Minimization problem on the Hardy-Sobolev inequality and on the Sobolev embedding for Lebesgue space with variable exponent

（Hardy-Sobolev 不等式に関連する最小化問題及び
変数指数Lebesgue空間に関するSobolev埋め込みについて）

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1 Introduction

The main topic of this thesis is the Hardy-Sobolev inequality

$$\mu_{s,\lambda}^N(\Omega)\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)} \leq \int_\Omega (|\nabla u|^2+\lambda u^2)dx.$$

The bounded embedding $H^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |x|^{-s}dx)$ is non-compact when $0 \in \overline{\Omega}$, and thus existence of the minimizer of the largest possible constant $\mu_{s,\lambda}^N(\Omega)$ is non-trivial. Concerning this problem, the position of the origin in bounded domain Ω plays a crucial role. Moreover, non-invariance of the scale of the domain is also important. This is completely different from the best constant on the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |x|^{-s}dx)$. Related to the scale, we can see that the inequality with the pair (λ^2, Ω) is equivalent to that with the pair $(1, \lambda\Omega)$, and thus we can control the scale by the positive parameter λ.

In section 2, we consider interior singularity case, that is $0 \in \Omega$. In this case, we show existence and non-existence of the minimizer depending on the parameter λ. More precisely, the borderline exists uniquely and the minimizer exists when λ is less than the borderline, and does not exists when λ is greater than the borderline.

The situation of boundary singularity case ($0 \in \partial\Omega$) is more complicated. The mean curvature at the origin is important role. In the positive mean curvature case, the existence result is obtained. We study the non-positive mean curvature case in section 3 and section 4. In this case, assuming that the dimension is greater than 3, we prove existence and non-existence of the minimizer depending on the parameter λ. These results are same as section 2 and different from that in the positive case. In addition, if λ is on the borderline, we can obtain existence result. Thus the minimizer exists if and only if λ is less than or equal to the borderline. In order to prove these results, we use technique of the blow-up analysis. We consider the asymptotic behavior of the least-energy solutions for the corresponding elliptic equation as $\lambda \to \infty$. By obtaining the fine properties of the least-energy solutions, we can obtain the asymptotic behavior of $\mu_{s,\lambda}^N(\Omega)$. We can show existence and non-existence result by using the behavior of $\mu_{s,\lambda}^N(\Omega)$. In section 3, we prove uniqueness of the minimizer other than these above results. We can prove this result by considering the asymptotic behavior of the least-energy solutions as $\lambda \to 0$.

Section 5 is a joint work with Megumi Sano (Osaka City University). In section 5, we change topic and consider the embedding $H^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. The function space $L^{q(\cdot)}(\Omega)$ is the Lebesgue space with a variable exponent. In bounded domain case, many researchers have studied so far. In section 5, we study compactness and non-compactness of the embedding $W_{rad}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$. Related to the property of a variable exponent q, we found the borderline on compactness and non-compactness explicitly in a certain sense. As an application of our results, we show existence of the solution for quasilinear elliptic problem involving a variable exponent.

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Contents

2 Minimization problem on the Hardy-Sobolev inequality

Abstract

We study minimization problems on Hardy-Sobolev type inequality. We consider the case where singularity is in interior of bounded domain $\Omega \subset \mathbb{R}^N$. The attainability of best constants for Hardy-Sobolev type inequalities with boundary singularities have been studied so far, for example [5] [6] [9] etc.... According to their results, the mean curvature of $\partial\Omega$ at singularity affects the attainability of the best constants. In contrast with boundary singularity case, in interior singularity case it is well known that the best Hardy-Sobolev constant

$$\mu_s(\Omega) := \left\{ \int_\Omega |\nabla u|^2 dx \middle| u \in H_0^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\}$$

is never achieved for all bounded domain Ω. We can see that the position of singularity on domain is related to the existence of minimizer. In this section, we consider the attainability of the best constant for the embedding $H^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |x|^{-s}dx)$ for bounded domain Ω with $0 \in \Omega$. In this problem, scaling invariance doesn't hold and we can not obtain information of singularity like mean curvature.

2.1 Introduction

We study minimization problems on the Hardy-Sobolev type inequalities. Let $N \geq 3$, Ω is a bounded domain in $\mathbb{R}^N$, $\partial\Omega$ satisfies the cone property, $0 < s < 2$, and $2^*(s) := 2(N-s)/(N-2)$. Unless otherwise stated, we assume that $0 \in \Omega$. The Hardy-Sobolev inequality asserts that there exists a positive constant C such that

$$C \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega |\nabla u|^2 dx \tag{1}$$

for all $u \in H_0^1(\Omega)$. For $s = 0$, the inequality (1) is called Sobolev inequality and for $s = 2$, the inequality (1) is called Hardy inequality.

In the non-singular case ($s = 0$), it is well known that the best Sobolev constant S is independent of domain Ω and S is never achieved for all bounded domains. But if $\Omega = \mathbb{R}^N$ and $H_0^1(\Omega)$ is replaced by the function space of $u \in L^{2N/(N-2)}(\Omega)$ with $\nabla u \in L^2(\Omega)$, then S is achieved by the function $u(x) = c(1+|x|^2)^{(2-N)/2}$ and hence the value $S = N(N-2)\pi[\Gamma(N/2)/\Gamma(N)]^{2/N}$ explicitly (see [1], [13] and [16]).

In the case of $s = 2$, the best constant for the Hardy inequality is $[(N-2)/2]^2$ and this constant is never achieved for all bounded domains and $\mathbb{R}^N$. This fact suggests that it is possible to improve this inequality. For example Brezis and Vazquez [2], many people research the optimal inequality of (1). In other words, the best remainder term for (1) is studied actively.

In the case of $0 < s < 2$, the best Hardy-Sobolev constant is defined by

$$\mu_s(\Omega) := \left\{ \int_\Omega |\nabla u|^2 dx \middle| u \in H_0^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\}.$$

This constant has some similar properties to these of the best Sobolev constant. Indeed, due to scaling invariance, $\mu_s(\Omega)$ is independent of Ω, and thus $\mu_s := \mu_s(\Omega) = \mu_s(\mathbb{R}^N)$ is not attained for all bounded domains. If $\Omega = \mathbb{R}^N$, then μ_s is attained by

$$y_a(x) = [a(N-s)(N-2)]^{\frac{N-2}{2(2-s)}} \left(a + |x|^{2-s}\right)^{\frac{2-N}{2-s}}$$

for some $a > 0$ and hence

$$\mu_s = (N-2)(N-s) \left(\frac{\omega_{N-1}}{2-s} \frac{\Gamma^2(\frac{N-s}{2-s})}{\Gamma(\frac{2(N-s)}{2-s})} \right)^{\frac{2-s}{N-s}} \tag{2}$$

(see [9] and [13]) where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$.

In boundary singularity case, the result of the attainability for $\mu_s(\Omega)$ is quite different from that in the situation of $0 \in \Omega$. By Ghoussoub and Robert [6], it has proved that if Ω has smooth boundary and the mean curvature of $\partial\Omega$ at 0 is negative, then the extremal of $\mu_s(\Omega)$ exists for all $N \geq 3$. Recently, Lin and Wadade [14] have studied the following minimization problem;

$$\mu_{s,p}^\lambda(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 dx + \lambda \left(\int_\Omega |u|^p dx \right)^{\frac{2}{p}} \middle| u \in H_0^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\}$$

where $\lambda \in \mathbb{R}$ and $2 \leq p \leq 2N/(N-2)$. Furthermore, as related results, Hsia, Lin and Wadade [10] studied the existence of the solution of double critical elliptic equations related with $\mu_{s,2^*}^\lambda(\Omega)$, that is, they have showed the existence of the solution for

$$\begin{cases} -\Delta u + \lambda u^{2^*-1} + \frac{u^{2^*(s)-1}}{|x|^s} = 0, \quad u > 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

under the appropriate conditions where $2^* = 2N/(N-2)$. To prove these results, we use the theorem of Egnell [4]. He showed that the existence of the extremal for $\mu_s(\Omega)$ if Ω is a half space $\mathbb{R}_+^N$ or an open cone. The open cone $\mathcal{C}$ is written of the form $\mathcal{C} := \{x \in \mathbb{R}^N | x = r\theta,\ \theta \in \Sigma\}$ where Σ is connected domain on the unit sphere $\mathcal{S}^{N-1}$ in $\mathbb{R}^N$. By this result, we can see that $\mu_s(\mathcal{C}) > \mu_s(\mathbb{R}^N)$ and there is a positive solution for

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(s)-1}}{|x|^s} \quad \text{in } \mathcal{C}, \\ u = 0 \quad \text{on } \partial\mathcal{C}, \quad \text{and} \quad u(x) = o(|x|^{2-N}) \text{ as } x \to \infty. \end{cases}$$

The Neumann case also has been studied. Let Ω has C^2 boundary and the mean curvature of $\partial\Omega$ at 0 is positive. Ghoussoub and Kang [5] have showed that there is a least energy solution for

$$\begin{cases} -\Delta u + \lambda u = \frac{|u|^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for $N \geq 3$, $\lambda > 0$.

Like these results, if $0 \in \partial\Omega$, we can use the benefit of the mean curvature of $\partial\Omega$ at 0 to show the results. However if $0 \in \Omega$, we cannot obtain the information of singularity such the mean curvature, and the fact causes some technical difficulties.

In this section, we consider the attainability for the following minimization problem

$$\mu_{s,\lambda}^N(\Omega) := \inf \left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2) dx \middle| u \in H^1(\Omega), \ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\},$$

where λ is a positive parameter. This parameter means the scale of Ω. Actually $\mu_{s,1}^N(\sqrt{\lambda}\Omega) = \mu_{s,\lambda}^N(\Omega)$ and attainability of $\mu_{s,1}^N(\sqrt{\lambda}\Omega)$ is equivalent to attainability of $\mu_{s,\lambda}^N(\Omega)$. The main theorem is as follows:

Theorem 2.1. *There exist a positive constant $\lambda_* = \lambda_*(\Omega)$ such that the following statements hold.*

(I) $\mu_{s,\lambda}^N(\Omega)$ *is attained for any* $0 < \lambda < \lambda_*$

(II) $\mu_{s,\lambda}^N(\Omega)$ *is not attained for any* $\lambda > \lambda_*$.

(III) We have

$$\lambda_* \geq \mu_s \left(\int_\Omega |x|^{-s} dx \right)^{\frac{2}{2^*(s)}} |\Omega|^{-1}$$

where μ_s is defined in (2), $|\Omega|$ is the N-dimensional Lebesgue measure of domain Ω.

The rest of this section is organized as follows. In 2.2 we introduce three lemmas to prove Theorem 2.1. Then in 2.3 we prove Theorem 2.1 using the lemmas in 2.2. In Section 2.4, as an application, we consider the case when the singularity is on the boundary of domain. Then we introduce a new result concerning the attainability of $\mu_{s,\lambda}^N(\Omega)$ in boundary singularity case.

2.2 Preparation

In this subsection, we prepare some lemmas to prove Theorem 2.1.

Lemma 2.2. *(i)* $\mu_{s,\lambda}^N(\Omega) \leq \mu_s$ *holds for any* $\lambda > 0$.

(ii) $\mu^N_{s,\lambda}(\Omega)$ *is continuous and non-decreasing with respect to* λ.

(iii) $\lim_{\lambda\to 0}\mu^N_{s,\lambda}(\Omega) = 0$.

Proof. We show (i). For $\varepsilon > 0$ we set

$$U_\varepsilon(x) = (\varepsilon + |x|^{2-s})^{\frac{2-N}{2-s}}, \quad u_\varepsilon(x) = \phi(x)U_\varepsilon(x), \quad v_\varepsilon = \frac{u_\varepsilon}{\left(\int_\Omega \frac{u_\varepsilon^{2^*(s)}}{|x|^s}dx\right)^{\frac{1}{2^*(s)}}},$$

where $\phi \in C_c^\infty(\Omega)$ is a cut-off function such that $0 \le \phi \le 1$ in Ω, $\phi = 1$ in $B_R(0)$, $\phi = 0$ in $\Omega \setminus B_{2R}(0)$. Due to Lemma 11.1 in [9], we have

$$\int_\Omega |\nabla v_\varepsilon|^2 dx = \mu_s + O(\varepsilon^{\frac{N-2}{2-s}}), \quad \int_\Omega v_\varepsilon^2 dx = \begin{cases} O(\varepsilon^{\frac{2}{2-s}}) & (N \ge 5) \\ O(\varepsilon^{\frac{2}{2-s}}\log\frac{1}{e}) & (N = 4) \\ O(\varepsilon^{\frac{1}{2-s}}) & (N = 3) \end{cases},$$

as $\varepsilon \to 0$. Hence we obtain (i). (ii) is obtained immediately by the definition of $\mu^N_{s,\lambda}(\Omega)$. In order to prove (iii) we use a constant. Set $C = \left(\int_\Omega |x|^{-s}dx\right)^{-1/2^*(s)}$. Then C satisfies the constraint of $\mu^N_{s,\lambda}(\Omega)$ and

$$\mu^N_{s,\lambda}(\Omega) \le \lambda \int_\Omega C^2 dx = \lambda|\Omega| \left(\int_\Omega |x|^{-s}dx\right)^{-\frac{2}{2^*(s)}}.$$

Letting $\lambda \to 0$ and hence we obtain (iii). □

Lemma 2.3. *There exists a positive constant* C *which depends on only* Ω *such that*

$$\mu_s \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u|^2 dx + C\int_\Omega u^2 dx \quad (u \in H^1(\Omega)). \qquad (3)$$

Before beginning the proof, we make a remark. H. Jaber [12] has shown that the following theorem.

Theorem 2.4. *([12]) If* (M, g) *is a compact Riemannian manifold without boundary and* $0 \in M$*, there is a constant* $C = C(M, g)$ *such that*

$$\mu_s \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x,0)^s}dv_g\right)^{\frac{2}{2^*(s)}} \le \int_M |\nabla u|^2 dv_g + C\int_\Omega u^2 dv_g \quad (u \in H^1(M))$$

where d_g *is the Riemannian distance on* M.

Different from Theorem 2.4, Ω is bounded domain of $\mathbb{R}^N$ and therefore Ω has a boundary, thus we can show the inequality (3) simply.

Proof. Let $0 \in \Omega_1 \subset \Omega_2 \subset \Omega$ and these two subdomain are taken suitable again later. A cut-off function is defined by ϕ which satisfies

$$\phi \in C_c^\infty(\Omega), \quad 0 \le \phi \le 1 \text{ in } \Omega, \quad \phi = 1 \text{ on } \Omega_1, \quad \phi = 0 \text{ on } \Omega \setminus \Omega_2.$$

Here, we construct a partition of unity $\eta_1,\ \eta_2$ defined by

$$\eta_1 := \frac{\phi^2}{\phi^2 + (1-\phi)^2}, \quad \eta_2 := \frac{(1-\phi)^2}{\phi^2 + (1-\phi)^2}.$$

Note that $\eta_1^{\frac{1}{2}},\ \eta_2^{\frac{1}{2}} \in C^2(\Omega)$ by the definition. We may assume that $u \in C^\infty(\Omega) \cap H^1(\Omega)$ by density. We have

$$\begin{aligned}
\mu_s \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} &= \mu_s \left\| u^2 \right\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \\
&= \mu_s \left\| \sum_{i=1}^{2} \eta_i u^2 \right\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \\
&\le \mu_s \sum_{i=1}^{2} \left\| \eta_i u^2 \right\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \\
&= \mu_s \sum_{i=1}^{2} \left(\int_\Omega \frac{|\eta_i^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
&= I_1 + I_2.
\end{aligned}$$

We estimate I_1, I_2 for each.

For I_1, since $\mathrm{supp}\eta_1 \subset \Omega$ we can use the Hardy-Sobolev inequality. We get that

$$\begin{aligned}
I_1 &= \mu_s \left(\int_\Omega \frac{|\eta_1^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla(\eta_1^{\frac{1}{2}} u)|^2 dx \\
&= \int_\Omega |\nabla u|^2 \eta_1 dx + \int_\Omega \nabla(\eta_1^{\frac{1}{2}}) \cdot \nabla(\eta_1^{\frac{1}{2}} u^2) dx.
\end{aligned}$$

Since $\eta_1^{\frac{1}{2}} \in C^2(\Omega)$ we may integrate by parts the second term and hence we obtain

$$I_1 \le \int_\Omega |\nabla u|^2 \eta_1 dx - \int_\Omega \Delta(\eta_1^{\frac{1}{2}}) \eta_1^{\frac{1}{2}} u^2 dx \tag{4}$$

For I_2, since $0 \notin \mathrm{supp}\eta_2$ and taking account to that $\eta = 0$ on Ω_1 we have

$$\begin{aligned}
I_2 &= \mu_s \left(\int_\Omega \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} = \mu_s \left(\int_{\Omega \setminus \Omega_1} \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
&\leq \mu_s \cdot a \left(\int_{\Omega \setminus \Omega_1} |\eta_2^{\frac{1}{2}} u|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \\
&\leq \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} \left(\int_{\Omega \setminus \Omega_1} |\eta_2^{\frac{1}{2}} u|^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\leq \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} S(\Omega, \Omega_1)^{-1} \int_{\Omega \setminus \Omega_1} |\nabla (\eta_2^{\frac{1}{2}} u)|^2 dx \\
&= \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} S(\Omega, \Omega_1)^{-1} \int_{\Omega} |\nabla (\eta_2^{\frac{1}{2}} u)|^2 dx
\end{aligned}$$

where $a := \mathrm{dist}(0, \partial\Omega_1)^{-2s/2^*(s)}$ and

$$S(\Omega, \Omega_1) := \inf \left\{ \int_{\Omega \setminus \Omega_1} |\nabla u|^2 dx \middle| u \in H^1(\Omega),\ u = 0 \text{ on } \partial\Omega_1,\ \int_{\Omega \setminus \Omega_1} |u|^{2^*} dx = 1 \right\}.$$

Here, let us take $\Omega_0 \subset \Omega_1$. It is clearly that $a \leq \mathrm{dist}(0, \partial\Omega_0)^{-2s/2^*(s)}$. On the other hand, for $u \in H^1(\Omega \setminus \Omega_1)$ such that $u = 0$ on $\partial\Omega_1$, we define $v \in H^1(\Omega \setminus \Omega_0)$ by

$$v := \begin{cases} u & \text{in } \Omega \setminus \Omega_1 \\ 0 & \text{in } \Omega_1 \setminus \Omega_0. \end{cases}$$

By identifying $u \in H^1(\Omega \setminus \Omega_1)$ with $v \in H^1(\Omega \setminus \Omega_0)$ concerning the calculation of the Sobolev quotient, we may see that

$$\{u \in H^1(\Omega \setminus \Omega_1) | u = 0 \text{ on } \partial\Omega_1\} \subset \{u \in H^1(\Omega \setminus \Omega_0) | u = 0 \text{ on } \partial\Omega_0\}.$$

Hence we obtain $S(\Omega, \Omega_1) \geq S(\Omega, \Omega_0)$. Consequently, if Ω_1 is sufficiently large, a and $S(\Omega, \Omega_1)^{-1}$ is bounded from above uniformly. By choosing Ω_1 and Ω_2 close to Ω we obtain

$$I_2 \leq \frac{1}{2} \int_\Omega |\nabla (\eta_2^{\frac{1}{2}} u)|^2 dx.$$

Therefore

$$I_2 \leq \int_\Omega |\nabla u|^2 \eta_2 dx + \int_\Omega |\nabla \eta_2^{\frac{1}{2}}|^2 u^2 dx. \tag{5}$$

Here, since $\eta_1^{\frac{1}{2}},\ \eta_2^{\frac{1}{2}} \in C^2(\Omega)$ there is a positive constant C such that

$$\max_{x \in \Omega} |\Delta(\eta_1^{\frac{1}{2}})| \leq \frac{C}{2}, \quad \max_{x \in \Omega} |\nabla \eta_2^{\frac{1}{2}}|^2 \leq \frac{C}{2}. \tag{6}$$

This constant depends on only Ω.

Consequently (4), (5) and (6) yield that

$$\mu_s \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le I_1 + I_2 \le \int_\Omega |\nabla u|^2 dx + C \int_\Omega u^2 dx.$$

□

Lemma 2.5. *The following statements hold true;*

(i) If $\mu^N_{s,\lambda}(\Omega) < \mu_s$, then $\mu^N_s(\Omega)$ is attained.

(ii) If there exist λ_* such that $\mu^N_{s,\lambda_*}(\Omega) = \mu_s$, then $\mu^N_{s,\lambda}(\Omega)$ is not attained for all $\lambda > \lambda_*$.

Proof of Lemma 2.5 (i). Assume $\{u_n\}_{n=1}^\infty \subset H^1(\Omega)$ is a minimizing sequence of $\mu^N_{s,\lambda}(\Omega)$. By the constraint of $\mu^N_{s,\lambda}(\Omega)$ we have

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = 1 \tag{7}$$

for all $n \in \mathbb{N}$ and which implies

$$\int_\Omega (|\nabla u_n|^2 + \lambda u_n^2) dx = \mu^N_{s,\lambda}(\Omega) + o(1) \quad (n \to \infty). \tag{8}$$

Thus u_n is bounded in $H^1(\Omega)$. So we can suppose, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^1(\Omega) \\ u_n &\to u \quad \text{in } L^p(\Omega) \quad (1 \le p < 2^*) \\ u_n &\to u \quad \text{in } L^q(\Omega, |x|^{-s}dx) \quad (1 \le q < 2^*(s)) \\ u_n &\to u \quad \text{a.e. in } \Omega \end{aligned}$$

as $n \to \infty$.

For this limit function u, we show that $u \not\equiv 0$ a.e. in Ω. Assume that $u \equiv 0$ a.e. in Ω. By the inequality (3) in Lemma 2.3,

$$\mu_s \left(\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u_n|^2 dx + C \int_\Omega u_n^2 dx \tag{9}$$

holds for all n. Thus (7), (8), (9) and $u_n \to 0$ in $L^2(\Omega)$ yield

$$\mu_s \le \mu^N_{s,\lambda}(\Omega) + o(1).$$

Letting $n \to \infty$, we obtain $\mu_s \le \mu^N_{s,\lambda}(\Omega)$ and which is a contradiction in the assumption of $\mu^N_{s,\lambda}(\Omega) < \mu_s$. Consequently $u \not\equiv 0$.

By the theorem of Brezis and Lieb (see [3]), we obtain

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} dx + o(1)$$

and it follows that

$$\begin{aligned}
1 &= \left(\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\
&= \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} + o(1) \\
&\le \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} + \left(\int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} + o(1).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} + \left(\int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\
\le\ & \frac{\int_\Omega(|\nabla u|^2 + \lambda u^2)dx}{\mu_{s,\lambda}^N(\Omega)} + \frac{\int_\Omega(|\nabla(u_n - u)|^2 + \lambda(u_n - u)^2dx}{\mu_{s,\lambda}^N(\Omega)} \\
=\ & \frac{\int_\Omega(|\nabla u_n|^2 + \lambda u_n^2)dx}{\mu_{s,\lambda}^N(\Omega)} + o(1) \\
=\ & 1 + o(1).
\end{aligned}$$

Hence there exist a limit and we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\
=\ & \lim_{n\to\infty}\left[\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} + \left(\int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}\right] \\
=\ & 1.
\end{aligned}$$

By the equality condition of the above, we get either

$$u \equiv 0 \quad \text{a.e. in } \Omega \quad \text{or} \quad u_n \to u \not\equiv 0 \quad \text{in } L^{2^*(s)}(\Omega, |x|^{-s}dx).$$

Since $u \not\equiv 0$ we obtain $u_n \to u \not\equiv 0$ in $L^{2^*(s)}(\Omega, |x|^{-s}dx)$ and $\int_\Omega |u|^{2^*(s)}/|x|^s dx = 1$. Hence this u is the minimizer of $\mu_s^N(\Omega)$. □

Proof of Lemma 2.5 (ii). We assume that $\lambda > \lambda_*$, u is a minimizer of $\mu_{s,\lambda}^N(\Omega)$ and derive a contradiction. We have

$$\mu_{s,\lambda}^N(\Omega) = \int_\Omega (|\nabla u|^2 + \lambda u^2)dx > \int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda_* u^2)dx \ge \mu_{s,\lambda_*}^N(\Omega).$$

By Lemma 2.2 and the assumption $\mu_{s,\lambda_*}^N(\Omega) = \mu_s$ we have

$$\mu_s \ge \mu_{s,\lambda}^N(\Omega) > \mu_{s,\lambda_*}^N(\Omega) = \mu_s.$$

This is a contradiction. □

2.3 Proof of Theorem 2.1

In this section, we prove Theorem 2.1.

Proof. Define a positive constant λ_* by

$$\lambda_* = \inf\{C > 0 \mid \text{Inequality (3) holds.}\}$$

By the definition of λ_* we have

$$\begin{aligned} \mu^N_{s,\lambda}(\Omega) < \mu_s \qquad &\text{if} \quad \lambda < \lambda_* \\ \mu^N_{s,\lambda}(\Omega) = \mu_s \qquad &\text{if} \quad \lambda \geq \lambda_* \end{aligned}$$

Consequently by Lemma 2.5 we obtain (i) and (ii). Finally, from the definition of λ_* we have

$$\begin{aligned} \lambda_* &= \sup\left\{ \frac{\mu_s(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx)^{\frac{2}{2^*(s)}} - \int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} \middle| u \in H^1(\Omega) \setminus \{0\} \right\} \\ &\geq \mu_s \left(\int_\Omega |x|^{-s} dx \right)^{\frac{2}{2^*(s)}} |\Omega|^{-1}. \end{aligned}$$

□

2.4 Singularity on the boundary

Throughout this subsection, we assume that $0 \in \partial\Omega$. A situation where $\partial\Omega \in C^2$ and the mean curvature of $\partial\Omega$ at 0 is positive has studied by [5]. In this subsection we assume that $\partial\Omega$ is "flat" near 0, that is for Ω after rotation there exists $r > 0$ such that $B_r(0) \cap \Omega = B_r^+(0) := B_r(0) \cap \mathbb{R}^N_+$, where $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N | x_n > 0\}$ is a half space. This condition is a special case of vanishing of the mean curvature of $\partial\Omega$ at 0. We show the following results by using the strategy in 2.2 and 2.3.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^N$ be bounded domain, $0 \in \partial\Omega$ and $\partial\Omega$ is flat near 0. Then there exists a positive constant $\lambda_{**} = \lambda_{**}(\Omega)$ such that the following statements hold;*

(I) $\mu^N_{s,\lambda}(\Omega)$ is attained for any $0 < \lambda < \lambda_{**}$

(II) $\mu^N_{s,\lambda}(\Omega)$ is not attained for any $\lambda > \lambda_{**}$.

(III) We have

$$\lambda_{**} \geq \frac{\mu_s}{2^{\frac{2-s}{N-s}}} \left(\int_\Omega |x|^{-s} dx \right)^{\frac{2}{2^*(s)}} |\Omega|^{-1}$$

We prove the theorem in the same way as in Section 2 and Section 3. Different from the proof of Theorem 2.1, we need the following lemma instead of Lemma 2.3.

Lemma 2.7. *There is a positive constant C depends on only Ω such that*

$$\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u|^2 dx + C\int_\Omega u^2 dx \quad (u \in H^1(\Omega)). \tag{10}$$

Proof. By the hypothesis of Ω we take a constant $r > 0$ such that $B_r(0) \cap \Omega = B_r^+(0)$. For $u \in H^1(\Omega)$ we have

$$\begin{aligned}\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} &= \left(\int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s}dx + \int_{\Omega\setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\ &\le \left(\int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} + \left(\int_{\Omega\setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\ &= J_1 + J_2.\end{aligned}$$

For $u \in H^1(B_r^+(0))$, $\tilde{u} \in H^1(B_r(0))$ is defined by the even reflection for the direction x_N, that is,

$$\tilde{u}(x', x_N) := \begin{cases} u(x', x_N) & \text{if } 0 \le x_N < 1 \\ u(x', -x_N) & \text{if } -1 < x_N < 0. \end{cases}$$

Concerning J_1, by Lemma 2.3 we have

$$\begin{aligned} J_1 &= \left(\int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\ &= \left(\frac{1}{2}\right)^{\frac{2}{2^*(s)}}\left(\int_{B_r(0)} \frac{|\tilde{u}|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \\ &\le \left(\frac{1}{2}\right)^{\frac{2}{2^*(s)}}\mu_s^{-1}\left(\int_{B_r(0)} |\nabla\tilde{u}|^2 dx + C_1\int_{B_r(0)} \tilde{u}^2 dx\right) \\ &= \left(\frac{1}{2}\right)^{\frac{2}{2^*(s)}}\mu_s^{-1}\cdot 2\left(\int_{B_r^+(0)} |\nabla u|^2 dx + C_1\int_{B_r^+(0)} u^2 dx\right) \\ &= \left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\right)^{-1}\left(\int_{B_r^+(0)} |\nabla u|^2 dx + C_1\int_{B_r^+(0)} u^2 dx\right)\end{aligned}$$

for some positive constant C_1 depends on only $B_r(0)$.

Next, we estimate J_2. Let $\delta > 0$ for sufficiently small. We consider $\{\phi_i\}_{i=1}^m$ a partition of unity on $\overline{\Omega \setminus B_r^+(0)}$ such that $\phi_i^{\frac{1}{2}} \in C^1$ and $|\mathrm{supp}\phi_i| \le \delta$ for all i.

Since $|x|^{-s} \leq r^{-s}$ for $x \in \Omega \setminus B_r^+(0)$ we have

$$\begin{aligned} J_2 &= \left(\int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \sum_{i=1}^m \left(\int_{\Omega \setminus B_r^+(0)} \frac{|\phi_i^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\ &\leq r^{-\frac{2s}{2^*(s)}} \sum_{i=1}^m \left(\int_{\Omega \setminus B_r^+(0)} |\phi_i^{\frac{1}{2}} u|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}}. \end{aligned}$$

By Hölder inequalities it follows that

$$\begin{aligned} \left(\int_{\Omega \setminus B_r^+(0)} |\phi_i^{\frac{1}{2}} u|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} &\leq |\mathrm{supp}\phi_i|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} \|\phi_i^{\frac{1}{2}} u\|^2_{L^{2^*}(\Omega \setminus B_r^+(0))} \\ &\leq \delta^{\frac{2}{2^*(s)} - \frac{2}{2^*}} \|\phi_i^{\frac{1}{2}} u\|^2_{L^{2^*}(\Omega \setminus B_r^+(0))} \end{aligned}$$

for each $i \in \mathbb{N}$. Since δ is sufficiently small, by using the Sobolev inequalities (If necessary we use the Sobolev inequalities of mixed boundary condition version.) we have

$$J_2 \leq \left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}} \right)^{-1} \cdot \frac{1}{2} \sum_{i=1}^m \int_{\Omega \setminus B_r^+(0))} |\nabla(\phi_i^{\frac{1}{2}} u)|^2 dx.$$

Consequently we have

$$J_2 \leq \left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}} \right)^{-1} \left(\int_{\Omega \setminus B_r^+(0)} |\nabla u|^2 dx + C_2 \int_{\Omega \setminus B_r^+(0)} u^2 dx \right).$$

for some positive constant C_2 depends on only $\Omega \setminus B_r^+(\Omega)$. Combining the estimates of J_1 and J_2 we obtain

$$\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \right)^{\frac{2}{2^*(s)}} \leq J_1 + J_2 \leq \left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}} \right)^{-1} \left(\int_\Omega |\nabla u|^2 dx + C \int_\Omega u^2 dx \right)$$

for some positive constant C depends on Ω. □

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3 Asymptotic behavior of the least-energy solutions of a semilinear elliptic equation with the Hardy-Sobolev critical exponent

Abstract

We investigate the existence, the non-existence and the asymptotic behavior of the least-energy solutions of a semilinear elliptic equation with the Hardy-Sobolev critical exponent. In the boundary singularity case, it is known that the mean curvature of the boundary at origin plays a crucial role on the existence of the least-energy solutions. In this section, we study the relation between the asymptotic behavior of the solutions and the mean curvature at origin.

3.1 Introduction

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ bounded domain with smooth boundary, $0 < s < 2$, $2^*(s) = 2(N-s)/(N-2)$ and λ be a positive parameter. In this section we assume that $0 \in \partial\Omega$. We study the existence, the non-existence and the asymptotic behavior as $\lambda \to \infty$ of the least-energy solutions of

$$\begin{cases} -\Delta u + \lambda u = \frac{u^{2^*(s)-1}}{|x|^s}, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{11}$$

The existence of the least-energy solution of (11) is equivalent to the existence of the minimizer for the corresponding minimization problem

$$\mu_{s,\lambda}^N(\Omega) = \inf\left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2)dx \middle| u \in H^1(\Omega), \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1 \right\}. \tag{12}$$

Actually, if the minimizer u_λ for $\mu_{s,\lambda}^N(\Omega)$ exists then $v_\lambda := \mu_{s,\lambda}^N(\Omega)^{(N-2)/(4-2s)}u_\lambda$ is a least-energy solution of (11) and vise versa.

Minimization problems and semilinear elliptic equations on the Hardy-Sobolev type inequality have been studied extensively by many authors. The Dirichlet case, that is, concerning the attainability for

$$\mu_s^D(\Omega) = \inf\left\{ \int_\Omega |\nabla u|^2 dx \middle| u \in H_0^1(\Omega), \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1 \right\}$$

is studied in [8–11, 13, 15]. In the interior singularity case, the remainder term of the Hardy-Sobolev inequality is studied by [16]. The optimal Hardy-Sobolev inequality on compact Riemannian manifold is also studied due to [14].

In the Neumann case, we have obtained some results. In the interior singularity case, the existence and non-existence results of the minimizer for $\mu_{s,\lambda}^N(\Omega)$ are obtained by [12]. In the boundary singularity case, some results are due to [5, 8, 12]. Due to these results, the attainability for $\mu_{s,\lambda}^N(\Omega)$ is different for each situation. In both the Dirichlet case and the Neumann case, the position of 0 on Ω affects the attainability for the best constant.

There are many results on the least-energy solutions of the Neumann problem

$$\begin{cases} -d\Delta u + u = u^p, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{13}$$

where $d > 0$ is a constant. It is shown that the least-energy solution of (13) exists by [1, 23] and so on. Moreover, by for instance [3, 4, 24, 25] Lin-Ni's conjecture is studied, that is, they investigate that for d sufficiently large whether the solution of (13) is only constant or not.

The asymptotic behavior of the least-energy solutions as $d \to 0$ is studied particularly by [2, 17–21]. In the subcritical case $1 < p < (N+2)/(N-2)$, the least-energy solution has only one maximum point and this point lies on the boundary. Moreover, this maximum point approaches the boundary point of maximum mean curvature as $d \to 0$ and the peak is bounded from above uniformly with respect to d. On the other hand, in the critical case $p = (N+2)/(N-2)$, it is proved that peak is at most one and blows up on a boundary point. By [21] we know that the asymptotic behavior of the best constant for the embedding $H^1(\Omega) \subset L^{2N/(N-2)}(\Omega)$, that is,

$$S_d^N(\Omega) = \inf \left\{ \int_\Omega (|\nabla u|^2 + \frac{1}{d} u^2) dx \middle| u \in H^1(\Omega), \int_\Omega |u|^{\frac{2N}{N-2}} dx = 1 \right\}$$

as $d \to 0$. On the asymptotic behavior of the least-energy solutions of (13) and S_d^N the mean curvature of $\partial\Omega$ plays a crucial role.

Our main purpose of this section is to investigate the asymptotic behavior of the least-energy solutions of (11) as $\lambda \to \infty$. In [5, 8], the existence of the least energy solutions of (11) is guaranteed for any $\lambda > 0$ if the mean curvature of $\partial\Omega$ at 0 is positive. Thus it is natural that we investigate the asymptotic behavior of the least-energy solutions of (11). However in the case when the mean curvature at 0 is non-positive, the existence of the least-energy solutions of (11) is not studied so far. As our second purpose of this section we obtain the answer of this problem through the investigation into the asymptotic behavior.

This section is organized as follows. In 3.2 we prepare the useful facts and some lemmas. In 3.3 we consider the asymptotic behavior of the least-energy solution of (11). In 3.4 we consider the behavior of $\mu_{s,\lambda}^N(\Omega)$ as $\lambda \to \infty$. Throughout these two subsections we assume the existence of the least-energy solutions of (11) for any Ω. In 3.5 we show some results on the minimization problem of $\mu_{s,\lambda}^N(\Omega)$.

Remark 3.1. *Since the nonlinear term in (11) has a singularity at* 0*, solutions are not classical solutions. Indeed, if* $u \in H^1(\Omega)$ *is a weak solution of (11) by the elliptic regularity theory* $u \in C^2_{loc}(\overline{\Omega} \setminus \{0\})$ *and* $u \in C^{0,\alpha}(\overline{\Omega})$ *(see [5, 9]). Therefore we should regard* $\partial/\partial\nu$ *as the bounded linear operator from* $W^{2,p}(\Omega)$ *to* $L^p(\partial\Omega)$ *at* 0*.*

3.2 Preliminaries

In this section we prepare some useful facts.

We recall that some facts about a diffeomorphism straightening a boundary portion around a point $P \in \partial\Omega$, which was introduced in [17–20]. Through translation and rotation of the coordinate system we may assume that P is the origin and inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N-axis. In a neighborhood $\mathcal{N}$ around P, there exists a smooth function $\psi(x'),\ x' = (x_1, \ldots, x_{N-1})$ such that $\partial\Omega \cap \mathcal{N}$ can be represented by

$$x_N = \psi(x') = \frac{1}{2}\sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)$$

where $\alpha_1, \ldots, \alpha_{N-1}$ are the principal curvatures of $\partial\Omega$ at P. For $y \in \mathbb{R}^N$ with $|y|$ sufficiently small, we define a mapping $x = \Phi(y) = (\Phi_1(y), \ldots, \Phi_N(y))$ by

$$\Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial\psi}{\partial x_j}(y') & j = 1, \ldots, N-1 \\ y_N + \psi(y') & j = N. \end{cases}$$

The differential map $D\Phi$ is

$$D\Phi(y) = \begin{pmatrix} \delta_{ij} - \dfrac{\partial^2\psi}{\partial x_i \partial x_j}(y')y_N & -\dfrac{\partial\psi}{\partial x_i}(y') \\ \dfrac{\partial\psi}{\partial x_j}(y') & 1 \end{pmatrix}_{1 \le i,j \le N-1}$$

and near $y = 0$

$$|J\Phi(y)| = |\mathrm{det} D\Phi(y)| = 1 - (N-1)H(P)y_N + O(|y|^2).$$

We write as $\Psi(x) = (\Psi_1(x), \ldots, \Psi_N(x))$ instead of the inverse map $\Phi^{-1}(x)$. $B_r(a)$ denotes a open ball with center a and radius r. In addition, suppose $B_r = B_r(0)$ and $B_r^+ = \{y \in B_r | y_N > 0\}$.

We consider the function as

$$U(x) = \left(1 + \frac{|x|^{2-s}}{(N-s)(N-2)}\right)^{-\frac{N-2}{2-s}}. \tag{14}$$

Note that $U(0) = 1$ and U is a minimizer for

$$\mu_s := \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \middle| u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\} \tag{15}$$

which is the best constant for the Hardy-Sobolev inequality. For U define the scaling function by

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x}{\varepsilon}\right).$$

We have the following lemma regarding $\mu_{s,\lambda}^N(\Omega)$.

Lemma 3.2. *(i) $\mu_{s,\lambda}^N(\Omega)$ is continuous and non-decreasing with respect to λ.*

(ii) For any $\lambda > 0$, $\mu^N_{s,\lambda}(\Omega) \le \mu_s/2^{(2-s)/(N-s)}$.

(iii) $\lim_{\lambda\to 0}\mu^N_{s,\lambda}(\Omega) = 0$.

Proof. We show only part (ii).

For given $\phi \in C^1(\Omega\cap\mathcal{N}_0)$ we set $\tilde{\phi}(y) = \phi(\Phi(y))$, where $\mathcal{N}_0$ is a neighborhood around 0 such that $\Omega\cap\mathcal{N}_0 = \Phi(B^+_\delta)$. If $\tilde{\phi}(y)$ is a radially symmetric function, we have

$$\begin{aligned}\int_{\Omega\cap\mathcal{N}_0} |\nabla\phi(x)|^2 dx &= \frac{\omega_{N-1}}{2}\int_0^\delta r^{N-1}|\tilde{\phi}'|^2(r)dr \\ &\quad -\frac{(N-1)\pi^{\frac{N-1}{2}}}{(N+1)\Gamma(\frac{N+1}{2})}H(0)\int_0^\delta r^N|\tilde{\phi}'|^2(r)dr \\ &\quad +\int_0^\delta O(r^{N+1})|\tilde{\phi}'|^2(r)dr, \end{aligned} \tag{16}$$

$$\int_{\Omega\cap\mathcal{N}_0} |\phi(x)|^2 dx = \frac{\omega_{N-1}}{2}\int_0^\delta r^{N-1}\tilde{\phi}^2(r)dr + \int_0^\delta O(r^N)|\phi^2|(r)dr, \tag{17}$$

$$\begin{aligned}\int_{\Omega\cap\mathcal{N}_0} \frac{\phi^{2^*(s)}}{|x|^s}dx &= \frac{\omega_{N-1}}{2}\int_0^\delta r^{N-s-1}\tilde{\phi}^{2^*(s)}(r)dr \\ &\quad -(N-1)\left[1-\frac{s}{2(N+1)}\right]\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})}H(0)\int_0^\delta r^{N-s}\phi^{2^*(s)}dr \\ &\quad +\int_0^\delta O(r^{N-s+1})\tilde{\phi}^{2^*(s)}dr, \end{aligned} \tag{18}$$

where ω_{N-1} is the surface area of a unit sphere. Set a cut-off function $\eta(y) = \eta(|y|)$ such that support of η is in B_δ and $\eta = 1$ in $B_{\delta/2}$. Choosing $\eta(y)U_\varepsilon(y)$ as $\tilde{\phi}$ in (16), (17) and (18) and hence we obtain

$$\begin{aligned}&\frac{\int_\Omega(|\nabla(\eta U_\varepsilon)|^2dx + \lambda|\eta U_\varepsilon|^2)dx}{\left(\int_\Omega \frac{|\eta U_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}} \\ &= \begin{cases} \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}}\mu_s - c_1H(0)\varepsilon + [\lambda(c_2+O(\varepsilon|\log\varepsilon|)) + O(\varepsilon)]\,\varepsilon^2 & (N\ge 5) \\ \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}}\mu_s - c_1H(0)\varepsilon + \left[\lambda\left(c_2+O\left(|\log\varepsilon|^{-1}\right)\right) + O(1)\right]\varepsilon^2|\log\varepsilon| & (N=4) \\ \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}}\mu_s - c_1H(0)\varepsilon|\log\varepsilon| + [\lambda(c_2+O(\varepsilon)) + O(1)]\,\varepsilon & (N=3) \end{cases}\end{aligned}$$

where c_1, c_2 are positive constants which depend only on N. Tending ε to 0 and we obtain the estimate of part (ii). □

Lemma 3.3. *We have either*

(i) There exist $\tilde{\lambda}$ such that for $\lambda \geq \tilde{\lambda}$

$$\mu_{s,\lambda}^N(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s, \tag{19}$$

or

(ii) For all λ the equality (19) does not hold and

$$\lim_{\lambda \to \infty} \mu_{s,\lambda}^N(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s. \tag{20}$$

where μ_s is defined by (15).

To prove this lemma, we prepare one proposition.

Proposition 3.4. *Fix $\varepsilon > 0$ sufficiently small. Then there exists a positive constant $C = C(\varepsilon)$ such that for $u \in H^1(\Omega)$*

$$\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \left(\int_\Omega \frac{u^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)} \leq (1+\varepsilon) \int_\Omega |\nabla u|^2 dx + C \int_\Omega u^2 dx. \tag{21}$$

Proof of Proposition 3.4. We choose small constant $\delta > 0$, $r > 0$ and V which is a neighborhood around 0 such that

$$x_N = \psi_0(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2), \quad |\nabla \psi_0(x')| \leq \delta \quad \text{on} \quad \partial\Omega \cap V,$$

and $\{(x', x_N - \psi_0) | (x', x_N) \in \Omega \cap V\} = B_r^+$.

Due to [12] there exists a positive constant $C = C(B_r)$ such that

$$\mu_s \left(\int_{B_r} \frac{u^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)} \leq \int_{B_r} |\nabla u|^2 dx + C \int_{B_r} u^2 dx. \tag{22}$$

By the transformation $y' = x'$, $y_N = x_N - \psi_0(x')$ and the inequality (22), it follows that

$$\begin{aligned} & \mu_s \left(\int_{\Omega \cap V} \frac{|u|^{2^*(s)}}{(|x'|^2 + |x_N - \psi_0|^2)^{s/2}} dx\right)^{2/2^*(s)} \\ = & \ \mu_s \left(\frac{1}{2} \int_{B_r^+} \frac{|\hat{u}|^{2^*(s)}}{|y|^s} dy\right)^{2/2^*(s)} \\ \leq & \left(\frac{1}{2}\right)^{2/2^*(s)} \int_{B_r^+} (|\nabla_y \hat{u}|^2 + C\hat{u}^2) dy \\ \leq & \ 2^{\frac{2-s}{N-s}} \left(1 + (N-1)\delta + \delta^2\right) \int_{\Omega \cap V} |\nabla_x u|^2 + C\hat{u}^2 dx \end{aligned}$$

where $\hat{u}(y) = u(y', y_N + \psi_0)$. On the other hand, if $|x|$ sufficiently small

$$(|x'|^2 + |x_N - \psi_0|^2)^{s/2} = (|x|^2 - 2\psi_0 x_N + \psi_0^2)^{s/2} \leq (1 + C_0|x|)|x|^s.$$

Now, we may assume that diamV < C$_1\delta$ for some C_1. Consequently taking ε such that

$$1 + \varepsilon = \frac{1 + (N-2)\delta + \delta^2}{1 + C_0 C_1 \delta}$$

and we obtain

$$\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \left(\int_{\Omega \cap V} \frac{u^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)} \leq (1+\varepsilon) \int_{\Omega \cap V} |\nabla u|^2 dx + C \int_{\Omega \cap V} u^2 dx.$$

In $\overline{\Omega \setminus V}$, taking into account that $|x|^{-s}$ has not a singularity and we have

$$\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \left(\int_{\Omega \setminus V} \frac{u^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)} \leq (1+\varepsilon) \int_{\Omega \setminus V} |\nabla u|^2 dx + C \int_{\Omega \cap V} u^2 dx.$$

The detail of calculations is in [12]. Hence we obtain (21). □

Proof of Lemma 3.3. If there exist $\tilde{\lambda}$ such that (19) holds, then by part (i) and part (ii) of Lemma 3.2 we can prove part (i).

Assume that for all $\lambda > 0$, the equality (19) does not hold. For any $\varepsilon > 0$ and $\lambda > 0$, there exist $u_{\lambda,\varepsilon}$ such that

$$\mu_{s,\lambda}^N(\Omega) \geq \int_\Omega |\nabla u_{\lambda,\varepsilon}|^2 dx + \lambda \int_\Omega u_{\lambda,\varepsilon}^2 dx - \varepsilon$$

We choose $\lambda = \lambda(\varepsilon)$ such that $\lambda \to \infty$ as $\varepsilon \to 0$ and $\lambda \geq C$ where C is given in Proposition 3.4. From the above inequality and (21) we have

$$0 \leq \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - \mu_{s,\lambda}^N(\Omega) \leq \varepsilon \int_\Omega |\nabla u_{\lambda,\varepsilon}|^2 dx + \varepsilon \leq \varepsilon \left\{1 + \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s + \varepsilon\right\}.$$

Hence tending ε to 0 and we obtain the equality (20). □

By the next lemma we can see the relation between the value of $\mu_{s,\lambda}^N(\Omega)$ and the existence of the minimizer of $\mu_{s,\lambda}^N(\Omega)$.

Lemma 3.5. *(i) If $\mu_{s,\lambda}^N(\Omega) < \mu_s/2^{(2-s)/(N-s)}$ then $\mu_{s,\lambda}^N(\Omega)$ is attained.*

(ii) If there exist a positive constant $\tilde{\lambda}$ such that $\mu_{s,\tilde{\lambda}}^N(\Omega) = \mu_s/2^{(2-s)/(N-s)}$ then $\mu_{s,\lambda}^N(\Omega)$ is not attained for all $\lambda > \tilde{\lambda}$.

Proof. (i) proved by the proof of Proposition 2.1 in [5].

We prove (ii). Let $\lambda > \tilde{\lambda}$ and u_λ be a minimizer of $\mu^N_{s,\lambda}(\Omega)$. Then we have

$$\begin{aligned}\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s = \mu^N_{s,\tilde{\lambda}}(\Omega) &\leq \int_\Omega (|\nabla u_\lambda|^2 + \tilde{\lambda} u_\lambda^2) dx \\ &< \int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2) dx = \mu^N_{s,\lambda}(\Omega) \leq \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s.\end{aligned}$$

This is a contradiction. □

3.3 Asymptotic behavior I

In this section and the next section we assume that the least-energy solution of (11) exists.

We investigate the asymptotic behavior of the least-energy solution of (11) as $\lambda \to \infty$. In order to prove Theorem 3.6, we apply the strategy in [17–20] to the equation (11). We assume v_λ is a least-energy solution of (11) and define α_λ and β_λ as

$$\alpha_\lambda = \|v_\lambda\|_{L^\infty(\Omega)} = v_\lambda(x_\lambda), \quad \beta_\lambda = \alpha_\lambda^{-\frac{2}{N-2}}.$$

Theorem 3.6. *We obtain the following results;*

(i) For all $x \in \Omega$, $v_\lambda(x) \to 0$,

(ii) $\alpha_\lambda^{\frac{4}{N-2}}/\lambda = (\lambda\beta^2)^{-1} \to \infty$,

(iii) $|x_\lambda| = o(\beta_\lambda)$

as $\lambda \to \infty$. For any $\varepsilon > 0$ and $\delta > 0$ there exists a positive constant λ_0 such that for all $\lambda > \lambda_0$

(iv) $\left| \dfrac{v_\lambda(x)}{\alpha_\lambda} - U\left(\dfrac{\Psi_\lambda(x)}{\beta_\lambda}\right) \right| < \varepsilon \quad$ in $\Omega \cap B_{\beta_\lambda \delta}$,

(v) $v_\lambda \leq 2\varepsilon \lambda^{\frac{N-2}{(4-2s)}} \exp(-\gamma_0 \xi(x) \lambda^{\frac{1}{2}}) \quad$ in $\Omega \setminus B_\delta$,

where U is defined in (14), $\xi(x) = \min\{\eta_0, \mathrm{dist}(x, \partial\Omega \cap B_\delta)\}$, $\eta_0 = \eta_0(\Omega)$ and $\gamma_0 = \gamma_0(\Omega, \varepsilon)$ are positive constants.

Lemma 3.7. *There exist a positive constant C which is independent of λ such that*

$$\frac{\alpha_\lambda^{\frac{4}{N-2}}}{\lambda} \geq C.$$

Proof. For simplicity, we write $v = v_\lambda$ and $\alpha = \alpha_\lambda$ for each. C_0, C_1, C_2, C_3 are positive constants which depends only on domain Ω. We have

$$\int_\Omega \nabla v \nabla \phi dx + \lambda \int_\Omega v\phi dx \leq \alpha^{2^*(s)-2} \int_\Omega \frac{v\phi}{|x|^s} dx \tag{23}$$

for all $\phi \in H^1(\Omega)$ satisfying $\phi \geq 0$. For $\beta \geq 1$, we define a function $H \in C^1([0,\infty))$ by setting $H(t) = t^\beta$ and $G(t) := \int_0^t |H'(s)|^2 ds = \frac{\beta^2}{2\beta-1} t^{2\beta-1}$. We easily find that

$$vG'(v) \geq G(v). \tag{24}$$

Replacing ϕ in (23) by $G(v)$ we have

$$\int_\Omega \nabla v \nabla G(v) dx + \lambda \int_\Omega v G(v) dx \leq \alpha^{2^*(s)-2} \int_\Omega \frac{vG(v)}{|x|^s} dx.$$

The chain rule, the definition of G and (24) yield

$$\int_\Omega |\nabla H(v)|^2 dx + \lambda \frac{\beta^2}{2\beta-1} \int_\Omega H(v)^2 dx \leq \alpha^{2^*(s)-2} \int_\Omega \frac{|vH'(v)|^2}{|x|^s} dx \tag{25}$$

For $\lambda \frac{\beta^2}{2\beta-1} \geq 1$, by the Hardy-Sobolev inequality it follows that

$$\mu_s^N(\Omega) \left(\int_\Omega \frac{H(v)^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega |\nabla H(v)|^2 dx + \lambda \frac{\beta^2}{2\beta-1} \int_\Omega H(v)^2 dx \tag{26}$$

where $\mu_s^N(\Omega) := \inf \left\{ \int_\Omega (|\nabla u|^2 + u^2) dx | u \in H^1(\Omega), \int_\Omega |u|^{2^*(s)}/|x|^s dx = 1 \right\}$. Since

$$H(v) = v^\beta, \quad vH'(v) = \beta v^\beta \tag{27}$$

Combining (25), (26) and (27) we have

$$\|v\|^2_{L^{2^*(s)\beta}(\Omega,|x|^{-s}dx)} \leq C_0^{\frac{1}{\beta}} \alpha^{(2^*(s)-2)\frac{1}{\beta}} \beta^{\frac{2}{\beta}} \|v\|^2_{L^{2\beta}(\Omega,|x|^{-s}dx)}$$

For $m = 0, 1, 2, \cdots$ we define $\beta_{m+1} = (2^*(s)/2)^m$, then we have

$$\begin{aligned} & \|v\|^2_{L^{2^*(s)\beta_{m+1}}(\Omega,|x|^{-s}dx)} \\ \leq \; & C_0^{\frac{1}{\beta_{m+1}}} \alpha^{(2^*(s)-2)\frac{1}{\beta_{m+1}}} \beta_{m+1}^{\frac{2}{\beta_{m+1}}} \|v\|_{L^{2\beta_{m+1}}(\Omega,|x|^{-s}dx)} \\ = \; & \prod_{l=0}^m C_0^{\frac{1}{2(2^*(s)/2)^l}} \alpha^{(2^*(s)-2)\frac{1}{(2^*(s)/2)^l}} \left(\frac{2^*(s)}{2} \right)^{l \frac{1}{(2^*(s)/2)^l}} \|v\|_{L^2(\Omega,|x|^{-s}dx)} \end{aligned} \tag{28}$$

Note that

$$\sum_{l=0}^\infty \left(\frac{2^*(s)}{2} \right)^{-l} = \lim_{m\to\infty} \frac{1 - \left(\frac{2^*(s)}{2} \right)^{-m-1}}{1 - \left(\frac{2^*(s)}{2} \right)^{-1}} = \frac{2^*(s)}{2^*(s)-2},$$

$$\sum_{l=0}^\infty l \left(\frac{2^*(s)}{2} \right)^{-l} \leq \sum_{l=0}^\infty (l+1) \left(\frac{2^*(s)}{2} \right)^{-l} \leq \frac{2^*(s)}{(2^*(s)-2)^2}.$$

Tending $m \to \infty$ in (28), and thus

$$\|v\|^2_\infty \leq C_1 \alpha^{(2^*(s)-2)\frac{2^*(s)}{2^*(s)-2}} \|v\|^2_{L^2(\Omega,|x|^{-s}dx)} = C_1 \alpha^{2^*(s)} \|v\|^2_{L^2(\Omega,|x|^{-s}dx)}.$$

Using the Hölder inequality we have

$$\begin{aligned} \|v\|^2_{L^2(\Omega,|x|^{-s}dx)} &= \int_\Omega \frac{v^2}{|x|^s}dx \le \left(\int_\Omega \frac{v^2}{|x|^2}dx\right)^{s/2}\left(\int_\Omega v^2 dx\right)^{1-s/2} \\ &< C_2\left(\int_\Omega v^2dx\right)^{1-s/2}. \end{aligned}$$

Consequently

$$\begin{aligned} \frac{1}{C_1C_2} &\le \alpha^{2^*(s)-2}\left(\int_\Omega v^2dx\right)^{1-s/2} \\ &= \left(\alpha^{\frac{4}{N-2}}\int_\Omega v^2dx\right)^{1-s/2} \\ &\le C_3\left(\frac{\alpha^{\frac{4}{N-2}}}{\lambda}\right)^{1-s/2}. \end{aligned}$$

Therefore we obtain

$$\frac{\alpha^{\frac{4}{N-2}}}{\lambda} > C.$$

□

Lemma 3.8.

$$|x_\lambda| = O(\beta_\lambda)$$

Proof. Step 1. First of all, we show that $\mathrm{dist}(x_\lambda, \partial\Omega) = O(\beta_\lambda)$. We assume that

$$\lim_{\lambda\to\infty}\frac{\mathrm{dist}(x_\lambda,\partial\Omega)}{\beta_\lambda} = \infty \tag{29}$$

and derive a contradiction. Assume that λ_k is positive increasing sequence such that $\lambda_k \to \infty$ as $k \to \infty$. By the assumption of (29) we may take a positive constant R such that

$$|B_R(0)| > \frac{1}{2}S_N(\Omega)^{-1}\mu_s^{\frac{N-s}{2-s}} \quad \text{and} \quad x_{\lambda_k} + \beta_{\lambda_k}z \in \Omega \quad \text{for all} \quad z \in B_{3R}(0) \tag{30}$$

where $|B_R(0)|$ is N-dimensional volume of $B_R(0)$ and

$$S^N(\Omega) = \inf\left\{\int_\Omega(|\nabla u|^2 + u^2)dx \middle| \int_\Omega |u|^{\frac{2N}{N-2}}dx = 1\right\}$$

is the best constant of the critical Sobolev embedding. We set

$$w_k(z) := \frac{v_{\lambda_k}(x_{\lambda_k} + \beta_{\lambda_k}z)}{\alpha_{\lambda_k}} \quad z \in B_{3R}(0).$$

Since $v_{\lambda_k} \in C^2_{loc}(\overline{\Omega}\setminus\{0\})$ $\tilde{v}_k$ satisfies

$$-\Delta w_k + \lambda\beta^2 w_k = \frac{w_k^{2^*(s)-1}}{\left|\frac{x_{\lambda_k}}{\beta_{\lambda_k}} + z\right|^s} \quad \text{in } B_{3R}(0).$$

Note that from (29) and Lemma 3.7

$$\lambda_k \beta_{\lambda_k} \to C, \quad \left|\frac{x_{\lambda_k}}{\beta_{\lambda_k}} + z\right|^{-s} = o(1) \quad \text{as} \quad k \to \infty \quad \text{for} \quad z \in B_{3R(0)}. \tag{31}$$

By using the elliptic regularity theory there exists w such that

$$w \in C^2(B_R(0)), \quad w_k \to w \quad \text{in} \quad C^2(B_R(0))$$

and

$$-\Delta w + Cw = 0 \quad \text{in} \quad B_R(0).$$

In addition $0 \le w(z) \le 1$ in $B_R(0)$ and $w(0) = 1$ since $\tilde{v}_k(0) = 1$. By the strong maximum principle $w \equiv 1$. But

$$\begin{aligned} |B_R(0)| &= \int_{B_R(0)} w^{\frac{2N}{N-2}} dz = \lim_{k\to\infty} \int_{B_R(0)} w_k^{\frac{2N}{N-2}} dz = \lim_{k\to\infty} \int_{B_{\beta_{\lambda_k} R(x_{\lambda_k})}} v_{\lambda_k}^{\frac{2N}{N-2}} dx \\ &\le \lim_{k\to\infty} \int_\Omega v_{\lambda_k}^{\frac{2N}{N-2}} dx \le \lim_{k\to\infty} S_N(\Omega)^{-1} \int_\Omega \left(|\nabla v_{\lambda_k}^2 + v_{\lambda_k}^2\right) dx \\ &= \frac{1}{2} S_N(\Omega)^{-1} \mu_s^{\frac{N-s}{2-s}} \end{aligned}$$

which contradicts the choice of R in (30).

Step 2. To end of the proof of this lemma we show that

$$x_\lambda \not\to x \quad \text{for all} \quad x \in \partial\Omega \setminus \{0\}.$$

We assume that there exists a point $x_0 \in \partial\Omega \setminus \{0\}$ such that $|x_\lambda - x_0| = O(\beta_\lambda)$ and derive a contradiction.

By translation and rotation of the coordinate system we may consider the equation

$$\begin{cases} -\Delta v_\lambda + \lambda v_\lambda = \frac{v_\lambda^{2^*(s)-1}}{|a_0 + x|^s} & \text{in } \Omega \\ \frac{\partial v_\lambda}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

and $x_\lambda \to 0$, where $a_0 \in \partial\Omega \setminus \{0\}$. Set $\lambda_k \to \infty$ and $x_{\lambda_k} \to 0$ as $k \to \infty$. For δ small sufficiently put $\hat{v}_{\lambda_k}(y) = v_{\lambda_k}(\Phi(y))$ for $y \in \overline{B}_{2\delta}^+$ and

$$\tilde{v}_{\lambda_k} = \begin{cases} \hat{v}_{\lambda_k}(y) & y \in \overline{B}_{2\delta}^+ \\ \hat{v}_{\lambda_k}(y', -y_N) & (y', -y_N) \in \overline{B}_{2\delta}^+. \end{cases}$$

We define a function w_k $(k = 1, 2, \ldots)$ by

$$w_k(z) = \frac{\tilde{v}_{\lambda_k}(Q_{\lambda_k} + \beta_{\lambda_k} z)}{\alpha_\lambda} \qquad z \in B_{\delta/\beta_{\lambda_k}}$$

where $Q_{\lambda_k} = \Psi(x_{\lambda_k}) = (q'_{\lambda_k}\beta_{\lambda_k}, q^N_{\lambda_k}\beta_{\lambda_k})$, $Q_{\lambda_k}/\beta_{\lambda_k} \to Q_\infty = (q'_\infty, q^N_\infty)$ as $k \to \infty$. By Step 1, $|Q_\infty| < \infty$.

We take a positive constant R such that

$$|B_R(0)| > S_N(\Omega)^{-1}\mu^{\frac{N-s}{2-s}}$$

in the same way as Step 1. Set a function ξ_k as

$$\xi_k(z) = \begin{cases} \Phi(Q_{\lambda_k} + \beta_{\lambda_k} z) & (z_N \geq -q^N_{\lambda_k}) \\ \Phi((q'_{\lambda_k} + z')\beta_{\lambda_k}, -(q^N_{\lambda_k} + z_N)\beta_{\lambda_k}) & (z_N < -q^N_{\lambda_k}). \end{cases}$$

Then w_k satisfies

$$-\sum_{i,j=1}^{N} a^k_{ij}(z)\frac{\partial^2 w_k}{\partial z_i \partial z_j} + \beta_{\lambda_k}\sum_{j=1}^{N} b^k_j(z)\frac{\partial w_k}{\partial z_j} + \lambda_k \beta^2_{\lambda_k} w_k = \frac{w_k^{2^*(s)-1}}{\left|\frac{a_0+\xi_k}{\beta_{\lambda_k}}\right|^s}$$

in $B_R(0) \setminus \{z_N = -q^N_{\lambda_k}\}$, where $a^k_{ij},\ b^k_j$ is defined as follows (here definitions is same as those in Step 2 in the section 4 in [20]):

$$a_{ij}(y) = \sum_{k=1}^{N} \frac{\partial \Psi_i}{\partial x_k}(\Phi(y))\frac{\partial \Psi_j}{\partial x_k}(\Phi(y)) \quad 1 \leq i,j \leq N \tag{33}$$

$$b_j(y) = (\Delta\Psi_j)(\Phi(y)) \quad 1 \leq j \leq N. \tag{34}$$

Then define

$$a^k_{ij}(z) = \begin{cases} a_{ij}(Q_{\lambda_k} + \beta_{\lambda_k} z) & z_N \geq -q^N_{\lambda_k}, \\ (-1)^{\delta_{iN}+\delta_{jN}} a_{ij}((q'_{\lambda_k} + z')\beta_{\lambda_k}, -(q^N_{\lambda_k} + z_N)\beta_{\lambda_k}) & z_N < q^N_{\lambda_k}, \end{cases}$$

$$b^k_j(z) = \begin{cases} b_j(Q_{\lambda_k} + \beta_{\lambda_k} z) & z_N \geq -q^N_{\lambda_k}, \\ (-1)^{\delta_{jN}} b_j((q'_{\lambda_k} + z')\beta_{\lambda_k}, -(q^N_{\lambda_k} + z_N)\beta_{\lambda_k}) & z_N < -q_{\lambda_k}. \end{cases}$$

By applying the elliptic regularity theory in [20] and arguing in the same manner as in Step 1 we have

$$w \in C^2(B_R(0)), \quad w_k \to w \quad \text{in} \quad C^2(B_R(0))$$

and $w \equiv 1$. It follows that

$$|B_R(0)| = \int_{B_R} w^{\frac{2N}{N-2}} dz \leq \lim_{k\to\infty} 2\int_\Omega v_{\lambda_k}^{\frac{2N}{N-2}} dz \leq S_N(\Omega)^{-1}\mu_s^{\frac{N-s}{2-s}}.$$

This contradicts the choice of R. □

Proof of Theorem 3.6 (ii), (iii), (iv). We can see $x_\lambda \to 0$ from Lemma 3.8. Put $k \to \infty$ and define λ_k, x_{λ_k}, $\hat{v}_{\lambda_k}$, $\tilde{v}_{\lambda_k}$, Q_{λ_k}, w_k and ξ_k respectively as those in Step 2 of the proof of Lemma 3.8. w_k satisfies

$$-\sum_{i,j=1}^{N} a^k_{ij}(z)\frac{\partial^2 w_k}{\partial z_i \partial z_j} + \beta_{\lambda_k}\sum_{j=1}^{N} b^k_j(z)\frac{\partial w_k}{\partial z_j} + \lambda_k \beta^2_{\lambda_k} w_k = \frac{w_k^{2^*(s)-1}}{\left|\frac{\xi_k}{\beta_{\lambda_k}}\right|^s}$$

in $B_{2\delta/\beta_{\lambda_k}}(0) \setminus \{z_N = -q^N_{\lambda_k}\}$. By the definition of ξ_k we have $|\xi_k/\beta_{\lambda_k}| \to |Q_\infty + z|$.

For any $L > 0$ and some $r > N/2$ by the Hölder inequality we have

$$\int_{B_L(-Q_\infty)} \left(\frac{w_k^{2^*(s)-1}}{\left|\frac{\xi_k}{\beta_{\lambda_k}}\right|^s} \right)^r dz < C(L) < \infty. \tag{35}$$

By applying the elliptic regularity theory in [20] there exists a function w such that

$$w \in C^2_{loc}(\overline{B_L(-Q_\infty)} \setminus \{-Q_\infty\}), \quad w_k \to w \quad \text{in} \quad C^{0,\alpha}(\overline{B_L(-Q_\infty)}) \cap H^1(B_L(-Q_\infty)).$$

Moreover, w satisfies $w(0) = 1$ and $w \in D^{1,2}(\mathbb{R}^N)$. In fact

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w|^2 dz &= \lim_{L\to\infty} \int_{B_L} |\nabla w|^2 dz \\ &\le \lim_{L\to\infty} \lim_{k\to\infty} 2 \int_\Omega (|\nabla v_k|^2 + \lambda_k v_k^2) dx \\ &\le \mu_s^{\frac{N-s}{2-s}}. \end{aligned}$$

Thus

$$w \in C^2_{loc}(\mathbb{R}^N \setminus \{-Q_\infty\}), \quad w_k \to w \quad \text{in} \quad C^{0,\alpha}_{loc}(\mathbb{R}^N) \cap H^1_{loc}(\mathbb{R}^N),$$

and w is a weak solution of

$$-\Delta w + Cw = \frac{w^{2^*(s)-1}}{|(Q_\infty + z)|^s} \quad \text{in} \quad \mathbb{R}^N, \tag{36}$$

where C is defined in (31). Define the function $f : \mathbb{R}^N \setminus \{-Q_\infty\} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, u) = \frac{|u|^{2^*(s)-2}u}{|Q_\infty + z|^s} - Cu.$$

Then we can see w and f satisfy the all conditions of the following proposition:

Proposition 3.9 (Claim 5.3 in [7])**.** *Let $f \in C^0((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R})$ and let $u \in D^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\} \cap H^1_{loc}(\mathbb{R}^N \setminus \{0\})$ be a weak solution of*

$$-\Delta_p u = f(x, u) \quad \text{in} \quad \mathbb{R}^N,$$

where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is p-Laplacian. Difine $F(x,u) := \int_0^u f(x,s)ds$ and assume that $F \in C^1((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R})$. Moreover, along the solution u, assume that $uf(\cdot,u)$, $F(\cdot,u)$ and $x \cdot (\nabla_x F)(\cdot,u) \in L^1(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} \left[\frac{N-p}{p} uf(x,u) - NF(x,u) - x \cdot (\nabla_x F)(\cdot,u) \right] dx = 0.$$

Applying this proposition to (36) and we obtain $C = 0$. Furthermore we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{w^{2^*(s)}}{|(Q_\infty + z)|^s} dz &\leq \lim_{k\to\infty} 2\int_\Omega \frac{v_{\lambda_k}^{2^*(s)}}{|x|^s} dx \\ &= \lim_{k\to\infty} \mu^N_{s,\lambda_k}(\Omega)^{\frac{N-s}{2-s}} \\ &= \mu_s^{\frac{N-s}{2-s}}.\end{aligned}$$

Hence w is a minimizer of μ_s. Since $0 \leq w \leq 1$ and $w(0) = 1$, we obtain $w = U$ and $Q_\infty = 0$. Therefore part (ii) and (iii) is proved.

For $z \in B_{\delta/\beta_{\lambda_k}}$ we set

$$\tilde{w}_k(z) = \frac{\tilde{v}_{\lambda_k}(\beta_{\lambda_k} z)}{\alpha_{\lambda_k}}. \tag{37}$$

Then since $Q_{\lambda_k}/\beta_{\lambda_k} \to 0$ as $k \to \infty$ we have

$$\tilde{w}_k \to U \quad \text{in} \quad C^{0,\alpha}_{loc}(\mathbb{R}^N) \cap H^1_{loc}(\mathbb{R}^N)$$

as $k \to \infty$. Hence part (iv) is obtained. □

Lemma 3.10. *We assume that $u \in H^1(\Omega)$ satisfy that $u \geq 0$ and*

$$\begin{cases} -\Delta u \leq \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{38}$$

Then for any $r > 0$ there exist positive constants $\mu = \mu(\Omega)$ and $C = C(\Omega, r)$ such that for any $Q \in \mathbb{R}^N$ we have

$$\sup_{x\in\Omega\cap B_r(Q)} u(x) \leq C \left(\int_{\Omega\cap B_{2r}(Q)} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{1}{2^*(s)}} \tag{39}$$

provided that

$$\int_{\Omega\cap B_{4r}(Q)} \frac{u^{2^*(s)}}{|x|^s} dx \leq \mu.$$

Proof. We prove Lemma 3.10 in the same way as the strategy of the proof of Lemma 2.13 in [18] . □

Proof of Theorem 3.6 (i). From Lemma 3.2, if u_λ is a minimizer for $\mu^N_{s,\lambda}(\Omega)$ then $\|u_\lambda\|_{L^2(\Omega)} = O(1/\lambda)$. Thus we have $u_\lambda(x) \to 0$ a.e. in Ω. Since $v_\lambda = \mu^N_{s,\lambda}(\Omega)^{(N-2)/(4-2s)} u_\lambda$ we have $v_\lambda(x) \to 0$ a.e. in Ω.

For all $x \in \Omega$, there exists a positive constant κ such that $0 \notin \overline{\Omega \cap B_{4\kappa}(x)}$. We have

$$\lim_{\lambda\to\infty} \int_{\Omega\cap B_{4\kappa}(x)} \frac{v_\lambda^{2^*(s)}}{|x|^s} dx = 0.$$

By Lemma 3.10 we obtain

$$v_\lambda(x) \le \sup_{x\in\Omega\cap B_\kappa(x)} v_\lambda(x) \le C\left(\int_{\Omega\cap B_{2\kappa}(x)} \frac{v_\lambda^{2^*(s)}}{|x|^s}dx\right)^{\frac{1}{2^*(s)}} \to 0$$

as $\lambda \to \infty$. □

Proof of Theorem 3.6 (v). For all $\varepsilon > 0$ and $\delta > 0$ by part (i) there exists $\lambda_0 > 0$ such that $v_\lambda(x) < \varepsilon$ in $\Omega \setminus B_\delta$ for all $\lambda > \lambda_0$. We set $w_\lambda = \lambda^{-(N-2)/(4-2s)}v_\lambda$, then w_λ satisfies

$$\begin{cases} -\frac{1}{\lambda}\Delta w_\lambda + w_\lambda = \frac{w_\lambda^{2^*(s)-1}}{|x|^s} & \text{in} \quad \Omega \\ \frac{\partial w_\lambda}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

For w_λ, we use the strategy of the proof of Theorem 2.3 (iii) in [20].

Proposition 3.11 (Lemma 4.2 in [6])**.** *Assume that $\varepsilon > 0$ and $\mathcal{A}$ is a domain. Let ϕ be a C^2 function satisfying $L\phi = \varepsilon^2\partial_i(a_{ik}\partial_k\phi) + q(x,\varepsilon)\phi = 0$ in $\mathcal{A}$, with $q(x,\varepsilon) < -a < 0$ in $\mathcal{A}$. Then there exists a positive constant $\mu = \mu(a_{ik}, a, \mathcal{A})$ such that*

$$|\phi(x)| \le 2(\sup|\phi(x)|)e^{-\frac{\mu\delta}{\varepsilon}},$$

where $\delta(x) = \mathrm{dist}(x, \partial\mathcal{A})$.

In the interior of $\Omega \setminus B_\delta$ we can apply Proposition 3.11 to w_λ directly. In the neighborhood around $\partial\Omega \setminus B_\delta$ we apply Proposition 3.11 to $\tilde{w}_\lambda = \lambda^{-(N-2)/(4-2s)}\tilde{v}_\lambda$, where $\tilde{v}_\lambda$ is defined in Step 2 of the proof of Lemma 3.8. Hence Theorem 3.6 (v) is proved. □

3.4 Asymptotic behavior II

In this section, we consider the asymptotic behavior of $\mu^N_{s,\lambda}(\Omega)$. Suppose v_λ is a least-energy solution of (11). Define for $f \in H^1(\Omega)$

$$Q_\lambda(f) = \frac{\int_\Omega(|\nabla f|^2 + \lambda f^2)dx}{\left(\int_\Omega \frac{|f|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}.$$

Theorem 3.12. *Assume that $N \ge 5$. There exist positive constants C_1 and C_2 such that as $\lambda \to \infty$*

$$\mu^N_{s,\lambda}(\Omega) = Q_\lambda(v_\lambda) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}}\mu_s - C_1H(0)\varepsilon + C_2\varepsilon^2\lambda + o(\varepsilon^2\lambda).$$

where and $H(0)$ is the mean curvature at 0

$$0 < \varepsilon = \begin{cases} O(1/\lambda) & H(0) > 0, \\ o(1/\lambda^{1/2}) & H(0) \le 0. \end{cases}$$

Proof. The approaches to prove Theorem 3.12 are very close to those in [21]. Therefore we omit the proof of Lemma 3.13 and Lemma 3.17.

Suppose that $\mathcal{N}_0$ is a neighborhood around 0 satisfying $\Omega \cap \mathcal{N}_0 = \Phi(B_{2\delta}^+)$. For $y \in B_{2\delta}^+$ put $\hat{v}_\lambda$ and $\tilde{v}_\lambda$ as in Step 2 of the proof of Lemma 3.8. By using (33) and (34) we define an elliptic operator L by

$$L = \sum_{i,j=1}^{N} \tilde{a}_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{j=1}^{N} \tilde{b}_j(y) \frac{\partial}{\partial y_j},$$

where

$$\tilde{a}_{ij}(z) = \begin{cases} a_{ij}(z) & z_N \geq 0, \\ (-1)^{\delta_{iN}+\delta_{jN}} a_{ij}(z', -z_N)) & z_N < 0, \end{cases}$$

$$\tilde{b}_j(z) = \begin{cases} b_j(z) & z_N \geq 0, \\ (-1)^{\delta_{jN}} b_j(z', -z_N)\beta_{\lambda_k}) & z_N < 0. \end{cases}$$

Since v_λ is a least-energy solution of (11) $\tilde{v}_\lambda$ satisfies

$$-L\tilde{v}_\lambda + \lambda \tilde{v}_\lambda = \frac{\tilde{v}_\lambda^{2^*(s)-1}}{|\Phi(y)|^s} \tag{40}$$

a.e. in $B_{2\delta}$. Set

$$\langle \nabla\phi, \nabla\psi \rangle_g = \sum_{i,j=1}^{N} \int_{B_\delta(0)} a_{ij}(y) \left(\frac{\partial \phi}{\partial y_j}(y) \frac{\partial \psi}{\partial y_k}(y) \right) |J\Phi| dy,$$

$$\langle \phi, \psi \rangle_\lambda = \langle \nabla\phi, \nabla\psi \rangle_g + \lambda \int_{B_\delta(0)} \phi\psi |J\Phi| dy,$$

$$\|\nabla\phi\|_g^2 = \langle \nabla\phi, \nabla\phi \rangle_g, \quad \|\phi\|_\lambda^2 = \langle \phi, \phi \rangle_\lambda.$$

From Theorem 3.6, we have

$$\lim_{\lambda\to\infty} \|\nabla\tilde{v}_\lambda\|_g^2 = \mu_s^{\frac{N-s}{2-s}}, \quad \lim_{\lambda\to\infty} \lambda \int_{B_\delta} \tilde{v}_\lambda^2 |J\Phi| dy = 0, \quad \lim_{\lambda\to\infty} \|\nabla\tilde{v}_\lambda - \nabla U_{\beta_\lambda}\|_g = 0.$$

Define the projection $P : H^1(B_\delta) \to H_0^1(B_\delta)$ by $u = Pv$ such that

$$Lu = Lv.$$

By the definition of L if $v(y', y_N) = v(y', -y_N)$ then $u(y', y_N) = u(y', -y_N)$. We set

$$h_\lambda = v_\lambda - Pv_\lambda, \quad \phi_\varepsilon = U_\varepsilon - PU_\varepsilon$$

and we can see by part (v) of Theorem 3.6 and the maximum principle

$$0 < h_\lambda = O(\varepsilon^{-\gamma\sqrt{\lambda}}) \quad \text{in } \overline{B}_\delta.$$

We can see

$$\phi_\varepsilon = \varepsilon^{\frac{N-2}{2}} \left(\varepsilon^{2-s} + \frac{\delta^{2-s}}{(N-s)(N-2)} \right)^{-\frac{N-2}{2-s}}.$$

Let

$$M = \{cPU_\varepsilon | c \in \mathbb{R}_+, 0 < \varepsilon \le 1\}, \quad \mathrm{dist}(u, M) = \inf_{\phi \in M} \|u - \phi\|_\lambda,$$

and

$$\mathcal{E}(\varepsilon, \lambda) = \left\{ \phi \in H_0^1(\Omega) \middle| \langle \phi, PU_\varepsilon \rangle_\lambda = \left\langle \phi, \frac{\partial}{\partial \varepsilon} PU_\varepsilon \right\rangle_\lambda = 0 \right\}.$$

We obtain the following lemma.

Lemma 3.13. *Suppose that $N \ge 5$. Then for λ sufficiently large* $\mathrm{dist}(Pv_\lambda, M)$ *is attained by $c_\lambda PU_\varepsilon$, where $\varepsilon = \varepsilon(\lambda)$. Moreover,*

$$\frac{\varepsilon}{\beta_\lambda} \to 1 \quad \textit{and} \quad c_\lambda \to 1$$

as $\lambda \to \infty$.

By this lemma we may write

$$Pv_\lambda = c_\lambda PU_\varepsilon + \omega_\lambda$$

where $\omega_\lambda \in \mathcal{E}(\varepsilon, \lambda)$ satisfying $\|\omega_\lambda\|_\lambda = o(1)$, $\|Pv_\lambda\|_\lambda^2 = c_\lambda^2 \|PU_\varepsilon\|_\lambda^2 + \|\omega_\lambda\|_\lambda^2$. Thus

$$v_\lambda = c_\lambda PU_\varepsilon + \omega_\lambda + h_\lambda.$$

We investigate the detail of the estimates for ω_λ.

Lemma 3.14. *We assume that $N \ge 5$ and $\varepsilon = \varepsilon(\lambda)$ is given in Lemma 3.13. Then there exists $\sigma > 0$ and λ_0 such that for all $\omega \in \mathcal{E}(\varepsilon, \lambda)$ and $\lambda > \lambda_0$ we have*

$$(2^*(s) - 1 + \sigma) \int_{B_\delta} \frac{U_\varepsilon^{2^*(s)-2} \omega^2}{|\Phi(y)|^s} |J\Phi| dy \le \|\omega\|_\lambda^2.$$

Proof. Suppose the above lemma does not hold. Then there exist sequences $\lambda_n \to \infty$, $\{\omega_n\} \subset \mathcal{E}(\varepsilon_n, \lambda_n)$ such that

$$(2^*(s) - 1 + o(1)) \int_{B_\delta} \frac{U_\varepsilon^{2^*(s)-2} \omega_n^2}{|\Phi(y)|^s} |D\Phi| dy \ge \|\omega_n\|_{\lambda_n}^2$$

where $\varepsilon_n = \varepsilon(\lambda_n)$. We may assume that $\|\omega_n\|_{\lambda_n} = 1$ without loss of generality. Define $\psi_n(z) = \varepsilon_n^{(N-2)/2} \omega_n(\varepsilon_n z)$ for $z \in B_\delta / \varepsilon_n$. Then we have

$$1 \le (2^*(s) - 1 + o(1)) \int_{B_{\delta/\varepsilon_n}} \frac{U^{2^*(s)-2} \psi_n^2}{|\frac{\Phi(\varepsilon_n z)}{\varepsilon_n}|^s} |D\Phi(\varepsilon_n z)| dz \tag{41}$$

On the other hand we have

$$\begin{aligned} 1 &= \|\omega_n\|_{\lambda_n}^2 \\ &\geq \sum_{i,j} \int_{B_\delta} a_{ij}(y) \left(\frac{\partial \omega_n}{\partial y_i}(y) \frac{\partial \omega_n}{\partial y_j}(y) \right) |D\Phi| dy + \lambda_n \int_{B_\delta} \omega_n^2 |D\Phi| dy \\ &\geq (1+o(1)) \int_{B_{\delta/\varepsilon_n}} |\nabla \psi_n(z)|^2 dz \end{aligned} \tag{42}$$

and

$$\begin{aligned} 1 &= \|\omega_n\|_{\lambda_n}^2 \\ &\geq C \left(\int_{B_\delta} \omega_n^{\frac{2N}{N-2}} |D\Phi| dy \right)^{\frac{N-2}{N}} \\ &= C(1+o(1)) \left(\int_{B_{\delta/\varepsilon_n}} \psi_n^{\frac{2N}{N-2}} dz \right)^{\frac{N-2}{N}}. \end{aligned}$$

Therefore after passing to a subsequence we have

$$\psi_n \to \psi_\infty \quad \text{weakly in } D_{loc}^{1,2}(\mathbb{R}^N), \quad \text{and} \quad \psi_n \to \psi_\infty \quad \text{strongly in } L_{loc}^2(\mathbb{R}^N).$$

We can see that

$$\langle \nabla \psi_\infty, \nabla U \rangle_{L^2(\mathbb{R}^N)} = 0, \quad \left\langle \nabla \psi_\infty, \nabla \left(\frac{\partial}{\partial \lambda} \Big|_{\lambda=1} U_\lambda \right) \right\rangle_{L^2(\mathbb{R}^N)} = 0. \tag{43}$$

Moreover from (41) and (42) it follows that

$$\int_{\mathbb{R}^N} |\nabla \psi_\infty|^2 dz \leq 1 \leq (2^*(s)-1) \int_{\mathbb{R}^N} \frac{U^{2^*(s)-2} \psi_\infty^2}{|z|^s} dz,$$

and hence

$$\frac{\int_{\mathbb{R}^N} |\nabla \psi_\infty|^2 dz}{\int_{\mathbb{R}^N} \frac{U^{2^*(s)-2} \psi_\infty^2}{|z|^s} dz} \leq 2^*(s) - 1. \tag{44}$$

However, (43) and (44) contradict the following lemma.

Lemma 3.15 ([22])**.** *We consider the eigenvalue problem:*

$$\begin{cases} -\Delta \psi = \mu \frac{U^{2^*(s)-1}}{|z|^s} \psi & \text{in} \quad \mathbb{R}^N, \\ \psi \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{45}$$

Then the first two eigenvalues of (45) are $\mu_1 = 1$, $\mu_2 = 2^*(s) - 1$ and the corresponding eigenfunction ψ_1 and ψ_2 satisfy

$$\psi_1 \in \operatorname{span}\{U_\varepsilon\} \quad \text{and} \quad \psi_2 \in \operatorname{span}\left\{ \frac{d}{d\varepsilon}\Big|_{\varepsilon=1} U_\varepsilon \right\}$$

respectively.

□

Recall that $Lh_\lambda = 0$ and $h_\lambda = O(\varepsilon^{-\gamma\sqrt{\lambda}})$. Multiplying (40) by ω_λ and integrating on B_δ by parts, we have

$$\|\omega_\lambda\|_\lambda^2 + O(\varepsilon^{-\gamma\sqrt{\lambda}})\|\omega_\lambda\|_\lambda = \int_{B_\delta} \frac{(c_\lambda PU_\varepsilon + h_\lambda + w_\lambda)^{2^*(s)-1} w_\lambda}{|\Phi(y)|^s} |D\Phi| dy.$$

For the right hand side we have

$$\begin{aligned} &\int_{B_\delta} \frac{(c_\lambda PU_\varepsilon + h_\lambda + w_\lambda)^{2^*(s)-1} w_\lambda}{|\Phi(y)|^s} |D\Phi| dy \\ = \quad & c_\lambda^{2^*(s)-1} \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s} |D\Phi| dy \\ & + (2^*(s)-1) c_\lambda^{2^*(s)-2} \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-2}\omega_\lambda^2}{|\Phi(y)|^s} |D\Phi| dy + O(\|\omega_\lambda\|_\lambda^\sigma + \varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda) \end{aligned}$$

where $\sigma = \min\{3, 2^*(s)\}$. Thus we have

$$\begin{aligned} & \|\omega_\lambda\|_\lambda^2 - (2^*(s)-1) c_\lambda^{2^*(s)-2} \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-2}\omega_\lambda^2}{|\Phi(y)|^s} |D\Phi| dy \\ = \quad & c_\lambda^{2^*(s)-1} \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s} |D\Phi| dy + O(\|\omega_\lambda\|_\lambda^\sigma + \varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda). \end{aligned} \tag{46}$$

Since $0 < PU_\varepsilon < U_\varepsilon$ and from Lemma 3.14 we have

$$\begin{aligned} \|\omega_\lambda\|_\lambda^2 \quad = \quad & \frac{2^*(s)-1+\sigma}{\sigma}(1+o(1)) \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s} |D\Phi| dy \\ & + O(\varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda). \end{aligned} \tag{47}$$

Set

$$\tilde{Q}_\lambda(f) := \frac{\|f\|_\lambda^2}{\left(\int_{B_\delta} \frac{f^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy\right)^{2/2^*(s)}}.$$

Lemma 3.16.

$$\begin{aligned} Q_\lambda(v_\lambda) \quad = \quad & \frac{1}{2^{\frac{2-s}{N-s}}} \tilde{Q}_\lambda(cPU) \\ & -(1+o(1)) \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s^{-\frac{N-2}{2-s}} \int_{B_\delta} \frac{PU^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s} |D\Phi| dy \\ & + O(e^{-\sqrt{\lambda}}\|\omega_\lambda\|_\lambda). \end{aligned}$$

Proof. From Theorem 3.6 it follows

$$\begin{aligned} Q_\lambda(v_\lambda) &= \frac{\int_{\Omega\cap\mathcal{N}_0} |\nabla v_\lambda|^2 + \lambda v_\lambda^2 dx}{\left(\int_{\Omega\cap\mathcal{N}_0} \frac{v_\lambda^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)}} + O(e^{-\gamma\sqrt{\lambda}}) \\ &= \frac{1}{2^{\frac{2-s}{N-s}}} \tilde{Q}_\lambda(\tilde{v}_\lambda) + O(\varepsilon^{-\gamma\sqrt{\lambda}} \|\omega_\lambda\|_\lambda) \end{aligned} \tag{48}$$

Since $\tilde{v}_\lambda = c_\lambda PU_\varepsilon + \omega_\lambda + h_\lambda$ we have

$$\|\tilde{v}_\lambda\|_\lambda^2 = \|c_\lambda PU_\varepsilon\|_\lambda^2 + \|\omega_\lambda\|_\lambda^2 + O(e^{-\gamma\sqrt{\lambda}}).$$

On the other hand,

$$\begin{aligned} &\left(\int_{B_\delta} \frac{\tilde{v}_\lambda^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy\right)^{2/2^*(s)} \\ = &\left(\int_{B_\delta} \frac{(c_\lambda PU_\varepsilon)^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy\right)^{2/2^*(s)} + \frac{2}{2^*(s)} \left(\int_{B_\delta} \frac{(c_\lambda PU_\varepsilon)^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy\right)^{2/2^*(s)-1} \\ &\times \int_{B_\delta} \frac{2^*(s)\,(c_\lambda PU_\varepsilon)^{2^*(s)-1}\,\omega_\lambda + \frac{2^*(s)(2^*(s)-1)}{2}\,(c_\lambda PU_\varepsilon)^{2^*(s)-2}\,\omega_\lambda^2}{|\Phi(y)|^s} |D\Phi| dy \\ &+O(\|\omega_\lambda\|_\lambda^\sigma + e^{-\sqrt{\lambda}}). \end{aligned}$$

Hence we obtain

$$\begin{aligned} &Q_\lambda(v_\lambda) \\ = &\frac{1}{2^{\frac{2-s}{N-s}}} \tilde{Q}_\lambda(c_\lambda PU_\varepsilon) \Bigg[1 + (1+o(1)) \\ &\times \left\{ \frac{\|\omega_\lambda\|_\lambda^2}{\|c_\lambda PU_\varepsilon\|_\lambda^2} - 2\frac{\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega}{|\Phi(y)|^s} |D\Phi| dy}{c_\lambda \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy} - (2^*(s)-1) \frac{\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-2}\omega_\lambda^2}{|\Phi(y)|^s} |D\Phi| dy}{c_\lambda^2 \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy} \right\} \Bigg] \\ &+O(\|\omega_\lambda\|_\lambda^\sigma + \varepsilon^{-\gamma\sqrt{\lambda}} \|\omega_\lambda\|_\lambda). \end{aligned}$$

Using (46), (47), (48), $c_\lambda = 1 + o(1)$, and

$$\lim_{\lambda\to\infty} \|PU_\varepsilon\|_\lambda^2 = \lim_{\lambda\to\infty} \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)}}{|\Phi(y)|^s} |D\Phi| dy = \mu_s^{\frac{N-s}{2-s}},$$

we obtain

$$\begin{aligned} Q_\lambda(v_\lambda) &= \frac{1}{2^{\frac{2-s}{N-s}}} \tilde{Q}_\lambda(c_\lambda PU_\varepsilon) \\ &\quad -(1+o(1)) \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s^{-\frac{N-2}{2-s}} \int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s} |D\Phi| dy \\ &\quad +O(\|\omega_\lambda\|_\lambda^\sigma + \varepsilon^{-\gamma\sqrt{\lambda}} \|\omega_\lambda\|_\lambda). \end{aligned}$$

□

Lemma 3.17.

$$\|\omega_\lambda\|_\lambda = O(e^{-\sqrt{\lambda}}) + \begin{cases} O\left(\varepsilon + \lambda\varepsilon^2\right) & (N \geq 7) \\ o(\lambda\varepsilon) & (N = 5, 6). \end{cases}$$

and

$$\int_{B_\delta} \frac{PU^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy = \begin{cases} O(\varepsilon^2 + \lambda^2\varepsilon^4) & (N \geq 7) \\ o(\lambda\varepsilon^2) & (N = 5, 6), \end{cases}$$

Hence

$$Q_\lambda(v_\lambda) = \frac{1}{2^{\frac{2-s}{N-s}}}\tilde{Q}_\lambda(c_\lambda PU_\varepsilon) + O(e^{-\sqrt{\lambda}}) + \begin{cases} O\left(\varepsilon^2 + \lambda^2\varepsilon^4\right) & (N \geq 7) \\ o(\lambda\varepsilon^2) & (N = 5, 6). \end{cases}$$

To end the proof of Theorem 3.12 we calculate $\tilde{Q}_\lambda(c_\lambda PU_\varepsilon)$. We replace $c_\lambda PU_\varepsilon$ by $\tilde{\phi}$ in (16), (17) and (18). When Ω satisfies $H(0) > 0$ we recall that v_λ exists and

$$Q_\lambda(v_\lambda) < \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s.$$

Consequently we have

$$Q_\lambda(v_\lambda) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - C_1 H(0)\varepsilon + C_2\lambda\varepsilon^2 + o(\lambda\varepsilon^2), \quad \varepsilon = \begin{cases} O(1/\lambda) & H(0) > 0, \\ o(1/\lambda^{1/2}) & H(0) \leq 0. \end{cases}$$

□

3.5 Minimization problem

Theorem 3.18. *Assume that $N \geq 5$ and Ω satisfies $H(0) \leq 0$. Then there exist $\lambda_* = \lambda_*(\Omega)$ such that*

(i) If $0 < \lambda < \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is attained.

(ii) If $\lambda > \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is not attained.

Proof. By Theorem 3.12 the minimizer of $\mu^N_{s,\lambda}(\Omega)$ does not exist for λ sufficiently large (if the minimizer exists, $\mu^N_{s,\lambda}(\Omega) > \mu_s/2^{(2-s)/(N-s)}$ and this contradicts (ii) in Lemma 3.2). Thus there exists $\lambda_* = \lambda_*(\Omega)$ such that part (i) of Lemma 3.3 holds true as $\tilde{\lambda} = \lambda_*$. Consequently from Lemma 3.5 we can prove (i) and (ii) immediately. □

The following theorem holds for all domains (we don't require the condition of the mean curvature at 0).

Theorem 3.19. *There exist $\lambda_{**} > 0$ such that if $\lambda < \lambda_{**}$ then the minimizer of $\mu^N_{s,\lambda}(\Omega)$ is unique.*

Proof. In order to prove this theorem we argue in the same way as [25]. Assume that v_λ is a least-energy solution of (11). Then

$$\int_\Omega \frac{v_\lambda^{2^*(s)}}{|x|^s}dx = \mu^N_{s,\lambda}(\Omega)^{\frac{N-s}{2-s}} \to 0 \quad \text{as} \quad \lambda \to 0.$$

From Lemma 3.10 we have $\|v_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$.

Set $\lambda_i \to 0$ as $i \to \infty$. Let u_i, v_i be the least-energy solutions of (11) when $\lambda = \lambda_i$ such that $\|u_i - v_i\|_{L^\infty(\Omega)} \neq 0$. Define $A_i = \|u_i - v_i\|_{L^\infty(\Omega)}$ and $z_i = A_i^{-1}(u_i - v_i)$. Then z_i satisfies $0 \leq z_i \leq 1$ in Ω, $\|z_i\|_{L^\infty(\Omega)} = 1$, and

$$\begin{cases} -\Delta z_i + \lambda z_i = \frac{u_i^{2^*(s)-1} - v_i^{2^*(s)-1}}{(u_i - v_i)|x|^s} z_i & \text{in} \quad \Omega, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Note that by the mean value theorem, we can see that

$$\frac{u_i^{2^*(s)-1} - v_i^{2^*(s)-1}}{(u_i - v_i)|x|^s} \to 0 \quad \text{as} \quad i \to \infty.$$

Thus by the elliptic regularity theory there exists $z_0 \in C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$ such that $z_i \to z_0$ in $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$ and

$$\begin{cases} -\Delta z_0 = 0 & \text{in} \quad \Omega, \\ \frac{\partial z_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Hence $z_0 \equiv 1$ since $\|z_i\|_{L^\infty(\Omega)} = 1$ for all i.

On the other hand, since u_i and v_i are solutions of (11) we have

$$\int_\Omega \frac{u_i^{2^*(s)-2} - v_i^{2^*(s)-2}}{|x|^s} u_i v_i dx = 0.$$

Since $u_i > 0$ and $v_i > 0$ we see $u_i - v_i$ changes the sign for all i. This is a contradiction. □

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4 A critical problem on the Hardy-Sobolev inequality in boundary singularity case

Abstract

We study a Neumann problem with the Hardy-Sobolev nonlinearity. In boundary singularity case, the impact of the mean curvature at singularity on existence of least-energy solution is well known. Existence and nonexistence of least-energy solution is studied by [9] except for lower dimension case. In this section, we improve this previous work. More precisely, we study four dimensional case and show existence of minimizer in critical case in some sense.

4.1 Introduction

Let $N \geq 3$, Ω be smooth bounded domain in $\mathbb{R}^N$, $0 < s < 2$, $2^*(s) = 2(N-s)/(N-2)$ and $\lambda > 0$. The bounded embedding from $H^1(\Omega)$ to $L^{2^*(s)}(\Omega, |x|^{-s}dx)$ leads to the Hardy-Sobolev inequality

$$\mu_{s,\lambda}^N(\Omega)\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)} \leq \int_\Omega (|\nabla u|^2 + \lambda u^2)dx,$$

where the constant $\mu_{s,\lambda}^N(\Omega)$ is the largest possible constant defined by

$$\mu_{s,\lambda}^N(\Omega) = \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}.$$

The Dirichlet case, that is, the minimization problem of

$$\mu_s^D(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}$$

is studied by many researchers in both interior singularity case and boundary singularity case. In interior singularity case, properties of $\mu_s^D(\Omega)$ is similar to the best constant of the Sobolev inequality. More precisely, $\mu_s^D(\Omega)$ is independent of Ω and never achieved. In boundary singularity case, [6–8] showed that minimizer exists when the mean curvature of $\partial\Omega$ at 0 is negative. In addition, [6] prove the nonexistence result under the assumption $T(\Omega) \subset \mathbb{R}_+^N$ for some rotation T, where $\mathbb{R}_+^N$ is a half-space. After these works [11] investigated a generalized minimization problem concerning $\mu_s^D(\Omega)$.

Existence and nonexistence of minimizer of $\mu_{s,\lambda}^N(\Omega)$ have been studied by [6,9]. In [6], they showed the existence of minimizer under the positivity of the mean curvature at 0. However the nonpositive mean curvature case was not dealt with in [6]. Recently, some part of this problem have been clarified by [9]. The result of nonpositive mean curvature case is completely different from that of positive mean curvature case. Concerning existence of minimizer of $\mu_{s,\lambda}^N(\Omega)$ in nonpositive mean curvature case we obtained the following result;

Theorem 4.1. *[9] Assume that $N \geq 5$ and the mean curvature of $\partial\Omega$ at 0 is nonpositive. Then there exists a positive constant $\lambda_* = \lambda_*(\Omega)$ such that*

(i) If $0 < \lambda < \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is attained.

(ii) If $\lambda > \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is not attained.

This situation is closed to the three dimensional case of the minimization problem introduced by Brezis and Nirenberg [3]:

$$S_\lambda(\Omega) := \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2) dx}{\left(\int_\Omega |u|^{\frac{2N}{N-2}} dx\right)^{(N-2)/N}}, \quad \lambda \in \mathbb{R}.$$

In [3] they proved that there exists $\tilde{\lambda}_*(\Omega) < 0$ such that $S_\lambda(\Omega)$ is attained when $\lambda < \tilde{\lambda}_*(\Omega)$ and $S_\lambda(\Omega)$ is not attained when $\lambda > \tilde{\lambda}_*(\Omega)$. In addition, it is also proved that if Ω is a ball, existence of a minimizer is equivalent to $\lambda < \lambda_*(\Omega)$. After that, by [5] this result was extended to the general bounded domain case.

Our main purpose of this section is to improve Theorem 4.1. More precisely, we investigate the case when $N = 4$ and the case when $\lambda = \lambda_*$.

What is related to the minimization problem $\mu^N_{s,\lambda}(\Omega)$ is the following elliptic equation:

$$\begin{cases} -\Delta u + \lambda u = \frac{u^{2^*(s)-1}}{|x|^s}, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{49}$$

Least-energy solution of (49) is defined by solution of (49) attaining $\mu^N_{s,\lambda}(\Omega)$ and thus existence of least-energy solution of (49) is equivalent to existence of minimizer of $\mu^N_{s,\lambda}(\Omega)$.

Asymptotic behavior of least-energy solution of (49) and $\mu^N_{s,\lambda}(\Omega)$ as $\lambda \to \infty$ have been studied in [9]. These studies are natural because a least-energy solution exists for any λ when the mean curvature at 0 is positive as we know. However, not only that, these studies play a crucial role in studying the minimization problem $\mu^N_{s,\lambda}(\Omega)$. Theorem 4.1 asserts that least-energy solution of (49) does not exist for sufficiently large λ when the mean curvature at 0 is nonpositive. In order to prove this fact, we need to investigate the asymptotic behavior of least-energy solution and $\mu^N_{s,\lambda}(\Omega)$ as $\lambda \to \infty$ under the assumption of existence of least-energy solution for any λ. This technique of the asymptotic analysis in [9] is used everywhere in this section.

Our main results is as follows:

Theorem 4.2. *Assume $N = 4$ and the mean curvature of $\partial\Omega$ at 0 is nonpositive. Then there exists a positive constant $\lambda_* = \lambda_*(\Omega)$ such that*

(i) If $0 < \lambda < \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is attained.

(ii) If $\lambda > \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is not attained.

Theorem 4.3. *Assume $N \geq 4$, the mean curvature at 0 is negative, and λ_* is a constant obtained by Theorem 4.1 or Theorem 4.2. Then $\mu^N_{s,\lambda_*}(\Omega)$ is attained.*

The approaches to prove these theorem are based on [1, 2, 9, 15–17]. The blow-up analysis for semilinear Neumann problem involving the Sobolev critical exponent was started by [1,2]. After these [15–17] studied the best constant of the Sobolev inequality, that is, they studied $\mu^N_{s,\lambda}(\Omega)$ in the case when $s = 0$, and [9] studied $\mu^N_{s,\lambda}(\Omega)$ in the case when $s \in (0, 2)$. In [15] the author studied the asymptotic analysis for the Neumann problem involving the mean curvature on the boundary condition. [16] studied minimization problem on exterior domain by using the techniques in [15]. In [17] they expanded some parts of the results of [15] into the four dimensional case. Finally [9] investigated the asymptotic analysis related to $\mu^N_{s,\lambda}(\Omega)$ as we mentioned before.

This section is organized as follows. In 4.2 we introduce some useful facts to prove Theorem 4.2 and Theorem 4.3. In 4.3 we prove Theorem 4.2. In 4.4 we prove Theorem 4.3.

4.2 Preliminaries

In this subsection we prepare some facts in advance to prove Theorem 4.2 and Theorem 4.3.

μ_s denotes the best constant of the Hardy-Sobolev inequality on $\mathbb{R}^N$, that is, μ_s is defined by

$$\mu_s := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \middle| u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\}. \tag{50}$$

We set

$$U(x) = \left(1 + \frac{|x|^{2-s}}{(N-s)(N-2)} \right)^{-\frac{N-2}{2-s}}, \tag{51}$$

which is a minimizer for μ_s. In addition, for $\varepsilon > 0$ we define a function by

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x}{\varepsilon}\right). \tag{52}$$

The following lemmas are introduced in [9].

Lemma 4.4. *(i)* $\mu^N_{s,\lambda}(\Omega)$ *is continuous and non-decreasing with respect to* λ.

(ii) For any $\lambda > 0$, $\mu^N_{s,\lambda}(\Omega) \le \mu_s / 2^{(2-s)/(N-s)}$.

(iii) $\lim_{\lambda \to 0} \mu^N_{s,\lambda}(\Omega) = 0$.

Lemma 4.5. *We have either*

(i) there exists $\tilde{\lambda}$ *such that for* $\lambda \ge \tilde{\lambda}$

$$\mu^N_{s,\lambda}(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s,$$

or

(ii) for all λ the equality of (i) does not hold and

$$\lim_{\lambda\to\infty} \mu^N_{s,\lambda}(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s.$$

Lemma 4.6. *(i) If $\mu^N_{s,\lambda}(\Omega) < \mu_s/2^{(2-s)/(N-s)}$ then $\mu^N_{s,\lambda}(\Omega)$ is attained.*

(ii) If there exists a positive constant $\tilde{\lambda}$ such that $\mu^N_{s,\tilde{\lambda}}(\Omega) = \mu_s/2^{(2-s)/(N-s)}$ then $\mu^N_{s,\lambda}(\Omega)$ is not attained for all $\lambda > \tilde{\lambda}$.

We recall some facts about a diffeomorphism straightening a boundary portion around a point $P \in \partial\Omega$, which was introduced in [10–14]. Through translation and rotation of the coordinate system we may assume that P is the origin and inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N-axis. In a neighborhood $\mathcal{N}$ around P, there exists a smooth function $\psi(x'),\ x' = (x_1, \dots, x_{N-1})$ such that $\partial\Omega \cap \mathcal{N}$ can be represented by

$$x_N = \psi(x') = \frac{1}{2}\sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)$$

where $\alpha_1, \dots, \alpha_{N-1}$ are the principal curvatures of $\partial\Omega$ at P. For $y \in \mathbb{R}^N$ with $|y|$ sufficiently small, we define a mapping $x = \Phi(y) = (\Phi_1(y), \dots, \Phi_N(y))$ by

$$\Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial\psi}{\partial x_j}(y') & j = 1, \dots, N-1 \\ y_N + \psi(y') & j = N. \end{cases}$$

The differential map $D\Phi$ is

$$D\Phi(y) = \begin{pmatrix} \delta_{ij} - \dfrac{\partial^2\psi}{\partial x_i \partial x_j}(y') y_N & -\dfrac{\partial\psi}{\partial x_i}(y') \\ \dfrac{\partial\psi}{\partial x_j}(y') & 1 \end{pmatrix}_{1\le i,j \le N-1}$$

and near $y = 0$

$$|J\Phi(y)| = |\det D\Phi(y)| = 1 - (N-1)H(P)y_N + O(|y|^2).$$

We write as $\Psi(x) = (\Psi_1(x), \dots, \Psi_N(x))$ instead of the inverse map $\Phi^{-1}(x)$.

$B_r(a)$ denotes a open ball with center a and radius r. In addition, suppose $B_r = B_r(0)$ and $B_r^+ = \{y \in B_r | y_N > 0\}$. Define $\mathcal{N}_0$ as a neighborhood around 0 such that $\Phi(B_\delta) = \mathcal{N}_0$.

4.3 Proof of Theorem 4.2

In this subsection we prove Theorem 4.2. In order to prove Theorem 4.2 we need the following proposition:

Proposition 4.7. *Suppose that there exists a least-energy solution of (49) for all λ. Then there exist positive constants C_1 and C_2 such that as $\lambda \to \infty$*

$$\mu^N_{s,\lambda}(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{4-s}} \mu_s - C_1 H(0)\varepsilon + C_2 \lambda \varepsilon^2 \log \frac{1}{\lambda^{1/2}\varepsilon} + O\left(\lambda\varepsilon^2 + \varepsilon^2 \log \frac{1}{\lambda^{1/2}\varepsilon}\right),$$

where $H(0)$ is the mean curvature at 0 and

$$0 < \varepsilon = o(1/\lambda^{1/2}).$$

Assuming that Proposition 4.7 holds, we obtain Theorem 4.2 immediately. Indeed, by Proposition 4.7 minimizer of $\mu^N_{s,\lambda}(\Omega)$ does not exist for λ sufficiently large (if the minimizer exists, $\mu^N_{s,\lambda}(\Omega) > \mu_s/2^{(2-s)/(N-s)}$ and this contradicts (ii) of Lemma 4.4). This implies the existence of $\lambda_* = \lambda_*(\Omega)$ such that part (i) of Lemma 4.5 holds as $\tilde{\lambda} = \lambda_*$. As a consequence (i) and (ii) follow from Lemma 4.6.

Proof of Proposition 4.7. We note that $2^*(s) = 4 - s$. Suppose that v_λ is a least-energy solution of (49), that is, v_λ satisfies

$$\begin{cases} -\Delta v_\lambda + \lambda v_\lambda = \frac{v_\lambda^{3-s}}{|x|^s}, \quad v_\lambda > 0 & \text{in} \quad \Omega, \\ \frac{\partial v_\lambda}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{53}$$

We define α_λ and β_λ as

$$\alpha_\lambda = \|v_\lambda\|_{L^\infty(\Omega)}, \quad \beta_\lambda = \frac{1}{\alpha_\lambda}.$$

From Theorem 3.1 in [9] we have

$$\lim_{\lambda\to\infty} \int_\Omega |\nabla v_\lambda|^2 dx = \frac{1}{2}\mu_s^{\frac{4-s}{2-s}}, \quad \lim_{\lambda\to\infty} \lambda \int_\Omega v_\lambda^2 dx = 0. \tag{54}$$

We set a cut-off function $\eta_\lambda(y) = \eta_\lambda(|y|)$ such that support of η_λ is in $B_{\delta/\lambda^{1/2}}$ and $\eta_\lambda = 1$ in $B_{\delta/2\lambda^{1/2}}$. We may assume that $|\nabla \eta_\lambda| \le C/\lambda^{1/2}$, $|D^2\eta_\lambda| \le C/\lambda$. For simplicity we write η. For η, a positive constant ε, and U_ε which is defined in (52) we set

$$V_\varepsilon(x) = \begin{cases} \eta(\Psi(x))U_\varepsilon(\Psi(x)) & (x \in \mathcal{N}_0), \\ 0 & (x \in \mathbb{R}^N \setminus \mathcal{N}_0). \end{cases}$$

Set

$$\langle \phi, \psi \rangle_\lambda = \int_\Omega \nabla\phi \nabla\psi dx + \lambda \int_\Omega \phi\psi dx, \quad \|\phi\|^2_\lambda = \langle \phi, \phi \rangle_\lambda .$$

In addition to (54), we have

$$\lim_{\lambda\to\infty} \|v_\lambda - V_{\beta_\lambda}\|_\lambda = 0.$$

Let

$$M = \{cV_\varepsilon | c \in \mathbb{R}_+, 0 < \varepsilon \leq 1\}, \quad \text{dist}(u, M) = \inf_{\phi \in M} \|u - \phi\|_\lambda,$$

and

$$\mathcal{E}(\varepsilon, \lambda) = \left\{ \phi \in H^1(\Omega) \middle| \langle \phi, V_\varepsilon \rangle_\lambda = \left\langle \phi, \frac{\partial}{\partial \varepsilon} V_\varepsilon \right\rangle_\lambda = 0 \right\}.$$

We have the following lemma corresponding to Lemma 4.2 in [9].

Lemma 4.8. *If λ is sufficiently large* $\text{dist}(v_\lambda, M)$ *is attained by cV_ε, where $c = c(\lambda)$, $\varepsilon = \varepsilon(\lambda)$. Moreover,*

$$\frac{\varepsilon}{\beta_\lambda} \to 1 \quad \textit{and} \quad c \to 1$$

as $\lambda \to \infty$.

From this lemma there exists $\omega_\lambda \in \mathcal{E}_{\varepsilon,\lambda}$ such that

$$v_\lambda = cV_\varepsilon + \omega_\lambda, \quad \|v_\lambda\|_\lambda^2 = \|cV_\varepsilon\|_\lambda^2 + \|\omega_\lambda\|_\lambda^2, \quad \|\omega_\lambda\|_\lambda = o(1). \tag{55}$$

By applying the proof of Lemma 4.3 in [9] we have the following lemma:

Lemma 4.9. *We assume $\varepsilon = \varepsilon(\lambda)$ is given in Lemma 4.9. Then there exists $\sigma > 0$ and λ_0 such that for all $\omega \in \mathcal{E}(\varepsilon, \lambda)$ and $\lambda > \lambda_0$ we have*

$$(3 - s + \sigma) \int_\Omega \frac{V_\varepsilon^{2-s} \omega^2}{|x|^s} dx \leq \|\omega\|_\lambda^2.$$

Multiplying (53) by ω_λ and integrating on Ω by parts, we have

$$\|\omega_\lambda\|_\lambda^2 = \int_\Omega \frac{(cV_\varepsilon + \omega_\lambda)^{3-s} w_\lambda}{|x|^s} dx.$$

For the right hand side we have

$$\begin{aligned} &\int_\Omega \frac{(cV_\varepsilon + \omega_\lambda)^{3-s} w_\lambda}{|x|^s} dx \\ = \quad & c^{3-s} \int_\Omega \frac{V_\varepsilon^{3-s} \omega_\lambda}{|x|^s} dx + (3 - s) c^{2-s} \int_\Omega \frac{V_\varepsilon^{2-s} \omega_\lambda^2}{|x|^s} dy + O(\|\omega_\lambda\|_\lambda^\gamma), \end{aligned}$$

where $\gamma = \min\{3, 4 - s\}$. Thus we have

$$\begin{aligned} &\|\omega_\lambda\|_\lambda^2 - (3 - s) c^{2-s} \int_\Omega \frac{V_\varepsilon^{2-s} \omega_\lambda^2}{|x|^s} dx \\ = \quad & c^{3-s} \int_\Omega \frac{V_\varepsilon^{3-s} \omega_\lambda}{|x|^s} dx + O(\|\omega_\lambda\|_\lambda^\gamma). \end{aligned} \tag{56}$$

From Lemma 4.9 it follows that

$$\|\omega_\lambda\|_\lambda^2 = \frac{3 - s + \sigma}{\sigma} (1 + o(1)) \int_\Omega \frac{V_\varepsilon^{3-s} \omega_\lambda}{|x|^s} dx + O(\|\omega_\lambda\|_\lambda^\gamma), \tag{57}$$

where we used that $c = 1 + o(1)$. Set

$$Q_\lambda(f) := \frac{\int_\Omega (|\nabla f|^2 + \lambda f^2) dx}{\left(\int_\Omega \frac{|f|^{4-s}}{|x|^s} dx\right)^{2/(4-s)}}.$$

By using (55) we have

$$\begin{aligned} Q_\lambda(v_\lambda) &= Q_\lambda(cV_\varepsilon)\left\{1 + \frac{\|\omega_\lambda\|_\lambda^2}{\|cV_\varepsilon\|_\lambda^2} - \frac{2\int_\Omega \frac{(cV_\varepsilon)^{3-s}\omega_\lambda}{|x|^s}dx}{\int_\Omega \frac{(cV_\varepsilon)^{4-s}}{|x|^s}dx} - \frac{(3-s)\int_\Omega \frac{(cV_\varepsilon)^{2-s}\omega_\lambda^2}{|x|^s}dx}{\int_\Omega \frac{(cV_\varepsilon)^{4-s}}{|x|^s}dx}\right\} \\ &\quad + O(\|\omega_\lambda\|_\lambda^\gamma) \end{aligned}$$

Recalling that $\varepsilon/\beta_\lambda \to 1$ as $\lambda \to \infty$, we can see

$$\lim_{\lambda\to\infty} \|cV_\varepsilon\|_\lambda^2 = \lim_{\lambda\to\infty} \int_\Omega \frac{(cV_\varepsilon)^{4-s}}{|x|^s} dx = \frac{1}{2}\mu_s^{\frac{4-s}{2-s}}.$$

Hence from (56) we have

$$Q(v_\lambda) = Q(cV_\varepsilon) - (1 + o(1))2^{\frac{2}{4-s}}\mu_s^{-\frac{2}{2-s}} \int_\Omega \frac{V_\varepsilon^{3-s}\omega_\lambda}{|x|^s} dx + O(\|\omega_\lambda\|_\lambda^\gamma). \qquad (58)$$

For $Q_\lambda(cV_\varepsilon)$ we apply the calculations in the proof of (ii) of Lemma 2.1 (see [9]). It follows that

$$Q_\lambda(cV_\varepsilon) = \left(\frac{1}{2}\right)^{\frac{2-s}{4-s}} \mu_s - C_1 H(0)\varepsilon + C_2\lambda\varepsilon^2 \log\frac{1}{\lambda^{1/2}\varepsilon} + O\left(\lambda\varepsilon^2 + \varepsilon^2\log\frac{1}{\lambda^{1/2}\varepsilon}\right) \qquad (59)$$

for some positive constants C_1 and C_2. In order to finish the proof we prove the following lemma in the same way as Step 3 of Section 4 in [16].

Lemma 4.10.

$$\int_\Omega \frac{V_\varepsilon^{3-s}\omega_\lambda}{|x|^s} dx = O\left(\lambda^{1/2}\varepsilon + \varepsilon\left(\log\frac{1}{\lambda^{1/2}\varepsilon}\right)^{1/2}\right)\|\omega_\lambda\|_\lambda.$$

This lemma and (57) yield that

$$\|\omega_\lambda\|_\lambda = O\left(\lambda^{1/2}\varepsilon + \varepsilon\left(\log\frac{1}{\lambda^{1/2}\varepsilon}\right)^{1/2}\right),$$

and thus

$$\int_\Omega \frac{V_\varepsilon^{3-s}\omega_\lambda}{|x|^s} dx = O\left(\lambda\varepsilon^2 + \varepsilon^2\log\frac{1}{\lambda^{1/2}\varepsilon}\right). \qquad (60)$$

Consequently Proposition 4.7 follows from (58), (59), and (60). □

4.4 Proof of Theorem 4.3

In this section we prove Theorem 4.3. Note that

$$\lim_{\lambda\to\lambda_*}\mu^N_{s,\lambda}(\Omega)=\mu^N_{s,\lambda_*}(\Omega)=\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}}\mu_s. \tag{61}$$

Proof of Theorem 1.3. For any $\lambda\in(0,\lambda_*)$ we write v_λ as a least-energy solution of (49). Recall that by the elliptic regularity theory $v_\lambda\in C^2_{loc}(\overline{\Omega}\setminus\{0\})$ and $v_\lambda\in C^{0,\alpha}(\overline{\Omega})$ (see [4, 7]).

Firstly, we claim that we only have to show the existence of a positive constant C such that $\|v_\lambda\|_{L^\infty(\Omega)}<C$ holds uniformly for λ near λ_*. Indeed, if $\|v_\lambda\|_{L^\infty(\Omega)}<C$ then there exists v such that

$$v_\lambda\rightharpoonup v \text{ weakly in } H^1(\Omega),\quad \text{and}\quad v_\lambda\to v \text{ strongly in } L^{2^*(s)}(\Omega,|x|^{-s}dx)$$

as $\lambda\to\lambda_*$. Since v_λ is a least-energy solution of (49) we have $v\not\equiv 0$. Therefore

$$\frac{\int_\Omega(|\nabla v|^2+\lambda_* v^2)dx}{\left(\int_\Omega\frac{|v|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}\le\lim_{\lambda\to\lambda_*}\frac{\int_\Omega(|\nabla v_\lambda|^2+\lambda v_\lambda^2)dx}{\left(\int_\Omega\frac{|v|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}=\mu^N_{s,\lambda_*}(\Omega)$$

and hence v is a minimizer of $\mu^N_{s,\lambda_*}(\Omega)$.

Next, we show that there exist a positive constant C such that $\|v_\lambda\|_{L^\infty(\Omega)}<C$ holds. Assume that λ_k is a sequence (a suitable subsequence is also written by λ_k) such that $\lambda_k\to\lambda_*$ as $k\to\infty$. We suppose that $\|v_{\lambda_k}\|_{L^\infty(\Omega)}\to\infty$ as $k\to\infty$ and derive a contradiction. Here, we define α_λ, β_λ, and $x_\lambda\in\overline{\Omega}$ as

$$\alpha_{\lambda_k}=\|v_{\lambda_k}\|_{L^\infty(\Omega)}=v_{\lambda_k}(x_{\lambda_k}),\quad \beta_{\lambda_k}=\alpha_{\lambda_k}^{-\frac{2}{N-2}}.$$

Step 1. We obtain the following results:

(i) $\int_\Omega v^2_{\lambda_k}\to 0$,

(ii) $|x_{\lambda_k}|=o(\beta_{\lambda_k})$,

as $k\to\infty$. For any $\varepsilon>0$ and $r>0$ there exists a k_0 such that for all $k>k_0$

(iii) $\left|\dfrac{v_{\lambda_k}(x)}{\alpha_{\lambda_k}}-U\left(\dfrac{\Psi(x)}{\beta_{\lambda_k}}\right)\right|<\varepsilon\quad$ in $\Omega\cap B_{r\beta_{\lambda_k}}$.

Proof. We obtain (ii), (iii) by applying the technique of the proof of (iii), (iv) of Theorem 3.1 in [9].

We prove (i). For any $R>0$, suppose that $\mathcal{N}_0^R$ denotes $\mathcal{N}_0$ defined in subsection 4.2 with $\delta=R$. From (iii) it follows that

$$\lim_{k\to\infty}\int_{\Omega^R_{\beta_{\lambda_k}}}\frac{v_{\lambda_k}^{2^*(s)}}{|x|^s}dx=\int_{B_R^+}\frac{U^{2^*(s)}}{|x|^s}dx=\frac{1}{2}\mu_s^{\frac{N-s}{2-s}}-\gamma(R), \tag{62}$$

where

$$\Omega^R_{\beta_{\lambda_k}} = \beta_{\lambda_k}(\Omega \cap \mathcal{N}_0^R), \quad \gamma(R) = \int_{\mathbb{R}^N_+ \setminus B^+_R} \frac{U^{2^*(s)}}{|x|^s} dx.$$

Since (61) we have

$$\lim_{k\to\infty} \int_{\Omega \setminus \Omega^R_{\beta_{\lambda_k}}} \frac{v_{\lambda_k}^{2^*(s)}}{|x|^s} dx = \gamma(R). \tag{63}$$

Clearly $\gamma(R) \to 0$ as $R \to \infty$. Here, we observe that

$$\int_\Omega v_{\lambda_k}^2 dx = \int_{\Omega^R_{\beta_{\lambda_k}}} v_{\lambda_k}^2 dx + \int_{\Omega \setminus \Omega^R_{\beta_{\lambda_k}}} v_{\lambda_k}^2 dx = I_1 + I_2.$$

For I_1 from (62) we have

$$I_1 \leq \left(\int_{\Omega^R_{\beta_{\lambda_k}}} |x|^{\frac{2s}{2^*(s)-2}} dx \right)^{\frac{2^*(s)-2}{2^*(s)}} \left(\int_{\Omega^R_{\beta_{\lambda_k}}} \frac{v_{\lambda_k}^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} = o(1),$$

as $k \to \infty$. For I_2 from (63) we have

$$\begin{aligned} I_2 &\leq \left(\int_{\Omega \setminus \Omega^R_{\beta_{\lambda_k}}} |x|^{\frac{2s}{2^*(s)-2}} dx \right)^{\frac{2^*(s)-2}{2^*(s)}} \left(\int_{\Omega \setminus \Omega^R_{\beta_{\lambda_k}}} \frac{v_{\lambda_k}^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\ &= C(\gamma(R) + o(1))^{\frac{2}{2^*(s)}} \end{aligned}$$

as $k \to \infty$. Letting $R \to \infty$ after $k \to \infty$, we obtain (i).

□

Step 2. We have as $k \to \infty$

$$\mu^N_{s,\lambda_k}(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - H(0)\varepsilon + \begin{cases} O(\varepsilon^2) & N \geq 5, \\ O(\varepsilon^2(\log\frac{1}{\varepsilon})^{\frac{4}{3}}) & N = 4, \end{cases} \tag{64}$$

where $\varepsilon = \varepsilon_k$ is a positive constant such that $\varepsilon \to 0$ as $k \to \infty$.

Proof. From Step 1 we have

$$\lim_{k\to\infty} \int_\Omega |\nabla v_{\lambda_k}|^2 dx = \frac{1}{2}\mu_s^{\frac{N-s}{2-s}}, \quad \lim_{k\to\infty} \lambda_k \int_\Omega v_{\lambda_k}^2 dx = 0, \quad \lim_{k\to\infty} \int_\Omega |\nabla (v_{\lambda_k} - U_{\beta_{\lambda_k}})|^2 dx = 0.$$

Therefore investigating the detail of the asymptotic behavior of $\mu^N_{s,\lambda_k}(\Omega)$ as $k \to \infty$ (see [9, 15–17], and Section 3 in this paper) we obtain (64). □

Consequently if the mean curvature of $\partial\Omega$ at 0 is negative (64) contradicts (ii) of Lemma 4.4, and which implies $\|v_{\lambda_k}\|_{L^\infty(\Omega)}$ is bounded uniformly. □

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5 Strauss's radial compactness and nonlinear elliptic equation involving a variable critical exponent

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Abstract

We study existence of a non-trivial solution of

$$-\Delta_p u(x) + u(x)^{p-1} = u(x)^{q(x)-1},\ u(x) \geq 0,\ x \in \mathbb{R}^N, \quad u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$$

under some conditions on $q(x)$, especially, $\liminf_{|x|\to\infty} q(x) = p$. Concerning this problem, we firstly consider compactness and non-compactness for the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$. We point out that the decaying speed of $q(x)$ at infinity plays an essential role on the compactness. Secondly, by applying the compactness result, we show the existence of a non-trivial solution of the elliptic equation.

5.1 Introduction and main results

In this article, we consider the following nonlinear elliptic equation

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{q(\cdot)-1}, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \end{cases} \tag{65}$$

for $1 < p < N$, where $\Delta_p u = \mathrm{div}\,(\,|\nabla u|^{p-2}\nabla u\,)$ is $p-$Laplacian, and variable exponent $q(x)$ is a measurable function satisfying $q(x) > p$, $\liminf_{|x|\to\infty} q(x) = p$.

$p(x)$-Laplacian type elliptic equation is one of the problems with variable exponent and this type equation on $\mathbb{R}^N$ is studied by many researchers in several subjects: multiplicity of solutions (see e.g. [1,13]), existence of solutions of equations involving several nonlinearities (see e.g. [2,12]), equations under periodic assumptions (see e.g. [11,26]) and so on. Moreover, existence of solutions of the equation (65) involving variable exponent touching the critical exponent, that is $\mathrm{ess\,sup}_{\mathbb{R}^N}\, q(x) = p^* := \frac{Np}{N-p}$, is studied by [4,23].

Concerning the critical exponent related to the Sobolev embedding in the whole space, not only p^*, another critical exponent exists and it is p. In the viewpoint of this, considering the case where $\mathrm{ess\,inf}_{\mathbb{R}^N}\, q(x) = p$ is natural as another critical case. However, even for p-Laplace equation there are no results in this case. Thus we study the problem (65) at the opening of this article. In this case, unlike the subcritical case, we need to overcome some difficulties to show the existence of a non-trivial solution of (65). We will explain them more precisely after Remark 5.5. Thus in advance of study of the equation (65), we consider the related embedding to the equation. Namely we study the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$.

We define the generalized Sobolev spaces $W^{k,p(x)}(\Omega)$ with variable exponents $p(x)$. For a domain $\Omega \subset \mathbb{R}^N$ and a function $p \in L^\infty(\Omega)$ with $p(x) \geq 1$ we set

$$L^{p(x)}(\Omega) = \left\{ u \text{ is a real measurable function on } \Omega \;\middle|\; \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$
$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \;\middle|\; D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k \right\}.$$

These $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are Banach spaces with the following norms:

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 \;\middle|\; \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \; \|u\|_{W^{k,p(x)}} = \|u\|_{p(x)} + \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(x)}.$$

When Ω is a bounded domain with the cone property, some results concerning the embedding of $W^{k,p(x)}(\Omega)$ are obtained by [14,17,20]. One of the results in [14] is the existence of the compact embedding. They consider the situation when $p(x)$ is uniformly continuous on $\overline{\Omega}$ and $1 < \operatorname{ess\,inf}_{\overline{\Omega}} p(x) \leq \operatorname{ess\,sup}_{\overline{\Omega}} p(x) < N/k$. Under this situation there exists a compact embedding from $W^{k,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ for $q(x)$ satisfying $p(x) \leq q(x)$ a.e. in Ω and $\operatorname{ess\,inf}_{\overline{\Omega}} p^*(x) - q(x) > 0$, where $p^*(x) = Np(x)/(N - kp(x))$. On the other hand, for $W^{1,p}(\Omega)$ Kurata and Shioji [17] consider the critical case, that is $\operatorname{ess\,sup}_{\overline{\Omega}} q(x) = p^*$. They showed that if there exist $x_0 \in \Omega, C_0 > 0, \eta > 0$, and $0 < \ell < 1$ such that $\operatorname{ess\,sup}_{\Omega \setminus B_\eta(x_0)} q(x) < p^*$ and

$$q(x) \leq p^* - \frac{C_0}{|\log|x - x_0||^\ell} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0),$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact. Conversely, if

$$q(x) \geq p^* - \frac{C_0}{|\log|x - x_0||} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0),$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ is not compact.

When $\Omega = \mathbb{R}^N$, Strauss [24] and Lions [18] showed that the radial Sobolev space $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ can be embedded to $L^q(\mathbb{R}^N)$ compactly for $q \in (p, p^*)$. In addition, related results are in [8, 10], and so on. In $p(x)$ case, under the same conditions as those of bounded domain case the compact embedding from $W^{1,p(x)}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is obtained for $q(x)$ satisfying $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) - p(x) > 0$ and $\operatorname{ess\,inf}_{\mathbb{R}^N} p^*(x) - q(x) > 0$ by [15]. On the other hand, the critical case, that is $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) - p(x) = 0$ or $\operatorname{ess\,inf}_{\mathbb{R}^N} p^*(x) - q(x) = 0$, has not been treated so far even if $p(x) \equiv p$.

In this paper, we fix $p(x) \equiv p$. Our first study is to obtain a sufficiently condition of compactness and non-compactness of the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ for variable exponent $q(x)$ satisfying $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$ and $\operatorname{ess\,sup}_{\mathbb{R}^N} q(x) = p^*$. Based on these results, as the second study we obtain a non-trivial solution of (65) under the compactness conditions with $\liminf_{|x|\to\infty} q(x) = p$.

Before introducing main results, we fix several notations. B_R denote a open ball centered 0 with radius R. ω_{N-1} is an area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Throughout this paper we assume that $q(x) \in L^\infty(\mathbb{R}^N)$ and $q(x) \geq 1$ for a.e. $x \in \mathbb{R}^N$. A letter C denotes various positive constant. If u is a radial function in $\mathbb{R}^N$, then we can write as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$ in $\mathbb{R}_+$. For simplicity we write $u(x) = u(|x|)$ with admitting some ambiguity.

Theorem 5.1. *(Non-compactness) If there exist positive constants R, C_0 and a open set Γ in $\mathbb{S}^{N-1}$ such that*

$$q(x) \leq p + \frac{C_0}{|\log|x||} \quad \text{for } x \in (R, +\infty) \times \Gamma, \tag{66}$$

then the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is not compact.

Theorem 5.2. *(Compactness) If there exist positive constants r, R, C_0, C_1, and $k, l \in (0,1)$ such that*

$$q(x) \leq p^* - \frac{C_0}{|\log|x||^k} \quad \text{for } x \in B_r, \tag{67}$$

$$q(x) \geq p + \frac{C_1}{|\log|x||^\ell} \quad \text{for } x \in \mathbb{R}^N \setminus B_R, \tag{68}$$

then the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact.

Remark 5.3. *In Theorem 5.2, we don't need the constraint $p \leq q(x) \leq p^*$. $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \subset L^{q(x)}(\mathbb{R}^N)$ holds whenever $q(x)$ satisfies $q(x) \leq p^*$ in B_r and $q(x) \geq p$ in $\mathbb{R}^N \setminus B_R$.*

Theorem 5.4. *Assume that $q(x)$ satisfies the hypotheses (67), (68) in Theorem 5.2 and $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. Then there exists a non-trivial weak solution $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ of (65) in the sense of*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla \phi + u^{p-1}\phi - u^{q(x)-1}\phi \right) dx = 0 \tag{69}$$

for any $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$.

Remark 5.5. *If $q(x)$ is a radially symmetric function satisfying the hypotheses of Theorem 5.4, then we can show that the weak solution u obtained in Theorem 5.4 satisfies $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$. Indeed, since u and $q(x)$ are radially symmetric, it follows that for all $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$*

$$\int_0^\infty \left(|u'(r)|^{p-2} u'(r) \phi'(r) + u^{p-1}\phi - u^{q(r)-1}\phi \right) r^{N-1} dr = 0,$$

where $r = |x|$. If for any $\psi \in C^\infty_c(\mathbb{R}^N)$ we consider the radial function $\Psi(r) = \int_{\omega \in \mathbb{S}^{N-1}} \psi(r\omega)\, dS_\omega$, then we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla \psi + u^{p-1}\psi - u^{q(x)-1}\psi \right) dx \\ &= \int_0^\infty \left(|u'(r)|^{p-2} u'(r) \Psi'(r) + u^{p-1}\Psi - u^{q(r)-1}\Psi \right) r^{N-1} dr = 0. \end{aligned}$$

Therefore we see that u satisfies (69) even for non-radial functions ϕ. Finally, by Corollary of Theorem 2 in [9] we have $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. And also, by Theorem 2.5.1 in [21] we have $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

We note the difficulties to obtain Theorem 5.4 caused by the condition $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$. *Ambrosetti-Rabinowitz condition (AR)* is well-known in order to obtain a non-trivial weak solution to the following problem by mountain pass method.

$$-\Delta_p u + |u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^N.$$

(AR) There are $\mu > p$ *and* $M > 0$ such that for $|u| \geq M, 0 < \mu F(x,u) \leq uf(x,u)$,

where $F(x,u) := \int_0^u f(x,s)ds$. Especially, condition (AR) has been used to establish not only the mountain pass structure of the energy functional but also the Palais-Smale condition. A weaker condition has also been considered, for instance, Liu-Wang [19] studied (SQ) which is called *super-quadratic condition.*

$$\text{(SQ)} \quad \lim_{|u|\to\infty} \frac{F(x,u)}{|u|^p} = \infty \quad \text{uniformly in } x \in \mathbb{R}^N.$$

However, assuming that the nonlinear term $u(x)^{q(x)-1}$ in (65) is a special case of the general nonlinear term $f(x,u)$, this does not satisfy even condition (SQ) when $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$. From these facts, it seems to be difficult to confirm whether the energy functional J (see Section 4) corresponding to (65) satisfies the Palais-Smale condition or not. In more detail, while the fact that bounded Palais-Smale sequence has a convergent subsequence is straightforward from Theorem 5.2, boundedness of all Palais-Smale sequence is non-trivial. Besides that, satisfying the mountain pass structure for J is not trivial since we can not apply the fibering map method directly.

To overcome these difficulties, in Section 3, we construct a solution of (65) as a limit of mountain pass solutions of some elliptic equations approaching (65) in the sense of energy functional. In Section 4, we show an another proof by using the variant of the mountain pass theorem. More precisely, by introducing the condition (C) (see Section 4) defined in [7] or [5] instead of the Palais-Smale condition, we obtain a solution of (65) in a different way from Section 3.

5.2 Compactness and non-compactness of the embedding

We prove Theorem 5.1 and Theorem 5.2. Before beginning the proof we recall the pointwise estimate and the compactness theorem introduced in [18], and [24] ($p = 2$). For the reader's convenience, the proofs are in Appendix.

Proposition 5.6. *For any $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ we have*

$$|u(x)| \leq \left(\frac{p}{\omega_{N-1}}\right)^{\frac{1}{p}} |x|^{-\frac{N-1}{p}} \|u\|^{\frac{p-1}{p}}_{L^p(\mathbb{R}^N)} \|\nabla u\|^{\frac{1}{p}}_{L^p(\mathbb{R}^N)}. \tag{70}$$

Proposition 5.7. *The embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$.*

Proof of Theorem 5.1. We shall show Theorem 5.1 in the same way as [17]. Set $r(x) = q(x) - p$ for $x \in \mathbb{R}^N$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a radial function satisfying $\phi \equiv 1$ on $B_{\frac{1}{2}}$ and $\mathrm{supp}\phi \subset B_1$. For $m \in \mathbb{N}$, we define $\phi_m(x) = m^{-\frac{N}{p}} \phi(\frac{x}{m})$. Then for any $m \in \mathbb{N}$ we obtain

$$\|\phi_m\|_{L^p(\mathbb{R}^N)} = \|\phi\|_{L^p(B_1)}, \quad \|\nabla \phi_m\|_{L^p(\mathbb{R}^N)} = m^{-1} \|\nabla \phi\|_{L^p(B_1)}.$$

Since $\{\phi_m\}_{m=1}^\infty$ is a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is reflexive (see e.g. Proposition 3.20. in [6]), there exist a weakly convergent subsequence $\{\phi_{m_j}\}_{j=1}^\infty$ and $\phi_\infty \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\phi_{m_j} \rightharpoonup \phi_\infty$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $j \to \infty$. By compactness of the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$ for $p < r < p^*$, we have $\phi_{m_j} \to \phi_\infty$ in $L^r(\mathbb{R}^N)$ and $\phi_{m_j} \to \phi_\infty$ a.e. in $\mathbb{R}^N$ which yields that $\phi_\infty \equiv 0$. On the other hand, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx &= \int_{B_m} m^{-\frac{N}{p}(p+r(x))} \left|\phi\left(\frac{x}{m}\right)\right|^{q(x)} dx \\
&= \int_{B_1} m^{-\frac{N}{p} r(my)} |\phi(y)|^{q(my)}\, dy \\
&\geq \int_{B_{\frac{1}{2}} \setminus B_{\frac{1}{4}}} m^{-\frac{N}{p} r(my)}\, dy.
\end{aligned}$$

Since Γ is open in $\mathbb{S}^{N-1}$, there exists a smooth subset $D \subset \mathbb{S}^{N-1}$ such that $D \subset \Gamma$. By using the polar coordinates as $y = s\omega\, (s > 0, \omega \in \mathbb{S}^{N-1})$ we obtain

$$\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx \geq \int_{s=\frac{1}{4}}^{\frac{1}{2}} \int_{\omega \in D} m^{-\frac{N}{p} r(ms\omega)}\, s^{N-1} ds dS_\omega.$$

By the assumption (66), we obtain $r(ms\omega) \leq C_0 |\log ms|^{-1}$ for large m, $s \in (1/4, 1/2)$, and $\omega \in D \subset \Gamma$. Moreover for $s \in (1/4, 1/2)$ and large m, it holds $\log ms = \log m + \log s \geq \frac{1}{2} \log m$ which yields that

$$r(ms\omega) \leq \frac{2C_0}{\log m}.$$

Therefore we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx &\geq \int_{s=\frac{1}{4}}^{\frac{1}{2}} \int_{\omega \in D} e^{-\frac{N}{p} \log m \, \frac{2C_0}{\log m}}\, s^{N-1} ds dS_\omega \\
&= \mathcal{H}^{N-1}(D)\, e^{-\frac{2C_0 N}{p}}\, \frac{2^{-N} - 4^{-N}}{N} > 0
\end{aligned}$$

for large m, where $\mathcal{H}^d$ is the $d-$dimensional Hausdorff measure. Thus, if we assume the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact, then we have $\int_{\mathbb{R}^N} |\phi_\infty|^{q(x)}\, dx > 0$ which contradicts $\phi_\infty \equiv 0$. Hence the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is not compact. □

Proof of Theorem 5.2. We assume that $r < R$ without loss of generality. Let $\{u_m\}_{m=1}^{\infty}$ be a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. We shall show the existence of a strongly convergence subsequence of $\{u_m\}_{m=1}^{\infty}$ in $L^{q(x)}(\mathbb{R}^N)$. By the reflexivity of $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, there exist a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ and $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_{m_j} \rightharpoonup u_0$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $j \to \infty$. Especially it also holds that $u_{m_j} \rightharpoonup u_0$ in $W^{1,p}(\mathbb{R}^N)$ as $j \to \infty$. And also, by Proposition 5.7 we have $u_{m_j} \to u_0$ in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ and

$$u_{m_j} \to u_0 \quad \text{a.e. in } \mathbb{R}^N \text{ as } j \to \infty. \tag{71}$$

Furthermore, $\{u_{m_j}|_{B_r}\}_{j=1}^{\infty} \subset W^{1,p}(B_r)$ is a bounded sequence and the embedding from $W^{1,p}(B_r)$ to $L^{q(x)}(B_r)$ is compact by the assumption (67) (see Remark 2 in [17]). Thus there exist a subsequence of $\{u_{m_j}|_{B_r}\}_{j=1}^{\infty}$ (we use $\{u_{m_j}|_{B_r}\}_{j=1}^{\infty}$ again for simplicity) and $v_0 \in L^{q(x)}(B_r)$ such that the followings hold true:

$$\begin{aligned} u_{m_j}|_{B_r} &\rightharpoonup v_0 \quad \text{in } W^{1,p}(B_r), \\ u_{m_j}|_{B_r} &\to v_0 \quad \text{in } L^{q(x)}(B_r), \\ u_{m_j}|_{B_r} &\to v_0 \quad \text{in } L^{p}(B_r), \\ u_{m_j}|_{B_r} &\to v_0 \quad \text{a.e. in } B_r \text{ as } j \to \infty. \end{aligned} \tag{72}$$

By (71) and (72), we can check that $u_0|_{B_r} = v_0$ a.e. in B_r which yields that

$$u_{m_j}|_{B_r} \to u_0|_{B_r} \quad \text{in } L^{q(x)}(B_r) \text{ as } j \to \infty. \tag{73}$$

In the similar way as above, we also obtain the followings

$$\begin{aligned} u_{m_j}|_{B_K \setminus B_r} &\rightharpoonup u_0|_{B_K \setminus B_r} \quad \text{in } W^{1,p}_{\mathrm{rad}}(B_K \setminus B_r), \\ u_{m_j}|_{B_K \setminus B_r} &\to u_0|_{B_K \setminus B_r} \quad \text{in } L^{q}(B_K \setminus B_r), \\ u_{m_j}|_{B_K \setminus B_r} &\to u_0|_{B_K \setminus B_r} \quad \text{a.e. in } B_K \setminus B_r \end{aligned} \tag{74}$$

for any $K > 0$ and any $q \geq 1$ as $j \to \infty$ since the embedding from $W^{1,p}_{\mathrm{rad}}(B_K \setminus B_r)$ to $L^q(B_K \setminus B_r)$ is compact for any K, q.

Set $v_{m_j} := u_{m_j} - u_0$. In order to make good use of (73) and (74) we divide $\int_{\mathbb{R}^N} |v_{m_j}(x)|^{q(x)} dx$ into three terms as follows:

$$\begin{aligned} & \int_{\mathbb{R}^N} |v_{m_j}(x)|^{q(x)} dx \qquad (75) \\ = \ & \int_{B_r} |v_{m_j}(x)|^{q(x)} dx + \int_{B_K \setminus B_r} |v_{m_j}(x)|^{q(x)} dx + \int_{\mathbb{R}^N \setminus B_K} |v_{m_j}(x)|^{q(x)} dx \\ =: \ & I_1(j) + I_2(j, K) + I_3(j, K), \end{aligned}$$

where K is sufficiently large.

Firstly, by (73) we have

$$I_1(j) = o(1) \text{ as } j \to \infty. \tag{76}$$

Next, for $I_2(j,K)$ we have

$$I_2(j,K) = \int_{B_K\setminus B_r} |v_{m_j}(x)|^{q(x)}dx \le \int_{B_K\setminus B_r} |v_{m_j}(x)|\,dx + \int_{B_K\setminus B_r} |v_{m_j}(x)|^{\|q\|_{L^\infty(\mathbb{R}^N)}}\,dx.$$

Thus, by (74) we obtain

$$I_2(j,K) = o(1) \text{ as } j\to\infty \text{ for fixed } K>0. \tag{77}$$

Finally we shall estimate $I_3(j,K)$. Since

$$|v_{m_j}(x)| \le \left(\frac{p}{\omega_{N-1}}\right)^{\frac{1}{p}} \|v_{m_j}\|_{W^{1,p}(\mathbb{R}^N)}|x|^{-\frac{N-1}{p}} \le C|x|^{-\frac{N-1}{p}}$$

by Proposition 5.6 and the boundedness of $\{v_{m_j}\}_{j=1}^\infty$, we can assume $|v_{m_j}(x)| \le 1$ for $x\in\mathbb{R}^N\setminus B_K$ with large K. Therefore by the assumption (68) we obtain

$$\begin{aligned}
I_3(j,K) &= \int_{\mathbb{R}^N\setminus B_K} |v_{m_j}|^{q(x)}dx \le \int_{\mathbb{R}^N\setminus B_K} |v_{m_j}|^{p+C_1(\log|x|)^{-\ell}}dx \\
&\le \sum_{n=2}^\infty \int_{B_{K^n}\setminus B_{K^{n-1}}} |v_{m_j}|^{p+C_1(n\log K)^{-\ell}}dx \\
&\le \sum_{n=2}^\infty \int_{B_{K^n}\setminus B_{K^{n-1}}} |v_{m_j}|^p \left(C|x|^{-\frac{N-1}{p}}\right)^{C_1(n\log K)^{-\ell}} dx \\
&\le C^{C_1(2\log K)^{-\ell}} \|v_{m_j}\|^p_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^\infty K^{-\frac{N-1}{p}(n-1)C_1(n\log K)^{-\ell}} \\
&\le C\sum_{n=2}^\infty \delta_1^{(n-1)^{1-\ell}} = C\sum_{n=1}^\infty \delta_1^{n^{1-\ell}},
\end{aligned}$$

where $\delta_1 = \delta_1(K) := K^{-\frac{N-1}{p}C_1(\log K)^{-\ell}} \to 0$ as $K\to\infty$. Since $\sum_{n=1}^\infty \delta_1^{n^{1-\ell}} = \delta_1 + \int_1^\infty \delta_1^{x^{1-\ell}}dx < \infty$ for each $\delta_1\in(0,1)$, we have

$$\sum_{n=1}^\infty \delta_1^{n^{1-\ell}} \to 0 \quad \text{as } K\to\infty.$$

Hence we have

$$I_3(j,K) = o(1) \text{ uniformly in } j \text{ as } K\to\infty. \tag{78}$$

We go back (75) and by (76), (77), and (78) we have

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} |v_{m_j}(x)|^{q(x)}dx = 0.$$

As a consequence we obtain $u_{m_j}\to u_0$ in $L^{q(x)}(\mathbb{R}^N)$. □

5.3 Approximation method : Proof of Theorem 5.4

In this section, we show Theorem 5.4 by using Theorem 5.2. First, we prepare the mountain pass theorem (Theorem 5.8) introduced in [22, 25], and so on which are based on [3]. Let V be a Banach space and $E \in C^1(V, \mathbb{R})$. We define a Palais-Smale sequence for E as $\{u_m\} \subset V$ satisfying $|E(u_m)| \leq c$ uniformly in m, and $E'(u_m) \to 0$ in V^*, where $E'(\cdot)$ is Fréchet derivative and V^* is the dual space of V. We say that E satisfies (P.-S.) condition if any Palais-Smale sequence has a strongly convergent subsequence.

Theorem 5.8 ([22, 25])**.** *Suppose $E \in C^1(V, \mathbb{R})$ satisfies (P.-S.) condition. Assume that*

(i) E(0)=0

(ii) There exist $\rho > 0,\ \alpha > 0$ such that $E(u) \geq \alpha$ for any $u \in V$ with $\|u\| = \rho$.

(iii) There exists $u_1 \in V$ such that $\|u_1\| \geq \rho$ and $E(u_1) < \alpha$.

Define

$$P = \{ p \in C([0,1], V) \mid p(0) = 0,\ p(1) = u_1 \}.$$

Then

$$\beta = \inf_{p \in P} \sup_{0 \leq t \leq 1} E(p(t))$$

is a critical value.

Proof of Theorem 5.4. *Step 1.* We may assume that R in the hypotheses of Theorem 5.2 is sufficiently large such that $\operatorname{ess\,inf}_{x \in B_R} q(x) = p + C_1(\log R)^{-\ell}$ without loss of generality. For $m \in \mathbb{N}$ let $\{R_m\}$ be a sequence such that $R_1 = R$, $R_m \to \infty$ as $m \to \infty$. Then we set functions as

$$q_m(x) = \begin{cases} q(x) & \text{if} \quad q(x) \geq p + C_1(\log R_m)^{-\ell}, \\ p + C_1(\log R_m)^{-\ell} & \text{if} \quad q(x) < p + C_1(\log R_m)^{-\ell}. \end{cases}$$

Define a functional J_m from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ by

$$J_m(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx - \int_{\mathbb{R}^N} \frac{1}{q_m(x)} u_+^{q_m(x)} dx.$$

We can check that $J_m \in C^1(W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$. Moreover, for each m, J_m satisfies as follows:

(i) J_m satisfies (P.-S.) condition.

(ii) $J_m(0) = 0$,

(iii) There exist positive constants α, ρ such that $J_m(u) \geq \alpha$ for any $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \rho$,

(iv) There exists $v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\|v\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$, $J_m(v) < \alpha$.

By Theorem 5.8 there exists a critical point $u_m \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ of J_m such that

$$J_m(u_m) = \beta_m,$$

where β_m is defined in the same way as β in Theorem 5.8. Thus u_m is a non-trivial weak solution of

$$-\Delta_p w + |w|^{p-2}w = w_+^{q_m(x)-1} \quad \text{in} \quad \mathbb{R}^N. \tag{79}$$

We can also see that $u_m \geq 0$ by multiplying both sides of (79) by $(u_m)_-$.

Proposition 5.9. *$\{u_m\}$ is bounded in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$.*

We will prove this proposition at last of this section.

Step 2. Since $\{u_m\}$ is a bounded sequence, there exists $u_0 \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ such that $u_m \rightharpoonup u_0$ weakly in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$. Put

$$G_m = \left\langle |\nabla u_m|^{p-2}\nabla u_m - |\nabla u_0|^{p-2}\nabla u_0, \nabla u_m - \nabla u_0 \right\rangle_{\mathbb{R}^N} + (u_m^{p-1} - u_0^{p-1})(u_m - u_0).$$

Then we have

$$\int_{\mathbb{R}^N} G_m dx = \int_{\mathbb{R}^N} (|\nabla u_m|^p + u_m^p)\, dx - \int_{\mathbb{R}^N} (|\nabla u_m|^{p-2}\nabla u_m \nabla u_0 + u_m^{p-1}u_0)\, dx + h_m,$$

where $h_m = \int_{\mathbb{R}^N} \left[|\nabla u_0|^{p-2}\nabla u_0(\nabla u_0 - \nabla u_m) + u_0^{p-1}(u_0 - u_m)\right] dx = o(1)$ as $m \to \infty$. Moreover, from (86) and (87) in the proof of Proposition 5.9 it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla u_m|^p + u_m^p)\, dx - \int_{\mathbb{R}^N} (|\nabla u_m|^{p-2}\nabla u_m \nabla u_0 + u_m^{p-1}u_0)\, dx \\
&= \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)-1} \left((u_m)_+ - u_0\right) dx \\
&\leq 2\, \|u_m^{q_m(x)-1}\|_{\frac{q(x)}{q(x)-1}} \|u_m - u_0\|_{q(x)} \\
&= 2\, \|u_m\|_{q(x)} \|u_m - u_0\|_{q(x)}
\end{aligned}$$

by the generalized Hölder inequality (see e.g. [16] Theorem 2.1). By the boundedness of $\{u_m\}$ in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ and Theorem 5.2 we have $\|u_m\|_{q(x)}\|u_m - u_0\|_{q(x)} = o(1)$ as $m \to \infty$. Hence

$$\int_{\mathbb{R}^N} G_m dx = o(1) \tag{80}$$

as $m \to \infty$. Recall that for $p \geq 1$, $a, b \in \mathbb{R}^d$ we have

$$\left\langle |b|^{p-2}b - |a|^{p-2}a, b - a \right\rangle \geq \begin{cases} 2^{2-p}|b-a|^p & \text{if} \quad p \geq 2, \\ (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} & \text{if} \quad 1 \leq p \leq 2. \end{cases}$$

From this inequality and (80) it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_m - \nabla u_0|^p + |u_m - u_0|^p) dx = o(1)$$

which is equivalent to $u_m \to u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$. Thus u_0 satisfies

$$-\Delta_p u_0 + u_0^{p-1} = u_0^{q(x)-1}, \; u_0 \geq 0 \quad \text{in} \quad \mathbb{R}^N.$$

Step 3. Finally, we have to show $u_0 \not\equiv 0$. From the boundedness of $\{u_m\}$ and Proposition 5.6, we see that $u_m \leq 1$ in $\mathbb{R}^N \setminus B_L$ for large L. Therefore we have

$$\int_{\mathbb{R}^N} (|\nabla u_m|^p + u_m^p) dx = \int_{\mathbb{R}^N} u_m^{q_m(x)} dx \leq \int_{\mathbb{R}^N} u_m^p dx + \int_{B_r} u_m^{p^*} dx + \int_{B_L \setminus B_r} u_m^{\|q\|_{L^\infty(\mathbb{R}^N)}} dx. \tag{81}$$

By the Sobolev inequality it follows that

$$\int_{B_r} u_m^{p^*} dx \leq \int_{\mathbb{R}^N} u_m^{p^*} dx \leq S^{-\frac{p^*}{p}} \left(\int_{\mathbb{R}^N} |\nabla u_m|^p dx \right)^{\frac{p^*}{p}}. \tag{82}$$

Moreover, we have

$$\begin{aligned}
\int_{B_L \setminus B_r} u_m^{\|q\|_{L^\infty(\mathbb{R}^N)}} dx &\leq C \left[\int_{B_L \setminus B_r} (|\nabla u_m|^p + |u_m|^p) dx \right]^{\frac{\|q\|_{L^\infty(\mathbb{R}^N)}}{p}} \\
&\leq C \left[\int_{B_L \setminus B_r} |\nabla u_m|^p + \left(\int_{B_L \setminus B_r} |u_m|^{p^*} dx \right)^{\frac{p}{p^*}} |B_L \setminus B_r|^{1-\frac{p}{p^*}} \right]^{\frac{\|q\|_{L^\infty(\mathbb{R}^N)}}{p}} \\
&\leq C \left(\int_{\mathbb{R}^N} |\nabla u_m|^p \right)^{\frac{\|q\|_{L^\infty(\mathbb{R}^N)}}{p}}.
\end{aligned} \tag{83}$$

Put $q_* := \min\{p^*, \|q\|_{L^\infty(\mathbb{R}^N)}\}$. From (81), (82), and (83), we obtain

$$C \leq \left(\int_{\mathbb{R}^N} |\nabla u_m|^p \right)^{\frac{q_* - p}{p}},$$

where we used that $u_m \not\equiv 0$. By Theorem 5.2 we have

$$\begin{aligned}
C &\leq \lim_{m\to\infty} \int_{\mathbb{R}^N} |\nabla u_m|^p dx \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^N} (-u_m^p + u_m^{q_m(x)}) dx \\
&\leq \int_{\mathbb{R}^N} u_0^{q(x)} dx.
\end{aligned}$$

Consequently we have $u_0 \not\equiv 0$. □

Proof of Proposition 5.9. We take a smooth radial function $\hat{u} > 0$ on $\mathbb{R}^N$. Since

$$\begin{aligned}
J_m(K\hat{u}) &\leq \frac{K^p}{p} \int_{\mathbb{R}^N} (|\nabla \hat{u}|^p + |\hat{u}|^p) \, dx - \int_{B_R} \frac{K^{q(x)}}{q(x)} \hat{u}_+^{q(x)} dx \\
&\leq \frac{K^p}{p} \int_{\mathbb{R}^N} (|\nabla \hat{u}|^p + |\hat{u}|^p) \, dx - \frac{K^{p+C_1(\log R)^\ell}}{\operatorname{ess\,sup}_{B_R} q(x)} \int_{B_R} \hat{u}_+^{q(x)} dx \to -\infty
\end{aligned}$$

as $K \to +\infty$, there exists $\hat{K} > 0$ independent of m such that $J_m(\hat{K}\hat{u}) < 0$. If we set $\hat{p}(t) = t\hat{K}\hat{u}$ for $t \in [0,1]$, then we see that

$$\hat{p} \in \hat{P} = \left\{ p \in C([0,1], W^{1,p}_{\text{rad}}(\mathbb{R}^N)) \;\Big|\; p(0) = 0,\ p(1) = \hat{K}\hat{u} \right\}.$$

Moreover, we have

$$\begin{aligned} \beta_m &= \inf_{p \in \hat{P}} \max_{0 \le t \le 1} J_m(p(t)) \le \max_{0 \le t \le 1} J_m(\hat{p}(t)) \\ &= \max_{0 \le t \le \hat{K}} \left[\frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla \hat{u}|^p + |\hat{u}|^p)\, dx - \int_{B_R} \frac{t^{q(x)}}{q(x)} \hat{u}_+^{q(x)} dx \right] \le C. \end{aligned} \tag{84}$$

On the other hand, since u_m is a critical point of J_m at β_m we have

$$\beta_m = \frac{1}{p} \int_{\mathbb{R}^N} (\, |\nabla u_m|^p + |u_m|^p \,)\, dx - \int_{\mathbb{R}^N} \frac{1}{q_m(x)} (u_m)_+^{q_m(x)} dx \tag{85}$$

and for any $\phi \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\, |\nabla u_m|^{p-2} \nabla u_m \nabla \phi \,+\, |u_m|^{p-2} u_m \phi \,)\, dx - \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)-1} \phi \, dx = 0. \tag{86}$$

In particular,

$$\int_{\mathbb{R}^N} (|\nabla u_m|^p + |u_m|^p \,)\, dx - \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)} dx = 0. \tag{87}$$

From (84), (85), and (87), it follows that

$$\int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)} \right) (u_m)_+^{q_m(x)} dx \le C.$$

Furthermore, by $q(x) \le q_m(x)$ we have

$$\int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q(x)} \right) (u_m)_+^{q_m(x)} dx \le C. \tag{88}$$

Thus for any $L > 0$ there exists a positive constant $C(L)$ such that

$$\int_{B_L} (u_m)_+^{q_m(x)} dx \le C(L). \tag{89}$$

Here, we take a constant $R_0 > R$ sufficiently large (This R_0 will be chosen again later) and we have

$$\|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} \le C(R_0) + \int_{\mathbb{R}^N \setminus B_{R_0}} (u_m)_+^{q_m(x)}\, dx \tag{90}$$

by (87) and (89). Set $\delta = C_1(\log R_0)^{-\ell}$ and $A_{n,m} := \{x \in B_{R_0^n} \setminus B_{R_0^{n-1}} \mid q_m(x) \leq p + \delta\}$. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{R_0}} (u_m)_+^{q_m(x)} dx \\
&\leq \int_{\{x \in \mathbb{R}^N \mid q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx + \int_{\{x \in \mathbb{R}^N \setminus B_{R_0} \mid q_m(x) \leq p+\delta\}} (u_m)_+^{q_m(x)} dx \\
&\leq \int_{\{x \in \mathbb{R}^N \mid q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx + \sum_{n=2}^{\infty} \int_{A_{n,m}} (u_m)_+^{p+C_1(n\log R_0)^{-\ell}} dx + \sum_{n=2}^{\infty} \int_{A_{n,m}} (u_m)_+^{p+\delta} dx \\
&=: L_1 + L_2 + L_3,
\end{aligned}$$

where third inequality comes from the assumption (68). We shall estimate L_1, L_2, and L_3. For L_1, by (88) we have

$$L_1 \leq \left(\frac{1}{p} - \frac{1}{p+\delta}\right)^{-1} \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)}\right) (u_m)_+^{q_m(x)} dx = C. \tag{91}$$

In order to estimate L_2 and L_3, we prepare an estimate of $\|u_m\|_{L^p(A_{n,m})}$. For each $n, m \in \mathbb{N}$ we have

$$\int_{A_{n,m}} u_m^p \, dx \leq 2\|u_m\|^p_{L^{q_m(x)}(A_{n,m})} \| 1 \|_{L^{r_m(x)}(A_{n,m})}$$

by the generalized Hölder inequality, where $r_m(x) := \frac{q_m(x)}{q_m(x)-p}$. Now we assume $\|u_m\|_{L^{q_m(x)}(A_{n,m})} > 1$ and $\| 1 \|_{L^{r_m(x)}(A_{n,m})} > 1$ (If not, the proof is much simpler). By Proposition 2.2. in [15] we have

$$\|u_m\|_{L^{q_m(x)}(A_{n,m})} \leq \left(\int_{A_{n,m}} u_m^{q_m(x)} \, dx\right)^{\left(\operatorname{ess.inf}_{x \in A_{n,m}} q_m(x)\right)^{-1}} \leq \left(\int_{A_{n,m}} u_m^{q_m(x)}\right)^{\frac{1}{p+(n\log R_0)^{-\ell}}}.$$

Since

$$\int_{A_{n,m}} u_m^{q_m(x)} \leq \left(\frac{1}{p} - \frac{1}{p+(n\log R_0)^{-\ell}}\right)^{-1} \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)}\right) u_m^{q_m(x)} \, dx \leq C(n\log R_0)^{\ell},$$

we obtain

$$\|u_m\|_{L^{q_m(x)}(A_{n,m})} \leq C(n\log R_0)^{\frac{\ell}{p+(n\log R_0)^{-\ell}}}. \tag{92}$$

In the same way as above, we have

$$\| 1 \|_{L^{r_m(x)}(A_{n,m})} \leq \left(\int_{A_{n,m}} dx\right)^{\left(\operatorname{ess.inf}_{x \in A_{n,m}} r_m(x)\right)^{-1}} \leq |\, A_{n,m} \,|^{\frac{1}{1+pC_1^{-1}(\log R_0)^{\ell}}} \leq C R_0^{\frac{n}{1+pC_1^{-1}(\log R_0)^{\ell}}}, \tag{93}$$

where the second inequality comes from

$$\text{ess.inf}_{x\in A_{n,m}} r_m(x) = 1 + \frac{p}{\text{ess.sup}_{x\in A_{n,m}} q_m(x) - p} \geq 1 + \frac{p}{\delta} = 1 + pC_1^{-1}(\log R_0)^{\ell}.$$

From (92) and (93) we obtain

$$\int_{A_{n,m}} u_m^p\, dx \leq CR_0^{\frac{n}{1+pC_1^{-1}(\log R_0)^{\ell}}} (n\log R_0)^{\frac{p\ell}{p+(n\log R_0)^{-\ell}}}. \tag{94}$$

For L_2, by using (94) and Proposition 5.6, we have

$$\begin{aligned}
L_2 &\leq C\sum_{n=2}^{\infty} \|u_m\|_{W^{1,p}(\mathbb{R}^N)}^{C_1(n\log R_0)^{-\ell}} R_0^{\left(-\frac{N-1}{p}\right)(n-1)C_1(n\log R_0)^{-\ell}} \int_{A_{n,m}} u_m^p\, dx \\
&\leq C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} R_0^{-\frac{(N-1)C_1}{2^{\ell}p}(n-1)^{1-\ell}(\log R_0)^{-\ell}} \left(\int_{A_{n,m}} u_m^p\right)^{\frac{C_1(n\log R_0)^{-\ell}}{p}} \\
&\leq C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} R_0^{-\frac{(N-1)C_1}{2^{\ell}p}(n-1)^{1-\ell}(\log R_0)^{-\ell} + \frac{C_1 n^{1-\ell}(\log R_0)^{-\ell}}{p+p^2C_1^{-1}(\log R_0)^{\ell}}} (n\log R_0)^{\frac{\ell C_1}{p+(n\log R_0)^{-\ell}}(n\log R_0)^{-\ell}} \\
&= C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} \delta_1(n,R_0)^{(n-1)^{1-\ell}} \delta_2(n,R_0).
\end{aligned}$$

Since

$$\delta_2(n,R_0) = (n\log R_0)^{\frac{\ell C_1}{p+(n\log R_0)^{-\ell}}(n\log R_0)^{-\ell}} \to 1 \text{ as } n\to\infty \text{ or } R_0\to\infty,$$

there exists a positive constant $\tilde{C}$ which is independent of n and R_0 such that

$$\delta_2(n,R_0) \leq \tilde{C}. \tag{95}$$

On the other hand, for large R_0 we obtain

$$\delta_1(n,R_0) = R_0^{-\frac{C_1}{p}(\log R_0)^{-\ell}\left[\frac{(N-1)}{2^{\ell}} - \frac{1}{1+pC_1^{-1}(\log R_0)^{\ell}}\left(\frac{n}{n-1}\right)^{1-\ell}\right]} \leq R_0^{-\frac{C_1(N-1)}{2^{\ell+1}p}(\log R_0)^{-\ell}}$$

which yields that

$$\delta_1 = \delta_1(n,R_0) \to 0 \text{ uniformly in } n \text{ as } R_0\to\infty. \tag{96}$$

From (95) and (96) we have

$$L_2 \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \sum_{n=2}^{\infty} \delta_1^{(n-1)^{1-\ell}} = o(1)\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \text{ as } R_0\to\infty$$

in the same way as the proof of Theorem 5.2. Thus for sufficiently large R_0 we have

$$L_2 \leq \frac{1}{3}\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p. \tag{97}$$

In the same way as L_2, we obtain the estimate of L_3 for large R_0 as follows.

$$\begin{aligned}
L_3 &\leq C\|u_m\|^{\delta}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{\left(-\frac{N-1}{p}\right)(n-1)\delta} \int_{A_{n,m}} u_m^p dx \\
&\leq C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{-\frac{N-1}{p}C_1(n-1)(\log R_0)^{-\ell}} \left(\int_{A_{n,m}} u_m^p dx\right)^{\frac{C_1}{p}(\log R_0)^{-\ell}} \\
&\leq C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{-\frac{N-1}{p}C_1(n-1)(\log R_0)^{-\ell}+\frac{n}{1+pC_1^{-1}(\log R_0)^{\ell}}\frac{C_1}{p}(\log R_0)^{-\ell}} (n\log R_0)^{\frac{\ell C_1}{p(\log R_0)^{\ell}+n-\ell}} \\
&\leq C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{-\frac{N-1}{2p}C_1(n-1)(\log R_0)^{-\ell}} (n\log R_0)^{\frac{\ell C_1}{p(\log R_0)^{\ell}+n-\ell}} \\
&\leq C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{-\frac{N-1}{4p}C_1(n-1)(\log R_0)^{-\ell}},
\end{aligned}$$

where the last inequality comes from

$$(n\log R_0)^{\frac{\ell C_1}{p(\log R_0)^{\ell}+n-\ell}} = o\left(R_0^{\frac{N-1}{4p}C_1(n-1)(\log R_0)^{-\ell}}\right) \text{ as } n\to\infty \text{ or } R_0\to\infty.$$

Therefore for sufficiently large R_0 we have

$$L_3 \leq \frac{1}{3}\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)}. \tag{98}$$

From (90), (91), (97), and (98) we have

$$\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \leq C + \frac{2}{3}\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)}.$$

As a consequence u_m is bounded. □

5.4 Mountain pass theorem under the condition (C) : Proof of Theorem 5.4

In this section, we show Theorem 5.4 by a different method from Section 3.

Cerami [7] and Bartolo-Benci-Fortunato [5] have proposed a variant of (P.-S.) condition. In this paper, we use the condition (C) introduced by [5, 7] and the mountain pass theorem under the condition (C) (Theorem 5.11). Let V be a real Banach space and $E \in C^1(V,\mathbb{R})$. First, we define the condition (C) based on [5, 7].

Definition 5.10 ([7], [5] Definition 1.1.). *We say that E satisfies the condition (C) in $(c_1, c_2), (-\infty \leq c_1 < c_2 \leq +\infty)$, if*

(i) every bounded sequence $\{u_k\} \subset E^{-1}((c_1,c_2))$, for which $\{E(u_k)\}$ is bounded and $E'(u_k) \to 0$, possesses a convergent subsequence, and

(ii) for any $c \in (c_1, c_2)$ there exist $\sigma, \rho, \alpha > 0$ such that $[c-\sigma, c+\sigma] \subset (c_1, c_2)$ and for any $u \in E^{-1}([c-\sigma, c+\sigma])$ with $\|u\| \geq \rho, \|E'(u)\|_* \|u\| \geq \alpha$.

Theorem 5.11 (Mountain pass theorem under the condition (C))**.** *Let E satisfy the condition (C) in $(0, +\infty)$. Assume that*

(i) E(0)=0

(ii) There exist $\rho > 0,\ \alpha > 0$ such that $E(u) \geq \alpha$ for any $u \in V$ with $\|u\| = \rho$.

(iii) There exists $u_1 \in V$ such that $\|u_1\| \geq \rho$ and $E(u_1) < \alpha$.

Define

$$P = \{ p \in C([0,1], V) \mid p(0) = 0,\ p(1) = u_1 \}.$$

Then

$$\beta = \inf_{p \in P} \sup_{0 \leq t \leq 1} E(p(t)) \geq \alpha$$

is a critical value.

For $c \in \mathbb{R}$, we set

$$E_c = \{ u \in V \mid E(u) < c \},\ K_c = \{ u \in V \mid E'(u) = 0, E(u) = c \}.$$

Note that Theorem 5.11 can be shown in the same way as the proof of Theorem 6.1 in p.109 in [25] by substituting the following deformation theorem under the condition (C) for Theorem 3.4 in p.83 in [25].

Theorem 5.12 ([5] Theorem 1.3.)**.** *Let E satisfy the condition (C) in (c_1, c_2). If $\beta \in (c_1, c_2)$ and N is any neighborhood of K_β, there exist a bounded homeomorphism η of V onto V and constants $\overline{\varepsilon} > \varepsilon > 0$ such that $[\beta - \overline{\varepsilon}, \beta + \overline{\varepsilon}] \subset (c_1, c_2)$, satisfying the following properties*

(I) $\eta(E_{\beta+\varepsilon} \setminus N) \subset E_{\beta-\varepsilon}$

(II) $\eta(E_{\beta+\varepsilon}) \subset E_{\beta-\varepsilon}$ if $K_\beta = \emptyset$

(III) $\eta(u) = u$ if $|E(u) - \beta| \geq \overline{\varepsilon}$.

We set a energy functional from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ as

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) - \int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx.$$

We can check that $J \in C^1(W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$.

Proposition 5.13. *Assume that $q(x)$ satisfies the hypotheses (67), (68) in Theorem 5.2 and $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. Then J satisfies the condition (C) on $\mathbb{R}$.*

Proof. We take $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ arbitrary. First, we shall show that J satisfies (i) in Definition 5.10. Let $\{u_m\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ be a bounded sequence satisfying that $J(u_m) \in (c_1, c_2)$ and $\|J'(u_m)\|_* \to 0$ as $m \to +\infty$. Then the following holds true for any $\phi \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} (\,|\nabla u_m|^{p-2}\nabla u_m \nabla\phi \,+\, |u_m|^{p-2}u_m\phi\,)\,dx - \int_{\mathbb{R}^N} (u_m)_+^{q(x)-1}\phi\,dx = o(1). \quad (99)$$

In particular, since $\{u_m\}$ is bounded it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_m|^p + |u_m|^p\,)\,dx - \int_{\mathbb{R}^N} (u_m)_+^{q(x)} dx = o(1). \quad (100)$$

Likewise since $\{u_m\}$ is bounded, there exists a subsequence written as $\{u_m\}$ for simplicity and $u_0 \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ such that $u_m \rightharpoonup u_0$ weakly in $W^{1,p}(\mathbb{R}^N)$. Put

$$G_m = \big\langle |\nabla u_m|^{p-2}\nabla u_m - |\nabla u_0|^{p-2}\nabla u_0, \nabla u_m - \nabla u_0\big\rangle_{\mathbb{R}^N} + (u_m^{p-1} - u_0^{p-1})(u_m - u_0)$$

as in Section 3. In the same way as Step 2 in the proof of Theorem 5.4 in Section 3 by substituting (99), (100) for (86), (87) respectively we have

$$\int_{\mathbb{R}^N} G_m\,dx = o(1)$$

as $m \to \infty$ by Theorem 5.2. Recalling that

$$\big\langle |b|^{p-2}b - |a|^{p-2}a, b - a\big\rangle \geq \begin{cases} 2^{2-p}|b-a|^p & \text{if} \quad p \geq 2, \\ (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} & \text{if} \quad 1 \leq p \leq 2, \end{cases}$$

and consequently we have

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla(u_m - u_0)|^p + |u_m - u_0|^p)\,dx \leq C \lim_{m\to\infty} \int_{\mathbb{R}^N} G_m\,dx = 0.$$

This implies that $u_m \to u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$.

Next, we shall show (ii). For any $c \in (c_1, c_2)$, we take some σ with $[c - \sigma, c + \sigma] \subset (c_1, c_2)$. We will choose suitable $\rho > 0$ again later. By deriving a contradiction, we show that there exists $\alpha > 0$ such that for any $u \in J^{-1}([c - \sigma, c + \sigma])$ with $\|u\| \geq \rho$, $\|J'(u)\|_*\|u\| \geq \alpha$. We assume that there exists $\{u_m\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ such that $u_m \in J^{-1}([c - \sigma, c + \sigma])$ with $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$, and $\|J'(u_m)\|_*\|u_m\|_{W^{1,p}(\mathbb{R}^N)} =: \alpha_m \to 0$ as $m \to +\infty$. Since $J'(u_m)u_m \to 0$ as $m \to +\infty$, we have

$$\left| \, \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (u_m)_+^{q(x)}\,dx \right| \leq \alpha_m$$

which yields that

$$\begin{aligned} c + \sigma &\geq J(u_m) \\ &\geq \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q(x)}\right)(u_m)_+^{q(x)} dx - \alpha_m. \end{aligned} \quad (101)$$

Moreover, in the same way as the proof of Proposition 5.9, for large m we have

$$\int_{A_n} u_m^p\,dx \le C R_0^{\overline{\frac{n}{1+pC_1^{-1}(\log R_0)^{\ell}}}} (n\log R_0)^{\overline{\frac{p\ell}{p+(n\log R_0)^{-\ell}}}}, \tag{102}$$

where $A_n := \{x \in B_{R_0^n} \setminus B_{R_0^{n-1}} \,|\, q(x) \le p+\delta\}$ for $n \ge 2$, and R_0 is the same as the proof of Proposition 5.9. By substituting (101), (102) for (88), (94), we obtain the following estimates:

$$\begin{aligned}\|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} - \alpha_m &\le \int_{B_{R_0}} (u_m)_+^{q(x)}dx + \int_{\mathbb{R}^N\setminus B_{R_0}} (u_m)_+^{q(x)}dx \\ &\le C(R_0)(c+\sigma+\alpha_m) + \frac{2}{3}\|u_m\|^p_{W^{1,p}(\mathbb{R}^N)},\end{aligned}$$

where $C(R_0)$ is a positive constant independent of ρ. Therefore we have

$$\begin{aligned}\|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} &\le 3\{\,\alpha_m + C(R_0)\,(c+\sigma+\alpha_m)\,\} \\ &\le 3\{\,1 + C(R_0)\,(c_2+1)\,\}\end{aligned} \tag{103}$$

for large m. If we choose sufficiently large ρ satisfying $\rho > 3^{1/p}\{\,1+C(R_0)(c_2+1)\,\}^{1/p}$, then we see that (103) contradicts $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \ge \rho$.

The proof of Proposition 5.13 is now complete. □

Proposition 5.14. *Assume that $q(x)$ satisfies the hypotheses (67), (68) in Theorem 5.2 and* $\operatorname{ess\,inf}_{x\in B_R} q(x) > p$. *Then J has the mountain pass geometry, that is J satisfies (i), (ii) and (iii) in Theorem 5.11.*

Proof. (i) is obvious. We prove (ii). Let S be the best constant of the Sobolev inequality : $S\|v\|^p_{L^{p^*}(\mathbb{R}^N)} \le \|\nabla v\|^p_{L^p(\mathbb{R}^N)}$ for $v \in C_c^\infty(\mathbb{R}^N)$. Set $q^* = \max\{p^*, p^2, \|q\|_{L^\infty(\mathbb{R}^N)}\}$. Note that $q^* \ge p^* > pN/(N-1)$. For $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx &\le \frac{1}{p}\int_{\mathbb{R}^N} |u|^p + \frac{1}{p}\left[\int_{B_r} |u|^{p^*}dx + \int_{\mathbb{R}^N\setminus B_r} |u|^{q^*}dx\right] \\ &\le \frac{1}{p}\int_{\mathbb{R}^N}|u|^p dx + \frac{1}{p}\left[\left(S^{-1}\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{\frac{p^*}{p}} + \|u\|^{q^*\frac{p-1}{p}}_{L^p(\mathbb{R}^N)} \|\nabla u\|^{\frac{q^*}{p}}_{L^p(\mathbb{R}^N)} K(r)\right] \\ &\le \frac{1}{p}\int_{\mathbb{R}^N}|u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx\left[S^{-\frac{p^*}{p}}\gamma^{p^*-p} + K(r)\gamma^{q^*-p}\right],\end{aligned}$$

where $K(r) = (p/\omega_{N-1})^{q^*/p}\int_{\mathbb{R}^N\setminus B_r} |x|^{-q^*(N-1)/p}dx < \infty$ and the second inequality comes from Proposition 5.6. From this if γ is sufficiently small, we have

$$J(u) \ge \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx\left[1 - S^{-\frac{p^*}{p}}\gamma^{p^*-p} - K(r)\gamma^{q^*-p}\right] > 0. \tag{104}$$

For $\{u_m\} \subset W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and γ satisfying $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$ and (104), we assume that $J(u_m) \to 0$ and derive a contradiction. From (104) it follows that $\int_{\mathbb{R}^N} |\nabla u_m|^p dx \to 0$. In addition, for sufficiently large R we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{1}{q(x)}(u_m)_+^{q(x)}dx &\leq \frac{1}{p}\left(\int_{B_r} |u_m|^{q(x)}dx + \int_{B_R\setminus B_r} |u_m|^{q(x)}dx + \int_{\mathbb{R}^N\setminus B_R} |u_m|^{q(x)}dx\right) \\
&\leq \frac{1}{p}\left[\int_{B_r} |u_m|^{p^*} + \int_{B_R} |u_m|^p dx + \int_{\mathbb{R}^N\setminus B_r} |u_m|^{q^*} + \int_{\mathbb{R}^N\setminus B_R} |u_m|^{p+C_1(\log|x|)^{-\ell}}dx\right] \\
&= \frac{1}{p}(H_1 + H_2 + H_3 + H_4).
\end{aligned}$$

By using the estimates in the calculation of $\int_{\mathbb{R}^N} (u)_+^{q(x)}/q(x)dx$ to show (104) we have $H_1 = o(1)$ and $H_3 = o(1)$ as $m \to \infty$. For H_2 we have

$$H_2 \leq |B_R|^{1-\frac{p}{p^*}} S^{-1} \int_{\mathbb{R}^N} |\nabla u_m|^p = o(1).$$

We can show that H_4 is bounded uniformly for m and $H_4 \to 0$ as $R \to \infty$ in the same way as the estimate of $I_3(j,K)$ in the proof of Theorem 2. Therefore

$$\int_{\mathbb{R}^N} \frac{1}{q(x)}|u_m|^{q(x)}dx \to 0$$

as $m \to \infty$, and which implies $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \to 0$ since $J(u_m) \to 0$ as $m \to \infty$. This contradicts $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$.

Finally, we prove (iii). We take a smooth radial function v such that $\|v\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$, $v > 0$ in B_R, where R is in the hypothesis (68). Recalling that $\underline{q} := \mathrm{ess}\ \inf_{x\in B_R} q(x) > p$. By taking sufficiently large t we have

$$\begin{aligned}
J(tv) &= \frac{t^p}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p)dx - \int_{\mathbb{R}^N} \frac{t^{q(x)}}{q(x)} v_+^{q(x)}dx \\
&\leq \frac{t^p}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p)dx - t^{\underline{q}}\int_{B_R} \frac{1}{q(x)} v_+^{q(x)}dx \\
&< 0.
\end{aligned}$$

Since $\|tv\|_{W^{1,p}(\mathbb{R}^N)} > \gamma$ we prove (iii). □

Proof of Theorem 5.4. From Proposition 5.13, Proposition 5.14, and Theorem 5.11, we can show the existence of a non-trivial critical point $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ which is a weak solution to $-\Delta_p u + |u|^{p-2}u = u_+^{q(x)-1}$ in $\mathbb{R}^N$. Then we also see that $u \geq 0$ in $\mathbb{R}^N$. □

5.5 Appendix

In this section we show Proposition 5.6 and Proposition 5.7.

Proof of Proposition 5.6. It is sufficiently to show (70) holds for $f \in C_c^\infty(\mathbb{R}^N)$ with radially symmetric. We have that

$$r^{N-1}|f(r)|^p = -\int_r^\infty \frac{d}{ds}\left(s^{N-1}|f(s)|^p\right)ds.$$

By direct calculation we have

$$(s^{N-1}|f(s)|^p)' = (N-1)s^{N-2}|f(s)|^p + ps^{N-1}|f(s)|^{p-2}f(s)f(s)'.$$

Thus it follows

$$\begin{aligned} r^{N-1}|f(r)|^p &= -(N-1)\int_r^\infty s^{N-2}|f(s)|^p ds - p\int_r^\infty s^{N-1}|f(s)|^{p-2}f(s)f(s)'ds \\ &\leq p\int_r^\infty s^{N-1}|f(s)|^{p-1}|f(s)'|ds \\ &\leq \frac{p}{\omega_{N-1}}\|f\|_{L^p(\mathbb{R}^N)}^{p-1}\|\nabla f\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Consequently (70) follows immediately. □

Proof of Proposition 5.7. By (70) we have

$$\int_{\mathbb{R}^N\setminus B_R}|u|^q dx \leq C_u\int_{\mathbb{R}^N\setminus B_R}|x|^{-\frac{N-1}{p}q} = C_u\int_R^\infty r^{-(N-1)\left(\frac{q}{p}-1\right)}dr,$$

where $C_u = \left(\frac{p}{\omega_{N-1}}\right)^{q/p}\|u\|_{L^p(\mathbb{R}^N)}^{q(p-1)/p}\|\nabla u\|_{L^p(\mathbb{R}^N)}^{q/p}$. When $(N-1)(q/p-1) > 1$, that is, $q > pN/(N-1)$ we have

$$\int_{\mathbb{R}^N\setminus B_R}|u|^q dx \leq C_u R^{-(N-1)\left(\frac{q}{p}-1\right)+1}.$$

Let $\{u_m\}$ be a sequence such that $u_m \rightharpoonup 0$ weakly in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Firstly we show that the case of $q \in (pN/(N-1), p^*)$. In this case we have

$$\int_{\mathbb{R}^N}|u_m|^q dx \leq \int_{B_R}|u_m|^q dx + C_{u_m}R^{-(N-1)\left(\frac{q}{p}-1\right)+1}.$$

Since C_{u_m} is bounded from above uniformly, letting $m \to \infty$ and $R \to \infty$ we have $u_m \to 0$ strongly in $L^q(\mathbb{R}^N)$.

Next, for $q \in (p, pN/(N-1)]$ using interpolation of L^q space, we have

$$\|u_m\|_{L^q(\mathbb{R}^N)} \leq \|u_m\|_{L^p(\mathbb{R}^N)}^{\lambda}\|u_m\|_{L^r(\mathbb{R}^N)}^{1-\lambda},$$

where $r \in (pN/(N-1), p^*)$. Since $\|u_m\|_{L^r(\mathbb{R}^N)} \to 0$ and $\|u_m\|_{L^p(\mathbb{R}^N)}$ is bounded we have $\|u_m\|_{L^q(\mathbb{R}^N)} \to 0$. □

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