The topology of toric origami manifolds with acyclic proper faces

(真の面が非輪状であるトーリック折り紙多様体のトポロジー)

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Abstract

Toric origami manifolds, introduced by A. Cannas da Silva, V. Guillemin and A. R. Pires, are generalizations of symplectic toric manifolds (or toric symplectic manifolds). Delzant's famous theorem tells us that there is a bijection between the set of compact connected symplectic toric manifolds and the set of Delzant polytopes. Cannas da Silva, V. Guillemin and A. R. Pires generalized this classification theorem to toric origami manifolds in [7]. They showed that there is a bijection between the set of toric origami manifolds and the set of origami templates. It is well known that many topological invariants, such as Betti numbers, cohomology rings and equivariant cohomology rings of symplectic toric manifolds, can be expressed in terms of the Delzant polytopes. Hence a natural question is how about toric orgami manifolds. When M is orientable and the orbit space of M/T is contractible, Holm and Pires study the topology of M in [12]. In this thesis we mainly study the topology of orientable toric manifolds such that every proper face of the orbit space is acyclic but the orbit space itself may be arbitrary. In the last part of this thesis, we make some observations about the non-orientable case.

Acknowledgement

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Contents

Chapter 1

Introduction

A symplectic toric manifold M is a compact connected symplectic manifold of dimension $2n$ with an effective Hamiltonian action of a compact n-dimensional torus T. Delzant's famous work tells us that there is a bejection between the set of compact connected symplectic toric manifolds and the set of nonsingular polytopes called Delzant polytopes. A Delzant polytope is the image of the moment map of a symplectic toric manifold. Delzant theorem connects the geometrical objects and the combinatorial objects, and many topological information of symplectic toric manifolds can be read from the corresponding combinatorial data, such as Betti numbers, cohomology rings, T-equivariant cohomology rings and so on. Recently, A. Cannas da Silva, V. Guillemin and A. R. Pires introduced a new geometrical object, toric origami manifolds in [7]. This new object is a generalization of symplectic toric manifolds. They also introduced combinatorial counterparts, origami templates, of toric origami manifolds, as Delzant polytopes are the combinatorial counterparts of symplectic toric manifolds. In [7] they generalized Delzant theorem to toric origami manifolds, i.e., they constructed a bijection between the set of toric origami manifolds and the set of origami templates through moment maps.

The construction of toric origami manifolds comes from folded symplectic manifolds, generalizations of symplectic manifolds [8]. A folded symplectic form on a $2n$-dimensional manifold M is a closed 2-form ω whose top power ω^n vanishes transversally on a subset Z and whose restriction to points in Z has maximal rank. The transversality condition implies that Z is either an empty set or a codimension-one submanifold of M, called the fold. If Z is an empty set, then M is a genuine symplectic manifold. Hence folded symplectic manifolds are generalization of symplectic manifolds. The maximality of the restriction of ω to Z implies the existence of a line field, the kernel of ω, on Z. If the line field is the vertical bundle of some principal S^1-fibration $Z \to X$, then ω is called an *origami form*. Similarly to the symplectic case, we can also define Hamiltonian actions and moment maps for origami manifolds. A toric origami manifold is a compact connected origami manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of a torus T. Roughly speaking, the combinatorial counterpart, an origami template, of a toric origami manifold is a collection of Delzant polytopes with some gluing conditions. A natural question is to describe the topological invariants such as Betti numbers, cohomology ring and T-equivariant cohomology ring of a toric origami manifold M in terms of the corresponding origami template; see [7] and [12].

In [12], Holm and Pires showed that if the folding hypersurface of M is coörientable, then the T-action on M is locally standard and the orbit space M/T is a manifold with corners. What is more, if we assume that each face of M/T is acyclic, then we can apply the general result of [15]. The Betti numbers can be expressed by the h-vector of the orbit space M/T, $H_T^*(M) \cong \mathbb{Z}[M/T]$, and $H^*(M) \cong \mathbb{Z}[M/T]/(\theta_1, ..., \theta_n)$, where $\mathbb{Z}[M/T]$ is the face ring of M/T, and $(\theta_1, ..., \theta_n)$ is the linear system of parameters given by characteristic functions on M/T. In [12], Holm and Pires discussed the topology of toric origami manifolds in a different way under the assumption that each face of M/T is acyclic.

In this thesis, we study the topology of toric origami manifolds in the case when each proper face of M/T is acyclic but M/T is arbitrary. Much of this work is based on the joint project with A. Ayzenberg, M. Masuda and S. Park [2].

This thesis is organized as follows. In Chapter 2 we first review the basic definition and properties of toric origami manifolds and origami templates. Then we state A. Cannas da Silva, V. Guillemin and A. R. Pires' classification theorem for toric origami manifolds.

In Chapter 3 we study the topology of orientable toric origami manifolds whose proper faces are acyclic. In Section 3.1 we give a formula to express the Betti numbers of M in terms of the face numbers of M/T and the first Betti number of M/T. In Section 3.2 we a give a formula to calculate the equivariant cohomology ring of M in terms of the face ring of M/T and the cohomology ring of M/T. In Section 3.3 we study the restriction map $\iota^*\colon H_T^{2j}(M) \to H^{2j}(M)$ by Serre spectral sequence. It is well-known that when M is a symplectic toric manifold, ι^* is surjective, but when M is a toric origami manifold, ι^* is not surjective in general.

Under the assumption that each proper face of M/T is acyclic, we show that except in degree 2, ι^* is surjective. In Section 3.4, we study the product structure of $H^*(M)$ by the ring homomorphism

$$\bar{\iota}^* \colon H_T^*(M)/(\pi^*(H^2(BT))) \to H^*(M)$$

induced from the restriction map

$$\iota^* \colon H_T^*(M) \to H^*(M).$$

In Section 3.5 we apply the arguments in Section 3.4 to 4 dimensional case. In Section 3.6, we make some observations on non-acyclic cases.

In Chapter 4, we study the topology of non-orientable toric origami manifolds. In Section 4.1 we study the cohomology groups of non-orientable toric origami manifolds with coörientable folding hypersurface under the assumption that each proper face of M/T is acyclic. We give a formula to express the cohomology groups of M in terms of the face numbers of M/T and the first Betti number of M/T. In Section 4.2, we study the topology of the simplest type of non-orientable toric origami manifolds with non-coörientable folding hypersurface. We express their cohomology groups by their corresponding orientable toric origami manifolds and T-invariant divisors corresponding to the folded facet.

Chapter 2

Toric origami manifolds

2.1 Folded symplectic manifolds

First, let us review the definition of symplectic manifolds.

Definition. A symplectic form on a smooth manifold M is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$, where nondegeneracy means that for any $q \in M$

$$\omega_q : \; T_q(M) \times T_q(M) \to \mathbb{R}$$

is nondegenerate. We call (M, ω) is a symplectic manifold.

If M is a symplectic manifold, then M is of even dimension $2n$, and ω^n never vanishes. Hence a symplectic manifold is always orientable. Moreover, if M is compact, then ω^n is a nonzero element in $H^{2n}(M)$, which implies that ω is nonzero in $H^2(M)$. Hence for a compact symplectic manifold M, $H^2(M) \neq 0$.

Example 2.1.1. On $\mathbb{R}^{2n}$, $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ is a symplectic form, where $(x_1, ..., x_n, y_1, ..., y_n)$ is the coordinate of $\mathbb{R}^{2n}$.

Example 2.1.2. Let M be a compact Riemann surface, then the area form on M is a symplectic form on M.

Theorem 2.1.1 (Darboux)**.** *Let (M, ω) be a symplectic manifold and p be a point in M. Then there is a coordinate chart $(U, x_1, x_2, ..., x_n, y_1, ..., y_n)$ centered at p such that on U*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

If we allow ω^n to vanish in some place, then we have the following definition of folded symplectic manifolds.

Definition. A folded symplectic form on a $2n$ dimensional manifold M is a closed 2-form ω satisfying the following two conditions:

1. ω^n vanishes transversally on a submanifold $i : Z \hookrightarrow M$;

2. $i^*\omega$ has maximal rank, i.e. $(i^*\omega)^{n-1}$ does not vanish.

We call (M, ω) a folded symplectic manifold and the submanifold Z is called the folding hypersurface or fold.

We know that $\omega^n : M \to \wedge^{2n}T^*M$ is a section of the line bundle $\wedge^{2n}T^*M$ over M. "Vanishes transversally" means that ω^n is transversal to the zero section. Hence, if $(\omega^n)^{-1}(0) \neq \emptyset$, then $Z = (\omega^n)^{-1}(0)$ is a

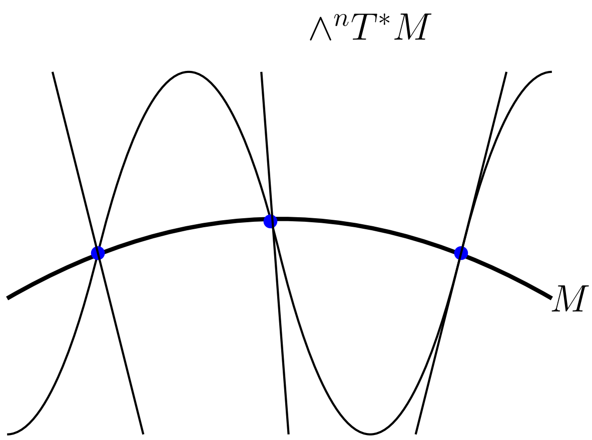


Figure 2.1: The blue parts denote the fold Z

codimension 1 submanifold of M. This is why we call Z "folding hypersurface". However, if $(\omega^n)^{-1}(0) = \emptyset$ then ω is a genuine symplectic form on M, so folded symplectic manifolds are generalization of symplectic manifolds.

For $p \in Z$,

$$(i^*\omega)_p : T_pZ \times T_pZ \to \mathbb{R}$$

is a bilinear 2-form. "Maximal rank" means that this 2-form has rank $2n-2$, so $(i^*\omega)_p$ has one-dimensional kernel.

Remark 2.1.1. The first condition does not imply the second condition; see [8].

Example 2.1.3. On $\mathbb{R}^{2n}$, $\omega = x_1 dx_1 \wedge dy_1 + \sum_{k=2}^n dx_k \wedge dy_k$ is a folded symplectic form, since

$$\omega^n = n! x_1 dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n$$

and it vanishes transversally on the hyperplane $Z = \{x_1 = 0\}$.

Example 2.1.4. For $n > 1$, $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ cannot be a symplectic manifold, since $H^2(S^{2n}) = 0$ for $n > 1$, but S^{2n} admits a folded symplectic form $\omega_0 = (\omega_{\mathbb{C}^n} \oplus 0)|_{S^{2n}}$, where

$$\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2} \sum_{k=1}^n dz_k \wedge d\overline{z}_k.$$

It is not difficult to check that ω_0^n vanishes transversally on

$$Z = S^{2n-1} \subset \mathbb{C}^n \oplus 0.$$

Example 2.1.5. The $\mathbb{Z}_2$ action on $\mathbb{C}^n \oplus \mathbb{R}$ given by

$$(z_1, ..., z_n, h) \mapsto (-z_1, ..., -z_n, -h),$$

induces a $\mathbb{Z}_2$ action, antipodal action, on S^{2n}. Then it is not difficult to see that ω_0 given in the last example is $\mathbb{Z}_2$-invariant, so it induces a folded symplectic form $\widetilde{\omega_0}$ on $\mathbb{R}P^{2n}$ with the fold $\mathbb{R}P^{2n-1} = \{[x_1 : y_1 : ..., x_n : y_n : 0]\}$, where $x_i + \sqrt{-1}y_i = z_i$, and $\sum_{i=1}^{n}(x_i^2 + y_i^2) = 1$.

The above example shows that a folded symplectic manifold can be non-orientable, so this is another difference between folded symplectic manifolds and symplectic manifolds.

Similarly to the case of symplectic manifolds, we also have Darboux's theorem for folded symplectic manifolds (see [7], [8]): If (M, ω) is a folded symplectic manifold with the fold Z, then for any $p \in Z$, there is a coordinate chart centered at p where the form ω is

$$x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + ... + dx_n \wedge dy_n.$$

2.2 Origami manifolds

Since $i^*\omega$ has maximal rank, for any $p \in Z$, $i^*\omega$ has one-dimensional kernel: the line field V on Z, called the null foliation. If we require that Z is a principal circle bundle over a compact space B and the tangent bundle TZ along the fiber direction coincides with the null foliation, then we say that (M, ω) is an origami manifold.

Definition. An origami manifold is a folded symplectic manifold (M, ω) whose null foliation is fibrating with oriented circle fibers, π, over a compact base B. The form ω is called an origami form and the null foliation, i.e., the vertical bundle of π is called the null fibration.

$$Z \downarrow^{\pi} B$$

Example 2.2.1. Let $(\mathbb{R}^{2n}, \omega)$ be the folded symplectic manifold discussed in Example 2.1.3, then it is not an origami manifold, since the fold Z is neither a circle bundle over some space nor compact.

Example 2.2.2. Let (S^{2n}, ω_0) be the folded symplectic manifold discussed in Example 2.1.4, then it is an origami manifold. In fact,

$$\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k = r_1 dr_1 \wedge d\theta_1 + ... r_n dr_n \wedge d\theta_n,$$

where $(r_1, \theta_1, ..., r_n, \theta_n)$ is the polar coordinate system of $\mathbb{C}^n$. The null foliation is the Hopf fibration since

$$\imath_{\frac{\partial}{\partial\theta_1}+...+\frac{\partial}{\partial\theta_n}} \omega_0 = -r_1 dr_1 - ... - r_n dr_n$$

vanishes on Z, so we have $S^1 \hookrightarrow S^{2n-1} \twoheadrightarrow \mathbb{C}P^{n-1}$.

Example 2.2.3. The folded symplectic manifold $(\mathbb{R}P^{2n}, \widetilde{\omega}_0)$ discussed in Example 2.1.5 also admits an origami structure. The null fibration is

$$S^1 \hookrightarrow \mathbb{R}P^{2n-1} \twoheadrightarrow \mathbb{C}P^{n-1}.$$

Definition. Two (oriented) origami manifolds (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ are symplectomorphic if there is a (orientation-preserving) diffeomorphism $\rho : M \to \widetilde{M}$ such that $\rho^* \widetilde{\omega} = \omega$

Definition. Let M be an origami manifold. We say that the folding hypersurface Z of M is coörientable, if each component of Z has an orientable neighborhood.

2.3 Toric origami manifolds

Definition. Let G be a Lie group. We say that the action G on an origami manifold (M, ω) is symplectomorphic, if there is a group homomorphism $\psi : G \to \mathrm{Diff}(M)$, such that $\psi(g)^*(\omega) = \omega$ for each $g \in G$. Moreover, we say this action is effective if $\mathrm{Ker}(\psi) = 1$.

Definition. The action of a Lie group G on an origami manifold (M, ω) is Hamiltonian if it admits a moment map, $\mu : M \to \mathfrak{g}^* = (\mathrm{Lie}(G))^*$, that is,

1. μ collects Hamiltonian functions, i.e., for each $X \in \mathfrak{g} := \mathrm{Lie}(G)$ $d\langle \mu, X\rangle = \imath_{X^\sharp}\omega,$, where $X^\sharp$ is the vector field generated by X;
2. μ is equivariant with respect to the given action of G on M and the coadjoint action of G on the dual vector space $\mathfrak{g}^*$, i.e., the following diagram is commutative.

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (Ad_g^*)} \\ M & \xrightarrow{\mu} & \mathfrak{g}^* \end{array}$$

Definition. A toric origami manifold (M, ω, T, μ) is a compact connected origami manifold (M, ω) equipped with an effective Hamiltonian action of a torus T with $\dim T = \frac{1}{2}\dim M$ and with a choice of a corresponding moment map μ.

Remark 2.3.1. When the fold $Z = \emptyset$, (M, ω, T, μ) is a symplectic toric manifold (or a toric symplectic manifold), so toric origami manifolds are generalization of symplectic toric manifolds.

Example 2.3.1. Let (S^4, ω_0) be the origami manifold discussed in Example 2.2.2. Then $T = (S^1)^2$ acts on $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$ by

$$(t_1, t_2) \cdot (z_1, z_2, r) = (t_1 z_1, t_2 z_2, r)$$

with moment map

$$\mu(z_1, z_2, r) = (\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2}).$$

Thus (S^4, ω_0, T, μ) is a toric origami manifold.

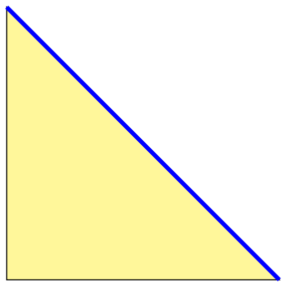


Figure 2.2: The image $\mu(Z)$ of the folding hypersurface (the equator) is the hypotenuse

2.4 Delzant Theorem

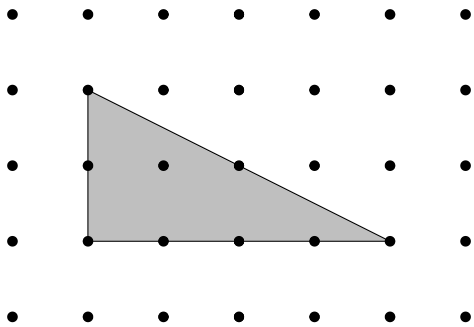
A famous theorem of Delzant [10] tells us that there is a one-to-one correspondence between the set of compact toric symplectic manifolds and the set of Delzant polytopes. Before discussing about Delzant's result, first let us review the definition of Delzant polytopes.

Definition. A polytope of dimension n in $\mathbb{R}^n$ is *Delzant* if:

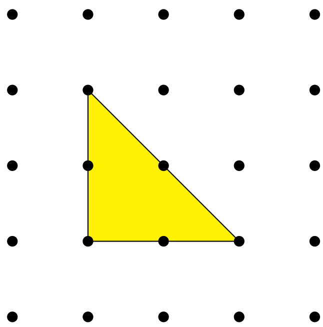
- it is simple, i.e., there are n edges meeting at each vertex;

- it is rational, i.e., each edge meeting at vertex p is of the form $p + tu_i \quad t \geq 0$, where $u_i \in \mathbb{Z}^n$;
- it is smooth, i.e., for each vertex, these $u_1, ..., u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Example 2.4.1. In the following pictures, the right polytope is a Delzant polytope, but the left one is not.



rational but not smooth



Delzant

Theorem 2.4.1 (Delzant [10]). *There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{compact toric} \\ \textit{symplectic manifolds} \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \{\textit{Delzant polytopes}\}$$

$$(M, \omega, T^n, \mu) \longmapsto \mu(M).$$

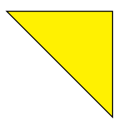
Example 2.4.2. Let ω be the Fubini-Study form on $\mathbb{C}P^2$. Then the T^2-action on $\mathbb{C}P^2$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

has moment map

$$\mu[z_0 : z_1 : z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right).$$

$$(\mathbb{C}P^2, \omega, T^2, \mu) \longrightarrow$$



2.5 Origami templates

Recently, Cannas da Silva, Guilemin and Pires generlized Delzant Theorem to toric origami manifolds. Before stating their classification theorem, we need to give the definition of origami templates. Roughly speaking, an origami template is a collection of Delzant polytopes which satisfy some compatibility conditions.

Definition. An n-dimensional *origami template* is a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a (nonempty) finite collection of n-dimensional *Delzant* polytopes and $\mathcal{F}$ is a collection of facets and pairs of facets of polytopes in $\mathcal{P}$ satisfying the following properties:

- for each pair $\{F_1, F_2\} \in \mathcal{F}$, the corresponding polytopes in $\mathcal{P}$ agree near those facets;
- if a facet F occurs in $\mathcal{F}$, either by itself or as a member of a pair, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;
- the topological space constructed from the disjoint union $\sqcup \Delta_i$, $\Delta_i \in \mathcal{P}$, by identifying facet pairs in $\mathcal{F}$ is connected.

If we denote the elements in $\mathcal{P}$ by vertexes, and the elements in $\mathcal{F}$ by edges, then we can give an equivalent definition of origami templates by graphs.

Let $\mathcal{D}_n$ denote the set of all Delzant polytopes in $\mathbb{R}^n$ (w.r.t. a given lattice), $\mathcal{F}_n$ — the set of all their facets and G a connected graph (loops and multiple edges are allowed) with the vertex set V and the edge set E.

Definition. An n-dimensional *origami template* consists of a connected graph G, called the template graph, and a pair of maps $\Psi_V \colon V \to \mathcal{D}_n$ and $\Psi_E \colon E \to \mathcal{F}_n$ such that:

1. If $e \in E$ is an edge of G with endpoints $v_1, v_2 \in V$, then $\Psi_E(e)$ is a facet of both polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$, and these polytopes coincide near $\Psi_E(e)$ (this means that there exists an open neighborhood U of $\Psi_E(e)$ in $\mathbb{R}^n$ such that $U \cap \Psi_V(v_1) = U \cap \Psi_V(v_2)$);

2. If $e_1, e_2 \in E$ are two edges of G adjacent to $v \in V$, then $\Psi_E(e_1)$ and $\Psi_E(e_2)$ are disjoint facets of $\Psi(v)$.

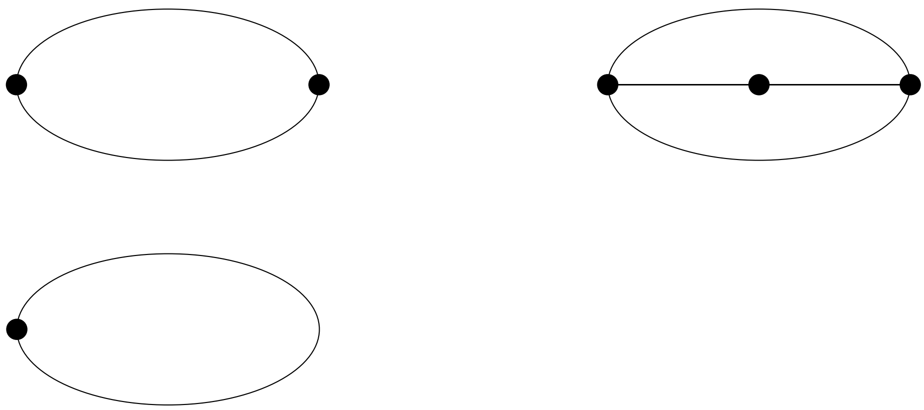
The facets of the form $\Psi_E(e)$ for $e \in E$ are called the fold facets of the origami template.

Denote by $|(G, \Psi_V, \Psi_E)|$ the topological space, constructed from the disjoint union $\bigsqcup_{v \in V} \Psi_V(v)$ by identifying facets $\Psi_E(e) \subset \Psi_V(v_1)$ and $\Psi_E(e) \subset \Psi_V(v_2)$ for any edge $e \in E$ with endpoints v_1, v_2.

Definition. An origami template (G, Ψ_V, Ψ_E) is called coörientable if the graph G has no loops, i.e., all eges have different endpoints.

Definition. Let $G = (V, E)$ be a graph, where V and E are the sets of vertexes and edges respectively. We say that $G = (V, E)$ is 2-colorable, if there is a function $f : V \to \{0, 1\}$ such that $f(i) \neq f(j)$ whenever $\{i, j\} \in E$.

Example 2.5.1. In the following, the first graph is 2-colorable, but the other two are not.

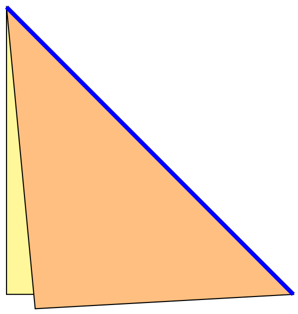


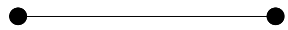
Definition. An origami template (G, Ψ_V, Ψ_E) is called orientable if the template graph G is 2-colorable.

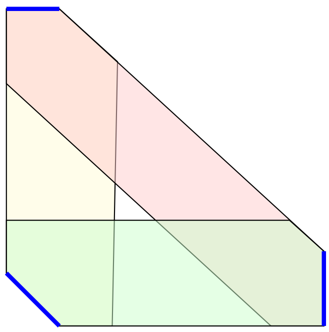
It is not difficult to see that if (G, Ψ_V, Ψ_E) is orientable, then so is the resulting space $|(G, \Psi_V, \Psi_E)|$.

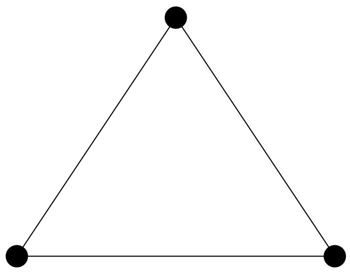
If a graph is 2-colorable, then it has no loops. Hence an orientable origami template is always coörientable, but the converse is not true as is shown in the following example.

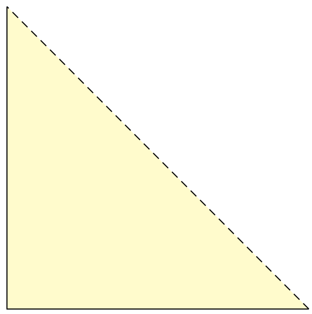
Example 2.5.2. In the following pictures, we draw the origami templates on the left side and their associated template graphs on the right side. We use the blue line and dashed line to denote the fold facets. The first and the second origami templates are coörientable but the third one is not. Although the second one is coörientable, it is not orientable, since its template graph is not 2-colorable.

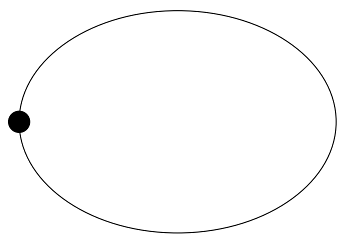












2.6 The classification of toric origami manifolds

After the preliminary in the last section, we can talk about Cannas da Silva, Guillemin and Pires' classification theorem for toric origami manifolds.

Theorem 2.6.1 ([7]). *Toric origami manifolds are classified by origami templates up to equivariant symplectomorphism preserving the moment maps. More specifically, at the level of symplectomorphism classes (on the left hand side), there is a one-to-one correpondence*

$$\{2n\text{-}diml\ toric\ origami\ manifolds\} \longrightarrow \{n\text{-}diml\ origami\ templates\}$$

$$(M^{2n}, \omega, T^n, \mu) \longmapsto \mu(M).$$

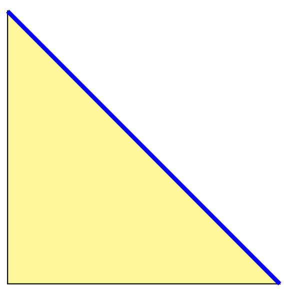
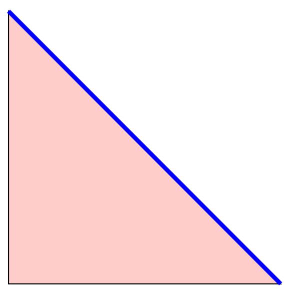
Moreover, oriented toric origami manifolds correspond to oriented origami templates and coöriented toric origami manifolds correspond to coöriented origami templates.

Example 2.6.1. Consider the toric origami manifold discussed in Example 2.3.1.

When $0 \leq r \leq 1$, the image of the moment map

$$\mu(z_1, z_2, r) = (\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2})$$

is a triangle and we color it by yellow and its hypotenuse by blue.

When $-1 \leq r \leq 0$, the image of μ is also a triangle and we color it by red and its hypotenuse also by blue. Now we have two copies of triangles with the smae hypotenuse, the image of the equator under the map μ. If we glue these two triangles along their hypotenuses, then we can obtain an origami template, and the resulting space of this origami template is homeomorphic to the orbit space S^4/T^2 as a manifold with corner.

Remark 2.6.1. Since $\mu : M \to \mathrm{Lie}(T)^*$ is equivariant, it induces a map $M/T \to \mu(M)$. We know that, when (M, ω, T^n, μ) is a symplectic toric manifold, then the orbit space M/T^n is homeomorphic to $\mu(M)$. However, this is not true in general when M is a toric origami manifold. For instance, consider the folded symplectic manifold $(T^2, \omega = \sin\theta_1 d\theta_1 \wedge d\theta_2)$, where the coordinates on the torus are $\theta_1, \theta_2 \in [0, 2\pi]$. The circle action on θ_2 coordinate is the usual rotation and $\mu = -\cos\theta_1$, so (T^2, ω, S^1, μ) is a toric origami manifold. It is not difficult to see that the image of μ is an interval while the orbit space T^2/S^1 is homeomorphic to S^1. However, the orbit space M/T^n is always homeomorphic to the resulting space of the associated origami template as a manifold with corners; see [7].

Chapter 3

On the topology of toric origami manifolds

In this chapter we will discuss the topological properties of toric origami manifolds. It is well-known that the cohomology ring and equivariant cohomology ring of a symplectic toric manifolds can be expressed in terms of the corresponding *Delzant polytope*, so a natural question is to describe the topological invariants of a toric origami manifold in terms of corresponding origami template. In general, toric origami manifolds are not simply connected, so it is more difficult to calculate their topological invariants than the case of symplectic toric manifolds. For the case that M is orientable and the folding hypersurface Z is connected, $H^*(M)$ was studied by Cannas da silva, Guillemin and Pires in [7]. Later, Holm and Pires in [12] studied the case that M is orientable and each face of M/T is acyclic. In this chapter we will discuss the topology of orientable toric origami manifolds for the case that each proper face of M/T is acyclic but M/T can be arbitary.

3.1 Betti numbers of toric origami manifolds with acyclic proper faces

Let M be an orientable toric origami manifold of dimension $2n$ with a fold Z. Let F be the corresponding folded facet in the origami template of M and let B be the symplectic toric manifold corresponding to F. The normal line bundle of Z to M is trivial so that an invariant closed tubular neighborhood of Z in M can be identified with $Z \times [-1, 1]$. We set

$$\tilde{M} := M - \mathrm{Int}(Z \times [-1, 1]).$$

This has two boundary components which are copies of Z. We close $\tilde{M}$ by gluing two copies of the disk bundle associated to the principal S^1-bundle $Z \to B$ along their boundaries. The resulting closed manifold (possibly disconnected), denoted M', is again a toric origami manifold by [7] and the graph associated to M' is the graph associated to M with the edge corresponding to the folded facet F removed.

Let G be the graph associated to the origami template of M and let $b_1(G)$ be its first Betti number. We assume that $b_1(G) \geq 1$. A folded facet in the origami template of M corresponds to an edge of G. We choose an edge e in a (non-trivial) cycle of G and let F, Z and B be respectively the folded facet, the fold and the symplectic toric manifold corresponding to the edge e. Then M' is connected and since the graph G' associated to M' is the graph G with the edge e removed, we have $b_1(G') = b_1(G) - 1$.

Two copies of B lie in M' as closed submanifolds, denoted B_+ and

B_-. Let N_+ (resp. N_-) be an invariant closed tubular neighborhood of B_+ (resp. B_-) and Z_+ (resp. Z_-) be the boundary of N_+ (resp. N_-). Note that $M' - \mathrm{Int}(N_+ \cup N_-)$ can naturally be identified with $\tilde{M}$, so that

$$\tilde{M} = M' - \mathrm{Int}(N_+ \cup N_-) = M - \mathrm{Int}(Z \times [-1, 1])$$

and

$$M' = \tilde{M} \cup (N_+ \cup N_-), \qquad \tilde{M} \cap (N_+ \cup N_-) = Z_+ \cup Z_-, \quad (3.1.1)$$

$$M = \tilde{M} \cup (Z \times [-1, 1]), \qquad \tilde{M} \cap (Z \times [-1, 1]) = Z_+ \cup Z_-. \quad (3.1.2)$$

Remark 3.1.1. It follows from (3.1.1) and (3.1.2) that

$$\chi(M') = \chi(\tilde{M}) + 2\chi(B), \quad \chi(M) = \chi(\tilde{M})$$

and hence $\chi(M') = \chi(M) + 2\chi(B)$. Note that this formula holds without the acyclicity assumption (made later) on proper faces of M/T.

We shall investigate relations among the Betti numbers of $M, M', \tilde{M}, Z$ and B. The spaces $\tilde{M}$ and Z are auxiliary ones and our aim is to find relations among the Betti numbers of M, M' and B. In the following, all cohomology groups and Betti numbers are taken with $\mathbb{Z}$-coefficients unless otherwise stated but the reader will find that the same argument works over any field.

Lemma 3.1.1. *The Betti numbers of Z and B have the relation*

$$b_{2i}(Z) - b_{2i-1}(Z) = b_{2i}(B) - b_{2i-2}(B)$$

for any i.

Proof. Since $\pi\colon Z \to B$ is a principal S^1-bundle and $H^{odd}(B) = 0$, the

Gysin exact sequence for the principal S^1-bundle splits into a short exact

$$0 \to H^{2i-1}(Z) \to H^{2i-2}(B) \to H^{2i}(B) \xrightarrow{\pi^*} H^{2i}(Z) \to 0 \quad \text{for any } i \tag{3.1.3}$$

and this implies the lemma. □

Lemma 3.1.2. *The Betti numbers of $\tilde{M}$, M', and B have the relation*

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M') - b_{2i-1}(M') - 2b_{2i-2}(B)$$

for any i.

Proof. We consider the Mayer-Vietoris exact sequence in cohomology for the triple $(M', \tilde{M}, N_+ \cup N_-)$:

$$\begin{aligned}
&\to H^{2i-2}(M') \to H^{2i-2}(\tilde{M}) \oplus H^{2i-2}(N_+ \cup N_-) \to H^{2i-2}(Z_+ \cup Z_-) \\
&\xrightarrow{\delta^{2i-2}} H^{2i-1}(M') \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(N_+ \cup N_-) \to H^{2i-1}(Z_+ \cup Z_-) \\
&\xrightarrow{\delta^{2i-1}} H^{2i}(M') \to H^{2i}(\tilde{M}) \oplus H^{2i}(N_+ \cup N_-) \to H^{2i}(Z_+ \cup Z_-) \\
&\xrightarrow{\delta^{2i}} H^{2i+1}(M') \to
\end{aligned}$$

Since the inclusions $B = B_\pm \mapsto N_\pm$ are homotopy equivalences and $Z_\pm = Z$, the restriction homomorphism $H^q(N_+ \cup N_-) \to H^q(Z_+ \cup Z_-)$ above can be replaced by $\pi^* \oplus \pi^* \colon H^q(B) \oplus H^q(B) \to H^q(Z) \oplus H^q(Z)$ which is surjective for even q from the sequence (3.1.3). Therefore, δ^{2i-2} and δ^{2i} in the exact sequence above are trivial. It follows that

$$\begin{aligned}
b_{2i-1}(M') - b_{2i-1}(\tilde{M}) - 2b_{2i-1}(B) + 2b_{2i-1}(Z) \\
-b_{2i}(M') + b_{2i}(\tilde{M}) + 2b_{2i}(B) - 2b_{2i}(Z) = 0.
\end{aligned}$$

Here $b_{2i-1}(B) = 0$ because B is a symplectic toric manifold, and $2b_{2i-1}(Z) + 2b_{2i}(B) - 2b_{2i}(Z) = 2b_{2i-2}(B)$ by Lemma 3.1.1. Using these identities, the identity above reduces to the identity in the lemma. □

Next we consider the Mayer-Vietoris exact sequence in cohomology for the triple $(M, \tilde{M}, Z \times [-1,1])$:

$$\to H^{2i-2}(M) \to H^{2i-2}(\tilde{M}) \oplus H^{2i-2}(Z \times [-1,1]) \to H^{2i-2}(Z_+ \cup Z_-)$$
$$\to H^{2i-1}(M) \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(Z \times [-1,1]) \to H^{2i-1}(Z_+ \cup Z_-)$$
$$\to H^{2i}(M) \quad \to H^{2i}(\tilde{M}) \oplus H^{2i}(Z \times [-1,1]) \qquad \to H^{2i}(Z_+ \cup Z_-) \to$$

We make the following assumption:

$(*)$ The restriction map $H^{2j}(\tilde{M}) \oplus H^{2j}(Z \times [-1,1]) \to H^{2j}(Z_+ \cup Z_-)$ in the Mayer-Vietoris sequence above is surjective for $j \geq 1$.

Note that the restriction map above is not surjective when $j = 0$ because the image is the diagonal copy of $H^0(Z)$ in this case and we will see in Lemma 3.1.5 below that the assumption $(*)$ is satisfied when every proper face of M/T is acyclic.

Lemma 3.1.3. *Suppose that the assumption* $(*)$ *is satisfied. Then*
$b_2(\tilde{M}) - b_1(\tilde{M}) = b_2(M) - b_1(M) + b_2(B)$,
$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) - b_{2i-1}(M) + b_{2i}(B) - b_{2i-2}(B)$ *for* $i \geq 2$.

Proof. By the assumption $(*)$, the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, Z \times [-1,1])$ splits into short exact sequences:

$$0 \to H^0(M) \to H^0(\tilde{M}) \oplus H^0(Z \times [-1,1]) \to H^0(Z_+ \cup Z_-)$$
$$\to H^1(M) \to H^1(\tilde{M}) \oplus H^1(Z \times [-1,1]) \to H^1(Z_+ \cup Z_-)$$
$$\to H^2(M) \to H^2(\tilde{M}) \oplus H^2(Z \times [-1,1]) \to H^2(Z_+ \cup Z_-) \to 0$$

and for $i \geq 2$

$$0 \to H^{2i-1}(M) \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(Z \times [-1,1]) \to H^{2i-1}(Z_+ \cup Z_-)$$
$$\to H^{2i}(M) \quad \to H^{2i}(\tilde{M}) \oplus H^{2i}(Z \times [-1,1]) \qquad \to H^{2i}(Z_+ \cup Z_-) \to 0.$$

The former short exact sequence above yields

$$b_2(\tilde{M}) - b_1(\tilde{M}) = b_2(M) - b_1(M) + b_2(Z) - b_1(Z) + 1$$

while the latter above yields

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) - b_{2i-1}(M) + b_{2i}(Z) - b_{2i-1}(Z) \quad \text{for } i \geq 2.$$

Here $b_{2i}(Z) - b_{2i-1}(Z) = b_{2i}(B) - b_{2i-2}(B)$ for any i by Lemma 3.1.1, so our lemma follows. □

Lemma 3.1.4. *Suppose that the assumption* $(*)$ *is satisfied and* $n \geq 2$. *Then*

$$b_1(M') = b_1(M) - 1, \quad b_2(M') = b_2(M) + b_2(B) + 1,$$
$$b_{2i+1}(M') = b_{2i+1}(M) \quad \textit{for } 1 \leq i \leq n-2.$$

Proof. It follows from Lemma 3.1.2 and Lemma 3.1.3 that

$$b_{2i}(M') - b_{2i-1}(M') = b_{2i}(M) - b_{2i-1}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \text{for } i \geq 2. \tag{3.1.4}$$

Take $i = n$ in (3.1.4) and use Poincaré duality. Then we obtain

$$b_0(M') - b_1(M') = b_0(M) - b_1(M) + b_0(B)$$

which reduces to the first identity in the lemma. This together with the first identity in Lemma 3.1.3 implies the second identity in the lemma.

Similarly, take $i = n - 1 (\geq 2)$ in (3.1.4) and use Poincaré duality. Then we obtain

$$b_2(M') - b_3(M') = b_2(M) - b_3(M) + b_0(B) + b_2(B).$$

This together with the second identity in the lemma implies $b_3(M') = b_3(M)$.

Take i to be $n-i$ in (3.1.4) (so $2 \le i \le n-2$) and use Poincaré duality. Then we obtain

$$b_{2i}(M') - b_{2i+1}(M') = b_{2i}(M) - b_{2i+1}(M) + b_{2i-2}(B) + b_{2i}(B).$$

This together with (3.1.4) implies

$$b_{2i+1}(M') - b_{2i-1}(M') = b_{2i+1}(M) - b_{2i-1}(M) \quad \text{for } 2 \le i \le n-2.$$

Since we know $b_3(M') = b_3(M)$, this implies the last identity in the lemma. $\square$

The following is a key lemma.

Lemma 3.1.5. *Suppose that every proper face of M/T is acyclic. Then the homomorphism $H^{2j}(\tilde{M}) \to H^{2j}(Z_+ \cup Z_-)$ induced from the inclusion is surjective for $j \ge 1$, in particular, the assumption $(*)$ is satisfied.*

Proof. Since $B_+ \cup B_-$ is a deformation retract of $N_+ \cup N_-$, the following diagram is commutative:

$$\begin{array}{ccc} H^{2j}(M') & \longrightarrow & H^{2j}(B_+ \cup B_-) \\ \big\downarrow & & \big\downarrow \pi_\pm^* \\ H^{2j}(\tilde{M}) & \longrightarrow & H^{2j}(Z_+ \cup Z_-) \end{array}$$

where $\pi_\pm \colon Z_+ \cup Z_- \to B_+ \cup B_-$ is the projection and the other homomorphisms are induced from the inclusions. By (3.1.3) $\pi_\pm^*$ is surjective, so it suffices to show that the homomorphism $H^{2j}(M') \to H^{2j}(B_+ \cup B_-)$ is surjective for $j \ge 1$.

The inverse image of a codimension j face of M'/T by the quotient map $M' \to M'/T$ is a codimension $2j$ closed orientable submanifold of M' and defines an element of $H_{2n-2j}(M')$ so that its Poincaré dual yields an element of $H^{2j}(M')$. The same is true for $B = B_+$ or B_-.

Note that $H^{2j}(B)$ is additively generated by τ_K's where K runs over all codimension j faces of $F = B/T$.

Set $F_\pm = B_\pm/T$, which are copies of the folded facet $F = B/T$. Let K_+ be any codimension j face of F_+. Then there is a codimension j face L of M'/T such that $K_+ = L \cap F_+$. We note that $L \cap F_- = \emptyset$. Indeed, if $L \cap F_- \neq \emptyset$, then $L \cap F_-$ must be a codimension j face of F_-, say H_-. If H_- is the copy K_- of K_+, then L will create a codimension j non-acyclic face of M/T which contradicts the acyclicity assumption on proper faces of M/T. Therefore, $H_- \neq K_-$. However, $F_\pm$ are respectively facets of some Delzant polytopes, say $P_\pm$, and the neighborhood of F_+ in P_+ is same as that of F_- in P_- by definition of an origami template (although P_+ and P_- may not be isomorphic). Let $\bar{H}$ and $\bar{K}$ be the codimension j faces of P_- such that $\bar{H} \cap F = H_-$ and $\bar{K} \cap F = K_-$. Since $H_- \neq K_-$, the normal cones of $\bar{H}$ and $\bar{K}$ are different. However, these normal cones must agree with that of L because $L \cap F_+ = K_+$ and $L \cap F_- = H_-$ and the neighborhood of F_+ in P_+ is same as that of F_- in P_-. This is a contradiction.

The codimension j face L of M'/T associates an element $\tau_L \in H^{2j}(M')$. Since $L \cap F_+ = K_+$ and $L \cap F_- = \emptyset$, the restriction of τ_L to $H^{2j}(B_+ \cup B_-) = H^{2j}(B_+) \oplus H^{2j}(B_-)$ is $(\tau_{K_+}, 0)$, where $\tau_{K_+} \in H^{2j}(B_+)$ is associated to K_+. Since $H^{2j}(B_+)$ is additively generated by τ_{K_+}'s where K_+ runs over all codimension j faces of F_+, for each element $(x_+, 0) \in H^{2j}(B_+) \oplus H^{2j}(B_-) = H^{2j}(B \cup B_-)$, there is an element $y_+ \in H^{2j}(M')$ whose restriction image is $(x_+, 0)$. The same is true for each element $(0, x_-) \in H^{2j}(B_+) \oplus H^{2j}(B_-)$. This implies the lemma. $\square$

Finally we obtain the following.

Theorem 3.1.1. *Let M be an orientable toric origami manifold of dimension $2n$ ($n \geq 2$) such that every proper face of M/T is acyclic. Then*

$$b_{2i+1}(M) = 0 \quad \text{for } 1 \leq i \leq n-2. \tag{3.1.5}$$

Moreover, if M' and B are as above, then

$$b_1(M') = b_1(M) - 1 \text{ (hence } b_{2n-1}(M') = b_{2n-1}(M) - 1 \text{ by Poincaré duality}),$$
$$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \text{for } 1 \leq i \leq n-1. \tag{3.1.6}$$

Finally, $H^*(M)$ is torsion free.

Proof. We have $b_1(M') = b_1(M) - 1$ by Lemma 3.1.4. Therefore, if $b_1(M) = 1$, then $b_1(M') = 0$, that is, the graph associated to M' is acyclic and hence $b_{odd}(M') = 0$ by [12] (or [15]). This together with Lemma 3.1.4 shows that $b_{2i+1}(M) = 0$ for $1 \leq i \leq n-2$ when $b_1(M) = 1$. If $b_1(M) = 2$, then $b_1(M') = 1$ so that $b_{2i+1}(M') = 0$ for $1 \leq i \leq n-2$ by the observation just made and hence $b_{2i+1}(M) = 0$ for $1 \leq i \leq n-2$ by Lemma 3.1.4. Repeating this argument, we see (3.1.5).

The relations in (3.1.6) follows from Lemma 3.1.4 and (3.1.4) together with the fact $b_{2i+1}(M) = 0$ for $1 \leq i \leq n-2$.

As we remarked before Lemma 3.1.1, the arguments developed in this section work with any field coefficients, in particular with $\mathbb{Z}/p$-coefficients for any prime p, and hence (3.1.5) and (3.1.6) hold for Betti numbers with $\mathbb{Z}/p$-coefficients, so the Betti numbers of M with $\mathbb{Z}$-coefficients agree with the Betti numbers of M with $\mathbb{Z}/p$-coefficients for any prime p. This implies that $H^*(M)$ has no torsion. □

As for $H^1(M)$, we have a clear geometrical picture.

Proposition 3.1.1. *Let M be an orientable toric origami manifold of dimension $2n$ ($n \geq 2$) such that every proper face of M/T is acyclic. Let $Z_1, \dots, Z_{b_1}$ be folds in M such that the graph associated to the origami template of M with the b_1 edges corresponding to $Z_1, \dots, Z_{b_1}$ removed is a tree. Then $Z_1, \dots, Z_{b_1}$ freely generate $H_{2n-1}(M)$, equivalently, their Poincaré duals $z_1, \dots, z_{b_1}$ freely generate $H^1(M)$. Furthermore, all the products generated by $z_1, \dots, z_{b_1}$ are trivial because $Z_1, \dots, Z_{b_1}$ are disjoint and the normal bundle of Z_j is trivial for each j.*

Proof. We will prove the proposition by induction on b_1. When $b_1 = 0$, the proposition is trivial; so we may assume $b_1 \geq 1$. Let Z and M' be as before. Since $b_1(M') = b_1 - 1$, there are folds $Z_1, \dots, Z_{b_1-1}$ in M' such that $Z_1, \dots, Z_{b_1-1}$ freely generate $H_{2n-1}(M')$ by induction assumption. The folds $Z_1, \dots, Z_{b_1-1}$ are naturally embedded in M and we will prove that these folds together with Z freely generate $H_{2n-1}(M)$.

We consider the Mayer-Vietoris exact sequence for a triple $(M, \tilde{M}, Z \times [-1,1])$:

$$0 \to H_{2n}(M) \xrightarrow{\partial_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{2*}} H_{2n-1}(\tilde{M}) \oplus H_{2n-1}(Z \times [-1,1])$$
$$\to H_{2n-1}(M) \xrightarrow{\partial_*} H_{2n-2}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{2*}} H_{2n-2}(\tilde{M}) \oplus H_{2n-2}(Z \times [-1,1])$$

where ι_1 and ι_2 are the inclusions. Since $\iota_1^*\colon H^{2n-2}(\tilde{M}) \to H^{2n-2}(Z_+ \cup Z_-)$ is surjective by Lemma 3.1.5, $\iota_{1*}\colon H_{2n-2}(Z_+ \cup Z_-) \to H_{2n-2}(\tilde{M})$ is injective when tensored with $\mathbb{Q}$. However, $H^*(Z)$ has no torsion in odd degrees because $H^{2i-1}(Z)$ is a subgroup of $H^{2i-2}(B)$ for any i by (3.1.3) and $H^*(B)$ is torsion free. Therefore, $H_*(Z)$ has no torsion in even degrees. Therefore, $\iota_{1*}\colon H_{2n-2}(Z_+ \cup Z_-) \to H_{2n-2}(\tilde{M})$ is injective without tensoring with $\mathbb{Q}$ and hence the above exact sequence reduces to this short exact sequence:

$$0 \to H_{2n}(M) \xrightarrow{\partial_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{2*}} H_{2n-1}(\tilde{M}) \oplus H_{2n-1}(Z \times [-1,1])$$
$$\to H_{2n-1}(M) \to 0.$$

Noting $\partial_*([M]) = [Z_+] - [Z_-]$ and $\iota_{2*}([Z_\pm]) = [Z]$, one sees that the above short exact sequence implies an isomorphism

$$\iota_*\colon H_{2n-1}(\tilde{M}) \cong H_{2n-1}(M) \tag{3.1.7}$$

where $\iota\colon \tilde{M} \to M$ is the inclusion map.

We consider the Mayer-Vietoris exact sequence for a triple $(M', \tilde{M}, N_+ \cup N_-)$:

$$0 \to H_{2n}(M') \xrightarrow{\partial'_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{3*}} H_{2n-1}(\tilde{M}) \oplus H_{2n-1}(N_+ \cup N_-)$$
$$\to H_{2n-1}(M') \xrightarrow{\partial'_*} H_{2n-2}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*} \oplus \iota_{3*}} H_{2n-2}(\tilde{M}) \oplus H_{2n-2}(N_+ \cup N_-)$$

where ι_3 is the inclusion map of the unit sphere bundle in $N_+ \cup N_-$. Note that $H_{2n-1}(N_+ \cup N_-) = H_{2n-1}(B_+ \cup B_-) = 0$ and $\iota_{1*}\colon H_{2n-2}(Z_+ \cup Z_-) \to H_{2n-2}(\tilde{M})$ is injective as observed above. Therefore, the above exact sequence reduces to this short exact sequence:

$$0 \to H_{2n}(M') \xrightarrow{\partial'_*} H_{2n-1}(Z_+ \cup Z_-) \xrightarrow{\iota_{1*}} H_{2n-1}(\tilde{M}) \xrightarrow{\iota_*} H_{2n-1}(M') \to 0.$$

Here $\partial_*([M]) = [Z_+] - [Z_-]$ and $H_{2n-1}(M')$ is freely generated by $Z_1, \dots, Z_{b_1-1}$ by induction assumption. Therefore, the above short exact sequence implies that

$$H_{2n-1}(\tilde{M}) \text{ is freely generated by } Z_1, \dots, Z_{b_1-1} \text{ and } Z_+ \text{ (or } Z_-).$$

This together with (3.1.7) completes the induction step and proves the lemma. □

Next we describe $b_{2i}(M)$ in terms of the face numbers of M/T and $b_1(M)$. Let $\mathcal{P}$ be the simplicial poset dual to $\partial(M/T)$. As usual, we define

$$\begin{aligned} f_i &= \text{ the number of } (n-1-i)\text{-faces of } M/T \\ &= \text{ the number of } i\text{-simplices in } \mathcal{P} \quad \text{for } i = 0, 1, \ldots, n-1 \end{aligned}$$

and the h-vector $(h_0, h_1, \ldots, h_n)$ by

$$\sum_{i=0}^{n} h_i t^{n-i} = (t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i}. \tag{3.1.8}$$

Example 3.1.1. Let (M, ω, T, μ) be a toric origami and the following picture is the associated origami template, whose resulting space is homeomorphic to M/T as a manifold with corners. It has 8 vertexes

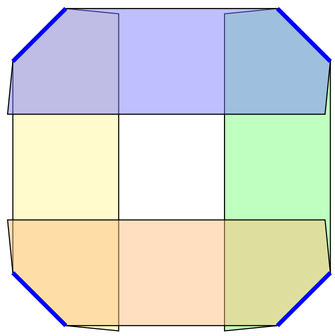


Figure 3.1: The origami template with four polygons

and 8 edges so $f_1 = 8$ and $f_0 = 8$. Hence the f-vector is $(f_0, f_1) = (8, 8)$ and the h-vector is $(h_0, h_1, h_2) = (1, 6, 1)$.

Theorem 3.1.2. *Let M be an orientable toric origami manifold of dimension $2n$ such that every proper face of M/T is acyclic. Let b_j be the j-th Betti number of M and $(h_0, h_1, \ldots, h_n)$ be the h-vector of M/T. Then*

$$\sum_{i=0}^{n} b_{2i} t^i = \sum_{i=0}^{n} h_i t^i + b_1 (1 + t^n - (1-t)^n),$$

in other words, $b_0 = h_0 = 1$ *and*

$$b_{2i} = h_i - (-1)^i \binom{n}{i} b_1 \quad \textit{for } 1 \le i \le n-1,$$

$$b_{2n} = h_n + (1 - (-1)^n) b_1.$$

Remark 3.1.2. We have $h_n = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i} f_i$ by (3.1.8) and $\chi(\partial(M/T)) = \sum_{i=0}^{n-1} (-1)^i f_i$ because every proper face of M/T is acyclic. Therefore, $h_n = (-1)^n - (-1)^n \chi(\partial(M/T))$. Since $b_{2n} = 1$, it follows from the last identity in Theorem 3.1.2 that

$$\chi(\partial(M/T)) - \chi(S^{n-1}) = ((-1)^n - 1) b_1.$$

Moreover, since $b_{2i} = b_{2n-2i}$, we have

$$\begin{aligned} h_{n-i} - h_i &= (-1)^i ((-1)^n - 1) b_1 \binom{n}{i} \\ &= (-1)^i (\chi(\partial(M/T)) - \chi(S^{n-1})) \binom{n}{i} \quad \text{for } 0 \le i \le n. \end{aligned}$$

These are generalized Dehn-Sommerville relations for $\partial(M/T)$ (or for the simplicial poset $\mathcal{P}$), see [21, p. 74] or [5, Theorem 7.44].

For a manifold Q of dimension n with corners (or faces), we define the f-polynomial and h-polynomial of Q by

$$f_Q(t) = t^n + \sum_{i=0}^{n-1} f_i(Q) t^{n-1-i}, \qquad h_Q(t) = f_Q(t-1)$$

as usual.

Lemma 3.1.6. *The h-polynomials of M'/T, M/T, and F have the relation* $h_{M'/T}(t) = h_{M/T}(t) + (t+1) h_F(t) - (t-1)^n$. *Therefore*

$$t^n h_{M'/T}(t^{-1}) = t^n h_{M/T}(t^{-1}) + (1+t) t^{n-1} h_F(t^{-1}) - (1-t)^n.$$

Proof. In the proof of Lemma 3.1.5 we observed that no facet of M'/T intersects with both F_+ and F_-. This means that no face of M'/T

intersects with both F_+ and F_- because every face of M'/T is contained in some facet of M'/T. Noting this fact, one can find that

$$f_i(M'/T) = f_i(M/T) + 2f_{i-1}(F) + f_i(F) \quad \text{for } 0 \leq i \leq n-1$$

where F is the folded facet and $f_{n-1}(F) = 0$. Therefore,

$$\begin{aligned} f_{M'/T}(t) &= t^n + \sum_{i=0}^{n-1} f_i(M'/T)t^{n-1-i} \\ &= t^n + \sum_{i=0}^{n-1} f_i(M'/T)t^i + 2\sum_{i=0}^{n-1} f_{i-1}(F)t^{n-1-i} + \sum_{i=0}^{n-2} f_i(F)t^{n-1-i} \\ &= f_{M/T}(t) + 2f_F(t) + tf_F(t) - t^n. \end{aligned}$$

Replacing t by $t-1$ in the identity above, we obtain the former identity in the lemma. Replacing t by t^{-1} in the former identity and multiplying the resulting identity by t^n, we obtain the latter identity. □

Proof of Theorem 3.1.2. Since $\sum_{i=0}^{n} h_i(M/T)t^i = t^n h_{M/T}(t^{-1})$, Theorem 3.1.2 is equivalent to

$$\sum_{i=0}^{n} b_{2i}(M)t^i = t^n h_{M/T}(t^{-1}) + b_1(M)(1 + t^n - (1-t)^n). \tag{3.1.9}$$

We shall prove (3.1.9) by induction on $b_1(M)$. The identity (3.1.9) is well-known when $b_1(M) = 0$. Suppose that $k = b_1(M)$ is a positive integer and the identity (3.1.9) holds for M' with $b_1(M') = k-1$. Then

$$\begin{aligned} &\sum_{i=0}^{n} b_{2i}(M)t^i \\ =&1 + t^n + \sum_{i=1}^{n-1}(b_{2i}(M') - b_{2i}(B) - b_{2i-2}(B))t^i \quad \text{(by Theorem 3.1.1)} \end{aligned}$$

$$=\sum_{i=0}^{n} b_{2i}(M')t^i - (1+t)\sum_{i=0}^{n-1} b_{2i}(B)t^i + 1 + t^n$$

$$=t^n h_{M'/T}(t^{-1}) + b_1(M')(1 + t^n - (1-t)^n) - (1+t)t^{n-1}h_F(t^{-1}) + 1 + t^n$$

$$\text{(by (3.1.9) applied to } M')$$

$$=t^n h_{M/T}(t^{-1}) + b_1(M)(1 + t^n - (1-t)^n)$$

$$\text{(by Lemma 3.1.6 and } b_1(M') = b_1(M) - 1),$$

proving (3.1.9) for M. This completes the induction step and the proof of Theorem 3.1.2. □

Example 3.1.2. Consider the toric origami manifold discussed in Example 3.1.1. By Theorem 3.1.2 we have

$$b_0 = b_4 = 1, \quad b_1 = b_3 = 1 \quad \text{and} \quad b_2 = 8.$$

3.2 Equivariant cohomology and face ring

A torus manifold M of dimension $2n$ is an orientable connected closed smooth manifold with an effective smooth action of an n-dimensional torus T having a fixed point ([11]). An orientable toric origami manifold with acyclic proper faces in the orbit space has a fixed point, so it is a torus manifold. The action of T on M is called *locally standard* if every point of M has a T-invariant open neighborhood equivariantly diffeomorphic to a T-invariant open set of a faithful representation space of T. Then the orbit space M/T is a nice manifold with corners. The torus action on an orientable toric origami manifold is locally standard. In this section, we study the equivariant cohomology of a locally standard torus manifold with acyclic proper faces of the orbit space.

We review some facts from [15]. Let Q be a nice manifold with corners of dimension n. Let $\mathcal{R}$ be a ground commutative ring with unit. We denote by $G \vee H$ the unique minimal face of Q that contains both G and H. The face ring $\mathcal{R}[Q]$ of Q is a graded ring defined by

$$\mathcal{R}[Q] := \mathcal{R}[v_F : \ F \text{ a face}]/I_Q$$

where $\deg v_F = 2 \operatorname{codim} F$ and I_Q is the ideal generated by all elements

$$v_G v_H - v_{G \vee H} \sum_{E \in G \cap H} v_E.$$

For each vertex $p \in Q$ the restriction map s_p is defined as the quotient map

$$s_p \colon \mathcal{R}[Q] \to \mathcal{R}[Q]/(v_F : \ p \notin F)$$

and it is proved in [15, Proposition 5.5] that the image $s_p(\mathcal{R}[Q])$ is the polynomial ring $\mathcal{R}[v_{Q_{i_1}}, \dots, v_{Q_{i_n}}]$ where $Q_{i_1}, \dots, Q_{i_n}$ are the n different facets containing p.

Lemma 3.2.1 (Lemma 5.6 in [15]). *If every face of Q has a vertex, then the sum $s = \oplus_p s_p$ of restriction maps over all vertices $p \in Q$ is a monomorphism from $\mathcal{R}[Q]$ to the sum of polynomial rings.*

In particular, $\mathcal{R}[Q]$ has no nonzero nilpotent element if every face of Q has a vertex. It is not difficult to see that every face of Q has a vertex if every proper face of Q is acyclic.

Let M be a locally standard torus manifold. Then the orbit space M/T is a nice manifold with corners. Let $q \colon M \to M/T$ be the quotient map. Note that $M^\circ := M - q^{-1}(\partial(M/T))$ is the T-free part. The projection $ET \times M \to M$ induces a map $\bar{q} \colon ET \times_T M \to M/T$, where

ET denotes the total space of the universal principal T-bundle and $ET \times_T M$ denotes the orbit space of $ET \times M$ by the diagonal T-action on $ET \times M$. Similarly we have a map $\bar{q}^\circ \colon ET \times_T M^\circ \to M^\circ/T$. The exact sequence of the equivariant cohomology groups for a pair (M, M°) together with the maps $\bar{q}$ and $\bar{q}^\circ$ produces the following commutative diagram:

$$\begin{array}{ccccc} H_T^*(M, M^\circ) & \xrightarrow{\eta^*} & H_T^*(M) & \xrightarrow{\iota^*} & H_T^*(M^\circ) \\ & & {\scriptstyle \bar{q}^*}\uparrow & & \uparrow{\scriptstyle (\bar{q}^\circ)^*} \\ & & H^*(M/T) & \xrightarrow{\bar{\iota}^*} & H^*(M^\circ/T) \end{array}$$

where η, ι and $\bar{\iota}$ are the inclusions and $H_T^*(X, Y) := H^*(ET \times_T X, ET \times_T Y)$ for a T-space X and its T-subspace Y as usual. Since the T-action on M° is free and $\bar{\iota} \colon M^\circ/T \to M/T$ is a homotopy equivalence, we have graded ring isomorphisms

$$H_T^*(M^\circ) \xrightarrow{((\bar{q}^\circ)^*)^{-1}} H^*(M^\circ/T) \xrightarrow{(\bar{\iota}^*)^{-1}} H^*(M/T) \tag{3.2.1}$$

and the composition $\rho := \bar{q}^* \circ (\bar{\iota}^*)^{-1} \circ ((\bar{q}^\circ)^*)^{-1}$, which is a graded ring homomorphism, gives the right inverse of ι^*, so the exact sequence above splits. Therefore, η^* and $\bar{q}^*$ are both injective and

$$H_T^*(M) = \eta^*(H_T^*(M, M^\circ)) \oplus \rho(H_T^*(M^\circ)) \quad \text{as graded groups.} \tag{3.2.2}$$

Note that both factors at the right hand side above are graded subrings of $H_T^*(M)$ because η^* and ρ are both graded ring homomorphisms.

Let $\mathcal{P}$ be the poset dual to the face poset of M/T as before. Then $\mathbb{Z}[\mathcal{P}] = \mathbb{Z}[M/T]$ by definition.

Proposition 3.2.1. *Suppose every proper face of the orbit space M/T is acyclic, and the free part of the action gives a trivial principal bundle*

$M^\circ \to M^\circ/T$. Then $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}] \oplus \tilde{H}^*(M/T)$ as graded rings.

Proof. Let R be the cone of $\partial(M/T)$ and let $M_R = M_R(\Lambda)$ be the T-space $R \times T/\sim$ where we use the characteristic function Λ obtained from M for the identification $\sim$. Let M_R° be the T-free part of M_R. Since the free part of the action on M is trivial, we have $M - M^\circ = M_R - M_R^\circ$. Hence,

$$H_T^*(M, M^\circ) \cong H_T^*(M_R, M_R^\circ) \quad \text{as graded rings} \tag{3.2.3}$$

by excision. Since $H_T^*(M_R^\circ) \cong H^*(M_R^\circ/T) \cong H^*(R)$ and R is a cone, $H_T^*(M_R^\circ)$ is isomorphic to the cohomology of a point. Therefore,

$$H_T^*(M_R, M_R^\circ) \cong H_T^*(M_R) \quad \text{as graded rings in positive degrees.} \tag{3.2.4}$$

On the other hand, the dual decomposition on the geometric realization $|\mathcal{P}|$ of $\mathcal{P}$ defines a face structure on the cone P of $\mathcal{P}$. Let $M_P = M_P(\Lambda)$ be the T-space $P \times T/\sim$ defined as before. Then a similar argument to that in [9, Theorem 4.8] shows that

$$H_T^*(M_P) \cong \mathbb{Z}[\mathcal{P}] \quad \text{as graded rings} \tag{3.2.5}$$

(this is mentioned as Proposition 5.13 in [15]). Since every face of P is a cone, one can construct a face preserving degree one map from R to P which induces an equivariant map $f\colon M_R \to M_P$. Then a similar argument to the proof of Theorem 8.3 in [15] shows that f induces a graded ring isomorphism

$$f^*\colon H_T^*(M_P) \xrightarrow{\cong} H_T^*(M_R) \tag{3.2.6}$$

since every proper face of R is acyclic. It follows from (3.2.3), (3.2.4), (3.2.5) and (3.2.6) that

$$H_T^*(M, M^\circ) \cong \mathbb{Z}[\mathcal{P}] \quad \text{as graded rings in positive degrees.} \tag{3.2.7}$$

Thus, by (3.2.1) and (3.2.2) it suffices to prove that the cup product of any $a \in \eta^*(H_T^*(M, M^\circ))$ and any $b \in \rho(\tilde{H}_T^*(M^\circ))$ is trivial. Since $\iota^*(a) = 0$ (as $\iota^* \circ \eta^* = 0$), we have $\iota^*(a \cup b) = \iota^*(a) \cup \iota^*(b) = 0$ and hence $a \cup b$ lies in $\eta^*(H_T^*(M, M^\circ))$. Since $\rho(H_T^*(M^\circ)) \cong H^*(M/T)$ as graded rings by (3.2.1) and $H^m(M/T) = 0$ for a sufficiently large m, $(a \cup b)^m = \pm a^m \cup b^m = 0$. However, we know that $a \cup b \in \eta^*(H_T^*(M, M^\circ))$ and $\eta^*(H_T^*(M, M^\circ)) \cong \mathbb{Z}[\mathcal{P}]$ in positive degrees by (3.2.7). Since $\mathbb{Z}[\mathcal{P}]$ has no nonzero nilpotent element as remarked before, $(a \cup b)^m = 0$ implies $a \cup b = 0$. $\square$

As discussed in [15, Section 6], there is a homomorphism

$$\varphi\colon \mathbb{Z}[\mathcal{P}] = \mathbb{Z}[M/T] \to \hat{H}_T^*(M) := H_T^*(M)/H^*(BT)\text{-torsions.} \tag{3.2.8}$$

In fact, φ is defined as follows. For a codimension k face F of M/T, $q^{-1}(F) =: M_F$ is a connected closed T-invariant submanifold of M of codimension $2k$, and φ assigns $v_F \in \mathbb{Z}[M/T]$ to the equivariant Poincaré dual $\tau_F \in H_T^{2k}(M)$ of M_F. One can see that φ followed by the restriction map to $H_T^*(M^T)$ can be identified with the map s in Lemma 3.2.1. Therefore, φ is injective if every face of Q has a vertex as mentioned in [15, Lemma 6.4].

Proposition 3.2.2. *Let M be a torus manifold with a locally standard torus action. If every proper face of M/T is acyclic and the free part of action gives a trivial principal bundle, then the $H^*(BT)$-torsion submod-*

ule of $H_T^*(M)$ agrees with $\bar{q}^*(\tilde{H}^*(M/T))$, where $\bar{q}\colon ET\times_T M\to M/T$ is the map mentioned before.

Proof. First we prove that all elements in $\bar{q}^*(\tilde{H}^*(M/T))$ are $H^*(BT)$-torsions. We consider the following commutative diagram:

$$\begin{array}{ccc} H_T^*(M) & \xrightarrow{\psi^*} & H_T^*(M^T) \\ {\scriptstyle \bar{q}^*}\Big\uparrow & & \Big\uparrow \\ H^*(M/T) & \xrightarrow{\bar{\psi}^*} & H^*(M^T) \end{array}$$

where the horizontal maps ψ^* and $\bar{\psi}^*$ are restrictions to M^T and the right vertical map is the restriction of $\bar{q}^*$ to M^T. Since M^T is isolated, $\bar{\psi}^*(\tilde{H}^*(M/T))=0$. This together with the commutativity of the above diagram shows that $\bar{q}^*(\tilde{H}^*(M/T))$ maps to zero by ψ^*. This means that all elements in $\bar{q}^*(\tilde{H}^*(M/T))$ are $H^*(BT)$-torsions because the kernel of ψ^* are $H^*(BT)$-torsions by the Localization Theorem in equivariant cohomology.

On the other hand, since every face of M/T has a vertex, the map φ in (3.2.8) is injective as remarked above. Hence, by Proposition 3.2.1, there are no other $H^*(BT)$-torsion elements. □

We conclude this section with observation on the orientability of M/T.

Lemma 3.2.2. *Let M be a closed smooth manifold of dimension $2n$ with a locally standard smooth action of the n-dimensional torus T. Then M/T is orientable if and only if M is.*

Proof. Since M/T is a manifold with corners and M°/T is its interior, M/T is orientable if and only if M°/T is. On the other hand, M is orientable if and only if M° is. Indeed, since the complement of M° in M

is the union of finitely many codimension-two submanifolds, the inclusion $\iota\colon M^\circ \to M$ induces an epimorphism on their fundamental groups and hence on their first homology groups with $\mathbb{Z}/2$-coefficients. Then it induces a monomorphism $\iota^*\colon H^1(M;\mathbb{Z}/2) \to H^1(M^\circ;\mathbb{Z}/2)$ because $H^1(X;\mathbb{Z}/2) = \mathrm{Hom}(H_1(X;\mathbb{Z}/2);\mathbb{Z}/2)$. Since $\iota^*(w_1(M)) = w_1(M^\circ)$ and ι^* is injective, $w_1(M) = 0$ if and only if $w_1(M^\circ) = 0$. This means that M is orientable if and only if M° is.

Thus, it suffices to prove that M°/T is orientable if and only if M° is. But, since M°/T can be regarded as the quotient of an iterated free S^1-action, it suffices to prove the following general fact: for a principal S^1-bundle $\pi\colon E \to B$ where E and B are both smooth manifolds, B is orientable if and only if E is. First we note that the tangent bundle of E is isomorphic to the Whitney sum of the tangent bundle along the fiber $\tau_f E$ and the pullback of the tangent bundle of B by π. Since the free S^1-action on E yields a nowhere zero vector field along the fibers, the line bundle $\tau_f E$ is trivial. Therefore

$$w_1(E) = \pi^*(w_1(B)). \tag{3.2.9}$$

We consider the Gysin exact sequence for our S^1-bundle:

$$\to H^{-1}(B;\mathbb{Z}/2) \to H^1(B;\mathbb{Z}/2) \xrightarrow{\pi^*} H^1(E,\mathbb{Z}/2) \to H^0(B;\mathbb{Z}/2) \to .$$

Since $H^{-1}(B;\mathbb{Z}/2) = 0$, the exact sequence above tells us that the map $\pi^*\colon H^1(B;\mathbb{Z}/2) \to H^1(E;\mathbb{Z}/2)$ is injective. This together with (3.2.9) shows that $w_1(E) = 0$ if and only if $w_1(B) = 0$, proving the desired fact. $\square$

3.3 Serre spectral sequence

Let M be an orientable toric origami manifold M of dimension $2n$ such that every proper face of M/T is acyclic. Note that M°/T is homotopy equivalent to a graph, hence does not admit nontrivial torus bundles. Thus the free part of the action gives a trivial principal bundle $M^\circ \to M^\circ/T$, and we may apply the results of the previous section.

We consider the Serre spectral sequence of the fibration $\pi\colon ET \times_T M \to BT$. Since BT is simply connected and both $H^*(BT)$ and $H^*(M)$ are torsion free by Theorem 3.1.1, the E_2-terms are given as follows:

$$E_2^{p,q} = H^p(BT; H^q(M)) = H^p(BT) \otimes H^q(M).$$

Since $H^{odd}(BT) = 0$ and $H^{2i+1}(M) = 0$ for $1 \leq i \leq n-2$ by Theorem 3.1.1,

$$E_2^{p,q} \text{ with } p+q \text{ odd vanishes unless } p \text{ is even and } q = 1 \text{ or } 2n-1. \tag{3.3.1}$$

We have differentials

$$\to E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q-r+1} \to$$

and

$$E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}.$$

We will often abbreviate $d_r^{p,q}$ as d_r when p and q are clear in the context. Since

$$d_r(u \cup v) = d_r u \cup v + (-1)^{p+q} u \cup d_r v \quad \text{for } u \in E_r^{p,q} \text{ and } v \in E_r^{p',q'}$$

and d_r is trivial on $E_r^{p,0}$ and $E_r^{p,0} = 0$ for odd p,

$$d_r \text{ is an } H^*(BT)\text{-module map.} \tag{3.3.2}$$

Note that

$$E_r^{p,q} = E_\infty^{p,q} \quad \text{if } p < r \text{ and } q+1 < r \tag{3.3.3}$$

since $E_r^{a,b} = 0$ if either $a < 0$ or $b < 0$.

There is a filtration of subgroups

$$H_T^m(M) = \mathcal{F}^{0,m} \supset \mathcal{F}^{1,m-1} \supset \cdots \supset \mathcal{F}^{m-1,1} \supset \mathcal{F}^{m,0} \supset \mathcal{F}^{m+1,-1} = \{0\}$$

such that

$$\mathcal{F}^{p,m-p}/\mathcal{F}^{p+1,m-p-1} = E_\infty^{p,m-p} \quad \text{for } p = 0, 1, \ldots, m. \tag{3.3.4}$$

There are two edge homomorphisms. One edge homomorphism

$$H^p(BT) = E_2^{p,0} \to E_3^{p,0} \to \cdots \to E_\infty^{p,0} \subset H_T^p(M)$$

agrees with $\pi^*\colon H^*(BT) \to H_T^*(M)$. Since $M^T \neq \emptyset$, one can construct a cross section of the fibration $\pi\colon ET \times_T M \to BT$ using a fixed point in M^T. So π^* is injective and hence

$$d_r\colon E_r^{p-r,r-1} \to E_r^{p,0} \text{ is trivial for every } r \geq 2 \text{ and } p \geq 0, \tag{3.3.5}$$

which is equivalent to $E_2^{p,0} = E_\infty^{p,0}$. The other edge homomorphism

$$H_T^q(M) \twoheadrightarrow E_\infty^{0,q} \subset \cdots \subset E_3^{0,q} \subset E_2^{0,q} = H^q(M)$$

agrees with the restriction homomorphism $\iota^*\colon H_T^q(M) \to H^q(M)$. Therefore, ι^* is surjective if and only if the differential $d_r\colon E_r^{0,q} \to E_r^{r,q-r+1}$ is trivial for every $r \geq 2$.

We shall investigate the restriction homomorphism $\iota^*\colon H_T^q(M) \to H^q(M)$. Since M/T is homotopy equivalent to the wedge of $b_1(M)$ circles, $H_T^q(M)$ vanishes unless q is 1 or even by Proposition 3.2.1 while $H^q(M)$ vanishes unless q is $1, 2n-1$ or even in between 0 and $2n$ by Theorem 3.1.1.

Lemma 3.3.1. *The homomorphism $\iota^*\colon H^1_T(M) \to H^1(M)$ is an isomorphism (so $H^1(M) \cong H^1(M/T)$ by Proposition 3.2.1).*

Proof. By (3.3.5),

$$d_2\colon E_2^{0,1} = H^1(M) \to E_2^{2,0} = H^2(BT)$$

is trivial. Therefore $E_2^{0,1} = E_\infty^{0,1}$. On the other hand, $E_\infty^{1,0} = E_2^{1,0} = H^1(BT) = 0$. These imply the lemma. □

Since $H_T^{2n-1}(M) = 0$, The homomorphism $\iota^*\colon H_T^{2n-1}(M) \to H^{2n-1}(M)$ cannot be surjective unless $H^{2n-1}(M) = 0$.

Lemma 3.3.2. *The homomorphism $\iota^*\colon H_T^{2j}(M) \to H^{2j}(M)$ is surjective except for $j = 1$ and the rank of the cokernel of ι^* for $j = 1$ is $nb_1(M)$.*

Proof. Since $\dim M = 2n$, we may assume $1 \le j \le n$.

First we treat the case where $j = 1$. Since $H_T^3(M) = 0$, $E_\infty^{2,1} = 0$ by (3.3.4) and $E_\infty^{2,1} = E_3^{2,1}$ by (3.3.3). This together with (3.3.5) implies that

$$d_2\colon H^2(M) = E_2^{0,2} \to E_2^{2,1} = H^2(BT) \otimes H^1(M) \quad \text{is surjective.} \tag{3.3.6}$$

Moreover $d_3\colon E_3^{0,2} = \ker d_2 \to E_3^{3,0}$ is trivial since $E_3^{3,0} = 0$. Therefore, $E_3^{0,2} = E_\infty^{0,2}$ by (3.3.3). Since $E_\infty^{0,2}$ is the image of $\iota^*\colon H_T^2(M) \to H^2(M)$, the rank of $H^2(M)/\iota^*(H_T^2(M))$ is $nb_1(M)$ by (3.3.6).

Suppose that $2 \le j \le n-1$. We need to prove that the differentials

$$d_r\colon E_r^{0,2j} \to E_r^{r,2j-r+1}$$

are all trivial. In fact, the target group $E_r^{r,2j-r+1}$ vanishes. This follows from (3.3.1) unless $r = 2j$. As for the case $r = 2j$, we note that

$$d_2\colon E_2^{p,2} \to E_2^{p+2,1} \quad \text{is surjective for } p \geq 0, \tag{3.3.7}$$

which follows from (3.3.2) and (3.3.6). Therefore $E_3^{p+2,1} = 0$ for $p \geq 0$, in particular $E_r^{r,2j-r+1} = 0$ for $r = 2j$ because $j \geq 2$. Therefore $\iota^*\colon H_T^{2j}(M) \to H^{2j}(M)$ is surjective for $2 \leq j \leq n-1$.

The remaining case $j = n$ can be proved directly, namely without using the Serre spectral sequence. Let x be a T-fixed point of M and let $\varphi\colon x \to M$ be the inclusion map. Since M is orientable and φ is T-equivariant, the equivariant Gysin homomorphism $\varphi_!\colon H_T^0(x) \to H_T^{2n}(M)$ can be defined and $\varphi_!(1) \in H_T^{2n}(M)$ restricts to the ordinary Gysin image of $1 \in H^0(x)$, that is the cofundamental class of M. This implies the surjectivity of $\iota^*\colon H_T^{2n}(M) \to H^{2n}(M)$ because $H^{2n}(M)$ is an infinite cyclic group generated by the cofundamental class. □

3.4 On the ring structure

Let $\pi\colon ET \times_T M \to BT$ be the projection. Since $\pi^*(H^2(BT))$ maps to zero by the restriction homomorphism $\iota^*\colon H_T^*(M) \to H^*(M)$, ι^* induces a graded ring homomorphism

$$\bar{\iota}^*\colon H_T^*(M)/(\pi^*(H^2(BT))) \to H^*(M) \tag{3.4.1}$$

which is surjective except in degrees 2 and $2n-1$ by Lemma 3.3.2 (and bijective in degree 1 by Lemma 3.3.1). Here $(\pi^*(H^2(BT)))$ denotes the ideal in $H_T^*(M)$ generated by $\pi^*(H^2(BT))$. The purpose of this section is to prove the following.

Proposition 3.4.1. *The map* $\bar{\iota}^*$ *in* (3.4.1) *is an isomorphism except in degrees* 2, 4 *and* $2n-1$*. Moreover, the rank of the cokernel of* $\bar{\iota}^*$ *in degree* 2 *is* $nb_1(M)$ *and the rank of the kernel of* $\bar{\iota}^*$ *in degree* 4 *is* $\binom{n}{2}b_1(M)$*.*

The rest of this section is devoted to the proof of Proposition 3.4.1. We recall the following result, which was proved by Schenzel ([20], [21, p.73]) for Buchsbaum simplicial complexes and generalized to Buchsbaum simplicial posets by Novik-Swartz ([17, Proposition 6.3]). There are several equivalent definitions for Buchsbaum simplicial complexes (see [21, p.73]). A convenient one for us would be that a finite simplicial complex Δ is Buchsbaum (over a field $\Bbbk$) if for all $p \in |\Delta|$ and all $i < \dim|\Delta|$, $H_i(|\Delta|, |\Delta|\backslash\{p\}; \Bbbk) = 0$, where $|\Delta|$ denotes the realization of Δ. In particular, a triangulation Δ of a manifold is Buchsbaum over any field $\Bbbk$. A simplicial poset is a (finite) poset P that has a unique minimal element, $\hat{0}$, and such that for every $\tau \in P$, the interval $[\hat{0}, \tau]$ is a Boolean algebra. The face poset of a simplicial complex is a simplicial poset and one has the realization $|P|$ of P where $|P|$ is a regular CW complex, all of whose closed cells are simplices corresponding to the intervals $[\hat{0}, \tau]$. A simplicial poset P is Buchsbaum (over $\Bbbk$) if its order complex $\Delta(\overline{P})$ of the poset $\overline{P} = P\backslash\{\hat{0}\}$ is Buchsbaum (over $\Bbbk$). Note that $|\Delta(\overline{P})| = |P|$ as spaces since $|\Delta(\overline{P})|$ is the barycentric subdivision of $|P|$. See [17] and [21] for more details.

Theorem 3.4.1 (Schenzel, Novik-Swartz)**.** *Let* Δ *be a Buchsbaum simplicial poset of dimension* $n-1$ *over a field* $\Bbbk$*,* $\Bbbk[\Delta]$ *be the face ring of*

Δ *and let* $\theta_1, \dots, \theta_n \in \Bbbk[\Delta]_1$ *be a linear system of parameters. Then*

$$F(\Bbbk[\Delta]/(\theta_1, \dots, \theta_n), t) = (1-t)^n F(\Bbbk[\Delta], t) + \sum_{j=1}^{n} \binom{n}{j} \Big(\sum_{i=-1}^{j-2} (-1)^{j-i} \dim_{\Bbbk} \tilde{H}_i(\Delta) \Big) t^j$$

where $F(M, t)$ *denotes the Hilbert series of a graded module* M*.*

As is well-known, the Hilbert series of the face ring $\Bbbk[\Delta]$ satisfies

$$(1-t)^n F(\Bbbk[\Delta], t) = \sum_{i=0}^{n} h_i t^i.$$

We define h'_i for $i = 0, 1, \dots, n$ by

$$F(\Bbbk[\Delta]/(\theta_1, \dots, \theta_n), t) = \sum_{i=0}^{n} h'_i t^i,$$

following [17].

Remark 3.4.1. Novik-Swartz [17] introduced

$$h''_i := h'_i - \binom{n}{j} \dim_{\Bbbk} \tilde{H}_{j-1}(\Delta) = h_j + \binom{n}{j} \Big(\sum_{i=-1}^{j-1} (-1)^{j-i} \dim_{\Bbbk} \tilde{H}_i(\Delta) \Big)$$

for $1 \le i \le n-1$ and showed that $h''_j \ge 0$ and $h''_{n-j} = h''_j$ for $1 \le j \le n-1$.

We apply Theorem 3.4.1 to our simplicial poset $\mathcal{P}$ which is dual to the face poset of $\partial(M/T)$. For that we need to know the homology of the geometric realization $|\mathcal{P}|$ of $\mathcal{P}$. First we show that $|\mathcal{P}|$ has the same homological features as $\partial(M/T)$.

Lemma 3.4.1. *The simplicial poset* $\mathcal{P}$ *is Buchsbaum, and* $|\mathcal{P}|$ *has the same homology as* $\partial(M/T)$*.*

Proof. We give a sketch of the proof. Details can be found in [1, Lemma 3.14]. There is a dual face structure on $|\mathcal{P}|$, and there exists a face preserving map $g\colon \partial(M/T) \to |\mathcal{P}|$ mentioned in the proof of Proposition 3.2.1. Let F be a proper face of M/T and F' the corresponding face of $|\mathcal{P}|$. By induction on $\dim F$ we can show that g induces the isomorphisms $g_*\colon H_*(\partial F) \xrightarrow{\cong} H_*(\partial F')$, $g_*\colon H_*(F) \xrightarrow{\cong} H_*(F')$, and $g_*\colon H_*(F, \partial F) \xrightarrow{\cong} H_*(F', \partial F')$. Since F is an acyclic orientable manifold with boundary, we deduce, by Poincaré-Lefschetz duality, that $H_*(F', \partial F') \cong H_*(F, \partial F)$ vanishes except in degree $\dim F$. Note that F' is a cone over $\partial F'$ and $\partial F'$ is homeomorphic to the link of a nonempty simplex of $\mathcal{P}$. Thus the links of nonempty simplices of $\mathcal{P}$ are homology spheres, and $\mathcal{P}$ is Buchsbaum [17, Prop.6.2]. Finally, g induces an isomorphism of spectral sequences corresponding to skeletal filtrations of $\partial(M/T)$ and $|\mathcal{P}|$, thus induces an isomorphism $g_*\colon H_*(\partial(M/T)) \xrightarrow{\cong} H_*(|\mathcal{P}|)$. □

Lemma 3.4.2. *$|\mathcal{P}|$ has the same homology as $S^{n-1}\sharp b_1(S^1 \times S^{n-2})$ (the connected sum of S^{n-1} and b_1 copies of $S^1 \times S^{n-2}$).*

Proof. By Lemma 3.4.1 we only need to prove that $\partial(M/T)$ has the same homology groups as $S^{n-1}\sharp b_1(S^1 \times S^{n-2})$. Since M/T is homotopy equivalent to a wedge of circles, $H^i(M/T) = 0$ for $i \geq 2$ and hence the homology exact sequence of the pair $(M/T, \partial(M/T))$ shows that

$$H_{i+1}(M/T, \partial(M/T)) \cong H_i(\partial(M/T)) \quad \text{for } i \geq 2.$$

On the other hand, M/T is orientable by Lemma 3.2.2 and hence

$$H_{i+1}(M/T, \partial(M/T)) \cong H^{n-i-1}(M/T)$$

by Poincaré–Lefschetz duality, and $H^{n-i-1}(M/T) = 0$ for $n - i - 1 \geq 2$.

These show that

$$H_i(\partial(M/T)) = 0 \quad \text{for } 2 \le i \le n-3.$$

Thus it remains to study $H_i(\partial(M/T))$ for $i = 0, 1, n-2, n-1$ but since $\partial(M/T)$ is orientable (because M/T is orientable), it suffices to show

$$H_i(\partial(M/T)) \cong H_i(S^{n-1}\sharp b_1(S^1 \times S^{n-2})) \quad \text{for } i = 0, 1. \tag{3.4.2}$$

When $n \ge 3$, $S^{n-1}\sharp b_1(S^1 \times S^{n-2})$ is connected, so (3.4.2) holds for $i = 0$ and $n \ge 3$. Suppose that $n \ge 4$. Then $H^{n-2}(M/T) = H^{n-1}(M/T) = 0$, so the cohomology exact sequence for the pair $(M/T, \partial(M/T))$ shows that $H^{n-2}(\partial(M/T)) \cong H^{n-1}(M/T, \partial(M/T))$ and hence $H_1(\partial(M/T)) \cong H_1(M/T)$ by Poincaré–Lefschetz duality. Since M/T is homotopy equivalent to a wedge of b_1 circles, this proves (3.4.2) for $i = 1$ and $n \ge 4$. Assume that $n = 3$. Then $H_1(M/T, \partial(M/T)) \cong H^2(M/T) = 0$. We also know $H_2(M/T) = 0$. The homology exact sequence for the pair $(M/T, \partial(M/T))$ yields a short exact sequence

$$0 \to H_2(M/T, \partial(M/T)) \to H_1(\partial(M/T)) \to H_1(M/T) \to 0.$$

Here $H_2(M/T, \partial(M/T)) \cong H^1(M/T)$ by Poincaré–Lefschetz duality. Since M/T is homotopy equivalent to a wedge of b_1 circles, this implies (3.4.2) for $i = 1$ and $n = 3$.

It remains to prove (3.4.2) when $n = 2$. We use induction on b_1. The assertion is true when $b_1 = 0$. Suppose that $b_1 = b_1(M/T) \ge 1$. We cut M/T along a fold so that $b_1(M'/T) = b_1(M/T) - 1$, where M' is the toric origami manifold obtained from the cut, see Section 3.1. Then $b_0(\partial(M'/T)) = b_0(\partial(M/T)) - 1$. Since (3.4.2) holds for $\partial(M'/T)$

by induction assumption, this observation shows that (3.4.2) holds for $\partial(M/T)$. $\square$

Lemma 3.4.3. *For $n \geq 2$, we have*

$$\sum_{i=0}^{n} h_i' t^i = \sum_{i=0}^{n} b_{2i} t^i - n b_1 t + \binom{n}{2} b_1 t^2.$$

Proof. By Lemma 3.4.2, for $n \geq 4$, we have

$$\dim \tilde{H}_i(\mathcal{P}) = \begin{cases} b_1 & \text{if } i = 1,\ n-2, \\ 1 & \text{if } i = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{i=-1}^{j-2} (-1)^{j-i} \dim \tilde{H}_i(\mathcal{P}) = \begin{cases} 0 & \text{if } j = 1, 2, \\ (-1)^{j-1} b_1 & \text{if } 3 \leq j \leq n-1, \\ ((-1)^{n-1} + 1) b_1 & \text{if } j = n. \end{cases}$$

Then, it follows from Theorem 3.4.1 that

$$\begin{aligned} \sum_{i=0}^{n} h_i' t^i &= \sum_{i=0}^{n} h_i t^i + \sum_{j=3}^{n-1} (-1)^{j-1} b_1 \binom{n}{j} t^j + ((-1)^{n-1} + 1) b_1 t^n \\ &= \sum_{i=0}^{n} h_i t^i - b_1 (1-t)^n + b_1 (1 - nt + \binom{n}{2} t^2) + b_1 t^n \\ &= \sum_{i=0}^{n} h_i t^i + b_1 (1 + t^n - (1-t)^n) - n b_1 t + \binom{n}{2} b_1 t^2 \\ &= \sum_{i=0}^{n} b_{2i} t^i - n b_1 t + \binom{n}{2} b_1 t^2 \end{aligned}$$

where we used Theorem 3.1.2 at the last identity. This proves the lemma when $n \geq 4$. When $n = 3$,

$$\dim \tilde{H}_i(\mathcal{P}) = \begin{cases} 2b_1 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and the same argument as above shows that the lemma still holds for $n = 3$. When $n = 2$,

$$\dim \tilde{H}_i(\mathcal{P}) = \begin{cases} b_1 & \text{if } i = 0, \\ b_1 + 1 & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the same holds in this case too. □

Remark 3.4.2. One can check that

$$\sum_{i=1}^{n-1} h''_i t^i = \sum_{i=1}^{n-1} b_{2i} t^i - n b_1 (t + t^{n-1}).$$

Therefore, $h''_i = h''_i(\mathcal{P})$ is not necessarily equal to $b_{2i} = b_{2i}(M)$ although both are symmetric. This is not surprising because h''_i depends only on the boundary of M/T. It would be interesting to ask whether $h''_i(\mathcal{P}) \leq b_{2i}(M)$ when the face poset of $\partial(M/T)$ is dual to $\mathcal{P}$ and whether the equality can be attained for some such M (M may depend on i).

Now we prove Proposition 3.4.1.

Proof of Proposition 3.4.1. At first we suppose that $\Bbbk$ is a field. By Proposition 3.2.1 we have $\mathbb{Z}[\mathcal{P}] = H_T^{even}(M)$. The images of ring generators of $H^*(BT; \Bbbk)$ by π^* provide an h.s.o.p. $\theta_1, \dots, \theta_n$ in $H_T^{even}(M; \Bbbk) = \Bbbk[\mathcal{P}]$. This fact simply follows from the characterization of homogeneous systems of parameters in face rings given by [6, Th.5.4]. Thus we have

$$F(H_T^{even}(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk))),t) = \sum_{i=0}^{n} b_{2i}(M)t^i - nb_1t + \binom{n}{2}b_1t^2 \tag{3.4.3}$$

by Lemma 3.4.3. Moreover, the graded ring homomorphism in (3.4.1)

$$\bar{\iota}^*\colon \Bbbk[\mathcal{P}]/(\theta_1,\dots,\theta_n) = H_T^{even}(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk))) \to H^{even}(M;\Bbbk) \tag{3.4.4}$$

is surjective except in degree 2 as remarked at the beginning of this section. Therefore, the identity (3.4.3) implies that $\bar{\iota}^*$ in (3.4.4) is an isomorphism except in degrees 2 and 4. Finally, the rank of the cokernel of $\bar{\iota}^*$ in degree 2 is $nb_1(M)$ by Lemma 3.3.2 and the rank of the kernel of $\bar{\iota}^*$ in degree 4 is $\binom{n}{2}b_1$ by (3.4.3), proving Proposition 3.4.1 over fields.

Now we explain the case $\Bbbk=\mathbb{Z}$. The map $\pi^*\colon H^*(BT;\Bbbk)\to H_T^*(M;\Bbbk)$ coincides with the map $\pi^*\colon H^*(BT;\mathbb{Z})\to H_T^*(M;\mathbb{Z})$ tensored with $\Bbbk$, since both $H^*(BT;\mathbb{Z})$ and $H_T^*(M;\mathbb{Z})$ are $\mathbb{Z}$-torsion free. In particular, the ideals $(\pi^*(H^2(BT;\Bbbk)))$ and $(\pi^*(H^2(BT;\mathbb{Z}))\otimes\Bbbk) = (\pi^*(H^2(BT;\mathbb{Z})))\otimes \Bbbk$ coincide in $H_T^*(M;\Bbbk)\cong H_T^*(M;\mathbb{Z})\otimes\Bbbk$. Consider the exact sequence

$$(\pi^*(H^2(BT;\mathbb{Z}))) \to H_T^*(M;\mathbb{Z}) \to H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z}))) \to 0$$

The functor $-\otimes\Bbbk$ is right exact, thus the sequence

$$(\pi^*(H^2(BT;\mathbb{Z})))\otimes\Bbbk \to H_T^*(M;\mathbb{Z})\otimes\Bbbk \to H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z})))\otimes\Bbbk \to 0$$

is exact. These considerations show that

$$H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z})))\otimes\Bbbk \cong H_T^*(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk)))$$

Finally, the map

$$\bar{\iota}^*\colon H_T^*(M;\Bbbk)/(\pi^*(H^2(BT;\Bbbk))) \to H^*(M,\Bbbk)$$

coincides (up to isomorphism) with the map

$$\bar{\iota}^*\colon H_T^*(M;\mathbb{Z})/(\pi^*(H^2(BT;\mathbb{Z}))) \to H^*(M,\mathbb{Z}),$$

tensored with $\Bbbk$. The statement of Proposition 3.4.1 holds for any field thus holds for $\mathbb{Z}$. □

We conclude this section with some observations on the kernel of $\bar{\iota}^*$ in degree 4 from the viewpoint of the Serre spectral sequence. Recall

$$H_T^4(M) = \mathcal{F}^{0,4} \supset \mathcal{F}^{1,3} \supset \mathcal{F}^{2,2} \supset \mathcal{F}^{3,1} \supset \mathcal{F}^{4,0} \supset \mathcal{F}^{5,-1} = 0$$

where $\mathcal{F}^{p,q}/\mathcal{F}^{p+1,q-1} = E_\infty^{p,q}$. Since $E_2^{p,q} = H^p(BT) \otimes H^q(X)$, $E_\infty^{p,q} = 0$ for p odd. Therefore,

$$\operatorname{rank} H_T^4(M) = \operatorname{rank} E_\infty^{0,4} + \operatorname{rank} E_\infty^{2,2} + \operatorname{rank} E_\infty^{4,0},$$

where we know $E_\infty^{0,4} = E_2^{0,4} = H^4(M)$ and $E_\infty^{4,0} = E_2^{4,0} = H^4(BT)$. As for $E_\infty^{2,2}$, we recall that

$$d_2\colon E_2^{p,2} \to E_2^{p+2,1} \quad \text{is surjective for any } p \geq 0$$

by (3.3.7). Therefore, noting $H^3(M) = 0$, one sees $E_3^{2,2} = E_\infty^{2,2}$. It follows that

$$\operatorname{rank} E_\infty^{2,2} = \operatorname{rank} E_2^{2,2} - \operatorname{rank} E_2^{4,1} = nb_2 - \binom{n+1}{2} b_1.$$

On the other hand, $\operatorname{rank} E_\infty^{0,2} = b_2 - nb_1$ and there is a product map

$$\varphi\colon E_\infty^{0,2} \otimes E_\infty^{2,0} \to E_\infty^{2,2}.$$

The image of this map lies in the ideal $(\pi^*(H^2(BT))$ and the rank of the cokernel of this map is

$$nb_2 - \binom{n+1}{2} b_1 - n(b_2 - nb_1) = \binom{n}{2} b_1.$$

Therefore

$$\operatorname{rank} E_\infty^{0,4} + \operatorname{rank}\operatorname{coker}\varphi = b_4 + \binom{n}{2} b_1$$

which agrees with the coefficient of t^2 in $F(H_T^{even}(M)/(\pi^*(H^2(BT))), t)$ by (3.4.3). This suggests that the cokernel of φ could correspond to the kernel of $\bar{\iota}^*$ in degree 4.

3.5 4-dimensional case

In this section, we explicitly describe the kernel of $\bar{\iota}^*$ in degree 4 when $n = 2$, that is, when M is of dimension 4. In this case, $\partial(M/T)$ is the union of $b_1 + 1$ closed polygonal curves.

First we recall the case when $b_1 = 0$. In this case, $H_T^{even}(M) = H_T^*(M)$. Let $\partial(M/T)$ be an m-gon and $v_1, \ldots, v_m$ be the primitive edge vectors in the multi-fan of M, where v_i and v_{i+1} spans a 2-dimensional cone for any $i = 1, 2, \ldots, m$ (see [16]). Note that $v_i \in H_2(BT)$ and we understand $v_{m+1} = v_1$ and $v_0 = v_m$ in this section. Since $\{v_j, v_{j+1}\}$ is a basis of $H_2(BT)$ for any j, we have $\det(v_j, v_{j+1}) = \pm 1$.

Let $\tau_i \in H_T^2(M)$ be the equivariant Poincaré dual to the characteristic submanifold corresponding to v_i. Then we have

$$\pi^*(u) = \sum_{i=1}^{m} \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT), \tag{3.5.1}$$

where $\langle \ , \ \rangle$ denotes the natural pairing between cohomology and homology, (see [14] for example). We multiply both sides in (3.5.1) by τ_i. Then, since $\tau_i \tau_j = 0$ if v_i and v_j do not span a 2-dimensional cone, (3.5.1) turns into

$$0 = \langle u, v_{i-1}\rangle \tau_{i-1}\tau_i + \langle u, v_i\rangle \tau_i^2 + \langle u, v_{i+1}\rangle \tau_i\tau_{i+1} \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{3.5.2}$$

If we take u with $\langle u, v_i\rangle = 1$, then (3.5.2) shows that τ_i^2 can be expressed as a linear combination of $\tau_{i-1}\tau_i$ and $\tau_i\tau_{i+1}$. If we take $u = \det(v_i, \ )$, then u can be regarded as an element of $H^2(BT)$ because $H^2(BT) = \mathrm{Hom}(H_2(BT), \mathbb{Z})$. Hence (3.5.2) reduces to

$$\det(v_{i-1}, v_i)\tau_{i-1}\tau_i = \det(v_i, v_{i+1})\tau_i\tau_{i+1} \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{3.5.3}$$

Finally we note that $\tau_i\tau_{i+1}$ maps to the cofundamental class of M up to sign. We denote by $\mu \in H_T^4(M)$ the element (either $\tau_{i-1}\tau_i$ or $-\tau_{i-1}\tau_i$) which maps to the cofundamental class of M.

When $b_1 \geq 1$, the above argument works for each component of $\partial(M/T)$. In fact, according to [14], (3.5.1) holds in $H_T^*(M)$ modulo $H^*(BT)$-torsion but in our case there is no $H^*(BT)$-torsions in $H_T^{even}(M)$ by Proposition 3.2.2. Suppose that $\partial(M/T)$ consists of m_j-gons for $j = 1, 2, \ldots, b_1 + 1$. To each m_j-gon, we have the class $\mu_j \in H_T^4(M)$ (mentioned above as μ). Since μ_j maps to the cofundamental class of M, $\mu_i - \mu_j$ $(i \neq j)$ maps to zero in $H^4(M)$; so it is in the kernel of $\bar{\iota}^*$. The subgroup of $H_T^{even}(M)/(\pi^*(H^2(BT)))$ in degree 4 generated by $\mu_i - \mu_j$ $(i \neq j)$ has the desired rank b_1.

Example 3.5.1. Take the 4-dimensional toric origami manifold M corresponding to the origami template shown on fig. 3.2 (Example 3.15 of [7]). Topologically M/T is homeomorphic to $S^1 \times [0, 1]$ and the boundary of M/T as a manifold with corners consists of two closed polygonal

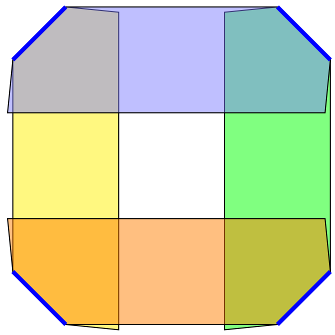


Figure 3.2: The origami template with four polygons

curves, each having 4 segments. The multi-fan of M is the union of two copies of the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the product torus action. Indeed, if v_1, v_2 are primitive edge vectors in the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ which spans a 2-dimensional cone, then the other primitive edge vectors $v_3, \ldots, v_8$ in the multi-fan of M are

$$v_3 = -v_1, \quad v_4 = -v_2, \quad \text{and} \quad v_i = v_{i-4} \quad \text{for } i = 5, \ldots, 8$$

and the 2-dimensional cones in the multi-fan are

$$\angle v_1v_2, \quad \angle v_2v_3, \quad \angle v_3v_4, \quad \angle v_4v_1,$$

$$\angle v_5v_6, \quad \angle v_6v_7, \quad \angle v_7v_8, \quad \angle v_8v_5,$$

where $\angle vv'$ denotes the 2-dimensional cone spanned by vectors v and v'. Note that

$$\tau_i\tau_j = 0 \text{ if } v_i, v_j \text{ do not span a 2-dimensional cone.} \tag{3.5.4}$$

We have

$$\pi^*(u) = \sum_{i=1}^{8} \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT). \tag{3.5.5}$$

Let $\{v_1^*, v_2^*\}$ be the dual basis of $\{v_1, v_2\}$. Taking $u = v_1^*$ or v_2^* , we see that

$$\tau_1 + \tau_5 = \tau_3 + \tau_7, \quad \tau_2 + \tau_6 = \tau_4 + \tau_8 \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{3.5.6}$$

Since we applied (3.5.5) for the basis $\{v_1^*, v_2^*\}$ of $H^2(BT)$, there is no other essentially new linear relation among τ_i's.

Now, multiply the equations (3.5.6) by τ_i and use (3.5.4). Then we obtain

$$\tau_i^2 = 0 \quad \text{for any } i,$$
$$(\mu_1 :=)\tau_1\tau_2 = \tau_2\tau_3 = \tau_3\tau_4 = \tau_4\tau_1,$$
$$(\mu_2 :=)\tau_5\tau_6 = \tau_6\tau_7 = \tau_7\tau_8 = \tau_8\tau_5 \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))).$$

Our argument shows that these together with (3.5.4) are the only degree two relations among τ_i's in $H_T^*(M)/(\pi^*(H^2(BT)))$. The kernel of

$$\bar{\iota}^*\colon H_T^{even}(M;\mathbb{Q})/(\pi^*(H^2(BT;\mathbb{Q}))) \to H^{even}(M;\mathbb{Q})$$

in degree 4 is spanned by $\mu_1 - \mu_2$.

3.6 Some observation on non-acyclic cases

The face acyclicity condition we assumed so far is not preserved under taking the product with a symplectic toric manifold N, but every face of codimension $\geq \frac{1}{2}\dim N + 1$ is acyclic. Motivated by this observation, we will make the following assumption on our toric origami manifold M of dimension $2n$:

> every face of M/T of codimension $\geq r$ is acyclic for some integer r.

Note that $r = 1$ in the previous sections. Under the above assumption, the arguments in Section 3.1 work to some extent in a straightforward way. The main point is that Lemma 3.1.5 can be generalized as follows.

Lemma 3.6.1. *The homomorphism $H^{2j}(\tilde{M}) \to H^{2j}(Z_+ \cup Z_-)$ induced from the inclusion is surjective for $j \geq r$.*

Using this lemma, we see that Lemma 3.1.3 turns into the following.

Lemma 3.6.2.

$$\sum_{i=1}^{r}(b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M})) = \sum_{i=1}^{r}(b_{2i}(M) - b_{2i-1}(M)) + b_{2r}(B)$$

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) - b_{2i-1}(M) + b_{2i}(B) - b_{2i-2}(B) \quad \textit{for } i \geq r+1.$$

Combining Lemma 3.6.2 with Lemma 3.1.2, Lemma 3.1.4 turns into the following.

Lemma 3.6.3.

$$b_1(M') = b_1(M) - 1, \quad b_{2r}(M') = b_{2r}(M) + b_{2r-2}(B) + b_{2r}(B),$$

$$b_{2i+1}(M') = b_{2i+1}(M) \quad \textit{for } r \leq i \leq n-r-1.$$

Finally, Theorem 3.1.1 is generalized as follows.

Theorem 3.6.1. *Let M be an orientable toric origami manifold of dimension $2n$ ($n \geq 2$) such that every face of M/T of codimension $\geq r$ is acyclic. Then*

$$b_{2i+1}(M) = 0 \quad \textit{for } r \leq i \leq n-r-1.$$

Moreover, if M' and B are as above, then

$$b_1(M') = b_1(M) - 1 \quad (\textit{hence } b_{2n-1}(M') = b_{2n-1}(M) - 1 \textit{ by Poincaré duality}),$$

$$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \textit{for } r \leq i \leq n-r.$$

Chapter 4

Towards non-orientable cases

4.1 The coörientable case

In this section we will discuss about the cohomology groups of a non-orientable toric origami manifold M with coörientable folding hypersurface. By Lemma 5.1 in [12], the T-action on M is locally standard, so M/T is a manifold with corners. In this section we assume that each proper face of M/T is acyclic. We will construct the orientation covering $\widehat{M}$ of M to study the cohomology groups of M. For this purpose, we construct the orientation covering for the associated origami template of M.

Construction. Let $G = (V, E)$ be the graph associated to M and (G, Ψ_V, Ψ_E) be the corresponding origami template. Then we can construct a new 2-colorable graph $\widehat{G} = (\widehat{V}, \widehat{E})$, the double covering of $G = (V, E)$, and its origami template is $(\widehat{G}, \Psi_{\widehat{V}}, \Psi_{\widehat{E}})$ by the following process. Set

$$V = \{v_1, ..., v_m\}.$$

1. Consider
$$\widehat{V} = W \sqcup U,$$
where
$$W = \{w_1, ..., w_m\},\ U = \{u_1, ..., u_m\},$$
are two sets. Let $f : W \to V$ and $g : U \to V$ be two maps such that $f(w_i) = v_i$ and $g(u_i) = v_i$;

2. $\widehat{E} := \{(w_i, u_j) \in W \times U \mid (f(w_i), g(u_j)) \in E\}$;

3. $\Psi_{\widehat{V}} := \widehat{V} \xrightarrow{f \sqcup g} V \xrightarrow{\Psi_v} \mathcal{D}_n$;

4. $\Psi_{\widehat{E}} : \widehat{E} \xrightarrow{\varphi} E \xrightarrow{\Psi_E} \mathcal{E}_n$, where $\varphi : (w_i, u_j) \mapsto (v_i, v_j)$.

Lemma 4.1.1. *Let $\mathcal{P}$ and $\widehat{\mathcal{P}}$ be the origami templates of $(G,\ \Psi_V,\ \Psi_E)$ and $(\widehat{G},\ \Psi_{\widehat{V}},\ \Psi_{\widehat{E}})$ respectively. Then there exists a map $\pi :\mid \widehat{\mathcal{P}} \mid \to \mid \mathcal{P} \mid$ which preserves the order of faces and is an orientation covering of $\mid \mathcal{P} \mid$.*

Proof. Recall that $\mid \mathcal{P} \mid = \sqcup(P_{v_k} \sqcup P_{v_l})/\sim$ where $(v_k, x) \sim (v_l, y)$ if and only if $(v_k, v_l) \in E$ and $x = y \in \Psi_E(v_k, v_l)$. $\mid \widehat{\mathcal{P}} \mid = \sqcup(P_{w_i} \sqcup P_{u_i})/\sim_2$ where $(w_i, x) \sim_2 (u_j, y)$ iff $(w_i, u_j) \in \widehat{E}$ and $x = y \in \Psi_{\widehat{E}}(w_i, u_j)$.

Since $\widehat{G}$ is 2-colorable, $\mid \widehat{\mathcal{P}} \mid$ is orientable as a manifold with corners. From the map $\sqcup(P_{w_i} \sqcup P_{u_i}) \xrightarrow{id \sqcup id} \sqcup P_{v_i}$, we have a well-defined continuous map
$$\sqcup(P_{w_i} \sqcup P_{u_i})/\sim_2 \longrightarrow (\sqcup P_{v_i})/\sim .$$
Namely,
$$\mid \widehat{\mathcal{P}} \mid \longrightarrow \mid \mathcal{P} \mid .$$
We denote this map by π. Note that the following diagram is commu-

tative

$$\begin{array}{ccc} \sqcup(P_{w_i} \sqcup P_{u_i}) & \xrightarrow{id \sqcup id} & \sqcup P_{v_i} \\ \widehat{q}\Big\downarrow & & \Big\downarrow q \\ \mid \widehat{\mathcal{P}} \mid & \xrightarrow{\pi} & \mid \mathcal{P} \mid \end{array}$$

where q and $\widehat{q}$ are the quotient maps. Then we can obtain that π is surjective since $q \circ (id \sqcup id)$ is surjective.

For any $[(v_i, x)] \in \mid \mathcal{P} \mid$, we have

$$\pi^{-1}[v_i, x] = \{[(w_i, x)],\ [(u_i, x)]\},$$

where "[ ]" means the equivalence classes in the corresponding quotient spaces. In fact, $[(w_i, x)] \neq [(u_i, x)]$ since $(v_i, v_i) \notin E$. Otherwise the origami template (G, Ψ_V, Ψ_E) is not coörientable. It is not difficult to check that π is a local homeomorphism, so $\pi : \mid \widehat{\mathcal{P}} \mid \to \mid \mathcal{P} \mid$ is an orientation covering map. Moreover π maps k-dim faces of $\widehat{\mathcal{P}}$ to k-dim faces of $\mathcal{P}$. This completes the proof. □

Lemma 4.1.2. *If $\mathcal{P}$ is coörientable and each proper face of $\mid \mathcal{P} \mid$ is acyclic, then each proper face of $\mid \widehat{\mathcal{P}} \mid$ is also acyclic.*

Proof. Let $\widehat{F}$ be a proper face of $\mid \widehat{\mathcal{P}} \mid$, then $\pi(\widehat{F})$ is also a proper face of $\mid \mathcal{P} \mid$. Set $\pi(\widehat{F}) = F$, then $\pi^{-1}(F)$ is a double covering of F. Since F is homotopy equivalent to wedge of circles and acyclic, F is simply connected. Thus $\pi^{-1}(F) = \widehat{F} \sqcup \widehat{F'}$, where $\widehat{F} \cong \widehat{F'} \cong F$ as manifolds with corners. Therefore, $\widehat{F}$ is acyclic.

□

We denote by $\widehat{M}$ the toric origami manifold corresponding to the origami template $\widehat{\mathcal{P}}$. Then it is not difficult to see that $\widehat{M}$ is an orientation covering of M and we denote the covering map by π.

Lemma 4.1.3. *The i-th cohomology group of M has the following form:*

$$H^i(M) = \mathbb{Z}^{b_i} \bigoplus (\mathbb{Z}_2)^{c_i}.$$

Proof. Consider the transfer homomorphism:

$$\tau^* : H^*(\widehat{M}) \to H^*(M).$$

Note that $\tau^* \circ \pi^* = 2$, where $\pi^* : H^*(M) \to H^*(\widehat{M})$ is induced from $\pi : \widehat{M} \to M$, so if $\alpha \in \ker \pi^*$, then $2\alpha = 0$. Each proper face of the orbit space $\widehat{M}/T$ is acyclic, so $H^*(\widehat{M})$ is torsion free by Theorem 3.1.1. Therefore, $H^i(M) = \mathbb{Z}^{b_i} \bigoplus (\mathbb{Z}_2)^{c_i}$, where $b_i, c_i \in \mathbb{N} \cup \{0\}$. □

By the above lemma and the universal coefficients theorem, it sufficient to consider $H^*(M; \mathbb{Q})$ and $H^*(M; \mathbb{Z}_2)$.

Lemma 4.1.4 ([3])**.** $b_i(\widehat{M}) = b_i(M) + b_{2n-i}(M)$.

Corollary 4.1.1. $b_{2i+1}(M) = 0 \quad$ *for* $1 \leq i \leq n-2$.

Proof. We know that $b_{2i+1}(\widehat{M}) = 0$ for $1 \leq i \leq n-2$ by Theorem 3.1.1, so $b_{2i+1}(M) = 0 =$ for $1 \leq i \leq n-2$ by Lemma 4.1.4. □

Since M is a toric origami manifold with coörientable folding hypersurface, topologically M is obtained by equivariant connected sums of toric symplectic manifolds along their T-invariant divisors. However for a non-orientable manifold, we can not apply Poincaré duality with $\mathbb{Q}$ coefficients.

To fix our notations, we recall the arguments at the beginning of section 3.1. Let M be a toric origami manifold of dimension $2n$ with coörientable folding hypersurface. Let Z be a component of the folding hypersurface, F be the corresponding folded facet in the origami template of M and let B be the symplectic toric manifold corresponding to F. The normal line bundle of Z to M is trivial so that an invariant closed tubular neighborhood of Z in M can be identified with $Z\times[-1,1]$. We set

$$\tilde{M} := M - \mathrm{Int}(Z \times [-1,1]).$$

This has two boundary components which are copies of Z. We close $\tilde{M}$ by gluing two copies of the disk bundle associated to the principal S^1-bundle $Z \to B$ along their boundaries. The resulting closed manifold (possibly disconnected), denoted M', is again a toric origami manifold.

Let G be the graph associated to the origami template of M. A folded facet in the origami template of M corresponds to an edge of G. We assume that $b_1(G) \geq 1$. We choose an edge e in a (non-trivial) cycle of G and let F, Z and B be respectively the folded facet, the fold and the symplectic toric manifold corresponding to the edge e. Then M' is connected and the graph G' associated to M' is nothing but the graph G with the edge e removed, so $b_1(G') = b_1(G) - 1$.

Two copies of B lie in M' as closed submanifolds, denoted B_+ and B_-. Let N_+ (resp. N_-) be an invariant closed tubular neighborhood of B_+ (resp. B_-) and Z_+ (resp. Z_-) be the boundary of N_+ (resp. N_-). Note that $M' - \mathrm{Int}(N_+ \cup N_-)$ can naturally be identified with $\tilde{M}$, so that

$$\tilde{M} = M' - \mathrm{Int}(N_+ \cup N_-) = M - \mathrm{Int}(Z \times [-1,1])$$

and

$$M' = \tilde{M} \cup (N_+ \cup N_-), \qquad \tilde{M} \cap (N_+ \cup N_-) = Z_+ \cup Z_-, \quad (4.1.1)$$

$$M = \tilde{M} \cup (Z \times [-1, 1]), \qquad \tilde{M} \cap (Z \times [-1, 1]) = Z_+ \cup Z_-. \quad (4.1.2)$$

We shall investigate relations among Betti numbers of $M, M', \tilde{M}, Z$ and B. The spaces $\tilde{M}$ and Z are auxiliary ones and our aim is to find relations between Betti numbers of M, M' and B.

Since the proof of Lemma 3.1.1 and Lemma 3.1.2 do not use Poincaré duality, we also have the following two equations.

$$b_{2i}(Z) - b_{2i-1}(Z) = b_{2i}(B) - b_{2i-2}(B) \quad \text{for any } i. \quad (4.1.3)$$

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M') - b_{2i-1}(M') - 2b_{2i-2}(B) \quad \text{for any } i. \quad (4.1.4)$$

Lemma 4.1.5. $b_{2n-1}(M) = b_1(M) - 1$.

Proof. By Lemma 4.1.4 we have

$$b_1(M) + b_{2n-1}(M) = b_1(\widehat{M}). \quad (4.1.5)$$

Let $\widehat{G}$ be the graph associated to $\widehat{M}$, then

$$|V(\widehat{G})| = 2|V(G)|, \quad |E(\widehat{G})| = 2|E(G)|, \quad (4.1.6)$$

where $V(\widehat{G})$ and $E(\widehat{G})$ denote the sets of vertexes and edges of $\widehat{G}$ respectively, and $V(G)$ and $E(G)$ denote the same sets for G.

Since

$$|E(G)| - |V(G)| + 1 = \dim H^1(G), \quad (4.1.7)$$

we have

$$|E(\widehat{G})| - |V(\widehat{G})| + 1 = \dim H^1(\widehat{G}). \quad (4.1.8)$$

Hence by (4.1.6), (4.1.7) and (4.1.8), we have

$$\dim H^1(\widehat{G}) = 2\dim H^1(G) - 1. \tag{4.1.9}$$

Since proper faces of $\widehat{M}/T$ and M/T are acyclic, $\widehat{M}$ and M must have fixed points. By Proposition 2.3 in [16], we have

$$\pi_1(\widehat{M}) \cong \pi_1(\widehat{M}/T) \cong \pi_1(\widehat{G}),$$

which implies that

$$\dim H_1(\widehat{M}) = \dim H_1(\widehat{G}).$$

Therefore

$$b_1(\widehat{M}) = \dim H^1(\widehat{G}). \tag{4.1.10}$$

Similarly, we have

$$b_1(M) = \dim H^1(G). \tag{4.1.11}$$

By (4.1.9), (4.1.10) and (4.1.11), we have

$$b_1(\widehat{M}) = 2b_1(M) - 1. \tag{4.1.12}$$

Then by (4.1.5), we have

$$b_{2n-1}(M) = b_1(M) - 1.$$

This completes the proof of the lemma. □

Lemma 4.1.6. $b_1(M') = b_1(M) - 1, \quad b_{2n}(M') - b_{2n-1}(M') = 1 - b_{2n-1}(M)$.

Proof. The first equation follows from

$$b_1(M) = b_1(G), \quad b_1(M') = b_1(G') \quad \text{and} \quad b_1(G) = b_1(G') + 1.$$

Next, we show the second equation.

Case 1: The case where M' is orientable. By Poincaré duality we have

$$b_{2n-1}(M') = b_1(M') = b_1(M) - 1. \tag{4.1.13}$$

Hence

$$b_{2n}(M') - b_{2n-1}(M') = 1 - (b_1(M) - 1) = 1 - b_{2n-1}(M), \tag{4.1.14}$$

so the last equality follows from Lemma 4.1.5.

Case 2: The case where M' is non-orientable. Then

$$b_{2n-1}(M') = b_1(M') - 1,$$

follows from Lemma 4.1.5. Note that when M' is non-orientable,

$$b_{2n}(M') = 0,$$

so

$$b_{2n}(M') - b_{2n-1}(M') = 1 - b_1(M') = 1 - (b_1(M) - 1) = 1 - b_{2n-1}(M).$$

This completes the proof of the lemma. □

Lemma 4.1.7. $b_2(M') = b_2(M) + b_2(B) + 1$.

Proof. Consider the Mayer-Vietoris exact sequence in cohomology for the triple $(M, \tilde{M}, Z \times [-1,1])$:

$$\begin{aligned}
&\to H^{2i-2}(M) \to H^{2i-2}(\tilde{M}) \oplus H^{2i-2}(Z \times [-1,1]) \to H^{2i-2}(Z_+ \cup Z_-) \\
&\to H^{2i-1}(M) \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(Z \times [-1,1]) \to H^{2i-1}(Z_+ \cup Z_-) \\
&\to H^{2i}(M) \quad \to H^{2i}(\tilde{M}) \oplus H^{2i}(Z \times [-1,1]) \qquad \to H^{2i}(Z_+ \cup Z_-) \to .
\end{aligned}$$

Case 1: $n \geqslant 3$. Since $H^3(M) = 0$ by Corollary 4.1.1, the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, Z \times [-1,1])$ splits into short exact sequences:

$$
\begin{aligned}
0 \to & H^0(M) \to H^0(\tilde{M}) \oplus H^0(Z \times [-1,1]) \to H^0(Z_+ \cup Z_-) \\
& \to H^1(M) \to H^1(\tilde{M}) \oplus H^1(Z \times [-1,1]) \to H^1(Z_+ \cup Z_-) \\
& \to H^2(M) \to H^2(\tilde{M}) \oplus H^2(Z \times [-1,1]) \to H^2(Z_+ \cup Z_-) \to 0.
\end{aligned}
$$

Hence we have

$$b_2(\tilde{M}) - b_1(\tilde{M}) = b_2(M) - b_1(M) + b_2(Z) - b_1(Z) + 1. \qquad (4.1.15)$$

By (4.1.3) we have

$$b_2(\tilde{M}) - b_1(\tilde{M}) = b_2(M) - b_1(M) + b_2(B). \qquad (4.1.16)$$

By (4.1.4) we obtain

$$b_2(\tilde{M}) - b_1(\tilde{M}) = b_2(M') - b_1(M') - 2b_0(B). \qquad (4.1.17)$$

By (4.1.16) and (4.1.17), we get

$$b_2(M) - b_1(M) = b_2(M') - b_1(M') - 2b_0(B) - b_2(B). \qquad (4.1.18)$$

Since $b_1(M') = b_1(M) - 1$, we have

$$b_2(M') = b_2(M) + b_2(B) + 1. \qquad (4.1.19)$$

Case 2: $n = 2$. Consider the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, Z \times [-1,1])$:

$$
\begin{aligned}
0 \to & H^0(M) \to H^0(\tilde{M}) \oplus H^0(Z \times [-1,1]) \to H^0(Z_+ \cup Z_-) \\
\to & H^1(M) \to H^1(\tilde{M}) \oplus H^1(Z \times [-1,1]) \to H^1(Z_+ \cup Z_-) \\
\to & H^2(M) \to H^2(\tilde{M}) \oplus H^2(Z \times [-1,1]) \to H^2(Z_+ \cup Z_-) \\
\to & H^3(M) \to H^3(\tilde{M}) \oplus H^3(Z \times [-1,1]) \to H^3(Z_+ \cup Z_-) \to 0.
\end{aligned}
$$

Since M is non-orientable, $H^4(M;\mathbb{Q}) = 0$. Hence the last term in the above exact sequence is 0. By the above exact sequence, we have

$$
b_2(\tilde{M}) - b_1(\tilde{M}) - b_3(\tilde{M}) = b_2(M) - b_1(M) - b_3(M) + b_2(Z) - b_1(Z). \tag{4.1.20}
$$

Note that $\tilde{M}$ is a manifold with boundary, so

$$
b_4(\tilde{M}) = 0. \tag{4.1.21}
$$

By (4.1.3), (4.1.4), (4.1.20) and (4.1.21) we have

$$
\begin{aligned}
& b_2(M') - b_1(M') - 2b_0(B) + b_4(M') - b_3(M') - 2b_2(B) \\
= & b_2(M) - b_1(M) - b_3(M) + b_2(B) - b_0(B).
\end{aligned} \tag{4.1.22}
$$

We know that when $n = 2$, $B = \mathbb{C}P^1$, so (4.1.22) reduces to

$$
b_2(M') - b_1(M') + b_4(M') - b_3(M') - 4 = b_2(M) - b_1(M) - b_3(M). \tag{4.1.23}
$$

By Lemma 4.1.6 and (4.1.23) we have

$$
b_2(M') = b_2(M) + 2 = b_2(M) + b_2(B) + 1.
$$

This completes the proof of the lemma. $\square$

Lemma 4.1.8.

$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B)$ *for* $2 \le i \le n-2$ *and* $n \geqslant 4$

$b_4(M') = b_4(M) + b_4(B) + b_2(B)$ *for* $n = 3$.

Proof. First, consider the case $n \geq 4$.

Since $H^{2i-1}(M;\mathbb{Q}) = 0$ for $2 \leq i \leq n-2$, we have

$$\begin{aligned} 0 \quad &\to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(Z \times [-1,1]) \to H^{2i-1}(Z_+ \cup Z_-) \\ \to H^{2i}(M) \to H^{2i}(\tilde{M}) \oplus H^{2i}(Z \times [-1,1]) \quad &\to H^{2i}(Z_+ \cup Z_-) \to 0, \end{aligned}$$

By the above exact sequence and (4.1.3) we have

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) + b_{2i}(B) - b_{2i-2}(B) \quad \text{for } 2 \leq i \leq n-2. \tag{4.1.24}$$

By the above equation and (4.1.4) we have

$$b_{2i}(M') - b_{2i-1}(M') - 2b_{2i-2}(B) = b_{2i}(M) + b_{2i}(B) - b_{2i-2}(B) \quad \text{for } 2 \leq i \leq n-2. \tag{4.1.25}$$

Since the folding hypersurface of M'/T is coörientable and each proper face of M'/T is acyclic, $b_{2i-1}(M') = 0$ for $2 \leq i \leq n-1$ by the same argument as in the proof of Corollary 4.1.1. Hence (4.1.25) reduces to

$$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \text{for } 2 \leq i \leq n-2. \tag{4.1.26}$$

Next, we consider the case $n = 3$.

By Corollary 4.1.1, the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, Z \times [-1,1])$ splits into

$$\begin{aligned} &\to 0 \to H^3(\tilde{M}) \oplus H^3(Z \times [-1,1]) \to H^3(Z_+ \cup Z_-) \\ &\to H^4(M) \to H^4(\tilde{M}) \oplus H^4(Z \times [-1,1]) \to H^4(Z_+ \cup Z_-) \\ &\to H^5(M) \to H^5(\tilde{M}) \oplus H^5(Z \times [-1,1]) \to H^5(Z_+ \cup Z_-) \\ &\to 0. \end{aligned}$$

Since M is non-orientable, $H^6(M;\mathbb{Q}) = 0$. Hence the last term in the above exact sequence is 0. Thus we have

$$b_4(\tilde{M})-b_3(\tilde{M})-b_5(\tilde{M}) = b_4(M)-b_5(M)+b_4(Z)-b_3(Z)-b_5(Z) \quad (4.1.27)$$

Note that $\tilde{M}$ is a manifold with boundary, so

$$b_6(\tilde{M}) = 0. \quad (4.1.28)$$

By (4.1.3), (4.1.4), (4.1.27) and (4.1.28), we have

$$\begin{aligned} &b_4(M') - b_3(M') - 2b_2(B) + b_6(M') - b_5(M') - 2b_4(B) \\ =&b_4(M) - b_5(M) + b_4(B) - b_2(B) - b_5(Z). \end{aligned} \quad (4.1.29)$$

Since $b_3(M') = 0$, $b_5(Z) = 1$ and $b_4(B) = 1$, (4.1.29) reduces to

$$b_4(M') - b_2(B) + b_6(M') - b_5(M') - 2 = b_4(M) - b_5(M). \quad (4.1.30)$$

We know that $b_6(M') - b_5(M') = 1 - b_5(M)$ by Lemma 4.1.6, so by (4.1.30), we have

$$b_4(M') = b_4(M) + b_4(B) + b_2(B).$$

This completes the proof of the lemma. □

Lemma 4.1.9.

$$b_{2n-2}(M') - b_{2n-4}(B) = b_{2n-2}(M) + 1$$

i.e.,

$$b_{2n-2}(M') = b_{2n-2}(M) + b_{2n-4}(B) + b_{2n-2}(B).$$

Proof. We consider two cases to prove the lemma.

Case 1: $n = 2$. In this case $B = \mathbb{C}P^1$, so the lemma follows from Lemma 4.1.7.

Case 2: $n \geq 3$. We know that $H^{2n-3}(M) = 0$ by Corollary 4.1.1 and $H^{2n}(M) = 0$ since M is non-orientable, so the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, Z \times [-1,1])$ splits into

$$\begin{aligned}
&\to 0 \to H^{2n-3}(\tilde{M}) \oplus H^{2n-3}(Z \times [-1,1]) \to H^{2n-3}(Z_+ \cup Z_-) \\
&\to H^{2n-2}(M) \to H^{2n-2}(\tilde{M}) \oplus H^{2n-2}(Z \times [-1,1]) \to H^{2n-2}(Z_+ \cup Z_-) \\
&\to H^{2n-1}(M) \to H^{2n-1}(\tilde{M}) \oplus H^{2n-1}(Z \times [-1,1]) \to H^{2n-1}(Z_+ \cup Z_-) \\
&\to 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&b_{2n-2}(\tilde{M}) - b_{2n-3}(\tilde{M}) - b_{2n-1}(\tilde{M}) \\
=&b_{2n-2}(M) - b_{2n-1}(M) - b_{2n-3}(Z) + b_{2n-2}(Z) - b_{2n-1}(Z).
\end{aligned} \tag{4.1.31}$$

Since $\tilde{M}$ is a manifold with boundary,

$$b_{2n}(\tilde{M}) = 0. \tag{4.1.32}$$

By (4.1.3), (4.1.4), (4.1.31) and (4.1.32), we have

$$\begin{aligned}
&b_{2n-2}(M') - b_{2n-3}(M') - 2b_{2n-4}(B) + b_{2n}(M') - b_{2n-1}(M') - 2b_{2n-2}(B) \\
=&b_{2n-2}(M) - b_{2n-1}(M) + b_{2n-2}(B) - b_{2n-4}(B) - b_{2n-1}(Z).
\end{aligned} \tag{4.1.33}$$

Note that $b_{2n-2}(B) = 1$, $b_{2n-1}(Z) = 1$, and $H^{2n-3}(M'; \mathbb{Q}) = 0$, so by (4.1.33) we obtain that

$$b_{2n-2}(M') + b_{2n}(M') - b_{2n-1}(M') - b_{2n-4}(B) = b_{2n-2}(M) - b_{2n-1}(M) + 2. \tag{4.1.34}$$

Hence the lemma follows from (4.1.34) and Lemma 4.1.6. This completes the proof of this lemma. □

In summary, by Corollary 4.1.1, Lemma 4.1.5, Lemma 4.1.7, Lemma 4.1.8, and Lemma 4.1.9 we obtain the following.

Lemma 4.1.10. *Let M be a non-orientable toric origami manifold of dimension $2n$ $(n \geq 2)$ with coörientable folding hypersurface such that every proper face of M/T is acyclic. Then*

$$\begin{aligned} & b_{2i+1}(M) = 0 \quad \text{for } 1 \leq i \leq n-2. \\ & b_1(M) = b_1(M/T), \quad b_{2n-1}(M) = b_1(M/T) - 1, \end{aligned} \tag{4.1.35}$$

Moreover, if M' and B are as above, then

$$\begin{aligned} & b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-2}(B) \quad \text{for } 1 \leq i \leq n-1, \\ & b_0(M) = 1, \quad b_{2n}(M) = 0. \end{aligned} \tag{4.1.36}$$

Since all the relations among $b_{2i}(M)$ and $b_{2i}(M')$ for $i \leq n-1$ are the same as (3.1.6) in Theorem 3.1.1, we have the following theorem.

Theorem 4.1.1. *Let M be a non-orientable toric origami manifold of dimension $2n$ $(n \geq 2)$ with coörientable folding hypersurface such that every proper face of M/T is acyclic. Let b_j be the j-th Betti number of M with $\mathbb{Q}$ coefficients and $(h_0, h_1, \dots, h_n)$ be the h-vector of M/T. Then*

$$b_{2i} = h_i - (-1)^i \binom{n}{i} b_1 \quad \text{for } 1 \leq i \leq n-1.$$

When we consider the Betti numbers of M with $\mathbb{Z}_2$ coefficients, we can use Poincaré duality for non-orientable manfolds, so all the arguments are the same as the orientable case.

Theorem 4.1.2. *Let M be a non-orientable toric origami manifold of dimension $2n$ ($n \geq 2$) with coörientable folding hypersurface such that every proper face of M/T is acyclic. Let b_j be the j-th Betti number of M with $\mathbb{Z}_2$ coefficients and $(h_0, h_1, \ldots, h_n)$ be the h-vector of M/T. Then*

$$\sum_{i=0}^{n} b_{2i} t^i = \sum_{i=0}^{n} h_i t^i + b_1(1 + t^n - (1-t)^n),$$

in other words, $b_0 = h_0 = 1$ and

$$b_{2i} = h_i - (-1)^i \binom{n}{i} b_1 \quad \text{for } 1 \leq i \leq n-1,$$

$$b_{2n} = h_n + (1 - (-1)^n) b_1,$$

$$b_1 = b_{2n-1} = b_1(M/T),$$

$$b_{2i+1} = 0 \qquad \text{for } 1 \leq i \leq n-2.$$

By Lemma 4.1.10, Theorem 4.1.1, Theorem 4.1.2 and the universal coefficients theorem, we obtain the following.

Theorem 4.1.3. *Let M be a non-orientable toric origami manifold of dimension $2n$ ($n \geq 2$) with coörientable folding hypersurface such that every proper face of M/T is acyclic. Let b_j be the j-th Betti number of M with $\mathbb{Z}$ coefficients and $(h_0, h_1, \ldots, h_n)$ be the h-vector of M/T. Then $H^i(M)$ is torsion free for $i \leq 2n-1$.*
Moreover,

$$b_0 = h_0 = 1,$$

$$b_{2i} = h_i - (-1)^i \binom{n}{i} b_1 \quad \text{for } 1 \leq i \leq n-1,$$

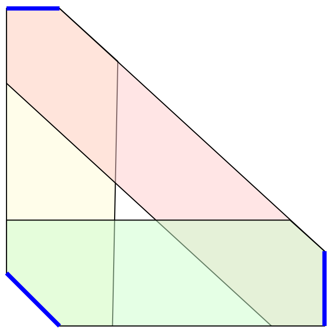
$$b_{2i-1}(M) = 0 \qquad \textit{for } 2 \le i \le n-1,$$

$$b_1 = b_1(M/T),$$

$$b_{2n-1} = b_1(M/T) - 1,$$

$$H^{2n}(M) \cong \mathbb{Z}_2.$$

Example 4.1.1. Consider the following origami template, and let M denote the toric origami manifold corresponding to this template.



From this template, we can see that the f-vector is

$$(f_0, f_1) = (6, 6),$$

so the h-vector is

$$(h_0, h_1, h_2) = (1, 4, 1).$$

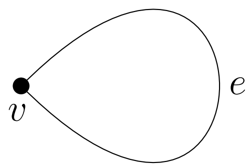
By Theorem 4.1.3, we have

$$H^0(M) = \mathbb{Z},\ H^1(M) = \mathbb{Z},\ H^2(M) = \mathbb{Z}^6,\ H^3(M) = 0,\ H^4(M) = \mathbb{Z}_2.$$

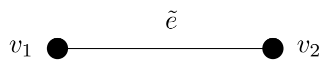
4.2 The non-coörientable case

In this section, we will discuss the cohomology groups of a non-orientable toric origami manifold M with a non-coörientable folding hypersurface Z and we assume connectedness of Z.

Let (G, Ψ_V, Ψ_E) denote the origami template of M such that the associated graph G has only one vertex v and a loop e.



Set $\Psi_V(v) = P$, $\Psi_E(e) = F$. Let M' and B be the symplectic toric manifolds corresponding to P and F respectively. Consider the graph $\widehat{G} = (\widehat{V}, \widehat{E})$, where $\widehat{V} = \{v_1, v_2\}$ and $\widehat{E} = \{\tilde{e}\} = \{(v_1, v_2)\}$.



Then we can construct a new origami template $(\widehat{G},\ \Psi_{\widehat{V}},\ \Psi_{\widehat{E}})$ such that

$$\Psi_{\widehat{V}}(v_1) = \Psi_{\widehat{V}}(v_2) = \Psi_V(v) = P$$

and

$$\Psi_{\widehat{E}}(\tilde{e}) = \Psi_E(e) = F.$$

Let $\widehat{M}$ be the toric origami manifold corresponding to the origami template $(\widehat{G},\ \Psi_{\widehat{V}},\ \Psi_{\widehat{E}})$. Since $\widehat{G}$ is 2-colorable, the origami template $(\widehat{G},\ \Psi_{\widehat{V}},\ \Psi_{\widehat{E}})$ is orientable. Hence $\widehat{M}$ is orientable by Theorem 2.6.1. Topologically, $\widehat{M}$ is just the equivariant connected sum of two copies of M' along the submanifold B. Let $\tilde{N}$ be an invariant closed tubular neighborhood of B in M' with boundary $\tilde{Z}$. Let N be an invariant closed tubular neighborhood of Z. By the radial blow-up operation in [7] for the non-coörientable case, $\tilde{Z}$ is a double covering but not an orientation covering of Z while $\widehat{M}$ is an orientation covering of M.

Set

$$\tilde{M} := M - \mathrm{Int}(N),$$

then

$$M = \tilde{M} \cup N, \qquad \tilde{M} \cap N = \tilde{Z}, \tag{4.2.1}$$

On the other hand,

$$\tilde{M} = M' - \mathrm{Int}(\tilde{N}),$$

and

$$M' = \tilde{M} \cup \tilde{N}, \qquad \tilde{M} \cap \tilde{N} = \tilde{Z}, \tag{4.2.2}$$

Since Z and $\tilde{Z}$ are orientable S^1-bundles over B, we obtain the following two equations by the same reason of Lemma 3.1.1

Lemma 4.2.1. *$b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) = b_{2i}(B) - b_{2i-2}(B)$ for any i.*

Lemma 4.2.2. *$b_{2i}(Z) - b_{2i-1}(Z) = b_{2i}(B) - b_{2i-2}(B)$ for any i.*

Since each face of the orbit space $\widehat{M}/T$ is acyclic, $H^*(\widehat{M})$ is torsion free by [15] or [12]. By the same argument as in the proof of Lemma 4.1.3, we have

$$H^i(M) = \mathbb{Z}^{b_i} \bigoplus (\mathbb{Z}_2)^{c_i}$$

for some $b_i, c_i \in \mathbb{N} \cup \{0\}$.

Hence, it is sufficient for us to consider $H^*(M;\mathbb{Q})$ and $H^*(M;\mathbb{Z}_2)$.

Proposition 4.2.1. *The $\mathbb{Q}$ coefficients Betti numbers between M and M' have the following relationship:*

$$b_{2i-1}(M) = 0 \quad \textit{for } 1 \leq i \leq n. \tag{4.2.3}$$

$$b_{2i}(M) = b_{2i}(M') - b_{2i-2}(B) \quad \textit{for } 1 \leq i \leq n. \tag{4.2.4}$$

Proof. Since each face of the orbit space $\widehat{M}/T$ is acyclic, $H^{2i-1}(\widehat{M}) = 0$ by [15] or [12]. Thus $H^{2i-1}(M;\mathbb{Q}) = 0$ by Lemma 4.1.4. Hence the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, N)$ splits into:

$$\begin{aligned} 0 &\to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(N) \to H^{2i-1}(\tilde{Z}) \\ &\to H^{2i}(M) \to H^{2i}(\tilde{M}) \oplus H^{2i}(N) \to H^{2i}(\tilde{Z}) \to 0. \end{aligned}$$

By the above short exact sequence, we have

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) + b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) - b_{2i}(N) + b_{2i-1}(N) \tag{4.2.5}$$

Since N is a line bundle over Z, N is homotopy equivalent to Z. Hence, we have

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M), \tag{4.2.6}$$

by Lemma 4.2.1 and Lemma 4.2.2. In fact,

$$b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) = b_{2i}(B) - b_{2i-2}(B)$$

and

$$b_{2i}(N) - b_{2i-1}(N) = b_{2i}(Z) - b_{2i-1}(Z) = b_{2i}(B) - b_{2i-2}(B),$$

so

$$b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) = b_{2i}(N) - b_{2i-1}(N).$$

Since $H^{2i-1}(M') = 0$ the Mayer-Vietoris exact sequence for the triple $(M', \tilde{M}, \tilde{N})$ splits into:

$$0 \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(\tilde{N}) \to H^{2i-1}(\tilde{Z})$$
$$\to H^{2i}(M') \to H^{2i}(\tilde{M}) \oplus H^{2i}(\tilde{N}) \to H^{2i}(\tilde{Z}) \to 0$$

We have

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M') + b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) - b_{2i}(\tilde{N}) + b_{2i-1}(\tilde{N}). \tag{4.2.7}$$

Since $\tilde{N}$ is homotopy equivalent to B, $b_{2i-1}(\tilde{N}) = 0$. Hence this proposition follows from Lemma 4.2.1, (4.2.6) and (4.2.7). □

Proposition 4.2.2. *The $\mathbb{Z}_2$ coefficients Betti numbers between of M and M' have the following relationship:*

$$b_{2i-1}(M) = b_{2i-2}(B) \quad \textit{for } 1 \leq i \leq n. \tag{4.2.8}$$

$$b_{2i}(M) = b_{2i}(M') \quad \textit{for } 1 \leq i \leq n. \tag{4.2.9}$$

Proof. Note that the map

$$H^{2i}(\tilde{M}) \to H^{2i}(\tilde{Z})$$

is surjective. In fact since B is a deformation retract of $\tilde{N}$, the following diagram is commutative:

$$\begin{array}{ccc} H^{2j}(M') & \longrightarrow & H^{2j}(B) \\ \downarrow & & \downarrow \pi^* \\ H^{2j}(\tilde{M}) & \longrightarrow & H^{2j}(\tilde{Z}) \end{array}$$

where $\pi : \tilde{Z} \to B$ is the projection and the other homomorphisms are induced from the inclusions. By (3.1.3) π^* is surjective, and since M' is a toric symplectic manifold, the homomorphism $H^{2j}(M') \to H^{2j}(B)$ is surjective. Hence

$$H^{2i}(\tilde{M}) \to H^{2i}(\tilde{Z})$$

is surjective. Therefore the Mayer-Vietoris exact sequence for the triple $(M, \tilde{M}, N)$ splits into:

$$\begin{aligned} 0 \to & H^{2i-1}(M) \to H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(N) \to H^{2i-1}(\tilde{Z}) \\ \to & H^{2i}(M) \to H^{2i}(\tilde{M}) \oplus H^{2i}(N) \to H^{2i}(\tilde{Z}) \to 0 \end{aligned}$$

Then we have

$$\begin{aligned} & b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) \\ = & b_{2i}(M) - b_{2i-1}(M) + b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) \\ - & b_{2i}(N) + b_{2i-1}(N). \end{aligned} \tag{4.2.10}$$

Note that N is homotopy equivalent to Z and Lemma 4.2.1 and Lemma 4.2.2 also hold for $\mathbb{Z}_2$ coefficients, so (4.2.10) reduces to

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M) - b_{2i-1}(M). \tag{4.2.11}$$

Since $H^{2i-1}(M'; \mathbb{Z}_2) = 0$, the Mayer-Vietoris exact sequence for the triple $(M', \tilde{M}, \tilde{N})$ splits into:

$$\begin{aligned} 0 \to & H^{2i-1}(\tilde{M}) \oplus H^{2i-1}(\tilde{N}) \to H^{2i-1}(\tilde{Z}) \\ \to & H^{2i}(M') \to H^{2i}(\tilde{M}) \oplus H^{2i}(\tilde{N}) \to H^{2i}(\tilde{Z}) \to 0 \end{aligned}$$

We have

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M') + b_{2i}(\tilde{Z}) - b_{2i-1}(\tilde{Z}) - b_{2i}(\tilde{N}) + b_{2i-1}(\tilde{N}). \tag{4.2.12}$$

$\tilde{N}$ is homotopy equivalent to B, so $H^{2i-1}(\tilde{N};\mathbb{Z}_2) = 0$ and $b_{2i}(\tilde{N}) = b_{2i}(B)$. Then by Lemma 4.2.1, we have

$$b_{2i}(\tilde{M}) - b_{2i-1}(\tilde{M}) = b_{2i}(M') - b_{2i-2}(B). \tag{4.2.13}$$

By (4.2.11) and (4.2.13) we obtain

$$b_{2i}(M) - b_{2i-1}(M) = b_{2i}(M') - b_{2i-2}(B). \tag{4.2.14}$$

We claim that for any i

$$b_{2n-2i}(M) = b_{2n-2i}(M'), \qquad b_{2n-2i-1}(M) = b_{2n-2i-2}(B). \tag{4.2.15}$$

We show (4.2.15) by induction on i.

When $i = 0$, $b_{2n}(M;\mathbb{Z}_2) = b_{2n}(M';\mathbb{Z}_2) = 1$, so by (4.2.14) we obtain that

$$b_{2n-1}(M) = b_{2n-2}(B).$$

Suppose that for $i \leq k$, we have

$$b_{2n-2i}(M) = b_{2n-2i}(M'), \qquad b_{2n-2i-1}(M) = b_{2n-2i-2}(B).$$

Then by Poincaré duality, we have

$$b_{2k+1}(M) = b_{2k}(B).$$

By (4.2.14), we have

$$b_{2k+2}(M) - b_{2k+1}(M) = b_{2k+2}(M') - b_{2k}(B).$$

Hence,

$$b_{2k+2}(M) = b_{2k+2}(M').$$

By Poincaré duality, we obtain

$$b_{2n-2k-2}(M) = b_{2n-2k-2}(M').$$

Using (4.2.14) again, we have

$$b_{2n-2k-3}(M) = b_{2n-2k-4}(B).$$

Therefore, for $i = k+1$, (4.2.15) also holds. This completes the proof of the proposition. $\square$

Example 4.2.1. The toric origami manifold corresponding to the left origami template is $M = \mathbb{R}P^4$ and it easy to see that $M' = \mathbb{C}P^2$ and $B = \mathbb{C}P^1$.

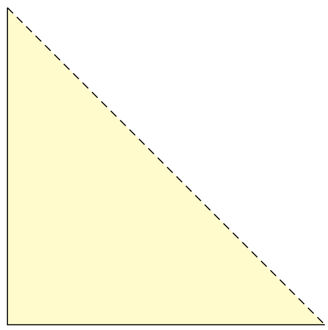


Figure 4.1: $M = \mathbb{R}P^4$

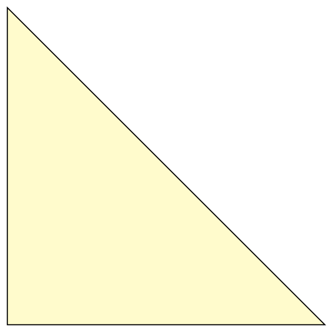


Figure 4.2: $M' = \mathbb{C}P^2$

For $\mathbb{Q}$ coefficients,

$$b_1(\mathbb{R}P^4) = b_3(\mathbb{R}P^4) = 0,$$
$$b_2(\mathbb{R}P^4) = b_2(\mathbb{C}P^2) - b_0(\mathbb{C}P^1) = 0,$$
$$b_4(\mathbb{R}P^4) = b_4(\mathbb{C}P^2) - b_2(\mathbb{C}P^1) = 0,$$
$$b_0(\mathbb{R}P^4) = b_0(\mathbb{C}P^2) = 1.$$

For $\mathbb{Z}_2$ coefficients,

$$b_1(\mathbb{R}P^4) = b_0(\mathbb{C}P^1) = 1,$$

$$b_3(\mathbb{R}P^4) = b_2(\mathbb{C}P^1) = 1,$$

$$b_2(\mathbb{R}P^4) = b_2(\mathbb{C}P^2) = 1,$$

$$b_4(\mathbb{R}P^4) = b_4(\mathbb{C}P^2) = 1,$$

$$b_0(\mathbb{R}P^4) = b_0(\mathbb{C}P^2) = 1.$$

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