

$$(-x_{a_1} - x_{a_2} - \cdots - x_{a_{j-1}})^j \cdot x_{b_{j+1}} \cdots x_{b_\ell} \tag{3.4.4}$$
$$= (x_{b_1} + x_{b_2} + \cdots + x_{b_\ell} + x_{a_j} + x_{a_{j+1}} + \cdots + x_{a_{n-\ell}} - \frac{n(n+1)}{2} t)^j \cdot x_{b_{j+1}} \cdots x_{b_\ell}.$$

Claim 1. The left hand side in (3.4.4) is a $\mathbb{Q}[t]$-linear combination of the x_T where the T are standard tableaux.

Proof. We expand the left hand side in (3.4.4). Then any monomial which appears in the expansion is of the form

$$x_{a_1}^{\alpha_1} \cdots x_{a_{j-1}}^{\alpha_{j-1}} x_{b_{j+1}} \cdots x_{b_\ell}$$

where $\sum_{i=1}^{j-1} \alpha_i = j$ and $\alpha_i \geq 0$. Note that $\alpha_i > 1$ for some i since $\sum_{i=1}^{j-1} \alpha_i = j$ and $\alpha_i \geq 0$. Therefore, using the relations (3.4.1), the monomial above turns into a sum of elements of the form

$$f(t) \cdot x_{c_1} \cdots x_{c_h}$$

where $h < \ell$, $1 \leq c_1 < \cdots < c_h \leq n$, and $f(t) \in \mathbb{Q}[t]$, and by the induction assumption the term above can be written as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau. This proves Claim 1. □

Claim 2. The right hand side in (3.4.4) can be written as a $\mathbb{Q}[t]$-linear combination of x_U and monomials x_T and $x_{U'}$ where the coefficient of x_U is equal to 1, T is a standard tableau on shape $(n-\ell, \ell)$ and U' is a permissible filling of $(n-\ell, \ell)$ such that each of the leftmost j columns are strictly increasing (i.e. $a_r < b_r, 1 \leq r \leq j$).

Proof. We expand the right hand side in (3.4.4). A monomial which appears in this expansion is of the form

$$x_{b_{p_1}}^{\beta_1}\cdots x_{b_{p_m}}^{\beta_m}x_{a_{q_1}}^{\alpha_1}\cdots x_{a_{q_h}}^{\alpha_h}x_{b_{j+1}}\cdots x_{b_\ell}$$

where $\sum_{i=1}^m \beta_i + \sum_{i=1}^h \alpha_i \leq j,\ \beta_i \geq 1,\ \alpha_i \geq 1$ and $1 \leq p_1 < \cdots < p_m \leq \ell,\ j \leq q_1 < \cdots < q_h \leq n-\ell$. It is enough to consider the case $\sum_{i=1}^m \beta_i + \sum_{i=1}^h \alpha_i = j$ since if $\sum_{i=1}^m \beta_i + \sum_{i=1}^h \alpha_i < j$ then it follows from the induction assumption that the above form can be written as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau. If $p_m \geq j+1$ or some β_i or α_i is more than 1, then it follows from the relations (3.4.1) and the induction assumption that the monomial above can be written as a linear combination of x_T's over $\mathbb{Q}[t]$ where T is a standard tableau. If $p_m \leq j$ and all β_i and α_i are equal to 1, then $h = j-m$ and the monomial above is of the form

$$x_{b_{p_1}}\cdots x_{b_{p_m}}x_{a_{q_1}}\cdots x_{a_{q_{j-m}}}x_{b_{j+1}}\cdots x_{b_\ell}$$

where $1 \leq p_1 < \cdots < p_m \leq j \leq q_1 < \cdots < q_{j-m} \leq n-\ell$. This monomial is associated to a permissible filling U' given by

$$U' = \begin{array}{|c|c|c|c|c|c|}\hline c_1 & \cdots & c_\ell & c_{\ell+1} & \cdots & c_{n-\ell} \\ \hline d_1 & \cdots & d_\ell \\ \cline{1-3}\end{array}$$

where

$$d_i = \begin{cases} b_{p_i} & \text{if } 1 \leq i \leq m, \\ \min\{\{a_{q_1},\cdots,a_{q_{j-m}},b_{j+1},\cdots,b_\ell\} - \{d_{m+1},\ldots,d_{i-1}\}\} & \text{if } m < i \leq \ell, \end{cases}$$

and

$$c_i = \min\{\{a_1,\cdots,a_{n-\ell},b_1,\cdots,b_j\} - \{a_{q_1},\cdots,a_{q_{j-m}},b_{p_1},\cdots,b_{p_m},c_1,\cdots,c_{i-1}\}\}$$

for $1 \leq i \leq n-\ell$. Note that $x_{U'} = x_U$ if and only if $m = j$, since $m = j \Leftrightarrow d_i = b_i$

for $1 \le i \le \ell$. We consider the case $m < j$. Since $j \le q_1$ and $a_j > b_j$ by (3.4.3), we have

$$c_i = \min\{\{a_1, \cdots, a_{j-1}, b_1, \cdots, b_j\} - \{b_{p_1}, \cdots, b_{p_m}, c_1, \cdots, c_{i-1}\}\}$$

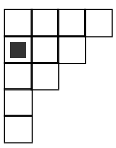
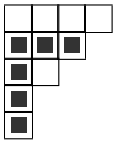
for $1 \le i \le j$. If $1 \le i \le m$, we have $c_i \le a_i < b_i \le b_{p_i} = d_i$. If $m < i \le j$, we have $c_i \le \max\{a_{j-1}, b_j\} < \min\{a_j, b_{j+1}\} \le d_i$ by (3.4.2), (3.4.3), and $j \le q_1$. Thus, U' is a permissible filling of $(n-\ell, \ell)$ such that each of the leftmost j columns are strictly increasing (i.e. $a_r < b_r, 1 \le r \le j$). This proves Claim 2. □

Claims 1 and 2 show that x_U can be written as a $\mathbb{Q}[t]$-linear combination of $x_{U'}$ and x_T, where U' and T are as above. Applying the above discussion for $x_{U'}$ in place of x_U, we see that $x_{U'}$ can be written as a $\mathbb{Q}[t]$-linear combination of $x_{U''}$ and x_T where U'' is a permissible filling of $(n-\ell, \ell)$ such that each of the leftmost $j+1$ columns are strictly increasing (i.e. $a_r < b_r, 1 \le r \le j+1$) and T is a standard tableau. Repeating this procedure, we can finally express x_U as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau.

(Case ii). If $\ell \ge k+1$, it follows from the relations (3.3.11) and the induction assumption that $x_{b_1} x_{b_2} \cdots x_{b_\ell}$ can be expressed as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau.

This completes the induction step and proves the proposition. □

Recall that for a box b in the ith row and jth column of a Young diagram λ, $h(i,j)$ denote the number of boxes in the hook formed by the boxes below b in the jth column, the boxes to the right of b in the ith row, and b itself.

Example. For the Young diagram  and the box in the $(2, 1)$ location, the hook is  and $h(2,1)=6$.

Lemma 3.4.2. *Let λ be a Young diagram. Let f^λ denote the number of standard tableaux on λ. Then*

$$\binom{n}{k} = \sum_{0\le \ell\le k} f^{(n-\ell,\ell)}.$$

Proof. We prove the lemma by induction on k. As the case $k=0$ is clear, we assume that $k\ge 1$ and that the lemma holds for $k-1$. We use the following hook length formula:

$$f^\lambda = \frac{n!}{\Pi_{(i,j)\in\lambda} h(i,j)}.$$

Using the induction assumption and the hook length formula, we have

$$\begin{aligned}\sum_{0\le \ell\le k} f^{(n-\ell,\ell)} &= \sum_{0\le \ell\le k-1} f^{(n-\ell,\ell)} + f^{(n-k,k)}\\ &= \binom{n}{k-1} + \frac{n!(n-2k+1)}{(n-k+1)!k!}\\ &= \binom{n}{k}.\end{aligned}$$

This completes the induction step and proves the lemma. □

It follows from Proposition 3.4.1 and Lemma 3.4.2 that

$$\operatorname{rank}_{\mathbb{Q}[t]} \mathbb{Q}[x_1,\dots,x_n,t]/I \le \sum_{0\le \ell\le k} f^{(n-\ell,\ell)} = \binom{n}{k}.$$

On the other hand, since the odd degree cohomology groups of $\mathcal{S}_\lambda$ vanish, we have an isomorphism $H^*_S(\mathcal{S}_\lambda;\mathbb{Q}) \cong \mathbb{Q}[t]\otimes H^*(\mathcal{S}_\lambda;\mathbb{Q})$ as $\mathbb{Q}[t]$-modules, and the cellular decomposition of $\mathcal{S}_\lambda$ given by Spaltenstein [33] (cf. also Hotta-Springer [18]) implies that

$$\dim H^*(\mathcal{S}_\lambda;\mathbb{Q}) = \binom{n}{\lambda} := \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_\ell!}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. These show

$$\mathrm{rank}_{\mathbb{Q}[t]} H_S^*(\mathcal{S}_{(n-k,k)};\mathbb{Q}) = \dim_{\mathbb{Q}} H^*(\mathcal{S}_{(n-k,k)};\mathbb{Q}) = \binom{n}{k}.$$

Therefore, we have

$$\mathrm{rank}_{\mathbb{Q}[t]}\mathbb{Q}[x_1,\ldots,x_n,t]/I \leq \mathrm{rank}_{\mathbb{Q}[t]} H_S^*(\mathcal{S}_{(n-k,k)};\mathbb{Q}).$$

This means that the epimorphism φ in (3.3.8) is actually an isomorphism, proving Theorem 3.3.3.

Chapter 4

The T^ℓ-equivariant cohomology rings of Springer varieties

In Chapter 4 we give an explicit presentation of the T^ℓ-equivariant cohomology rings of Springer varieties. We organize Chapter 4 as follows. In Section 4.1, we introduce a natural action of the ℓ-dimensional torus T^ℓ on the Springer variety $\mathcal{S}_\lambda$ and give the T^ℓ-fixed points $\mathcal{S}_\lambda^{T^\ell}$ of the Springer variety $\mathcal{S}_\lambda$. We construct an S_n-action on the equivariant cohomology group $H^*_{T^\ell}(\mathcal{S}_\lambda;\mathbb{Z})$ in Section 4.2 by using the localization technique which involves the equivariant cohomology of the T^ℓ-fixed points. We state the main theorem, Theorem 4.3.1, in Section 4.3, and prove it in Section 4.4 by using this S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda;\mathbb{Z})$. Our method of the proof is the T^ℓ-equivariant analogue of [37]. This is a joint work with Hiraku Abe in [2].

4.1 T^ℓ-fixed points of Springer varieties

We begin with a definition of type A nilpotent Springer varieties. We work with type A through out this chapter and hence omit it in the following. We first recall that the **(nilpotent) Springer variety** $\mathcal{S}_\lambda$ is given by

$$\mathcal{S}_\lambda = \{V_\bullet \in Flags(\mathbb{C}^n) \mid NV_i \subseteq V_{i-1} \text{ for all } 1 \leq i \leq n\}.$$

from Lemma 3.1.1.

The torus T^ℓ in (1.0.4) preserves the Springer variety $\mathcal{S}_\lambda$. Our goal in this section is to give the T^ℓ-fixed point set $\mathcal{S}_\lambda^{T^\ell}$.

We first recall that the T-fixed point set $Flags(\mathbb{C}^n)^T$ of the flag variety $Flags(\mathbb{C}^n)$ is given by the set of permutation flags;

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \dots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\}$$

where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{C}^n$ and S_n is the symmetric group on n letters $\{1, 2, \dots, n\}$, so we may identify $Flags(\mathbb{C}^n)^T$ with S_n. Namely,

$$w \leftrightarrow (\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \dots, e_{w(n)}\rangle = \mathbb{C}^n).$$

Let w be an element of S_n satisfying the following property:

the numbers between $\lambda_1 + \cdots + \lambda_{k-1} + 1$ and $\lambda_1 + \cdots + \lambda_k$ appear in the one-line notation of w as a subsequence in increasing order for each k with $1 \leq k \leq \ell$. (4.1.1)

Here, we write $\lambda_1 + \cdots + \lambda_{k-1} = 0$ when $k = 1$.

Example. We consider the case $n = 6$, $\ell = 3$, and $\lambda = (3, 2, 1)$. Using one-line notation, the following permutations

$$w_1 = 124365, \quad w_2 = 416253, \quad w_3 = 612435$$

satisfy condition (4.1.1). This is because the sequences $(1,2,3)$, $(4,5)$ and (6) appear in the one-line notations as a subsequence in increasing order.

Lemma 4.1.1. *The T^ℓ-fixed point set $\mathcal{S}_\lambda^{T^\ell}$ of the Springer variety $\mathcal{S}_\lambda$ is equal to the set of permutation flags*

$$\langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)} \rangle = \mathbb{C}^n$$

which satisfies condition (4.1.1)*. In particular, we may identify $\mathcal{S}_\lambda^{T^\ell}$ with*

$$S(\lambda) := \{w \in S_n \mid w \textit{ satisfies condition } (4.1.1)\}.$$

Proof. Let $V_\bullet$ be a permutation flag $V_i = \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)} \rangle$ for $i = 1, \cdots, n$ where $w \in S_n$ satisfies condition (4.1.1). Since $w(1)$ is equal to one of the numbers 1, $\lambda_1 + 1$, $\lambda_1 + \lambda_2 + 1$, $\ldots$, $\lambda_1 + \cdots + \lambda_{\ell-1} + 1$, we have $NV_1 \subseteq \{0\}$. If $w(1) = \lambda_1 + \cdots + \lambda_{k-1} + 1$ for some $1 \leq k \leq \ell$, then $w(2)$ is equal to one of the numbers 1 ,$\lambda_1 + 1, \ldots, \lambda_1 + \cdots + \lambda_{k-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_{\ell-1} + 1$. So we also have $NV_2 \subseteq V_1$. Continuing this argument, we have $NV_i \subseteq V_{i-1}$ for all $1 \leq i \leq n$, and it follows that $V_\bullet$ is an element of $\mathcal{S}_\lambda$. Also, $V_\bullet$ is clearly fixed by T^ℓ, so $V_\bullet$ is an element of $\mathcal{S}_\lambda^{T^\ell}$.

Conversely, let $V_\bullet$ be an element of $\mathcal{S}_\lambda^{T^\ell}$. Let v_1,v_2,$\ldots$,v_j be generators for V_j where $v_j = (x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)})^t$ in $\mathbb{C}^n$ for all j. Since we have

$$Nv_1 = (\underbrace{x_2^{(1)}, \cdots, x_{\lambda_1}^{(1)}, 0}_{\lambda_1}, \underbrace{x_{\lambda_1+2}^{(1)}, \cdots, x_{\lambda_1+\lambda_2}^{(1)}, 0}_{\lambda_2}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+2}^{(1)}, \cdots, x_n^{(1)}, 0}_{\lambda_\ell})^t,$$

the condition $NV_1 \subseteq V_0 = \{0\}$ implies that

$$v_1 = (\underbrace{x_1^{(1)}, 0, \cdots, 0}_{\lambda_1}, \underbrace{x_{\lambda_1+1}^{(1)}, 0, \cdots, 0}_{\lambda_2}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}^{(1)}, 0, \cdots, 0}_{\lambda_\ell})^t. \quad (4.1.2)$$

It follows that exactly one of $x_i^{(1)}$ $(i = 1, \lambda_1+1, \lambda_1+\lambda_2+1, \ldots, \lambda_1+\cdots+\lambda_{\ell-1}+1)$ which appear in (4.1.2) is nonzero. In fact, $V_\bullet$ is fixed by the T^ℓ-action and hence we have $h \cdot v_1 = v_1$ for arbitrary $h \in T^\ell$ where

$$h \cdot v_1 = (\underbrace{h_1 x_1^{(1)}, 0, \cdots, 0}_{\lambda_1}, \underbrace{h_2 x_{\lambda_1+1}^{(1)}, 0, \cdots, 0}_{\lambda_2}, \cdots\cdots, \underbrace{h_\ell x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}^{(1)}, 0, \cdots, 0}_{\lambda_\ell})^t.$$

Since each h_i runs over all nonzero complex numbers, only one of $x_i^{(1)}$ in (4.1.2) must be nonzero.

If $x_{\lambda_1+\cdots+\lambda_{k-1}+1}^{(1)}$ is nonzero for some $1 \leq k \leq \ell$, then we may assume that

$$v_1 = (0, \cdots, 0, 1, 0, \cdots, 0)^t,$$
$$v_j = (x_1^{(j)}, \cdots, x_{\lambda_1+\cdots+\lambda_{k-1}}^{(j)}, 0, x_{\lambda_1+\cdots+\lambda_{k-1}+2}^{(j)}, \cdots, x_n^{(j)})^t$$

for $2 \leq j \leq n$ where the $(\lambda_1 + \cdots + \lambda_{k-1} + 1)$-th component of v_1 is one. Since we have

$$Nv_2 = (\underbrace{x_2^{(2)}, \cdots, x_{\lambda_1}^{(2)}, 0}_{\lambda_1}, \underbrace{x_{\lambda_1+2}^{(2)}, \cdots, x_{\lambda_1+\lambda_2}^{(2)}, 0}_{\lambda_2}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+2}^{(2)}, \cdots, x_n^{(2)}, 0}_{\lambda_\ell})^t,$$

the condition $NV_2 \subseteq V_1$ implies that

$$v_2 = (\underbrace{x_1^{(2)}, 0, \cdots, 0}_{\lambda_1}, \cdots\cdots, \underbrace{0, x_{\lambda_1+\cdots+\lambda_{k-1}+2}^{(2)}, 0, \cdots, 0}_{\lambda_k}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}^{(2)}, 0, \cdots, 0}_{\lambda_\ell})^t. \quad (4.1.3)$$

Therefore, we see that the only one of $x_i^{(2)}$ $(i = 1, \lambda_1 + 1, \ldots, \lambda_1 + \cdots + \lambda_{k-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_{\ell-1} + 1)$ which appear in (4.1.3) is nonzero by an argument similar to that used above. Continuing this procedure, we conclude that $V_\bullet$ is a permutation flag whose permutation satisfies condition (4.1.1). In fact, we see that $V_\bullet$ forms

$$\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle = \mathbb{C}^n$$

for some $w \in S_n$, so it is enough to check that the permutation w satisfies condition (4.1.1). The number $w(1)$ is equal to one of the numbers 1, λ_1+1, $\lambda_1+\lambda_2+1$, $\ldots$, $\lambda_1+\cdots+\lambda_{\ell-1}+1$. If $w(1) = \lambda_1+\cdots+\lambda_{k-1}+1$, then $w(2)$ is equal to one of the numbers 1, λ_1+1, $\ldots, \lambda_1+\cdots+\lambda_{k-1}+2$, $\ldots$, $\lambda_1+\cdots+\lambda_{\ell-1}+1$ and so on. This means that for each $k=1,\ldots,\ell$ the numbers between $\lambda_1+\cdots+\lambda_{k-1}+1$ and $\lambda_1+\cdots+\lambda_k$ appear in the one-line notation of w as a subsequence in increasing order, so the permutation w satisfies condition (4.1.1). This completes the proof. □

Regarding a product of symmetric groups $S_{\lambda_1}\times S_{\lambda_2}\times\cdots\times S_{\lambda_\ell}$ as a subgroup of the symmetric group S_n, it follows from Lemma 4.1.1 that the T^ℓ-fixed points $\mathcal{S}_\lambda^{T^\ell}$ of the Springer variety $\mathcal{S}_\lambda$ are in one-to-one correspondence with the right cosets $S_{\lambda_1}\times S_{\lambda_2}\times\cdots\times S_{\lambda_\ell}\backslash S_n$. Namely, we have a bijection (cf. [7])

$$\theta : \mathcal{S}_\lambda^{T^\ell} = S(\lambda) \to S_{\lambda_1}\times S_{\lambda_2}\times\cdots\times S_{\lambda_\ell}\backslash S_n \quad ; \quad w \mapsto [w]. \tag{4.1.4}$$

4.2 An action of the symmetric group S_n on $H^*_{T^\ell}(\mathcal{S}_\lambda)$

In this section, we introduce an action of the symmetric group S_n on the equivariant cohomology $H^*_{T^\ell}(\mathcal{S}_\lambda)$ over $\mathbb{Z}$-coefficients by using the localization techniques similar to those in [39] and [11]. We will see that the projection map

$$\rho_\lambda : H^*_{T^n}(Flags(\mathbb{C}^n)) \to H^*_{T^\ell}(\mathcal{S}_\lambda)$$

induced from the inclusions of $\mathcal{S}_\lambda$ into $Flags(\mathbb{C}^n)$ and T^ℓ into T^n is an S_n-equivariant map. In particular, we consider the following commutative diagram

$$\begin{array}{ccc} H^*_{T^n}(Flags(\mathbb{C}^n)) & \xrightarrow{\iota_1} & H^*_{T^n}(Flags(\mathbb{C}^n)^{T^n}) = \bigoplus_{w \in S_n} \mathbb{Z}[t_1, \ldots, t_n] \\ \downarrow{\scriptstyle \rho_\lambda} & & \downarrow{\scriptstyle \pi} \\ H^*_{T^\ell}(\mathcal{S}_\lambda) & \xrightarrow{\iota_2} & H^*_{T^\ell}(\mathcal{S}_\lambda^{T^\ell}) = \bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell} \subseteq S_n} \mathbb{Z}[u_1, \ldots, u_\ell] \end{array} \tag{4.2.1}$$

where all the maps are induced from inclusion maps. We have $H^*_{T^n}(Flags(\mathbb{C}^n)^{T^n}) = \bigoplus_{w \in S_n} H^*(BT^n)$ and $H^*_{T^\ell}(\mathcal{S}_\lambda^{T^\ell}) = \bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} H^*(BT^\ell)$, and the identifications of $H^*(BT^n)$ and $H^*(BT^\ell)$ with the polynomial rings $\mathbb{Z}[t_1, \ldots, t_n]$ and $\mathbb{Z}[u_1, \ldots, u_\ell]$ respectively will be explained later in this section. All (equivariant) cohomology rings are assumed to be over $\mathbb{Z}$-coefficients unless otherwise specified.

It is known that $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ admit complex cellular decompositions ([33]), so the odd degree cohomology groups of $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ vanish. The path-connectedness of $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ together with this fact implies that $H^*_{T^n}(Flags(\mathbb{C}^n)) \cong H^*(BT^n) \otimes H^*(Flags(\mathbb{C}^n))$ as $H^*(BT^n)$-modules and $H^*_{T^\ell}(\mathcal{S}_\lambda) \cong H^*(BT^\ell) \otimes H^*(\mathcal{S}_\lambda)$ as $H^*(BT^\ell)$-modules. In particular, the equivariant cohomology $H^*_{T^n}(Flags(\mathbb{C}^n))$ and $H^*_{T^\ell}(\mathcal{S}_\lambda)$ are free modules over $H^*(BT^n)$ and $H^*(BT^\ell)$, respectively. Hence the maps ι_1 and ι_2 in (4.2.1) are injective (cf. [19] and [31]). The map π in (4.2.1) is clearly surjective. Also, the map ρ_λ in (4.2.1) is surjective. This is because the surjectivity of the restriction map $H^*(Flags(\mathbb{C}^n)) \to H^*(\mathcal{S}_\lambda)$ for ordinary cohomology was proved in [18], and hence the natural commutative diagram of the exact sequences ([27, Theorem

4.2.])

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (t_1, \ldots, t_n) & \longrightarrow & H^*_{T^n}(Flags(\mathbb{C}^n)) & \longrightarrow & H^*(Flags(\mathbb{C}^n)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \rho_\lambda} & & \downarrow & & \\
0 & \longrightarrow & (t_1, \ldots, t_n) & \longrightarrow & H^*_{T^\ell}(\mathcal{S}_\lambda) & \longrightarrow & H^*(\mathcal{S}_\lambda) & \longrightarrow & 0
\end{array}$$

shows the surjectivity of ρ_λ.

We construct S_n-actions on the three modules $H^*_{T^n}(Flags(\mathbb{C}^n))$, $\bigoplus_{w\in S_n} \mathbb{Z}[t_1, \ldots, t_n]$, and $\bigoplus_{w\in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \ldots, u_\ell]$ in (4.2.1) to construct an S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$.

Step 1: The S_n-action on $H^*_{T^n}(Flags(\mathbb{C}^n))$.

We first introduce the left action of the symmetric group S_n on the T^n-equivariant cohomology $H^*_{T^n}(Flags(\mathbb{C}^n))$. To do that, we consider the right S_n-action on the flag variety $Flags(\mathbb{C}^n)$ as follows. For any $V_\bullet \in Flags(\mathbb{C}^n)$, there exists $g \in U(n)$ so that $V_i = \bigoplus_{j=1}^{i} \mathbb{C}g(e_j)$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$. Then the right action of $w \in S_n$ on $Flags(\mathbb{C}^n)$ can be defined by

$$V_\bullet \cdot w = V'_\bullet \tag{4.2.2}$$

where $V'_i = \bigoplus_{j=1}^{i} \mathbb{C}g(e_{w(j)})$.

We recall an explicit presentation of the T^n-equivariant cohomology ring of the flag variety $Flags(\mathbb{C}^n)$. Let E_i be the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n) \times \mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T^n-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\bar{x}_i \in H^2_{T^n}(Flags(\mathbb{C}^n))$. Let $\mathbb{C}_i$ be the one dimensional representation of T^n through a map $T^n \to \mathbb{C}^*$ given by $(g_1, \ldots, g_n) \mapsto g_i$ where $\mathbb{C}^*$ acts on $\mathbb{C}_i = \mathbb{C}$ by the standard multiplication. We denote the first Chern class of the line bundle $ET^n \times_{T^n} \mathbb{C}_i$ over BT^n by $t_i \in H^2(BT^n)$. We may identify

$$H^*(BT^n) = \mathbb{Z}[t_1, \ldots, t_n].$$

Then $H^*_{T^n}(Flags(\mathbb{C}^n))$ is generated by $\bar{x}_1, \ldots, \bar{x}_n, t_1, \ldots, t_n$ as a ring. Defining a surjective ring homomorphism from the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n, t_1, \ldots, t_n]$ to the equivariant cohomology $H^*_{T^n}(Flags(\mathbb{C}^n))$ by sending x_i to $\bar{x}_i$ and t_i to t_i, its kernel $\tilde{I}$ is generated as an ideal by $e_i(x_1, \ldots, x_n) - e_i(t_1, \ldots, t_n)$ for all $1 \leq i \leq n$, where e_i is the i-th elementary symmetric polynomial. That is, we have an isomorphism

$$H^*_{T^n}(Flags(\mathbb{C}^n)) \cong \mathbb{Z}[x_1, \ldots, x_n, t_1, \ldots, t_n]/\tilde{I}. \tag{4.2.3}$$

This can be explained from the fact that we have $ET^n \times_{T^n} Flags(\mathbb{C}^n) = Flags(ET^n \times_{T^n} \mathbb{C}^n)$ where the right-hand-side is the flag bundle of the vector bundle $ET^n \times_{T^n} \mathbb{C}^n$, and then (4.2.3) follows from [5, Section 21] (cf. [12]).

Now the right action in (4.2.2) induces the following left action of the symmetric group S_n on $H^*_{T^n}(Flags(\mathbb{C}^n))$:

$$w \cdot \bar{x}_i = \bar{x}_{w(i)}, \ w \cdot t_i = t_i \tag{4.2.4}$$

for $w \in S_n$. This is because the pullback of the line bundle E_i/E_{i-1} under the right action in (4.2.2) is exactly the line bundle $E_{w(i)}/E_{w(i)-1}$ and the right action in (4.2.2) is T^n-equivariant.

Step 2: The S_n-action on $\bigoplus_{w \in S_n} \mathbb{Z}[t_1, \ldots, t_n]$.

We next define a left action of $v \in S_n$ on the direct sum $\bigoplus_{w \in S_n} \mathbb{Z}[t_1, \ldots, t_n]$ of the polynomial ring as follows:

$$(v \cdot f)|_w = f|_{wv} \tag{4.2.5}$$

where $w \in S_n$ and $f \in \bigoplus_{w \in S_n} \mathbb{Z}[t_1, \ldots, t_n]$.

Observe that the map ι_1 in (4.2.1) is the following mapping

$$\iota_1(\bar{x}_i)|_w = t_{w(i)}, \ \iota_1(t_i)|_w = t_i. \tag{4.2.6}$$

This is because the pullback of the line bundle $ET^n \times_{T^n} (E_i/E_{i-1}) \to ET^n \times_{T^n} Flags(\mathbb{C}^n)$ on $ET^n \times_{T^n} \{w\}$ is naturally isomorphic to the line bundle $ET^n \times_{T^n} \mathbb{C}_{w(i)}$ over BT^n appeared above, and hence (4.2.6) follows from the definition of $t_{w(i)}$. Note that it follows from (4.2.4), (4.2.5), and (4.2.6) that the map ι_1 is S_n-equivariant map, i.e. $w \cdot (\iota_1(f)) = \iota_1(w \cdot f)$ for any $f \in H^*_{T^n}(Flags(\mathbb{C}^n))$ and $w \in S_n$.

Step 3: The S_n-action on $\bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \dots, u_\ell]$.

We identify $H^*(BT^\ell)$ with a polynomial ring with ℓ variables. That is,

$$H^*(BT^\ell) = \mathbb{Z}[u_1, \dots, u_\ell]$$

where $u_i \in H^2(BT^\ell)$ is the first Chern class of the line bundle $ET^\ell \times_{T^\ell} \mathbb{C}_i$ over BT^ℓ. Here, $\mathbb{C}_i$ is the one dimensional representation of T^ℓ through a map $T^\ell \to \mathbb{C}^*$ given by $(h_1, \cdots, h_1, h_2, \cdots, h_2, \cdots\cdots, h_\ell, \cdots, h_\ell) \mapsto h_i$. We define the left action of $v \in S_n$ on the direct sum $\bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \dots, u_\ell]$ of the polynomial ring as follows. Recall that we have a bijection $\theta : \mathcal{S}_\lambda^{T^\ell} = S(\lambda) \to S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell} \backslash S_n$ given in (4.1.4), and there is a natural right action of S_n on the set of right cosets $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell} \backslash S_n$. So we can define a left S_n-action on $\bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \dots, u_\ell]$ by

$$(v \cdot f)|_w := f|_{\theta^{-1}(\theta(w)v)} \tag{4.2.7}$$

for $w \in \mathcal{S}_\lambda^{T^\ell}$ and $f \in \bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \dots, u_\ell]$.

Denote $\{1, 2, \dots, n\}$ by $[n]$. Let $p : [n] \to [\ell]$ be a map defined by

$$p(i) = k \tag{4.2.8}$$

if $\lambda_1 + \cdots + \lambda_{k-1} + 1 \le i \le \lambda_1 + \cdots + \lambda_k$ where $\lambda_1 + \cdots + \lambda_{k-1} = 0$ when $k = 1$.

Observe that the map π in (4.2.1) is the following mapping

$$\pi(f)|_w = f|_w(u_{p(1)}, \ldots, u_{p(n)}), \tag{4.2.9}$$

for $w \in \mathcal{S}_\lambda^{T^\ell}$ and $f \in \bigoplus_{w\in S_n} \mathbb{Z}[t_1, \ldots, t_n]$.

Remark. The map π is not S_n-equivariant map in general. For example, take $n = 3$ and $\lambda = (2, 1)$. In one-line notation, all the T^ℓ-fixed points $\mathcal{S}_\lambda^{T^\ell}$ of $\mathcal{S}_\lambda$ are the following

$$123,\ 132,\ 312.$$

If $v = 213 \in S_3$ and $f = (t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6)$ arranging by an order 123, 132, 213, 231, 312, 321 in S_3, then we have $v\cdot\pi(f) = (u_1, u_1^5, u_1^2)$ and $\pi(v\cdot f) = (u_1^3, u_1^5, u_1^2)$. Although the map π is not S_n-equivariant in general, we will see that the map ρ_λ in (4.2.1) is S_n-equivariant.

Step 4: The S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$.

Let $\bar{y}_i \in H^2_{T^\ell}(\mathcal{S}_\lambda)$ be the image $\rho_\lambda(\bar{x}_i)$ of $\bar{x}_i$ for each i, i.e. the T^ℓ-equivariant first Chern class of the i-th tautological line bundle restricted on $\mathcal{S}_\lambda$. We obtain the following lemma by the surjectivity of ρ_λ.

Lemma 4.2.1. *The T^ℓ-equivariant cohomology ring $H^*_{T^\ell}(\mathcal{S}_\lambda)$ is generated by $\bar{y}_1,\ldots,\bar{y}_n,u_1,\ldots,u_\ell$ as a ring where $\bar{y}_i$ is as above and $H^*(BT^\ell) = \mathbb{Z}[u_1, \ldots, u_\ell]$.*

It follows from (4.2.6), (4.2.9) and the commutative diagram in (4.2.1) that

$$\iota_2(\bar{y}_i)|_w = u_{p(w(i))} \text{ and } \iota_2(u_i)|_w = u_i. \tag{4.2.10}$$

Lemma 4.2.2. *For any $v \in S_n$ and $1 \le i \le n$, it follows that*

$$v \cdot (\iota_2(\bar{y}_i)) = \iota_2(\bar{y}_{v(i)}) \text{ and } v \cdot (\iota_2(u_i)) = \iota_2(u_i) \tag{4.2.11}$$

where the map ι_2 is in (4.2.1) *and $\bar{y}_i$ is as above.*

Proof. From (4.2.10) and (4.2.7), we have

$$(v \cdot (\iota_2(u_i)))|_w = \iota_2(u_i)|_{\theta^{-1}(\theta(w)v)} = u_i = \iota_2(u_i)|_w$$

for all $w \in S_n$. So the second equation holds. From (4.2.10) and (4.2.7) again, we have

$$(v \cdot (\iota_2(\bar{y}_i)))|_w = \iota_2(\bar{y}_i)|_{w'} = u_{p(w'(i))},$$
$$\iota_2(\bar{y}_{v(i)})|_w = u_{p(w(v(i)))}$$

where $w' = \theta^{-1}(\theta(w)v)$. Therefore, it is enough to prove $p(w'(i)) = p(wv(i))$. Since $[w'] = \theta(w)v = [wv]$ in $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell} \backslash S_n$, we have

$$\lambda_1 + \cdots + \lambda_{r-1} + 1 \leq w'(i) \leq \lambda_1 + \cdots + \lambda_r,$$
$$\lambda_1 + \cdots + \lambda_{r-1} + 1 \leq wv(i) \leq \lambda_1 + \cdots + \lambda_r$$

for some r. From the definition (4.2.8) of the map p, we have $p(w'(i)) = r = p(wv(i))$, and the first equation holds. We are done. □

Since the map ι_2 is injective, we obtain an S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ satisfying

$$w \cdot \bar{y}_i = \bar{y}_{w(i)} \text{ and } w \cdot u_i = u_i \tag{4.2.12}$$

for $w \in S_n$ from Lemma 4.2.1 and Lemma 4.2.2. This means that our S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ lifts the Springer's S_n-action on $H^*(\mathcal{S}_\lambda)$ (cf. [18]). Moreover, one can see that the map ρ_λ in (4.2.1) is an S_n-equivariant homomorphism by (4.2.4) and (4.2.12). We summarize the results in this section as follows.

Proposition 4.2.3. *There exists an S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ such that the map ρ_λ*

in (4.2.1) *is an S_n-equivariant homomorphism where the S_n-action on $H^*_{T^n}(Flags(\mathbb{C}^n))$ is given by* (4.2.4).

4.3 The main theorem in Chapter 4

In this section, we state our main theorem. For this purpose, let us clarify our notations. We set $p_\lambda(s) := \lambda_{n-s+1} + \lambda_{n-s+2} + \cdots + \lambda_\ell$ for $s = 1, \cdots, n$, namely the number of boxes strictly below than $(n-s)$-th row. We denote by $\check{\lambda}$ the transpose of λ, i.e. $\check{\lambda} = (\eta_1, \cdots, \eta_k)$ where $k = \lambda_1$ and $\eta_i = |\{j \mid \lambda_j \geq i\}|$ for $1 \leq i \leq k$. For indeterminates $y_1, \cdots, y_s$ and $a_1, a_2, \cdots$, let

$$e_d(y_1, \cdots, y_s | a_1, a_2, \cdots) := \sum_{r=0}^{d} (-1)^{d-r} e_r(y_1, \cdots, y_s) h_{d-r}(a_1, \cdots, a_{s+1-d}) \tag{4.3.1}$$

for $d \geq 0$ where e_i and h_i denote the i-th elementary symmetric polynomial and the i-th complete symmetric polynomial, respectively. In fact, this is the factorial Schur function ([28]) corresponding to the Young diagram consisting of the unique column of length d as shown in the next section (see Lemma 4.4.2). We also define a map $\phi_\lambda : [n] \to [\ell]$ by the condition

$$\begin{aligned}(u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) \\ = (\underbrace{u_1, \cdots, u_1}_{\lambda_1 - \lambda_2}, \underbrace{u_1, u_2, \cdots, u_1, u_2}_{2(\lambda_2 - \lambda_3)}, \cdots\cdots, \underbrace{u_1, u_2, \cdots, u_\ell, \cdots\cdots, u_1, u_2, \cdots, u_\ell}_{\ell(\lambda_\ell - \lambda_{\ell+1})})\end{aligned} \tag{4.3.2}$$

as ordered sequences where for each $1 \leq r \leq \ell$ the r-th subsequence of the right-hand-side consists of $(u_1, u_2, \cdots, u_r)$ repeated $(\lambda_r - \lambda_{r+1})$-times. Here, we denote $\lambda_{\ell+1} = 0$.

Let us define a ring homomorphism

$$\psi : \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell] \to H^*_{T^\ell}(\mathcal{S}_\lambda) \tag{4.3.3}$$

by sending y_i to $\bar{y}_i$ and u_i to u_i where $H^*(BT^\ell) = \mathbb{Z}[u_1, \cdots, u_\ell]$. Recall that $\bar{y}_i$ is the equivariant first Chern class of the tautological line bundle E_i/E_{i-1} over $Flags(\mathbb{C}^n)$ restricted to $\mathcal{S}_\lambda$ (see Section 4.2). This homomorphism ψ is a surjection by Lemma 4.2.1.

Theorem 4.3.1. *The map ψ in (4.3.3) induces a ring isomorphism*

$$H^*_{T^\ell}(\mathcal{S}_\lambda) \cong \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\widetilde{I}_\lambda$$

where $\widetilde{I}_\lambda$ is the ideal of the polynomial ring $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]$ generated by the polynomials $e_d(y_{i_1}, \cdots, y_{i_s} | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)})$ defined in (4.3.1) with ϕ_λ described in (4.3.2) for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$.

Remark. The argument in [37] treated the cohomology over $\mathbb{C}$-coefficients, but the argument works also over $\mathbb{Z}$ as well.

Remark. The ideal $\widetilde{I}_\lambda$ is the T^ℓ-equivariant analogue of so-called Tanisaki's ideal (it is written as $K_{\check{\lambda}}$ in [37]). Each generator of $\widetilde{I}_\lambda$ given above specializes to a generator of Tanisaki's ideal given in [37] after the evaluation $u_i = 0$ for all i.

4.4 Preliminaries from Grassmannians

To prove Theorem 4.3.1, we recall some facts about T^n-equivariant Schubert classes for Grassmannians and factorial Schur functions from [24] and [28] in this section. These facts will be used in the proof of Theorem 4.3.1.

4.4.1 T^n-equivariant Schubert classes for Grassmannians

Let s be a positive integer with $s \leq n$ and $Gr_s(\mathbb{C}^n)$ the Grassmannian of s-dimensional linear subspaces of $\mathbb{C}^n$. Let $U_\bullet$ be a complete flag in $\mathbb{C}^n$. For a Young diagram μ with at most s rows and $n-s$ columns, the Schubert variety corresponding to μ with respect to the reference flag $U_\bullet$ is an irreducible subvariety of $Gr_s(\mathbb{C}^n)$ defined by

$$X_\mu(U_\bullet) = \{V \in Gr_s(\mathbb{C}^n) \mid \dim(V \cap U_{n-s+i-\mu_i}) \geq i \text{ for all } 1 \leq i \leq s\}.$$

Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{C}^n$. For each permutation $w \in S_n$, we can think of w as a linear automorphism $w : \mathbb{C}^n \to \mathbb{C}^n$, by abuse of notation, which sends e_i to $e_{w(i)}$. In terms of the standard coordinate, this is given by $(z_1, \cdots, z_n) \mapsto (z_{w^{-1}(1)}, \cdots, z_{w^{-1}(n)})$. Again by abuse of notation, this naturally induces

$$w : Gr_s(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n) \quad ; \quad V \mapsto w(V), \tag{4.4.1}$$

$$w : Flags(\mathbb{C}^n) \to Flags(\mathbb{C}^n) \quad ; \quad V_\bullet \mapsto w(V_\bullet) \tag{4.4.2}$$

where $w(V_\bullet) := (w(V_0) \subset w(V_1) \subset \cdots \subset w(V_n))$. It is straightforward to see that

$$X_\mu(w(U_\bullet)) = w(X_\mu(U_\bullet)). \tag{4.4.3}$$

Let $F_\bullet$ and $\tilde{F}_\bullet$ be complete flags defined by $F_i := \langle e_1, \cdots, e_i\rangle$ and $\tilde{F}_i = \langle e_{n+1-i}, \cdots, e_n\rangle$, respectively. Then it is known that $X_\mu(\tilde{F}_\bullet) \cap X_\nu(F_\bullet) = \emptyset$ unless $\mu \subset \nu^*$ where $\nu^* = (n-s-\nu_s, \cdots, n-s-\nu_1)$ from [13, Section 9.4, Lemma 3]. If $U_\bullet = w(F_\bullet)$, then we have

$$X_\mu(w(\tilde{F}_\bullet)) \cap X_\nu(U_\bullet) = \emptyset \quad \text{unless } \mu \subset \nu^*. \tag{4.4.4}$$

The Grassmannian $Gr_s(\mathbb{C}^n)$ carries a T^n-action which is the restriction of the natural $\mathrm{GL}_n(\mathbb{C})$-action on $Gr_s(\mathbb{C}^n)$ regarding T^n as the diagonal subgroup of $\mathrm{GL}_n(\mathbb{C})$. Now, suppose that the flag $U_\bullet$ is T^n-invariant. Then the Schubert variety $X_\mu(U_\bullet)$ is a T^n-invariant irreducible subvariety of $Gr_s(\mathbb{C}^n)$, and hence we can consider the associated T^n-equivariant cohomology class $[X_\mu(U_\bullet)] \in H^*_{T^n}(Gr_s(\mathbb{C}^n))$ supported on $X_\mu(U_\bullet)$. This is called a T^ℓ-*equivariant Schubert class*. The above map $w : Flags(\mathbb{C}^n) \to Flags(\mathbb{C}^n)$ is equivariant with respective to the group homomorphism $\phi_w : T^n \to T^n$ given by $(g_1, \cdots, g_n) \mapsto (g_{w^{-1}(1)}, \cdots, g_{w^{-1}(n)})$. So it induces a group automorphism w^* of $H^*_{T^n}(Flags(\mathbb{C}^n))$, and we have

$$[X_\mu(w(F_\bullet))] = (w^{-1})^*[X_\mu(F_\bullet)]. \tag{4.4.5}$$

Note that the automorphism w^* of $H^*_{T^n}(Flags(\mathbb{C}^n))$ is in fact an $H^*(BT^n)$-algebra homomorphism with respect to the ring homomorphism $H^*(BT^n) = \mathbb{Z}[t_1, \cdots, t_n]$ given by $t_i \mapsto t_{w^{-1}(i)}$ where the identification $H^*(BT^n) = \mathbb{Z}[t_1, \cdots, t_n]$ is as in Section 4.2.

4.4.2 Factorial Schur functions

Let s be a positive integer. In [28], factorial Schur functions are defined as follows: for a Young diagram μ with at most s rows, the factorial Schur function associated to μ is defined to be

$$s_\mu(x_1, \cdots, x_s | a_1, a_2, \cdots) = \sum_T \prod_{\alpha \in \mu} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)})$$

as a polynomial in $\mathbb{Z}[x_1, \cdots, x_s] \otimes \mathbb{Z}[a_1, a_2, \cdots]$ where T runs over all semistandard tableaux of shape μ with entries in $\{1, \cdots, s\}$, $T(\alpha)$ is the entry of T in the cell $\alpha \in \mu$, and $c(\alpha) = j - i$ is the content of $\alpha = (i, j)$. This polynomial is symmetric in the x-variables.

For positive integers s and n with $s \le n$, let $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ be the projection defined by $p(V_\bullet) = V_s$. We recall that the T^n-equivariant Schubert class with respect to the standard flag $F_\bullet$ is represented by the factorial Schur function in the T^n-equivariant cohomology of $Flags(\mathbb{C}^n)$.

Proposition 4.4.1. ([29], [28], [24]) *For any Young diagram μ with at most s rows and $n-s$ columns, we have*

$$p^*[X_\mu(F_\bullet)] = s_\mu(-\bar{x}_1, \cdots, -\bar{x}_s | -t_n, \cdots, -t_1, 0, 0, \cdots)$$

in $H^*_{T^n}(Flags(\mathbb{C}^n))$.

For integers $0 \leq k \leq s$, let $\mu_{s,k} = (1, \cdots, 1, 0, \cdots, 0)$ with 1 repeated k-times and 0 repeated $(s-k)$-times. We will need the following lemma in the next section to prove Theorem 4.3.1.

Lemma 4.4.2. *For indeterminates $x_1, \cdots, x_s, a_1, a_2, \cdots$, we have*

$$s_{\mu_{s,k}}(x_1, \cdots, x_s | a_1, a_2, \cdots) = \sum_{r=0}^{k} (-1)^{k-r} e_r(x_1, \cdots, x_s) h_{k-r}(a_1, \cdots, a_{s+1-k})$$

where $\mu_{s,k}$ is as above.

Proof. We first find the coefficient of the monomial $x_1 \cdots x_r$ in $s_{\mu_{s,k}}(x|a)$. For each $I = (i_1, i_2, \cdots, i_{k-r})$ satisfying $r+1 \leq i_1 < i_2 < \cdots < i_{k-r} \leq s$, there is a summand in $s_{\mu_{s,k}}(x|a)$ corresponding to the standard tableau T_I of shape $\mu_{s,k}$ whose $(j,1)$-th entry is

$$\begin{cases} j & \text{if } 1 \leq j \leq r, \\ i_{j-r} & \text{if } r+1 \leq j \leq k. \end{cases}$$

The summand is of the form

$$(x_1 - a_1)(x_2 - a_1) \cdots (x_r - a_1)(x_{i_1} - a_{i_1 - r})(x_{i_2} - a_{i_2 - r - 1}) \cdots (x_{i_{k-r}} - a_{i_{k-r} - k + 1}),$$

and the contribution of the monomial $x_1 \cdots x_r$ from this polynomial is

$$(-1)^{k-r}(a_{i_1 - r} a_{i_2 - r - 1} \cdots a_{i_{k-r} - k + 1}) x_1 \cdots x_r.$$

Since the condition on I is equivalent to

$$1 \leq i_1 - r \leq i_2 - r - 1 \leq \cdots \leq i_{k-r} - k + 1 \leq s - k + 1,$$

we see that the coefficient of $x_1 \cdots x_r$ in $s_{\mu_{s,k}}(x_1, \cdots, x_s | a_1, a_2, \cdots)$ is

$$(-1)^{k-r} h_{k-r}(a_1, \cdots, a_{s-k+1}).$$

Recalling that $s_{\mu_{s,k}}(x_1, \cdots, x_s | a_1, a_2, \cdots)$ is symmetric in the x-variables, we conclude that the coefficient of $x_{j_1} \cdots x_{j_r}$ is $(-1)^{k-r} h_{k-r}(a_1, \cdots, a_{s-k+1})$ for any $1 \leq j_1 < \cdots < j_r \leq s$. Thus, the polynomial

$$(-1)^{k-r} e_r(x_1, \cdots, x_s) h_{k-r}(a_1, \cdots, a_{s-k+1})$$

gives the summand in $s_{\mu_{s,k}}(x_1, \cdots, x_s | a_1, a_2, \cdots)$ whose degree in the x-variables is r. □

4.5 Proof of the main theorem in Chapter 4

In this section, we prove Theorem 4.3.1. Our argument is the T^ℓ-equivariant version of [37]. Our first goal in this section is to show that

$$e_d(\bar{y}_{i_1}, \cdots, \bar{y}_{i_s} | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$$

in $H^*_{T^\ell}(\mathcal{S}_\lambda)$ for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$ which will be sated in Proposition 4.5.4.

Let us first consider the cases for $s < n$. Take a T^n-invariant complete flag $U_\bullet$ which refines the flag $(\cdots \subset N^2\mathbb{C}^n \subset N\mathbb{C}^n \subset \mathbb{C}^n)$. This is possible since N is in Jordan canonical form. We denote by $\bar{w}$ the element of S_n corresponding to $U_\bullet$, i.e. $U_\bullet = \bar{w} F_\bullet$ where $F_\bullet$ is the standard flag defined by $F_i = \langle e_1, \cdots, e_i \rangle$

for all $1 \leq i \leq n$. Recall that the map $p : [n] \to [\ell]$ is the map defined in (4.2.8).

Let $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ be the projection defined by $p(V_\bullet) = V_s$. Let $\mu_0 = (n-s, \cdots, n-s, 0, \cdots, 0)$ be a Young diagram with $n-s$ repeated $p_{\check{\lambda}}(s)$-times and 0 repeated $(s - p_{\check{\lambda}}(s))$-times. We recall the following fact from [37].

Proposition 4.5.1. *([37, Proposition 3])* $p(\mathcal{S}_\lambda) \subset X_{\mu_0}(U_\bullet)$

We reproduce the proof here for the convenience of the reader. For a complete flag $V_\bullet \in \mathcal{S}_\lambda$, we have $N^{n-s}\mathbb{C}^n \subset V_s$ because of the definition of $\mathcal{S}_\lambda$. With the fact that $\dim N^{n-s}\mathbb{C}^n = \operatorname{rank} N^{n-s} = p_{\check{\lambda}}(s)$, it follows that $N^{n-s}\mathbb{C}^n = U_{p_{\check{\lambda}}(s)}$. Hence, we obtain $\dim(V_s \cap U_i) = i$ for $i \leq p_{\check{\lambda}}(s)$. Also, it is easy to see that $\dim(V_s \cap U_{n-s+i}) \geq i$ for $i > p_{\check{\lambda}}(s)$ by elementary linear algebra.

Proposition 4.5.2. *$e_d(\bar{y}_{i_1}, \cdots, \bar{y}_{i_s} | u_{p(\bar{w}(1))}, \cdots, u_{p(\bar{w}(n))}) = 0$ in $H^*_{T^\ell}(\mathcal{S}_\lambda)$ for $1 \leq s < n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$.*

Proof. By the S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ constructed in Section 4.2, we may assume that $i_1 = 1, \cdots, i_s = s$. Proposition 4.5.1 shows that the projection $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ restricts to a map $\mathcal{S}_\lambda \to X_{\mu_0}(U_\bullet)$ which we will denote by $k : \mathcal{S}_\lambda \to X_{\mu_0}(U_\bullet)$. Together with this map, we obtain the following commutative diagram

$$\begin{array}{ccc} H^*_{T^n}(Flags(\mathbb{C}^n)) & \xleftarrow{p^*} & H^*_{T^n}(Gr_s(\mathbb{C}^n)) \\ \downarrow{\scriptstyle \rho_\lambda} & & \downarrow{\scriptstyle i^*} \\ H^*_{T^\ell}(\mathcal{S}_\lambda) & \xleftarrow{k^*} & H^*_{T^n}(X_{\mu_0}(U_\bullet)) \end{array} \tag{4.5.1}$$

where i^* and ρ_λ are the maps induced by the inclusions. Let $\mu_{s,d} = (1, \cdots, 1, 0, \cdots, 0)$ with 1 repeated d-times and 0 repeated $(s-d)$-times. This Young diagram has at most s rows and $n-s$ columns since we are assuming that $s < n$. Recall that the T^n-equivariant Schubert class $\tilde{S}_\mu = [X_\mu(\bar{w}\tilde{F}_\bullet)] \in H^*_{T^n}(Gr_s(\mathbb{C}^n))$ is supported on $X_\mu(\bar{w}\tilde{F}_\bullet)$. That is, $\tilde{S}_\mu$ is the image of a relative cohomology class in $H^*_{T^n}(Gr_s(\mathbb{C}^n), Gr_s(\mathbb{C}^n)\backslash X_\mu(\bar{w}\tilde{F}_\bullet))$. Now $\mu_{s,d} \not\subset \mu_0^*$ and (4.4.4) show that no cycle in $X_{\mu_0}(U_\bullet)$ intersects with $X_{\mu_{s,d}}(\bar{w}\tilde{F}_\bullet)$. That is, in the commutative diagram

$$\begin{array}{ccc} H^*_{T^n}(Gr_s(\mathbb{C}^n)) & \longleftarrow & H^*_{T^n}(Gr_s(\mathbb{C}^n), Gr_s(\mathbb{C}^n)\backslash X_\mu(\bar{w}\tilde{F}_\bullet)) \\ \downarrow{\scriptstyle i^*} & \swarrow & \\ H^*_{T^n}(X_{\mu_0}(U_\bullet)) & & \end{array}$$

induced by natural inclusions, the left-down map is a zero map. So it follows that $i^*\tilde{S}_{\mu_{s,d}} = 0$ for $d \geq s+1-p_{\check{\lambda}}(s)$. Thus, we obtain

$$\rho_\lambda(p^*\tilde{S}_{\mu_{s,d}}) = 0 \tag{4.5.2}$$

by the commutativity of the diagram (4.5.1). We note that, if $s = n$ which we excluded from our case, the equality (4.5.2) does not hold because we have $Gr_s(\mathbb{C}^n) = X_{\mu_0}(U_\bullet)$ in this case.

To relate (4.5.2) to the claim of Lemma 4.5.2, we write $\rho_\lambda(p^*\tilde{S}_{\mu_{s,d}})$ in terms of $\bar{y}_1, \cdots, \bar{y}_n$ and $u_1, \cdots, u_\ell$. Let us first describe $p^*\tilde{S}_{\mu_{s,d}}$ in terms of $\bar{x}_1, \cdots, \bar{x}_n$ and $t_1, \cdots, t_n$. Let $w \in S_n$ be a fixed permutation. Recall from (4.4.1) that we have the induced map $w : Flags(\mathbb{C}^n) \to Flags(\mathbb{C}^n)$ which satisfies that

$$w^*(t_i\alpha) = t_{w^{-1}(i)}w^*(\alpha) \tag{4.5.3}$$

for any $\alpha \in H^*_{T^n}(Flags(\mathbb{C}^n))$ and $i = 1, \cdots, n$ where the products are taken by the cup products via the canonical homomorphism $H^*(BT^n) \to H^*_{T^n}(Flags(\mathbb{C}^n))$. Similarly, w induces an automorphism of $w : Gr_s(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ (see (4.4.2)), and the projection map $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ is compatible with the automorphisms of $Flags(\mathbb{C}^n)$ and $Gr_s(\mathbb{C}^n)$, i.e., $p \circ w = w \circ p$ as maps $Flags(\mathbb{C}^n) \to$

$Gr_s(\mathbb{C}^n)$. Observe that

$$w^*\bar{x}_i = \bar{x}_i \tag{4.5.4}$$

since $w \in S_n$ naturally induces a map $E_i/E_{i-1} \to E_i/E_{i-1}$ which is a fiber-wise isomorphism.

For any Young diagram μ with at most s rows and $n-s$ columns, we have that $X_\mu(\bar{w}\tilde{F}_\bullet) = X_\mu(\bar{w}w_0F_\bullet)$ from the definition where $w_0 \in S_n$ is the longest element with respect to the Bruhat order. So it follows from (4.4.5), (4.5.3), (4.5.4), and Proposition 4.4.1 that

$$\begin{aligned} p^*\tilde{S}_\mu &= p^*((\bar{w}w_0)^{-1})^*[X_\mu(F_\bullet)] = ((\bar{w}w_0)^{-1})^*p^*[X_\mu(F_\bullet)] \\ &= s_\mu(-\bar{x}_1,\cdots,-\bar{x}_s|-t_{\bar{w}(1)},\cdots,-t_{\bar{w}(n)}). \end{aligned}$$

Now Lemma 4.4.2 together with the definition (4.3.1) shows that

$$p^*\tilde{S}_{\mu_{s,d}} = (-1)^d e_d(\bar{x}_1,\cdots,\bar{x}_s|t_{\bar{w}(1)},\cdots,t_{\bar{w}(n)}). \tag{4.5.5}$$

Thus, applying ρ_λ to this equality, we obtain

$$\rho_\lambda(p^*\tilde{S}_{\mu_{s,d}}) = (-1)^d e_d(\bar{y}_1,\cdots,\bar{y}_s|u_{p(\bar{w}(1))},\cdots,u_{p(\bar{w}(n))}) \tag{4.5.6}$$

because of (4.2.9), and we obtain the desired equality by (4.5.2). □

From now on, we take a specific choice of $\bar{w}$ as follows and we study the image of the Schubert classes $p^*\tilde{S}_\mu$ under ρ_λ. We choose $\bar{w}$ so that its one-line notation is given by

$$\bar{w} = J_1 \cdots J_\ell$$

where each J_r is a sequence

$$J_r = j_r^{(1)} j_r^{(2)} \cdots j_r^{(\lambda_r - \lambda_{r+1})}$$

consisting of sequences of the form

$$j_r^{(m)} = (\lambda_1 - \lambda_r) + m\ ,\ (\lambda_1 - \lambda_r) + \lambda_2 + m\ ,\ \ldots\ldots\ ,\ (\lambda_1 - \lambda_r) + \lambda_2 + \cdots + \lambda_r + m.$$

Note that $j_r^{(m)}$ is a sequence of length r and J_r is a sequence of length $r(\lambda_r - \lambda_{r+1})$. We define J_r to be the empty sequence if $\lambda_r = \lambda_{r+1}$. Recall that $\lambda_{\ell+1} = 0$ in our convention.

Lemma 4.5.3. *The complete flag $\bar{w}F_\bullet$ refines the flag $(0 \subset \cdots \subset N^2\mathbb{C}^n \subset N\mathbb{C}^n \subset \mathbb{C}^n)$.*

Proof. We list the numbers $1, 2, \cdots, n$ as in the one-line notation of $\bar{w}$:

$$j_1^{(1)} \cdots j_1^{(\lambda_1 - \lambda_2)}\ j_2^{(1)} \cdots j_2^{(\lambda_2 - \lambda_3)} \quad \cdots \quad j_\ell^{(1)} \cdots j_\ell^{(\lambda_\ell - \lambda_{\ell+1})}. \tag{4.5.7}$$

Then

$$N\mathbb{C}^n = \langle e_i \mid i \text{ is not in } j_\ell^{(\lambda_\ell - \lambda_{\ell+1})}\ \rangle.$$

If $\lambda_\ell - \lambda_{\ell+1} \geq 2$, then

$$N^2\mathbb{C}^n = \langle e_i \mid i \text{ is not in } j_\ell^{(\lambda_\ell - \lambda_{\ell+1} - 1)} j_\ell^{(\lambda_\ell - \lambda_{\ell+1})}\ \rangle,$$

if $\lambda_\ell - \lambda_{\ell+1} = 1$, then

$$N^2\mathbb{C}^n = \langle e_i \mid i \text{ is not in } j_{\ell-1}^{(\lambda_{\ell-1} - \lambda_\ell)} j_\ell^{(1)}\ \rangle$$

and so on. In general, for each k, the linear subspace $N^k\mathbb{C}^n$ is generated by the set of e_i for which i is not in r-th subsequence from the right in (4.5.7) for $1 \leq r \leq k$. This implies that the complete flag $\bar{w}F_\bullet$ refines the flag $(0 \subset \cdots \subset N^2\mathbb{C}^n \subset N\mathbb{C}^n \subset \mathbb{C}^n)$. □

Example. If $n = 16$ and $\lambda = (7, 5, 2, 2)$, then

$$\bar{w} = 1\ 2\ 3\ 8\ 4\ 9\ 5\ 10\ 6\ 11\ 13\ 15\ 7\ 12\ 14\ 16$$

where $J_1 = j_1^{(1)} j_1^{(2)} = 1\ 2$, $J_2 = j_2^{(1)} j_2^{(2)} j_2^{(3)} = 3\ 8\ 4\ 9\ 5\ 10$, J_3 is the empty sequence, and $J_4 = j_4^{(1)} j_4^{(2)} = 6\ 11\ 13\ 15\ 7\ 12\ 14\ 16$. Then $\bar{w}F_\bullet$ refines the flag $(\cdots \subset N^2\mathbb{C}^n \subset N\mathbb{C}^n \subset \mathbb{C}^n)$.

We now refine Proposition 4.5.2 by taking the explicit choice of $\bar{w}$ given above.

Proposition 4.5.4. *$e_d(\bar{y}_{i_1}, \cdots, \bar{y}_{i_s} | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$ in $H^*_{T^\ell}(\mathcal{S}_\lambda)$ for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$.*

Proof. We first consider the case for $s < n$. The map $p : [n] \to [\ell]$ defined in (4.2.8) takes each sequence $j_r^{(m)}$ to the sequence $1, \cdots, r$ since the k-th number of $j_r^{(m)}$ satisfies

$$\lambda_1 + \cdots + \lambda_{k-1} + 1 \leq (\lambda_1 - \lambda_r) + \lambda_2 + \cdots + \lambda_k + m \leq \lambda_1 + \cdots + \lambda_k.$$

This shows that $p \circ \bar{w}$ coincides with the map ϕ_λ defined in (4.3.2). From Proposition 4.5.2, we obtain the desired equality for the case $s < n$.

Next, we prove the claim for the case $s = n$. We have that $d \geq n+1-p_{\check{\lambda}}(n) =$

1 in this case. Observe that in $H^*_{T^n}(Flags(\mathbb{C}^n))$ we have

$$
\begin{aligned}
&e_d(\bar{x}_1, \cdots, \bar{x}_n | t_1, \cdots, t_n) \\
&\qquad = \sum_{r=0}^{d} (-1)^{d-r} e_r(\bar{x}_1, \cdots, \bar{x}_n) h_{d-r}(t_1, \cdots, t_{n+1-d}) \\
&\qquad = \sum_{r=0}^{d} (-1)^{d-r} e_r(t_1, \cdots, t_n) h_{d-r}(t_1, \cdots, t_{n+1-d})
\end{aligned}
$$

by the presentation given in (4.2.3). It is straightforward to check that this is equal to $e_d(t_{n+2-d}, \cdots, t_n)$ by comparing the degree d part of the product of the following generating functions with a formal variable z for elementary and complete symmetric polynomials :

$$
\begin{aligned}
\prod_{i=1}^{n}(1 - t_i z) &= \sum_{r=0}^{n} (-1)^r e_r(t_1, \cdots, t_n) z^r, \\
\prod_{i=1}^{n+1-d} \frac{1}{1 - t_i z} &= \sum_{r \geq 0} h_r(t_1, \cdots, t_{n+1-d}) z^r.
\end{aligned}
$$

Observe that $e_d(t_{n+2-d}, \cdots, t_n)$ is zero since the number of variables is greater than d. That is, the polynomial $e_d(\bar{x}_1, \cdots, \bar{x}_n | t_1, \cdots, t_n)$ vanishes in $H^*_{T^n}(Flags(\mathbb{C}^n))$, and hence we see that $e_d(\bar{y}_1, \cdots, \bar{y}_n | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$. □

Proof of Theorem 4.3.1:

Proposition 4.5.4 shows that the surjective homomorphism (4.3.3) induces a surjective ring homomorphism

$$
\bar{\psi} : \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots u_\ell]/\widetilde{I}_\lambda \longrightarrow H^*_{T^\ell}(\mathcal{S}_\lambda). \tag{4.5.8}
$$

In what follows, we prove that this is an isomorphism by thinking of both sides as $\mathbb{Z}[u_1, \cdots u_\ell]$-algebras. Namely, the ring on the left-hand-side admits the obvious multiplication by $u_1, \cdots, u_\ell$, and the ring on the right-hand-side has the canonical ring homomorphism $H^*(BT^\ell) \to H^*_{T^\ell}(\mathcal{S}_\lambda)$ with the identification

$H^*(BT^\ell) = \mathbb{Z}[u_1, \cdots u_\ell]$.

From the cellular decomposition given in [33], $H^*(\mathcal{S}_\lambda)$ is a free $\mathbb{Z}$-module and

$$\mathrm{rank}_{\mathbb{Z}} H^*(\mathcal{S}_\lambda) = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_\ell!}.$$

We denote the multinomial coefficient $\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_\ell!}$ by $\binom{n}{\lambda}$. Recall that we have $H^*_{T^\ell}(\mathcal{S}_\lambda) \cong \mathbb{Z}[u_1, \cdots, u_\ell] \otimes H^*(\mathcal{S}_\lambda)$ as $\mathbb{Z}[u_1, \cdots, u_\ell]$-modules as discussed in Section 4.2. So the rank of $H^*_{T^\ell}(\mathcal{S}_\lambda)$ over $\mathbb{Z}[u_1, \cdots, u_\ell]$ coincides with the rank of $H^*(\mathcal{S}_\lambda)$:

$$\mathrm{rank}_{\mathbb{Z}[u_1,\cdots,u_\ell]} H^*_{T^\ell}(\mathcal{S}_\lambda) = \mathrm{rank}_{\mathbb{Z}} H^*(\mathcal{S}_\lambda) = \binom{n}{\lambda}.$$

Hence, we complete our proof of Theorem 4.3.1 by the following Lemma and the fact that, for any commutative ring R with unit, a surjective homomorphism from an R-module to a free R-module of the same rank is an isomorphism.

□

Lemma 4.5.5. *Let $k = \binom{n}{\lambda}$ be the multinomial coefficient as above. Let $\Phi_1(y), \cdots, \Phi_k(y)$ be homogeneous polynomials in $\mathbb{Z}[y_1, \cdots, y_n]$ which give an additive basis of $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ where I_λ is generated by $e_d(y_{i_1}, \cdots, y_{i_s})$ for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\tilde{\lambda}}(s)$. If we think of $\Phi_1(y), \cdots, \Phi_k(y)$ as elements of $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$, then they generate $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-module.*

Remark. There exists a homogeneous $\mathbb{Z}$-basis of $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$. In fact, the argument in [37] works for the proof of the fact that we have a well-defined surjective map

$$\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda \to H^*(\mathcal{S}_\lambda) \tag{4.5.9}$$

which maps each y_i to the first Chern class of the tautological line bundle E_i/E_{i-1} over $Flags(\mathbb{C}^n)$ restricted to $\mathcal{S}_\lambda$. This surjectivity can also be explained as a non-equivariant limit of (4.5.8), namely we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots u_\ell]/\widetilde{I}_\lambda & \xrightarrow{\bar{\psi}} & H^*_{T^\ell}(\mathcal{S}_\lambda) \\ \downarrow & & \downarrow \\ \mathbb{Z}[y_1, \cdots, y_n]/I_\lambda & \longrightarrow & H^*(\mathcal{S}_\lambda) \end{array}$$

where the left-vertical map is the map $u_i \mapsto 0$ and the right-vertical map is the canonical forgetful map. It follows that the rank of $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ is less than or equal to $k = \binom{n}{\lambda}$ from the proof of Theorem 1 in [37] by replacing the "dim" to "rank" in the argument. Thus, the surjective map (4.5.9) is in fact an isomorphism from the fact mentioned above Lemma 4.5.5, and this implies that $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ is a free $\mathbb{Z}$-module.

Proof. It suffices to show that any monomial m of the variables $y_1, \cdots, y_n$ in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ can be written as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$. We prove this by induction on the degree d of m. The base case $d = 0$ is clear, i.e. $\Phi_i(y) = 1$ for some i. We assume that $d \geq 1$ and the claim holds for $d - 1$. Let θ be a homomorphism from $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ to $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ sending y_i to y_i and u_i to 0. This is well-defined since each generator $e_d(\bar{y}_1, \cdots, \bar{y}_s | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)})$ of $\tilde{I}_\lambda$ is mapped to the corresponding generator $e_d(y_{i_1}, \cdots, y_{i_s})$ of I_λ. By assumption, $\theta(m)$ can be written as a $\mathbb{Z}$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$, that is, we have

$$m - \sum_i a_i \Phi_i(y) \in \ker \theta$$

for some $a_i \in \mathbb{Z}$. Here, $\ker \theta$ is the ideal of $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ generated by $u_1, \cdots, u_\ell$. In fact, it follows that the image of I_λ in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$

is included in the ideal $(u_1, \cdots, u_\ell)$ of $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ from the following equation in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$:

$$e_d(y_{i_1}, \cdots, y_{i_s}) = -\sum_{0\le r<d} (-1)^{d-r} e_r(y_{i_1}, \cdots, y_{i_s}) h_{d-r}(u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(s+1-d)}).$$

Therefore, the monomial m can be written as

$$m = \sum_i a_i \Phi_i(y) + \sum_{j=1}^{\ell} f_j(y,u) u_j \tag{4.5.10}$$

for some polynomials $f_1(y,u), \cdots, f_\ell(y,u)$. Since m has degree d, we can replace the polynomials in the right-hand-side by their homogeneous components of degree d. Namely, we can assume that $\deg \Phi_i(y) = \deg f_j(y,u)+1 = d$. The induction hypothesis shows that each $f_j(y,u)$ is written as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$ since the degree of each monomial in y contained in $f_j(y,u)$ is less than d. Hence, the element m is written by a $\mathbb{Z}[u_1, \cdots, u_\ell]$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$ in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$, as desired. □

Chapter 5

The S-equivariant cohomology rings of regular nilpotent Hessenberg varieties

In Chapter 5 we give an explicit presentation of the S-equivariant cohomology rings of regular nilpotent Hessenberg varieties. Chapter 5 is organized as follows. We briefly recall the necessary background in Section 5.1. Our main theorem, Theorem 5.2.1, is formulated in Section 5.2. We sketch the outline of the proof in Section 5.3. We see that the cohomology rings of regular nilpotent Hessenberg varieties are Poincaré duality algebra from Corollaly 5.2.2 in Section 5.4. This is a joint work with Hiraku Abe, Megumi Harada and Mikiya Masuda in [1].

5.1 Background on Hessenberg varieties

In this section we briefly recall the terminology required to understand the statements of our main results; in particular we recall the definition of the regular nilpotent Hessenberg variety in type A in Chapter 1, denoted $\mathrm{Hess}(\mathsf{N}, h)$, along with a natural S-action on it. Since we work exclusively with type A in this chapter, we henceforth omit it from our terminology.

We first begin by recalling the definition of a Hessenberg function. A **Hessenberg function** is a function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i + 1) \geq h(i)$ for all $1 \leq i < n$. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \ldots, h(n) = n)$.

Definition. Let $\mathsf{N} : \mathbb{C}^n \to \mathbb{C}^n$ be the regular nilpotent linear operator, i.e. N has one Jordan block with eigenvalue 0. The **regular nilpotent Hessenberg variety** $\mathrm{Hess}(\mathsf{N}, h)$ is defined as the following subvariety of $Flags(\mathbb{C}^n)$:

$$\mathrm{Hess}(\mathsf{N}, h) := \{V_\bullet \in Flags(\mathbb{C}^n) \mid \mathsf{N}V_i \subseteq V_{h(i)} \text{ for all } i = 1, \ldots, n\}. \quad (5.1.1)$$

Next recall that the standard torus T in (1.0.2) naturally acts on the flag variety $Flags(\mathbb{C}^n)$. However, this T-action does not preserve the subvariety $\mathrm{Hess}(\mathsf{N}, h)$ in general. This problem can be rectified by considering instead the action of the one-dimensional subtorus S of T in (1.0.3), which does preserve $\mathrm{Hess}(\mathsf{N}, h)$ ([15, Lemma 5.1]). Recall that the T-fixed points $Flags(\mathbb{C}^n)^T$ of the flag variety $Flags(\mathbb{C}^n)$ can be identified with the permutation group S_n on n letters. More concretely, it is straightforward to see that the T-fixed points are the set

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\} \quad (5.1.2)$$

where $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$.

Since the S-fixed points $Flags(\mathbb{C}^n)^S$ of the flag variety $Flags(\mathbb{C}^n)$ is also given by the above set, we have

$$\mathrm{Hess}(\mathsf{N}, h)^S = \mathrm{Hess}(\mathsf{N}, h) \cap (Flags(\mathbb{C}^n))^T.$$

Therefore, we may view $\mathrm{Hess}(\mathsf{N}, h)^S$ as a subset of S_n.

5.2 Statement of the main theorem in Chapter 5

In this section we state the main result in Chapter 5. We first recall some notation and terminology. Let E_i denote the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n) \times \mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\tilde{\tau}_i \in H^2_T(Flags(\mathbb{C}^n))$. Let $\mathbb{C}_i$ denote the one dimensional representation of T through the map $T \to \mathbb{C}^*$ given by $diag(g_1, \ldots, g_n) \mapsto g_i$. In addition we denote the first Chern class of the line bundle $ET \times_T \mathbb{C}_i$ over BT by $t_i \in H^2(BT)$. It is well-known that the $t_1, \ldots, t_n$ generate $H^*(BT)$ as a ring and are algebraically independent, so we may identify $H^*(BT)$ with the polynomial ring $\mathbb{Q}[t_1, \ldots, t_n]$ as rings. Furthermore, it is known that $H^*_T(Flags(\mathbb{C}^n))$ is generated as a ring by the elements $\tilde{\tau}_1, \ldots, \tilde{\tau}_n, t_1, \ldots, t_n$. Indeed, by sending x_i to $\tilde{\tau}_i$ and the t_i to t_i we obtain the following isomorphism:

$$H^*_T(Flags(\mathbb{C}^n)) \cong \mathbb{Q}[x_1, \ldots, x_n, t_1, \ldots, t_n]/(e_i(x_1, \ldots, x_n) - e_i(t_1, \ldots, t_n) \mid 1 \leq i \leq n).$$

Here the e_i denote the degree-i elementary symmetric polynomials in the relevant variables. In particular, since the odd cohomology of the flag variety

$Flags(\mathbb{C}^n)$ vanishes, we additionally obtain the following:

$$H^*(Flags(\mathbb{C}^n)) \cong \mathbb{Q}[x_1, \ldots, x_n]/(e_i(x_1, \ldots, x_n) \mid 1 \leq i \leq n). \tag{5.2.1}$$

As mentioned in Section 5.1, in this manuscript we focus on a particular circle subgroup S of the usual maximal torus T. For this subgroup S, we denote the first Chern class of the line bundle $ES \times_S \mathbb{C}$ over BS by $t \in H^2(BS)$, where by $\mathbb{C}$ we mean the standard one-dimensional representation of S through the map $S \to \mathbb{C}^*$ given by $diag(g, g^2, \ldots, g^n) \mapsto g$. Analogous to the identification $H^*(BT) \cong \mathbb{Q}[t_1, \ldots, t_n]$, we may also identify $H^*(BS)$ with $\mathbb{Q}[t]$ as rings.

Consider the restricion homomorphism

$$H^*_T(Flags(\mathbb{C}^n)) \to H^*_S(\mathrm{Hess}(\mathsf{N}, h)). \tag{5.2.2}$$

Let τ_i denote the image of $\tilde{\tau}_i$ under (5.2.2). We next analyze some algebraic relations satisfied by the τ_i. For this purpose, we now introduce some polynomials $f_{i,j} = f_{i,j}(x_1, \ldots, x_n, t) \in \mathbb{Q}[x_1, \ldots, x_n, t]$.

First we define

$$p_i := \sum_{k=1}^{i} (x_k - kt) \quad (1 \leq i \leq n). \tag{5.2.3}$$

For convenience we also set $p_0 := 0$ by definition. Let (i, j) be a pair of natural numbers satisfying $n \geq i \geq j \geq 1$. These polynomials should be visualized as being associated to the (i, j)-th spot in an $n \times n$ matrix. Note that by assumption on the indices, we only define the $f_{i,j}$ for entries in the lower-triangular part of the matrix, i.e. the part at or below the diagonal. The definition of the $f_{i,j}$ is inductive, beginning with the case when $i = j$, i.e. the two indices are equal. In this case we make the following definition:

$$f_{j,j} := p_j \quad (1 \leq j \leq n). \tag{5.2.4}$$

Now we proceed inductively for the rest of the $f_{i,j}$ as follows: for (i,j) with $n \geq i > j \geq 1$ we define:

$$f_{i,j} := f_{i-1,j-1} + \big(x_j - x_i - t\big) f_{i-1,j}. \tag{5.2.5}$$

Again for convenience we define $f_{*,0} := 0$ for any $*$. Informally, we may visualize each $f_{i,j}$ as being associated to the lower-triangular (i,j)-th entry in an $n \times n$ matrix, as follows:

$$\begin{pmatrix} f_{1,1} & 0 & \cdots & \cdots & 0 \\ f_{2,1} & f_{2,2} & 0 & \cdots & \\ f_{3,1} & f_{3,2} & f_{3,3} & \ddots & \\ \vdots & & & & \\ f_{n,1} & f_{n,2} & \cdots & & f_{n,n} \end{pmatrix} \tag{5.2.6}$$

To make the discussion more concrete, we present an explicit example.

Example. Suppose $n = 4$. Then the $f_{i,j}$ have the following form.

$f_{i,i} = p_i \quad (1 \leq i \leq 4)$

$f_{2,1} = (x_1 - x_2 - t)p_1$

$f_{3,2} = (x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2$

$f_{4,3} = (x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2 + (x_3 - x_4 - t)p_3$

$f_{3,1} = (x_1 - x_3 - t)(x_1 - x_2 - t)p_1$

$f_{4,2} = (x_1 - x_3 - t)(x_1 - x_2 - t)p_1 + (x_2 - x_4 - t)\{(x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2\}$

$f_{4,1} = (x_1 - x_4 - t)(x_1 - x_3 - t)(x_1 - x_2 - t)p_1$

For general n, the polynomials $f_{i,j}$ for each (i,j)-th entry in the matrix (5.2.6) above can also be expressed in a closed formula in terms of certain polynomials $\Delta_{i,j}$ for $i \geq j$ which are determined inductively, starting on the main diagonal. As for the $f_{i,j}$, we think of $\Delta_{i,j}$ for $i \geq j$ as being associated to the (i,j)-th box in an $n \times n$ matrix. In what follows, for $0 < k \leq n-1$, we refer to the lower-triangular matrix entries in the (i,j)-th spots where $i - j = k$ as the k**-th**

lower diagonal. (Equivalently, the k-th lower diagonal is the "usual" diagonal of the lower-left $(n-k) \times (n-k)$ submatrix.) The usual diagonal is the 0-th lower diagonal in this terminology. We now define the $\Delta_{i,j}$ as follows.

1. First place the linear polynomial $x_i - it$ in the i-th entry along the 0-th lower (i.e. main) diagonal, so $\Delta_{i,i} := x_i - it$.

2. Suppose that $\Delta_{i,j}$ for the $k-1$-st lower diagonal have already been defined. Let (i,j) be on the k-th lower diagonal, so $i - j = k$. Define

$$\Delta_{i,j} := \left(\sum_{\ell=1}^{j} \Delta_{i-j+\ell-1,\ell} \right) (x_j - x_i - t).$$

In words, this means the following. Suppose $k = i - j > 0$. Then $\Delta_{i,j}$ is the product of $(x_j - x_i - t)$ with the sum of the entries in the boxes which are in the "diagonal immediately above the (i,j) box" (i.e. the boxes which are in the $(k-1)$-st lower diagonal), but we omit any boxes to the right of the (i,j) box (i.e. in columns $j+1$ or higher). Finally, the polynomial $f_{i,j}$ is obtained by taking the sum of the entries in the (i,j)-th box and any boxes "to its left" in the same lower diagonal. More precisely,

$$f_{i,j} = \sum_{k=1}^{j} \Delta_{i-j+k,k}. \tag{5.2.7}$$

We are now ready to state our main result.

Theorem 5.2.1. *Let n be a positive integer and $h : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ a Hessenberg function. Let $\mathrm{Hess}(\mathsf{N},h) \subset Flags(\mathbb{C}^n)$ denote the corresponding regular nilpotent Hessenberg variety equipped with the S-action described above. Then the restriction map*

$$H^*_T(Flags(\mathbb{C}^n)) \to H^*_S(\mathrm{Hess}(\mathsf{N},h))$$

is surjective. Moreover, there is an isomorphism of $\mathbb{Q}[t]$*-algebras*

$$H_S^*(\mathrm{Hess}(\mathsf{N}, h)) \cong \mathbb{Q}[x_1, \ldots, x_n, t]/I_h$$

sending x_i *to* τ_i *and* t *to* t *and we identify* $H^*(BS) = \mathbb{Q}[t]$*. Here the ideal* I_h *is defined by*

$$I_h := (f_{h(j),j} \mid 1 \leq j \leq n). \tag{5.2.8}$$

We can also describe the ideal I_h defined in (5.2.8) as follows. Any Hessenberg function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ determines a subspace of the vector space $M(n \times n, \mathbb{C})$ of matrices as follows: an (i, j)-th entry is required to be 0 if $i > h(j)$. If we represent a Hessenberg function h by listing its values $(h(1), h(2), \cdots, h(n))$, then the Hessenberg subspace can be described in words as follows: the first column (starting from the left) is allowed $h(1)$ non-zero entries (starting from the top), the second column is allowed $h(2)$ non-zero entries, et cetera. For example, if $h = (3, 3, 4, 5, 7, 7, 7)$ then the Hessenberg subspace is

$$\left\{ \begin{pmatrix} \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star \end{pmatrix} \right\} \subseteq M(7 \times 7, \mathbb{C}).$$

Then, using the association of the polynomials $f_{i,j}$ with the (i, j)-th entry of the matrix (5.2.6), the ideal I_h can be described as being "generated by the $f_{i,j}$ in the boxes at the bottom of each column in the Hessenberg space". For instance, in the $h = (3, 3, 4, 5, 7, 7, 7)$ example above, the generators are $\{f_{3,1}, f_{3,2}, f_{4,3}, f_{5,4}, f_{7,5}, f_{7,6}, f_{7,7}\}$.

Our main result generalizes previous known results.

Remark. Consider the special case $h = (2, 3, \ldots, n, n)$. In this case the corresponding regular nilpotent Hessenberg variety has been well-studied and it is called a **Peterson variety** Pet_n (of type A). Our result above is a generalization of the result in [11] which gives a presentation of $H^*_S(Pet_n)$. Indeed, for $1 \leq j \leq n-1$, we obtain from (5.2.5) and (5.2.3) that

$$\begin{aligned} f_{j+1,j} &= f_{j,j-1} + (x_j - x_{j+1} - t) f_{j,j} \\ &= f_{j,j-1} + (-p_{j-1} + 2p_j - p_{j+1} - 2t) p_j \end{aligned}$$

and since $f_{n,n} = p_n$ we have

$$\begin{aligned} H^*_S(Pet_n) &\cong \mathbb{Q}[x_1, \ldots, x_n, t] / \big(f_{j,j-1} + (-p_{j-1} + 2p_j - p_{j+1} - 2t)p_j,\ p_n \mid 1 \leq j \leq n-1\big) \\ &= \mathbb{Q}[x_1, \ldots, x_n, t] / \big((-p_{j-1} + 2p_j - p_{j+1} - 2t)p_j,\ p_n \mid 1 \leq j \leq n-1\big) \\ &\cong \mathbb{Q}[p_1, \ldots, p_{n-1}, t] / \big((-p_{j-1} + 2p_j - p_{j+1} - 2t)p_j \mid 1 \leq j \leq n-1\big) \end{aligned}$$

which agrees with [11]. (Note that we take by convention $p_0 = p_n = 0$.)

The main theorem above also immediately yields a computation of the ordinary cohomology ring. Indeed, since the odd degree cohomology groups of $\mathrm{Hess}(\mathsf{N}, h)$ vanish [38], by setting $t = 0$ we obtain the ordinary cohomology. Let $\check{f}_{i,j} := f_{i,j}(x, t = 0)$ denote the polynomials in the variables x_i obtained by setting $t = 0$. A computation then shows that

$$\check{f}_{i,j} = \sum_{k=1}^{j} x_k \prod_{\ell=j+1}^{i} (x_k - x_\ell).$$

(For the case $i = j$ we take by convention $\prod_{\ell=j+1}^{i}(x_k - x_\ell) = 1$.) We have the following.

Corollary 5.2.2. *Let the notation be as above. There is a ring isomorphism*

$$H^*(\mathrm{Hess}(\mathsf{N},h)) \cong \mathbb{Q}[x_1,\dots,x_n]/\check{I}_h$$

where $\check{I}_h := \big(\check{f}_{h(j),j} \mid 1 \le j \le n\big)$.

Remark. Consider the special case $h = (n,n,\dots,n)$. In this case the condition in (5.1.1) is vacuous and the associated regular nilpotent Hessenberg variety is the flag variety $Flags(\mathbb{C}^n)$. In this case we can explicitly relate the generators $\check{f}_{h(j)=n,j}$ of our ideal $\check{I}_h = \check{I}_{(n,n,\dots,n)}$ with the power sums $\mathsf{p}_r(x) = \mathsf{p}_r(x_1,\dots,x_n) := \sum_{k=1}^n x_k^r$, thus relating our presentation with the usual Borel presentation as in (5.2.1), see e.g. [13]. More explicitly, for r be an integer, $1 \le r \le n$, define

$$\mathsf{q}_r(x) = \mathsf{q}_r(x_1,\dots,x_n) := \sum_{k=1}^{n+1-r} x_k \prod_{\ell=n+2-r}^{n} (x_k - x_\ell).$$

Note that by definition $\mathsf{q}_r(x) = \check{f}_{n,n+1-r}$ so these are the generators of $\check{I}(n,n,\dots,n)$. The polynomials $\mathsf{q}_r(x)$ and the power sums $\mathsf{p}_r(x)$ can then be shown to satisfy the relations

$$\mathsf{q}_r(x) = \sum_{i=0}^{r-1} (-1)^i e_i(x_{n+2-r},\dots,x_n)\mathsf{p}_{r-i}(x). \tag{5.2.9}$$

Remark. In the usual Borel presentation of $H^*(Flags(\mathbb{C}^n))$, the ideal I of relations is taken to be generated by the elementary symmetric polynomials. The power sums p_r generate this ideal I when we consider the cohomology with $\mathbb{Q}$ coefficients, but this is not true with $\mathbb{Z}$ coefficients. Thus our main Theorem 5.2.1 does not hold with $\mathbb{Z}$ coefficients in the case when $h = (n,n,\dots,n)$, suggesting that there is some subtlety in the relationship between the choice of coefficients and the choice of generators of the ideal $I(h)$.

5.3 Sketch of the proof of the main theorem in Chapter 5

We now sketch the outline of the proof of the main result (Theorem 5.2.1) above. As a first step, we show that the elements τ_i satisfy the relations $f_{h(j),j} = f_{h(j),j}(\tau_1, \ldots, \tau_n, t) = 0$. The main technique of this part of the proof is (equivariant) localization, i.e. the injection

$$H_S^*(\mathrm{Hess}(\mathsf{N}, h)) \to H_S^*(\mathrm{Hess}(\mathsf{N}, h)^S). \tag{5.3.1}$$

Specifically, we show that the restriction $f_{h(j),j}(w)$ of each $f_{h(j),j}$ to an S-fixed point $w \in \mathrm{Hess}(\mathsf{N}, h)^S$ is equal to 0; by the injectivity of (5.3.1) this then implies that $f_{h(j),j} = 0$ as desired. This part of the argument is rather long and requires a technical inductive argument based on a particular choice of total ordering on $\mathrm{Hess}(\mathsf{N}, h)^S$ which refines a certain natural partial order on Hessenberg functions. Once we show $f_{h(j),j} = 0$ for all j, we obtain a well-defined ring homomorphism which sends x_i to τ_i and t to t:

$$\varphi_h : \mathbb{Q}[x_1, \ldots, x_n, t]/(f_{h(j),j} \mid 1 \leq j \leq n) \to H_S^*(\mathrm{Hess}(\mathsf{N}, h)). \tag{5.3.2}$$

We then show that the two sides of (5.3.2) have identical Hilbert series. This part of the argument is rather straightforward, following the techniques used in e.g. [11].

The next key step in our proof of Theorem 5.2.1 relies on the following two key ideas: firstly, we use our knowledge of the special case where the Hessenberg function h is $h = (n, n, \ldots, n)$, for which the associated regular nilpotent Hessenberg variety is the flag variety $Flags(\mathbb{C}^n)$, and secondly, we consider localizations of the rings in question with respect to $R := \mathbb{Q}[t] \backslash \{0\}$. For the following, for $h = (n, n, \ldots, n)$ we let $\mathcal{H} := \mathrm{Hess}(h = (n, n, \ldots, n)) = Flags(\mathbb{C}^n)$

denote the flag variety and let I denote the associated ideal $I(n, n, \ldots, n)$. In this case we know that the map $\varphi := \varphi_{(n,n,\ldots,n)}$ is surjective since the Chern classes τ_i are known to generate the cohomology ring of $Flags(\mathbb{C}^n)$. Since the Hilbert series of both sides are identical, we then know that φ is an isomorphism.

The following commutative diagram is crucial for the remainder of the argument.

$$\begin{array}{ccccc} R^{-1}\big(\mathbb{Q}[x_1, \ldots, x_n, t]/I\big) & \xrightarrow[\cong]{R^{-1}\varphi} & R^{-1}H_S^*(\mathcal{H}) & \xrightarrow[\cong]{} & R^{-1}H_S^*(\mathcal{H}^S) \\ \big\downarrow \scriptstyle{\text{surj}} & & \big\downarrow & & \big\downarrow \scriptstyle{\text{surj}} \\ R^{-1}\big(\mathbb{Q}[x_1, \ldots, x_n, t]/I_h\big) & \xrightarrow{R^{-1}\varphi_h} & R^{-1}H_S^*(\text{Hess}(\mathsf{N}, h)) & \xrightarrow[\cong]{} & R^{-1}H_S^*(\text{Hess}(\mathsf{N}, h)^S) \end{array}$$

The horizontal arrows in the right-hand square are isomorphisms by the localization theorem. Since φ is an isomorphism, so is $R^{-1}\varphi$. The rightmost and leftmost vertical arrows are easily seen to be surjective, implying that $R^{-1}\varphi_h$ is also surjective. A comparison of Hilbert series shows that $R^{-1}\varphi_h$ is an isomorphism. Finally, to complete the proof we consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Q}[x_1, \ldots, x_n, t]/I_h & \xrightarrow{\varphi_h} & H_S^*(\text{Hess}(\mathsf{N}, h)) \\ \big\downarrow \scriptstyle{\text{inj}} & & \big\downarrow \scriptstyle{\text{inj}} \\ R^{-1}\mathbb{Q}[x_1, \ldots, x_n, t]/I_h & \xrightarrow[\cong]{R^{-1}\varphi_h} & R^{-1}H_S^*(\text{Hess}(\mathsf{N}, h)) \end{array}$$

for which it is straightforward to see that the vertical arrows are injections. From this it follows that φ_h is an injection, and once again a comparison of Hilbert series shows that φ_h is in fact an isomorphism.

5.4 Applications

Lastly, we state that the cohomology rings of regular nilpotent Hessenberg varieties are Poincaré duality algebra.

Definition. [16, Definition 2.78] Suppose that $R = \oplus_{i=0}^{d} R_i$ is an Artinian algebra and R_0 is a field. We say R is a *Poincaré duality algebra* if the map

$$R_i \times R_{d-i} \to R_d$$

defined by the multiplication in R gives a perfect pairing for every $i = 0, \cdots, d$.

Corollary 5.4.1. *Let $h \in H_n$ be a Hessenberg function and let* $\mathrm{Hess}(\mathsf{N}, h)$ *denote the associated regular nilpotent Hessenberg variety. Then, with respect to the usual grading and multiplication in cohomology, the ordinary cohomology ring* $H^*(\mathrm{Hess}(\mathsf{N}, h))$ *is a Poincaré duality algebra.*

Proof. Let $R := \mathbb{Q}[x_1, \ldots, x_n]/\check{I}_h$. From Corolally 5.2.2, we prove that R is a Poincaré duality algebra. Since the set of units in R are precisely those of the form $a_0 + a_1 + a_2 + \cdots$ with $a_i \in R_i$ and $a_0 \neq 0$, the ring R is local with the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ (cf. [3, Proposition 1.6]). Moreover, we have $\mathfrak{m}^k = 0$ for some k, so R is also Artinian (cf. [3, Proposition 8.6]). Therefore, the homogeneous polynomials $\check{f}_{h(1),1}, \ldots, \check{f}_{h(n),n}$ form a regular sequence in $\mathbb{Q}[x_1, \ldots, x_n]$ (cf. [11, Proposition 5.1]). We also see that the homogeneous polynomials $\check{f}_{h(1),1}, \ldots, \check{f}_{h(n),n}$ form a regular sequence in the ring of formal power series $\mathbb{Q}[[x_1, \ldots, x_n]]$. Since the completion $\hat{R}$ of R is isomorphic to $\mathbb{Q}[[x_1, \ldots, x_n]]/(\check{f}_{h(1),1}, \ldots, \check{f}_{h(n),n})$ (cf. [3, Proposition 10.12, Proposition 10.15]), R is a complete intersection ring (cf. [26, Theorem 21.2]). In particular, R is Gorenstein (cf. [26, Theorem 21.3]). From Theorem 2.79 in [16], R is a Poincaré duality algebra. □

Chapter 6

The S-equivariant cohomology rings of Peterson varieties in all Lie types

In Chapter 6 we give an explicit presentation of the S-equivariant cohomology rings of Peterson varieties in all Lie types. Chapter 6 is organized as follows. We briefly recall the necessary background in Section 6.1. We derive the relevant quadratic relations in Section 6.2. In particular, a key computation is contained in Lemma 6.2.3. The main theorem, Theorem 6.3.1, is proven in Section 6.3. This is a joint work with Megumi Harada and Mikiya Masuda in [14].

6.1 Background on the Peterson variety

In this section we record some facts about Peterson varieties which we require in this manuscript.

Let G be a complex semisimple linear algebraic group of rank n. We fix B a Borel subgroup and T a maximal torus of G such that $T \subseteq B \subseteq G$. These choices then determine the following data:

- a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$,
- the associated Weyl group W,
- the associated Lie algebras $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$, and
- root spaces $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ for each root α.

Definition. Let E_α be a basis element of the root space $\mathfrak{g}_\alpha$ and let $\mathsf{N} = \sum_{\alpha \in \Delta} E_\alpha$, a regular nilpotent operator. In this setting we may define the **Peterson variety (associated to $\mathfrak{g}$)** as

$$Pet := \{gB \in G/B \mid Ad(g^{-1})(\mathsf{N}) \in \mathfrak{b} \oplus \bigoplus_{\alpha \in -\Delta} \mathfrak{g}_\alpha\}.$$

As is well-known, the maximal torus T acts on G/B by left multiplication. This action does not in general preserve the Peterson variety. However, using the homomorphism $\phi : T \to (\mathbb{C}^*)^n$ defined by $t \mapsto (\alpha_1(t), \ldots, \alpha_n(t))$ and defining S to be the connected component of the identity in

$$\phi^{-1}(\{(c, c, \ldots, c) \mid c \in \mathbb{C}^*\})$$

it can be seen that the restriction of the T-action on G/B to the subgroup S does preserve Pet ([15, Lemma 5.1]).

Next recall that the T-fixed points of G/B are in bijective correspondence with the Weyl group W of G. Moreover, since the S-fixed points Pet^S of the Peterson variety satisfy the relation

$$Pet^S = Pet \cap (G/B)^T$$

we may view Pet^S as a subset of the Weyl group W. Indeed, the fixed point set Pet^S may be described concretely as follows. For a subset K of the set Δ simple roots, let W_K denote the parabolic subgroup generated by K and let w_K denote the longest element of W_K. Then it is known [15, Proposition 5.8] that

$$Pet^S = \{w_K \mid K \subseteq \Delta\}.$$

Here and below we always use complex coefficients $\mathbb{C}$ for our cohomology rings and hence omit it from our notation. Let $\alpha_i : T \to \mathbb{C}^*$ be a homomorphism which thus determines a complex 1-dimensional representation of T. Let $ET \times_T \mathbb{C} \to BT$ be the corresponding complex line bundle and by slight abuse of notation we let $\alpha_i \in H^2(BT)$ also denote the corresponding first Chern class. With this notation in place we have

$$H^*(BT) = \mathbb{C}[\alpha_1, \ldots, \alpha_n].$$

Consider the 1-dimensional representation of the diagonal subgroup $\{(c, c, \ldots, c) : c \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^n$ obtained via the projection $(c, c, \ldots, c) \to c$. Composing with the restriction to S of the above homomorphism ϕ, we obtain a 1-dimensional representation of S and an associated line bundle $ES \times_S \mathbb{C} \to BS$ with first Chern class denoted $t \in H^2(BS)$. With this notation in place we have

$$H^*(BS) = \mathbb{C}[t].$$

Next we recall that the inclusion homomorphism $S \hookrightarrow T$ induces a homomorphism $\pi\colon H^*(BT) \to H^*(BS)$ and from the definition of ϕ we obtain

$$\pi(\alpha_i) = t \quad (i = 1, 2, \ldots, n). \tag{6.1.1}$$

We now consider the following commutative diagram

$$\begin{array}{ccc} H_T^*(G/B) & \longrightarrow & \bigoplus_{w \in (G/B)^T = W} H_T^*(w) \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\pi}\downarrow \\ H_S^*(Pet) & \longrightarrow & \bigoplus_{w \in Pet^S \subseteq W} H_S^*(w) \end{array} \tag{6.1.2}$$

where all the maps are induced from inclusions of subgroups or inclusions of subspaces. As is well-known, the odd cohomology $H^{odd}(G/B)$ of G/B vanishes. The same holds for the Peterson variety, i.e. $H^{odd}(Pet) = 0$ [30]. Thus we obtain that both horizontal maps in (6.1.2) are injective, and we may identify $H_T^*(G/B)$ (respectively $H_S^*(Pet)$) with its image under these maps. For $w \in (G/B)^T \cong W$ (respectively $w \in Pet^S \subseteq W$) and $f \in H_T^*(G/B)$ (respectively $f \in H_S^*(Pet)$) we will denote by $f(w)$ the restriction of f to the w-th factor $H_T^*(w) = H^*(BT) = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$ (resp. $H_S^*(w) = H^*(BS) = \mathbb{C}[t]$) in the direct products on the right hand sides of (6.1.2).

For $v \in W$, we let σ_v denote the corresponding equivariant Schubert class in $H_T^*(G/B)$, and let p_v denote its image $\rho(\sigma_v)$ in $H_S^*(Pet)$. We call p_v a **Peterson Schubert class** (associated to v). Let s_i be the simple reflection corresponding to a simple root α_i. The vertices of the Dynkin diagram corresponding to the set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is in 1-1 correspondence with Δ. Here and below, we assume to be fixed an ordering of the simple roots as given in [10, Figure 1] (which in turn agrees with the standard ordering in [20, p.58]). With respect to this ordering, given any subset $K = \{\alpha_{a_1}, \alpha_{a_2}, \ldots, \alpha_{a_k}\}$ of the simple roots with $a_1 < a_2 < \cdots < a_k$, we define an element v_K of W by the formula

$$v_K := s_{a_1} s_{a_2} \cdots s_{a_k}.$$

The Peterson Schubert classes p_{v_K} corresponding to the Weyl group elements v_K defined above satisfy the following property.

Proposition 6.1.1 (Theorem 3.5 in [10]). *The Peterson Schubert classes* $\{p_{v_K} \mid K \subseteq \Delta\}$ *form a* $\mathbb{C}[t]$*-module basis for* $H^*_S(Pet)$.

It follows from Proposition 6.1.1 that the ρ in (6.1.2) is surjective. It is well-known that the equivariant Schubert classes σ_{s_i} generate $H^*_T(G/B)$ as a $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$-algebra. From the surjectivity of the homomorphism ρ we immediately obtain the following.

Proposition 6.1.2. *The Peterson Schubert classes* p_{s_i} $(i = 1, 2, \ldots, n)$ *generate* $H^*_S(Pet)$ *as a* $\mathbb{C}[t]$*-algebra.*

Since the odd cohomology $H^{odd}(Pet)$ of the Peterson variety vanishes, we know that as a $\mathbb{C}[t]$-module the equivariant cohomology $H^*_S(Pet)$ is isomorphic to $\mathbb{C}[t] \otimes H^*(Pet)$. It is known [6, Theorem 3] that

$$\begin{aligned} F(H^*(Pet), s) &= (1+s^2)^n \\ F(H^*_S(Pet), s) &= \frac{(1+s^2)^n}{1-s^2} \end{aligned} \tag{6.1.3}$$

where the left hand sides denotes the Hilbert series of the graded rings $H^*(Pet)$ and $H^*_S(Pet)$ with respect to the variable s (of degree 1).

Fix an integer i with $1 \leq i \leq n$ and a subset $K \subseteq \Delta$. From Proposition 6.1.1 it follows that the product $p_{s_i} \cdot p_{v_K}$ can be written uniquely as a $\mathbb{C}[t]$-linear combination of the p_{v_J} (for $J \subseteq \Delta$). The so-called Monk's formula gives a concrete computation of the coefficients in this linear combination.

Theorem 6.1.3 (Monk's formula for Peterson varieties for all Lie types, Theorem 4.2 in [10])**.** *The Peterson Schubert classes satisfy the following relation:*

$$p_{s_i} \cdot p_{v_K} = p_{s_i}(w_K) \cdot p_{v_K} + \sum_{\substack{J \supset K \\ |J|=|K|+1}} c_{i,K}^J \cdot p_{v_J}$$

where the coefficient $c_{i,K}^J$ *are non-negative rational numbers. More specifically, we have*

$$c_{i,K}^J = (p_{s_i}(w_J) - p_{s_i}(w_K)) \cdot \frac{p_{v_K}(w_J)}{p_{v_J}(w_J)}.$$

Next we recall the so-called Giambelli's formula. From Proposition 6.1.2 it follows that each module generator p_{v_K} can be expressed as a polynomial (with $\mathbb{C}[t]$ coefficients) in the (ring) generators p_{s_i}. The Giambelli formula gives a concrete expression for this polynomial as follows.

Theorem 6.1.4 (Giambelli's formula for Peterson varieties for all Lie types, Theorem 5.5 in [10])**.** *Suppose* K *is a subset of the simple roots* Δ*. Assume that the Dynkin diagram corresponding to the subset* K *is connected. Then*

$$\frac{|K|!}{|\mathcal{R}(v_K)|} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}$$

where $|\mathcal{R}(v_K)|$ *denotes the number of distinct reduced-word expressions for* v_K*.*

Remark (cf. Theorem 5.3 in [10])**.** The connectedness assumption in the above theorem is not serious, in the following sense. Suppose $J, K \subseteq \Delta$ are two subsets of Δ such that their corresponding Dynkin diagrams are connected. Suppose, however, that $J \cup K$ has corresponding Dynkin diagram that is not connected. Then $p_{v_{J\cup K}}$ is simply the product of p_{v_J} and p_{v_K}, i.e.

$$p_{v_{J\cup K}} = p_{v_J} \cdot p_{v_K}.$$

6.2 Quadratic relations satisfied by the Peterson Schubert classes p_{s_i}

In this section, we derive certain quadratic relations satisfied by the cohomology-degree-2 Peterson Schubert classes p_{s_i} $(i = 1, 2, \ldots, n)$ by using Monk's formula (Theorem 6.1.3), Giambelli's formula (Theorem 6.1.4), and Billey's formula recalled below. We will then show in the next section that these relations are sufficient to determine the equivariant cohomology ring $H^*_S(Pet)$ of the Peterson variety.

Theorem 6.2.1 (Billey's formula, Theorem 4 in [4]). *Let $w \in W$ and fix a reduced word decomposition $w = s_{b_1} s_{b_2} \cdots s_{b_m}$ of w. Set $r(i, w) := s_{b_1} s_{b_2} \cdots s_{b_{i-1}}(\alpha_{b_i})$. For an equivariant Schubert class σ_v for $v \in W$ we have the following:*

$$\sigma_v(w) = \sum_{\substack{reduced\ words \\ v = s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_\ell}}}} \prod_{i=1}^{\ell} r(j_i, w).$$

We begin with some elementary computations involving Peterson Schubert classes. First, from Monk's formula (Theorem 6.1.3) applied to the case $K = \{\alpha_i\}$ and $v_K = s_i$ we obtain

$$p_{s_i}^2 = p_{s_i}(s_i) \cdot p_{s_i} + \sum_{j \neq i} c_i^j \cdot p_{v_{\{\alpha_i, \alpha_j\}}} \tag{6.2.1}$$

where

$$c_i^j = (p_{s_i}(w_{\{\alpha_i, \alpha_j\}}) - p_{s_i}(s_i)) \cdot \frac{p_{s_i}(w_{\{\alpha_i, \alpha_j\}})}{p_{v_{\{\alpha_i, \alpha_j\}}}(w_{\{\alpha_i, \alpha_j\}})}. \tag{6.2.2}$$

More specifically, since Theorem 6.2.1 implies that $\sigma_{s_i}(s_i) = \alpha_i$, from (6.1.1) we conclude

$$p_{s_i}(s_i) = t. \tag{6.2.3}$$

We record the following.

Lemma 6.2.2. *In* (6.2.1), *if* s_i *and* s_j *commute, then* $c_i^j = 0$.

Proof. Since s_i and s_j commute, we have $w_{\{\alpha_i,\alpha_j\}} = s_i s_j$. Moreover from Theorem 6.2.1 we can compute

$$\sigma_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_i}(s_i s_j) = \alpha_i.$$

From (6.1.1) we get $p_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = t$. Then the equations (6.2.2), (6.2.3) imply $c_i^j = 0$ as desired. □

In the case when s_i and s_j do not commute, the Dynkin diagram corresponding to the subset $K = \{\alpha_i, \alpha_j\}$ is connected, so Giambelli's formula (Theorem 6.1.4) yields

$$p_{v_{\{\alpha_i,\alpha_j\}}} = \frac{1}{2} p_{s_i} p_{s_j}. \tag{6.2.4}$$

In this case, the coefficient appearing in (6.2.1) can be expressed in terms of the Cartan matrix.

Lemma 6.2.3. *In* (6.2.1), *if* s_i *and* s_j *do not commute, then*

$$c_i^j = -\langle \alpha_i, \alpha_j \rangle$$

where $\langle \alpha_i, \alpha_j \rangle$ *denotes the* Cartan integer*.*

Proof. From (6.2.2), (6.2.3), (6.2.4) we can compute

$$c_i^j = \frac{2(p_{s_i}(w_{\{\alpha_i,\alpha_j\}}) - t)}{p_{s_j}(w_{\{\alpha_i,\alpha_j\}})} \tag{6.2.5}$$

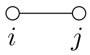
so it suffices to compute $p_{s_i}(w_{\{\alpha_i,\alpha_j\}})$ and $p_{s_j}(w_{\{\alpha_i,\alpha_j\}})$. In what follows we use the notation

$$a_{ij} := \langle \alpha_i, \alpha_j \rangle \ \ (i \neq j), \quad a := a_{ij} a_{ji}.$$

With this notation in place, note that by definition of the Cartan integers we have that the action of the simple reflections s_j on the simple roots α_j may be expressed as

$$s_j(\alpha_i) = \begin{cases} \alpha_i - a_{ij}\alpha_j & (i \neq j), \\ -\alpha_i & (i = j). \end{cases} \tag{6.2.6}$$

In order to prove the lemma, we consider each of the possible cases.

(i) In the case when the Dynkin diagram corresponding to $\{\alpha_i, \alpha_j\}$ is of the form  the order of $s_i s_j$ is 3 so we have

$$w_{\{\alpha_i,\alpha_j\}} = s_i s_j s_i = s_j s_i s_j.$$

Using Theorem 6.2.1 and (6.2.6) in this case we can compute that

$$\sigma_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_i}(s_i s_j s_i) = \alpha_i + s_i s_j(\alpha_i) = a\alpha_i - a_{ij}\alpha_j,$$
$$\sigma_{s_j}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_j}(s_j s_i s_j) = \alpha_j + s_j s_i(\alpha_j) = a\alpha_j - a_{ji}\alpha_i.$$

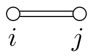
Then (6.1.1) implies

$$p_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = (a - a_{ij})t, \quad p_{s_j}(w_{\{\alpha_i,\alpha_j\}}) = (a - a_{ji})t.$$

Finally (6.2.5) yields

$$c_i^j = \frac{2(a - a_{ij} - 1)}{(a - a_{ji})} \tag{6.2.7}$$

and substituting $a = a_{ij}a_{ji}$, $a_{ij} = -1$ we obtain $c_i^j = -a_{ij}$ as desired.

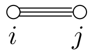
(ii) In the case  the order of $s_i s_j$ is 4 so we have

$$w_{\{\alpha_i,\alpha_j\}} = s_i s_j s_i s_j = s_j s_i s_j s_i.$$

Using the above together with Theorem 6.2.1 and (6.2.6) we may compute

$$\sigma_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_i}(s_is_js_is_j) = \alpha_i + s_is_j(\alpha_i) = a\alpha_i - a_{ij}\alpha_j,$$
$$\sigma_{s_j}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_j}(s_js_is_js_i) = \alpha_j + s_js_i(\alpha_j) = a\alpha_j - a_{ji}\alpha_i$$

which is the same as case (i) above. Thus (6.2.7) also holds in this case and since $a = a_{ij}a_{ji} = 2$ we obtain $c_i^j = -a_{ij}$ as required.

(iii) Finally, in the case 

the element s_is_j has order 6 and thus

$$w_{\{\alpha_i,\alpha_j\}} = s_is_js_is_js_is_j = s_js_is_js_is_js_i.$$

In this case we have $a = 3$ so Theorem 6.2.1 and (6.2.6) yield that

$$\sigma_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_i}(s_is_js_is_js_is_j) = \alpha_i + s_is_j(\alpha_i) + (s_is_j)^2(\alpha_i) = 4\alpha_i - 2a_{ij}\alpha_j,$$
$$\sigma_{s_j}(w_{\{\alpha_i,\alpha_j\}}) = \sigma_{s_j}(s_js_is_js_is_js_i) = \alpha_j + s_js_i(\alpha_j) + (s_js_i)^2(\alpha_j) = 4\alpha_j - 2a_{ji}\alpha_i.$$

Then from (6.1.1) we compute

$$p_{s_i}(w_{\{\alpha_i,\alpha_j\}}) = (4 - 2a_{ij})t, \quad p_{s_j}(w_{\{\alpha_i,\alpha_j\}}) = (4 - 2a_{ji})t.$$

Equation (6.2.5) then implies

$$c_i^j = \frac{2(3 - 2a_{ij})}{4 - 2a_{ji}}$$

and finally using that $a_{ij}a_{ji} = 3$ we get that $c_i^j = -a_{ij}$ as desired.

This completes the proof of the lemma. □

From the above considerations we obtain the following proposition.

Proposition 6.2.4. *In the equivariant cohomology ring $H_S^*(Pet)$ of the Peter-*

son variety, the following quadratic relations are satisfied:

$$\sum_{j=1}^{n} \langle \alpha_i, \alpha_j \rangle p_{s_i} p_{s_j} - 2t p_{s_i} = 0 \quad (1 \leq i \leq n).$$

Proof. If s_i and s_j commute then $\langle \alpha_i, \alpha_j \rangle = 0$, so by Lemma 6.2.2 the conclusion of Lemma 6.2.3 holds in this case. From this and (6.2.4) we see that (6.2.1) can be expressed as

$$p_{s_i}^2 = t \cdot p_{s_i} - \frac{1}{2} \sum_{j \neq i} \langle \alpha_i, \alpha_j \rangle p_{s_i} p_{s_j}.$$

Since $\langle \alpha_i, \alpha_i \rangle = 2$ for any i, the above equation can be re-written to be of the form given in the statement of the proposition. □

6.3 The main theorem in Chapter 6

Let $(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ be the Cartan matrix associated to a rank n semisimple Lie algebra $\mathfrak{g}$. Using the coefficients in the Cartan matrix, we define an ideal J in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, t]$ as follows:

$$J := \left(\sum_{j=1}^{n} \langle \alpha_i, \alpha_j \rangle x_i x_j - 2t x_i \mid 1 \leq i \leq n \right).$$

From Proposition 6.1.2 and Proposition 6.2.4 it then follows that the map sending x_i to p_{s_i} defines a surjective $\mathbb{C}[t]$-algebra homomorphism

$$\varphi : \mathbb{C}[x_1, \ldots, x_n, t]/J \twoheadrightarrow H_S^*(Pet). \tag{6.3.1}$$

Here $H^*(BS) = \mathbb{C}[t]$ and Pet denotes the Peterson variety associated to the Lie algebra $\mathfrak{g}$. Since $H^{odd}(Pet) = 0$, as a $H^*(BS)$-module we have $H_S^*(Pet) \cong$

$H^*(BS) \otimes H^*(Pet)$. Defining the ideal $\check{J}$ as

$$\check{J} = \left(\sum_{j=1}^{n} \langle \alpha_i, \alpha_j \rangle x_i x_j \mid 1 \leq i \leq n \right) \tag{6.3.2}$$

we then also have a surjective ring homomorphism

$$\check{\varphi} \colon \mathbb{C}[x_1, \ldots, x_n]/\check{J} \twoheadrightarrow H^*(Pet). \tag{6.3.3}$$

The following is the main theorem.

Theorem 6.3.1. *The maps φ and $\check{\varphi}$ of* (6.3.1) *and* (6.3.3) *are both isomorphisms.*

In order to prove Theorem 6.3.1 we use the theory of regular sequences. For reference we briefly recall the definition and a key property of regular sequences (cf. [11]).

Definition. Let R be a graded commutative algebra over $\mathbb{C}$ and let R_+ denote the positive-degree elements in R. Then a homogeneous sequence $\theta_1, \ldots, \theta_r \in R_+$ is a *regular sequence* if θ_k is a non-zero-divisor in the quotient ring $R/(\theta_1, \ldots, \theta_{k-1})$ for every $1 \leq k \leq r$. This is equivalent to saying that $\theta_1, \ldots, \theta_r$ is algebraically independent over $\mathbb{C}$ and R is a free $\mathbb{C}[\theta_1, \ldots, \theta_r]$-module.

It is a well-known fact (see for instance [36, p.35]) that a homogeneous sequence $\theta_1, \ldots, \theta_r \in R_+$ is a regular sequence if and only if

$$F(R/(\theta_1, \ldots, \theta_r), s) = F(R, s) \prod_{k=1}^{r} (1 - s^{\deg \theta_k}) \tag{6.3.4}$$

where $F(R/(\theta_1, \ldots, \theta_r), s)$ and $F(R, s)$ denote the Hilbert series of the graded rings $R/(\theta_1, \ldots, \theta_r)$ and R, respectively.

The following proposition gives a convenient characterization of regular sequences.

Proposition 6.3.2. *[11, Proposition 5.1] A sequence of positive-degree homogeneous elements $\theta_1, \dots, \theta_r$ in the polynomial ring $\mathbb{C}[z_1, \dots, z_r]$ is a regular sequence if and only if the solution set in $\mathbb{C}^r$ of the equations $\theta_1 = 0, \dots, \theta_r = 0$ consists only of the origin $\{0\}$.*

We can now prove our main theorem in Chapter 6.

Proof of Theorem 6.3.1. We first claim that if $\check{\varphi}$ is an isomorphism then it follows that φ is an isomorphism. To see this, suppose that $\check{\varphi}$ is an isomorphism. Then the sequence

$$\begin{aligned}\theta_i &:= \sum_{j=1}^{n} \langle \alpha_i, \alpha_j \rangle x_i x_j - 2t x_i \quad \text{for } 1 \leq i \leq n, \\ \theta_{n+1} &:= t\end{aligned}$$

in $\mathbb{C}[x_1, \dots, x_n, t]$ is regular, where $deg(x_i) = deg(t) = 2$. Indeed,

$$\begin{aligned}&F(\mathbb{C}[x_1, \dots, x_n, t]/(\theta_1, \dots, \theta_n, \theta_{n+1}), s) \\ =&F(\mathbb{C}[x_1, \dots, x_n]/\check{J}, s) \\ =&(1+s^2)^n \\ =&\frac{1}{(1-s^2)^{n+1}} \cdot (1-s^4)^n (1-s^2) \\ =&F(\mathbb{C}[x_1, \dots, x_n, t], s) \prod_{i=1}^{n+1} (1 - s^{\deg \theta_i})\end{aligned}$$

so this follows from (6.3.4). Note that a subsequence $\theta_1, \dots, \theta_n$ of a regular sequence $\theta_1, \dots, \theta_{n+1}$ is again a regular sequence, so from (6.3.4) and (6.1.3) we obtain

$$\begin{aligned} F(\mathbb{C}[x_1,\dots,x_n,t]/J,s) &= F(\mathbb{C}[x_1,\dots,x_n,t]/(\theta_1,\dots,\theta_n),s) \\ &= \frac{1}{(1-s^2)^{n+1}}\prod_{i=1}^{n}(1-s^{\deg\theta_i}) \\ &= \frac{(1+s^2)^n}{1-s^2} \\ &= F(H_S^*(Pet),s) \end{aligned}$$

from which it follows that φ is an isomorphism.

Thus it suffices to check that $\check{\varphi}$ is an isomorphism. We already know that $\check{\varphi}$ is surjective and from equation (6.1.3) we know that $F(H^*(Pet),s) = (1+s^2)^n$. Thus in order to show that $\check{\varphi}$ is injective it suffices to show that

$$F(\mathbb{C}[x_1,\dots,x_n]/\check{J},s) = (1+s^2)^n. \tag{6.3.5}$$

Note that by (6.3.4), the equality (6.3.5) is equivalent to the statement that $\sum_{j=1}^{n}\langle\alpha_i,\alpha_j\rangle x_ix_j \quad (1\le i\le n)$ is a regular sequence. Furthermore, by Proposition 6.3.2 , in order to prove (6.3.5) it in turn suffices to show that the zero set of the collection of quadratic equations

$$\sum_{j=1}^{n}\langle\alpha_i,\alpha_j\rangle x_ix_j = 0 \quad (1\le i\le n), \tag{6.3.6}$$

given by the generators of the ideal $\check{J}$ of (6.3.2) is $\{0\}$, i.e., the equations (6.3.6) have only the trivial solution.

Suppose in order to derive a contradiction that (6.3.6) has a non-trivial solution $(b_1,\dots,b_n)$. In particular, setting $I=\{i \mid b_i\neq 0\}$, we have $I\neq\emptyset$ and so since $b_i\neq 0$ for $i\in I$ we obtain from (6.3.6) that

$$\sum_{j\in I}\langle\alpha_i,\alpha_j\rangle b_j = 0 \quad (i\in I).$$

Since $(\langle\alpha_i, \alpha_j\rangle)_{i,j\in I}$ is a $|I| \times |I|$ square matrix which is again the Cartan matrix of a semisimple Lie algebra, it must be positive definite [21, section 2.4] and in particular non-singular. Thus the b_i must be 0 for $i \in I$, contradicting the assumption on I. Thus (6.3.6) has only the trivial solution, as desired. □

Remark. Theorem 6.3.1 is a generalization to all Lie types of the computation given in [11]. Indeed, the generators of the ideal given in [11] are the same as those given above, up to a scalar factor of 1/2.

Remark. In fact, Theorem 6.3.1 holds also with $\mathbb{Q}$ coefficients. Indeed, since both φ and $\check{\varphi}$ can be defined over $\mathbb{Z}$, if the maps become isomorphisms upon tensoring with $\mathbb{C}$ then they are also isomorphisms upon tensoring with $\mathbb{Q}$.

Bibliography

[1] H. Abe, M. Harada, T. Horiguchi, and M. Masuda, *The equivariant cohomology rings of regular nilpotent Hessenberg varieties in Lie type A: a research announcement*, Morfismos, vol. 18, no. 2 (2014), 51-65.

[2] H. Abe and T. Horiguchi, *The torus equivariant cohomology rings of Springer varieties*, arXiv:1404.1217.

[3] M. Atiyah and I. G. Macdonald, Commutative algebra. Addison-Wesley, Reading, Mass 1969.

[4] S. Billey, *Kostant polynomials and the cohomology of G/B*, Duke Math. J. 96 (1999), no. 1, 205-224.

[5] R. Bott and L. W. Tu, Differential Forms in Algebraic Topology, Springer-Verlag, New York, 1982.

[6] M. Brion and J. Carrell, *The equivariant cohomology ring of regular varieties*, Michigan Math. J. 52 (2004), no. 1, 189-203.

[7] J. Carrell, *Orbits of the Weyl group and a theorem of DeConcini and Procesi*, Compositio Math. 60 (1986), no. 1, 45-52.

[8] T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. of Math. (2) 100 (1974), 307-321.

[9] C. DeConcini and C. Procesi, *Symmetric functions, conjugacy classes and the flag variety*, Invent. Math. 64 (1981), no. 2, 203-219.

[10] E. Drellich, *Monk's rule and Giambelli's formula for Peterson varieties of all Lie types*, J. Algebraic Combin. 41 (2015), no. 2, 539-575.

[11] Y. Fukukawa, M. Harada and M. Masuda, *The equivariant cohomology rings of Peterson varieties*, J. Math. Soc. Japan 67 (2015), no. 3, 1147-1159.

[12] Y. Fukukawa, H. Ishida, M. Masuda, *The cohomology ring of the GKM graph of a flag manifold of classical type*, Kyoto J. Math. 54 (2014), no. 3, 653-677.

[13] W. Fulton, Young Tableaux: With Applications to Representation Theory and Geometry, London Math. Soc. Student Texts **35**, Cambridge Univ. Press, 1997.

[14] M. Harada, T. Horiguchi and M. Masuda, *The equivariant cohomology rings of Peterson varieties in all Lie types*, Canad. Math. Bull. 58 (2015), no. 1, 80-90.

[15] M. Harada and J. Tymoczko, *Poset pinball, GKM-compatible subspaces, and Hessenberg varieties*, arXiv:1007.2750.

[16] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe, The Lefschetz properties, Lecture Notes in Math. vol. 2080, Springer 2013.

[17] T. Horiguchi, *The* S^1*-equivariant cohomology rings of* $(n-k, k)$ *Springer varieties*, Osaka J. Math. 52 (2015), no. 4, 1051-1063.

[18] R. Hotta and T.A. Springer, *A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups*, Invent. Math. 41 (1977), no. 2, 113-127.

[19] W. Y. Hsiang, *On some fundamental theorems in cohomology theory of topological transformation groups*, Taita J. Math. 2 (1970), 61-87.

[20] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math. 9, Springer-Verlag, 1980.

[21] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge studies in advanced mathematics vol. 29, 1997.

[22] E. Insko, *Equivariant cohomology and local invariants of Hessenberg varieties*, Thesis (Ph.D.)-The University of Iowa. 2012. 122 pp.

[23] E. Insko and A. Yong, *Patch ideals and Peterson varieties*, Transform. Groups 17 (2012), no. 4, 1011-1036.

[24] A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. 119 (2003), no. 2, 221-260.

[25] B. Kostant, *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ*, Selecta Math. (N.S.) 2 (1996), no. 1, 43-91.

[26] H. Matsumura, Commutative ring theory, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989.

[27] M. Mimura and H. Toda, Topology of Lie Groups. I, II. Translated from the 1978 Japanese edition by the authors. Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991.

[28] A. I. Molev and B. E. Sagan, *A Littlewood-Richardson rule for factorial Schur functions*, Trans. Amer. Math. Soc. 351 (1999), no. 11, 4429-4443.

[29] A. Okounkov *Quantum immanants and higher Capelli identities*, Transform. Groups 1 (1996), no. 1-2, 99-126.

[30] M. Precup, *Affine pavings of Hessenberg varieties for semisimple groups*, Selecta Math. (N.S.) 19 (2013), no. 4, 903-922.

[31] D. Quillen, *The spectrum of an equivariant cohomology ring: I*, Ann. of Math. (2) 94 (1971), 549-572.

[32] K. Rietsch, *Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties*, J. Amer. Math. Soc. 16 (2003), no. 2, 363-392 (electronic).

[33] N. Spaltenstein, *The fixed point set of a unipotent transformation on the flag manifold*, Nederl. Akad. Wetensch. Proc. Ser. A 79=Indag. Math. 38 (1976), no. 5, 452-456.

[34] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. 36 (1976), 173-207.

[35] T. A. Springer, *A construction of representations of Weyl groups*, Invent. Math. 44 (1978), no. 3, 279-293.

[36] R. P. Stanley, Combinatorics and Commutative Algebra, Second Edition 1996, Birkhäuser, Boston.

[37] T. Tanisaki, *Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups*, Tôhoku Math. J. (2) 34 (1982), no. 4, 575-585.

[38] J. S. Tymoczko, *Linear conditions imposed on flag varieties*, Amer. J. Math. 128 (2006), no. 6, 1587-1604.

[39] J. S. Tymoczko, *Permutation actions on equivariant cohomology of flag varieties*, Toric topology, 365-384, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008.