On strongly quasi-hereditary algebras
(強準遺伝代数について)

理学研究科
数物系専攻

平成 30 年度

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Abstract

Ringel's right-strongly quasi-hereditary algebras are a special class of quasi-hereditary algebras introduced by Cline–Parshall–Scott. We give characterizations of these algebras in terms of heredity chains, right rejective subcategories and coreflective subcategories. As applications, we prove the following results. One is that any artin algebra of global dimension at most two is always right-strongly quasi-hereditary. The others are characterizations of Auslander algebras and Auslander–Dlab–Ringel algebras to be strongly quasi-hereditary. We show that the Auslander algebra of a representation-finite algebra A is strongly quasi-hereditary if and only if A is a Nakayama algebra. We show that the Auslander–Dlab–Ringel algebra of an artin algebra A is strongly quasi-hereditary if and only if the global dimension of A is at most two.

Acknowledgements

I would like to thank my supervisor, Professor Hyohe Miyachi.
I would like to express deepest gratitude to Professor Osamu Iyama for his insightful comments and permission to include detailed proof of results in [Iya03b]. I am also grateful to Takahide Adachi, Aaron Chan, Ryoichi Kase, Yoshiyuki Kimura, Hiroyuki Minamoto, Yasuaki Ogawa and Kentaro Wada for their valuable comments and discussions.
I would like to thank my family for their understanding and support. I wish to thank my father, Hikoshige Tsukamoto, for his kindness and cooperation. My final gratitude goes to my mother, Tomoko Tsukamoto, for always encouraging me.

Contents

1 Introduction

1.1 Background

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott [CPS88, PS88, Sco87] to study highest weight categories in the representation theory of semisimple complex Lie algebras and algebraic groups. Since the introduction of quasi-hereditary algebras, many classes of algebras arising from algebraic Lie theory naturally were shown to be quasi-hereditary. For example, Schur algebras and q-Schur algebras have quasi-hereditary structures (cf. [PW91, Theorem 11.5.2]).

On the other hand, quasi-hereditary algebras were widely studied by Dlab and Ringel from the viewpoint of the representation theory of algebras [DR89a, DR89b, DR89c, DR92]. They proved that any algebra of global dimension at most two have a quasi-hereditary structure [DR89c]. In particular, each Auslander algebra has a quasi-hereditary structure [DR89a]. Moreover Iyama showed finiteness of the representation dimension of artin algebras by using quasi-hereditary algebras. This theorem states that any artin algebra A can be written as eBe for some quasi-hereditary algebra B and an idempotent e of B [Iya03a, Iya03b]. One can regard this theorem as a generalization of [DR89b].

Motivated by Iyama's finiteness theorem, Ringel introduced the notion of *right-strongly quasi-hereditary* algebras from the viewpoint of highest weight categories and he proved that the algebra B in Iyama's finiteness theorem is not only quasi-hereditary, but even right-strongly quasi-hereditary. One of the advantages of right-strongly quasi-hereditary algebras is that they have better upper bound on global dimension than that of general quasi-hereditary algebras [Rin10, §4].

Recently right-strongly (resp. left-strongly) quasi-hereditary algebras are studied intensely. It was shown that various algebras have right-strongly (resp. left-strongly) quasi-hereditary structures. For example, cluster-tilted algebras for preprojective algebras [GLS07, IR11], Auslander–Dlab–Ringel algebras [Con16, Con17, CE18], nilpotent quiver algebras [ES17], matrix algebras of d-systems [Cou17], and so on [Eir17, HP17, KK18].

In this thesis, we discuss categorical aspects of right-strongly quasi-hereditary algebras following an approach in [Iya03b, Section 2], which is unpublished. In particular, we give a characterization of right-strongly quasi-hereditary algebras in terms of the following three notions (Theorem 3.35).

- right-strongly heredity chains (Definition 3.1),
- total right rejective chains (Definition 3.19),
- coreflective chains (Definition 3.33).

As an application, we sharpen a well-known result of Dlab–Ringel [DR89c, Theorem 2] stating that any artin algebra of global dimension at most two is quasi-hereditary. We prove that such an algebra is always right-strongly (resp. left-strongly) quasi-hereditary (Theorem 4.1). We give a detailed proof following the strategy of [Iya03b, Theorem 3.6].

In particular, Auslander algebras are right-strongly quasi-hereditary. We show that the Auslander algebra of a representation-finite algebra A is strongly quasi-hereditary if and only if A is a Nakayama algebra. By [Con16], Auslander–Dlab–Ringel algebras are left-strongly quasi-hereditary. We give several characterizations of Auslander–Dlab–Ringel algebras to be strongly quasi-hereditary.

1.2 Our results

Recall that right-strongly (resp. left-strongly) quasi-hereditary algebras are defined as quasi-hereditary algebras whose standard modules have projective dimension at most one (Definition 2.7). Our starting point is the following observation which gives a characterization of right-strongly (resp. left-strongly) quasi-hereditary algebras in terms of heredity chains.

Proposition 1.1 (Proposition 3.7). *Let A be an artin algebra. Then A is right-strongly (resp. left-strongly) quasi-hereditary if and only if there exists a heredity chain*

$$A = H_0 > H_1 > \cdots > H_n = 0$$

such that H_i is a projective right (resp. left) A-module for any $0 \leq i \leq n-1$.

We call such a heredity chain a *right-strongly* (resp. *left-strongly*) *heredity chain.*

Moreover we give categorical interpretations of right-strongly (resp. left-strongly) heredity chains. For an artin algebra A, there exists a bijection between idempotent ideals of A and full subcategories of the category $\mathsf{proj}\, A$ of finitely generated projective A-modules given by $AeA \mapsto \mathsf{add}\, eA$. This gives a bijection between chains of idempotent ideals of A and chains of full subcategories of $\mathsf{proj}\, A$. A key idea of this thesis is to translate properties of idempotent ideals into properties of full subcategories of $\mathsf{proj}\, A$.

For an artin algebra A and an arbitrary factor algebra B of A, we naturally regard $\mathsf{mod}\, B$ as a full subcategory of $\mathsf{mod}\, A$. In this case, each $X \in \mathsf{mod}\, A$ has a right (resp. left) ($\mathsf{mod}\, B$)-approximation of X which is monic (resp. epic) in $\mathsf{mod}\, A$. More generally, subcategories of an additive category with these properties are called *right* (resp. *left*) *rejective* subcategories in [Iya03a, Iya03b, Iya04]. They are a special class of *coreflective* (resp. *reflective*) *subcategories* (see Definition 3.29) appearing in the classical theory of localizations of abelian categories [Ste75].

Using the notion of right rejective (resp. left rejective, coreflective, reflective) subcategories, we introduce the notion of *total right rejective* (resp. *total left rejective, coreflective, reflective*) *chains* of an additive category (Definitions 3.19 and 3.33). The following main theorem in this thesis characterizes right-strongly (resp. left-strongly) quasi-hereditary algebras in terms of these chains.

Theorem 1.2 (Theorem 3.23 and 3.35). *Let A be an artin algebra and*

$$A = H_0 > H_1 > \cdots > H_n = 0 \tag{1-1}$$

a chain of idempotent ideals of A. For $0 \leq i \leq n-1$, we write $H_i = Ae_iA$, where e_i is an idempotent of A. Then the following statements are equivalent:

(i) (1-1) *is a right-strongly* (*resp. left-strongly*) *heredity chain.*

(ii) *The following chain is a total right* (*resp. left*) *rejective chain of* $\mathsf{proj}\, A$.

$$\mathsf{proj}\, A = \mathsf{add}\, e_0 A \supset \mathsf{add}\, e_1 A \supset \cdots \supset \mathsf{add}\, e_n A = 0.$$

(iii) (1-1) *is a heredity chain of* A *and the following chain is a coreflective* (*resp. reflective*) *chain of* $\mathsf{proj}\, A$.

$$\mathsf{proj}\, A = \mathsf{add}\, e_0 A \supset \mathsf{add}\, e_1 A \supset \cdots \supset \mathsf{add}\, e_n A = 0.$$

We apply total right (resp. left) rejective chains to study right-strongly (resp. left-strongly) quasi-hereditary algebras. We give the following result by [Iya03b, Theorem 3.6] and Theorem 1.2.

Theorem 1.3 (Theorem 4.1)**.** *Let* A *be an artin algebra. If* $\operatorname{gldim} A \leq 2$*, then* A *is a right-strongly* (resp. *left-strongly*) *quasi-hereditary algebra.*

An artin algebra which has a heredity chain such that it is a right-strongly heredity chain and a left-strongly heredity chain is called a *strongly quasi-hereditary algebra*. They have global dimension at most two [Rin10], but algebras with global dimension at most two are not necessarily strongly quasi-hereditary. Applying our results on rejective chains, we give the following characterization of Auslander algebras to be strongly quasi-hereditary.

Theorem 1.4 (Theorem 4.6)**.** *Let* A *be a representation-finite artin algebra and* B *the Auslander algebra of* A*. Then* B *is a strongly quasi-hereditary algebra if and only if* A *is a Nakayama algebra* (*see* [ARS95, §4.2] *for the definition of Nakayama algebras*).

Note that Theorem 1.4 can be obtained by a different method using a recent result [Eir17, Theorem 3].

Since the Auslander–Dlab–Ringel algebra of a Nakayama algebra coincides with the Auslander algebra of the Nakayama algebra, it is strongly quasi-hereditary algebra by Theorem 1.4. However, Auslander–Dlab–Ringel algebras are not necessarily strongly quasi-hereditary algebras. We give characterizations of strongly quasi-hereditary Auslander–Dlab–Ringel algebras by using Theorem 1.2.

Theorem 1.5 (Theorem 4.12)**.** *Let* A *be an artin algebra with Loewy length* $m \geq 2$ *and* $G := \bigoplus_{i=1}^{m} A/J(A)^i$*. Let* $B := \operatorname{End}_A(G)$ *be the Auslander–Dlab–Ringel algebra of* A*. Then the following statements are equivalent.*

(i) B *is a strongly quasi-hereditary algebra.*

(ii) $\operatorname{gldim} B = 2$.

(iii) *For any* $i \in I$, $P(i)J(A) \in \mathsf{add}\, G$.

Moreover we consider the Auslander–Dlab–Ringel algebras of semilocal modules introduced by [LX93]. Since any artin algebra is a semilocal module, it is a generalization of the Auslander–Dlab–Ringel algebra in the sense of Theorem 1.5. In [LX93], they prove that Auslander–Dlab–Ringel algebras of semilocal modules are quasi-hereditary. We sharpen this result.

Theorem 1.6 (Theorem 4.16)**.** *The Auslander–Dlab–Ringel algebra of any semilocal module is left-strongly quasi-hereditary.*

As an application, we give a tightly upper bound on global dimension of an Auslander–Dlab–Ringel algebra.

2 Preliminaries

Notation. For background materials in the representation theory of algebras, we refer to [ARS95, ASS06]. Let A be an artin algebra. Let $J(A)$ be the Jacobson radical of A. We denote by $\operatorname{gldim} A$ the global dimension of A. We write $\mathsf{mod}\, A$ for the category of finitely generated right A-modules and $\mathsf{proj}\, A$ for the full subcategory of $\mathsf{mod}\, A$ consisting of the finitely generated projective A-modules. For $M \in \mathsf{mod}\, A$, we denote by $\mathsf{add}\, M$ the full subcategory of $\mathsf{mod}\, A$ whose objects are direct summands of finite direct sums of copies of M.

We fix a complete set of representatives of isomorphism classes of simple A-modules $\{S(i) \mid i \in I\}$. For $i \in I$, we denote by $P(i)$ the projective cover of $S(i)$ and $I(i)$ the injective hull of $S(i)$. For $X \in \mathsf{mod}\, A$, we write $[X : S(i)]$ for the composition multiplicity of $S(i)$. We denote by $\mathbf{k}$ a field.

2.1 Quasi-hereditary algebras and highest weight categories

We start with recalling definitions of quasi-hereditary algebras and highest weight categories.

Definition 2.1 (Cline–Parshall–Scott [CPS88], Dlab–Ringel [DR89c])**.** Let A be an artin algebra.

(1) A two-sided ideal H of A is called *heredity* if it satisfies the following conditions:

  (a) H is an idempotent ideal (i.e. $H^2 = H$), or equivalently, there exists an idempotent e such that $H = AeA$ [DR89c, Statement 6];

  (b) H is projective as a right A-module;

  (c) $HJ(A)H = 0$.

(2) A chain of idempotent ideals of A

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$$

is called a *heredity chain* if H_i/H_{i+1} is a heredity ideal of A/H_{i+1} for $0 \leq i \leq n-1$.

(3) A is called a *quasi-hereditary algebra* if there exists a heredity chain of A.

Quasi-hereditary algebras are strongly related to highest weight categories defined below. In fact, an artin algebra A is quasi-hereditary if and only if there exists a partial order $\leq$ on I such that $(\mathsf{mod}\, A, \leq)$ is a highest weight category (see [CPS88, Theorem 3.6]).

Definition 2.2. Let $\leq$ be a partial order on the index set I of simple A-modules.

(1) For each $i \in I$, we denote by $\Delta(i)$ the maximal factor module of $P(i)$ whose composition factors have the form $S(j)$ for some $j \leq i$. The module $\Delta(i)$ is called the *standard module* corresponding to i.

(2) For each $i \in I$, we denote by $\nabla(i)$ the maximal submodule of $I(i)$ whose composition factors have the form $S(j)$ for some $j \leq i$. The module $\nabla(i)$ is called the *costandard module* corresponding to i.

Let $\Delta := \{\Delta(i) \mid i \in I\}$ be the set of standard modules. We denote by $\mathcal{F}(\Delta)$ the full subcategory of $\mathsf{mod}\, A$ whose objects are the modules which have a Δ-filtration, namely $M \in \mathcal{F}(\Delta)$ if and only if there exists a chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_l = 0$$

such that M_i/M_{i+1} is isomorphic to a module in Δ. For $M \in \mathcal{F}(\Delta)$, we denote by $(M : \Delta(i))$ the filtration multiplicity of $\Delta(i)$, which dose not depend on the choice of Δ-filtrations (cf. [Don98, A.1 (7)]). Dually, we define the full subcategory $\mathcal{F}(\nabla)$ of $\mathsf{mod}\, A$.

Definition 2.3 (Cline-Parshall-Scott [CPS88])**.** We say that a pair $(\mathsf{mod}\, A, \leq)$ is a *highest weight category* if there exists a short exact sequence

$$0 \to K(i) \to P(i) \to \Delta(i) \to 0$$

for any $i \in I$ with the following properties:

(a) $K(i) \in \mathcal{F}(\Delta)$ for any $i \in I$;

(b) if $(K(i) : \Delta(j)) \neq 0$, then we have $i < j$.

Remark 2.4. We can also give the definition of highest weight categories by using the notion of costandard modules. A pair $(\mathsf{mod}\, A, \leq)$ is a highest weight category if and only if there exists a short exact sequence

$$0 \to \nabla(i) \to I(i) \to I(i)/\nabla(i) \to 0$$

for any $i \in I$ with the following properties:

(a) $I(i)/\nabla(i) \in \mathcal{F}(\nabla)$ for any $i \in I$;

(b) if $(I(i)/\nabla(i) : \nabla(j)) \neq 0$, then we have $i < j$.

For a highest weight category $(\mathsf{mod}\, A, \leq)$ and a refinement $\leq'$ of $\leq$, it is clear that $(\mathsf{mod}\, A, \leq')$ is also a highest weight category whose standard modules coincide with those of $(\mathsf{mod}\, A, \leq)$. Therefore, without loss of generality, one can assume that the partial order $\leq$ on I is a total order.

To explain a connection between quasi-hereditary algebras and highest weight categories more explicitly, we introduce the following notion.

Definition 2.5 (Uematsu-Yamagata [UY90])**.** Let A be an artin algebra. A chain of idempotent ideals

$$A = H_0 > H_1 > \cdots > H_n = 0$$

is called *maximal* if the length of the chain is the number of simple modules.

Any heredity chain of an artin algebra can be refined to a maximal heredity chain [UY90, Proposition 1.3].

Let A be an artin algebra with simple A-modules $\{S(i) \mid i \in I\}$ and ε_i a primitive idempotent of A corresponding to $S(i)$. Then there is a bijection

$$\{\text{total orders on } I\} \overset{1:1}{\longleftrightarrow} \{\text{maximal chains of idempotent ideals}\}$$

given by setting $H_j := A(\varepsilon_{i_{j+1}} + \cdots + \varepsilon_{i_n})A$ and

$$(i_1 < i_2 < \cdots < i_n) \mapsto (A = H_0 > H_1 > \cdots > H_n). \tag{2-2}$$

Proposition 2.6 (Cline-Parshall-Scott [CPS88, §3])**.** *Let A be an artin algebra and $\leq$ a total order on I.*

(1) *A pair $(\mathsf{mod}\, A, \leq)$ is a highest weight category with standard modules $\{\Delta(i_1), \ldots, \Delta(i_n)\}$ if and only if the corresponding maximal chain of idempotent ideals is a heredity chain.*

(2) *If the condition in (1) are satisfied, then we have $H_j/H_{j+1} \cong \Delta(i_j)^{m_j}$ as right A-modules for some positive integer m_j.*

2.2 Right-strongly (resp. left-strongly) quasi-hereditary algebras

Ringel introduced the notion of right-strongly quasi-hereditary algebras as a special class of quasi-hereditary algebras. First we recall the definition of right-strongly quasi-hereditary algebras.

Definition 2.7 (Ringel [Rin10, §4])**.** Let A be an artin algebra and $\leq$ a partial order on I.

(1) We say that a pair $(A, \leq)$ (or simply A) is *right-strongly quasi-hereditary* if there exists a short exact sequence

$$0 \to K(i) \to P(i) \to \Delta(i) \to 0$$

for any $i \in I$ with the following properties:

(a) $K(i)$ is a direct summand of projective modules $P(j)$ with $i < j$;

(b) If $[\Delta(i)J : S(j)] \neq 0$, then we have $i < j$.

(2) We say that a pair $(A, \leq)$ (or simply A) is *left-strongly quasi-hereditary* if $(A^{\mathrm{op}}, \leq)$ is right-strongly quasi-hereditary.

We can easily check that Definition 2.7 is equivalent to the following condition which is frequently used in this thesis.

Proposition 2.8. *Let A be an artin algebra and $\leq$ a partial order on I. Then the following statements are equivalent.*

(i) *A pair* $(A, \leq)$ *is right-strongly quasi-hereditary.*

(ii) *There exists a short exact sequence*

$$0 \to K(i) \to P(i) \to \Delta(i) \to 0$$

for any $i \in I$ *with the following properties:*

(a) $K(i) \in \mathcal{F}(\Delta)$ *for all* $i \in I$;

(b) *if* $(K(i) : \Delta(j)) \neq 0$*, then we have* $i < j$;

(c) $K(i)$ *is a projective right* A*-module, or equivalently, the right* A*-module* $\Delta(i)$ *has projective dimension at most one.*

Proof. (i) $\Rightarrow$ (ii): This is clear by [Rin10, Proposition in §4].

(ii) $\Rightarrow$ (i): If $P(j)$ is a direct summand of $K(i)$, then $(K(i) : \Delta(i)) \neq 0$, and hence we have $i < j$. Since A is quasi-hereditary, we obtain that

$$[\Delta(i) : S(j)] = (\nabla(i) : \nabla(j)) + (I(i)/\nabla(i) : \nabla(j))$$

by $(I(i) : \nabla(j)) = [\Delta(i) : S(j)]$. If $j \neq i$, then $[\Delta(i)J : S(j)] = (I(i)/\nabla(i) : \nabla(j))$. Thus if $[\Delta(i)J : S(j)] \neq 0$, then we have $j > i$. If $j = i$, then

$$[\Delta(i) : S(i)] = (\nabla(i) : \nabla(i)) + (I(i)/\nabla(i) : \nabla(i)) = 1 + 0 = 1.$$

The proof is complete. $\square$

Since the properties (a) and (b) in Proposition 2.8 implies that $(\mathsf{mod}\, A, \leq)$ is a highest weight category, any right-strongly quasi-hereditary algebra is quasi-hereditary.

As before, for a right-strongly quasi-hereditary algebra $(A, \leq)$ and a refinement $\leq'$ of $\leq$, it is clear that $(A, \leq')$ is also a right-strongly quasi-hereditary algebra whose standard modules coincide with those of $(A, \leq)$. Therefore, without loss of generality, one can assume that the partial order $\leq$ on I is a total order.

In this thesis, for a quiver Q and arrows $\alpha : x \to y$ and $\beta : y \to z$ in Q, we denote by $\alpha\beta$ the composition. We denote by ϵ_i the idempotent corresponding to a vertex i.

Example 2.9. Let $n \geq 2$ be an integer and let A_n be the $\mathbf{k}$-algebra defined by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{i-1}}{\overset{\alpha_{i-1}}{\rightleftarrows}} i \underset{\beta_i}{\overset{\alpha_i}{\rightleftarrows}} i+1 \underset{\beta_{i+1}}{\overset{\alpha_{i+1}}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

with relations $\alpha_{i-1}\alpha_i$, $\beta_i\beta_{i-1}$, $\beta_{i-1}\alpha_{i-1} - \alpha_i\beta_i$ for $2 \leq i \leq n-1$ and $\beta_{n-1}\alpha_{n-1}$. The algebra A_n is Morita equivalent to a block of a Schur algebra (see [DR94, Erd93]).

If $n = 2$, then the indecomposable projective modules $P(i)$ have the following shapes:

$$\begin{array}{ccc} 1 & & 2 \\ 2 & & 1 \\ 1 & & \end{array}$$

For the total order $\{1 < 2\}$, we have $\Delta(1) = S(1)$ and $\Delta(2) = P(2)$. Hence A_2 is right-strongly quasi-hereditary.

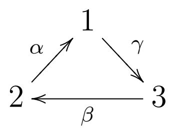
If $n > 2$, then the indecomposable projective modules $P(i)$ have the following shapes:

$$\begin{array}{c} 1 \\ 2 \\ 1 \end{array} \quad \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \quad \cdots \quad \begin{array}{ccc} & i & \\ i-1 & & i+1 \\ & i & \end{array} \quad \cdots \quad \begin{array}{c} n \\ n-1 \end{array}$$

Thus A_n is quasi-hereditary with respect to the total order $\{1 < 2 < \cdots < n\}$. However A_n is not right-strongly quasi-hereditary with respect to any order.

It is well known that $(A, \leq)$ is quasi-hereditary if and only if so is $(A^{\mathrm{op}}, \leq)$ (see [CPS88, Lemma 3.4] and [DR89c, Statement 9]). However, even if $(A, \leq)$ is right-strongly quasi-hereditary, it does not necessarily hold that $(A^{\mathrm{op}}, \leq)$ is right-strongly quasi-hereditary (or equivalently, $(A, \leq)$ is left-strongly quasi-hereditary). In fact, Ringel gave an example of a right-strongly quasi-hereditary algebra which is not left-strongly quasi-hereditary for any partial order on I (see [Rin10, A2 (1)]).

Example 2.10. Let A be the $\mathbf{k}$-algebra defined by the quiver



with relations $\alpha\gamma$ and $\beta\alpha$. Then the indecomposable projective A-modules $P(i)$ have the following shapes:

$$\begin{array}{c} 1 \\ 3 \\ 2 \end{array} \quad \begin{array}{c} 2 \\ 1 \\ \end{array} \quad \begin{array}{c} 3 \\ 2 \\ \end{array}$$

For the total order $\{1 < 2 < 3\}$, we have $\Delta(1) = S(1)$ and $\Delta(i) = P(i)$ for $i = 2, 3$. Hence A is right-strongly quasi-hereditary. On the other hand, the indecomposable projective A^{op}-modules have the following shapes:

$$\begin{array}{c} 1^{\mathrm{op}} \\ 2^{\mathrm{op}} \\ \end{array} \quad \begin{array}{c} 2^{\mathrm{op}} \\ 3^{\mathrm{op}} \\ 1^{\mathrm{op}} \end{array} \quad \begin{array}{c} 3^{\mathrm{op}} \\ 1^{\mathrm{op}} \\ \end{array}$$

For the total order $\{1 < 2 < 3\}$, we have $\Delta^{\mathrm{op}}(i) = S^{\mathrm{op}}(i)$ for $i = 1, 2$ and $\Delta^{\mathrm{op}}(3) = P^{\mathrm{op}}(3)$. Hence A is not left-strongly quasi-hereditary. However, for the total order $\{2 < 1 < 3\}$, A is not right-strongly quasi-hereditary but A is left-strongly quasi-hereditary.

3 Characterizations of right-strongly quasi-hereditary algebras

3.1 Right-strongly heredity chains

In this subsection, we give a characterization of right-strongly (resp. left-strongly) quasi-hereditary algebras in terms of heredity chains.

Definition 3.1. Let A be an artin algebra and

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0 \tag{3-3}$$

a chain of idempotent ideals.

(1) We call (3-3) a *right-strongly* (*resp. left-strongly*) *heredity chain* if the following conditions hold for any $0 \leq i \leq n-1$:

  (a) H_i is projective as a right (resp. left) A-module;

  (b) $(H_i/H_{i+1})J(A/H_{i+1})(H_i/H_{i+1}) = 0$.

(2) We call (3-3) a *strongly heredity chain* if the following conditions hold for any $0 \leq i \leq n-1$:

  (a) H_i is projective as a right A-module and as a left A-module;

  (b) $(H_i/H_{i+1})J(A/H_{i+1})(H_i/H_{i+1}) = 0$.

Proposition 3.2. *Any right-strongly* (*resp. left-strongly*) *heredity chain of A is a heredity chain.*

Proof. Let (3-3) be a right-strongly heredity chain. It is enough to show that H_i/H_{i+1} is projective as a right (A/H_{i+1})-module for any $0 \leq i \leq n-1$. Since (3-3) is a right-strongly heredity chain, we have that H_i is projective as a right A-module for any $0 \leq i \leq n-1$. Hence $H_i \otimes_A (A/H_{i+1}) = H_i/H_{i+1}$ is projective as a right (A/H_{i+1})-module for any $0 \leq i \leq n-1$. □

Example 3.3. Let A be an artin algebra. Then A is hereditary if and only if any chain of idempotent ideals of A is a strongly heredity chain.

Proof. The "only if" part is clear. By [DR89c, Theorem 1], A is hereditary if and only if any chain of idempotent ideals of A is a heredity chain. Therefore the "if" part follows. □

Example 3.4. Any heredity chain of length at most two is clearly a right-strongly (resp. left-strongly) heredity chain.

Example 3.5. Let A be the Auslander algebra of the truncated polynomial algebra $\mathbf{k}[x]/(x^n)$. Namely A is given by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{i-1}}{\overset{\alpha_{i-1}}{\rightleftarrows}} i \underset{\beta_i}{\overset{\alpha_i}{\rightleftarrows}} i+1 \underset{\beta_{i+1}}{\overset{\alpha_{i+1}}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

with relations $\beta_i\alpha_i - \alpha_{i+1}\beta_{i+1}$ $(1 \leq i \leq n-2)$ and $\beta_{n-1}\alpha_{n-1}$. Then

$$A > A(\varepsilon_2 + \cdots + \varepsilon_n)A > \cdots > A\varepsilon_n A > 0$$

is a strongly heredity chain of A. This example can be explained by Theorem 4.6.

We prepare the following easy observation.

Lemma 3.6. *Let A be an artin algebra and $A = H_0 > H_1 > \cdots > H_n = 0$ a chain of two-sided ideals. Then the following conditions are equivalent:*

(i) *H_i is projective as a right (resp. left) A-module for $0 \leq i \leq n-1$.*

(ii) *The projective dimension of H_i/H_{i+1} as a right (resp. left) A-module is at most one for $0 \leq i \leq n-1$.*

Proof. (i) ⇒ (ii): This is clear from the short exact sequence

$$0 \to H_{i+1} \to H_i \to H_i/H_{i+1} \to 0.$$

(ii) ⇒ (i): Since $0 \to H_1 \to H_0 \to H_0/H_1 \to 0$ is a short exact sequence such that $H_0 = A$ is a projective A-module, H_1 is also projective as a right A-module. Thus we obtain the assertion inductively. □

Now, we are ready to prove the following main observation in this subsection.

Proposition 3.7. *Let A be an artin algebra, $\leq$ a total order on I and*

$$A = H_0 > H_1 > \cdots > H_n = 0 \tag{3-4}$$

a maximal chain of idempotent ideals corresponding to $\leq$ by (2-2). Then $(A, \leq)$ is a right-strongly (resp. left-strongly) quasi-hereditary algebra if and only if (3-4) is a right-strongly (resp. left-strongly) heredity chain.

Proof. Both conditions imply that (3-4) is a heredity chain by Proposition 2.6 (1) and Proposition 3.2. Moreover we have an isomorphism

$$H_j/H_{j+1} \cong \Delta(i_j)^{m_j} \tag{3-5}$$

as right A-modules for some positive integer m_j by Proposition 2.6 (2).

By (3-5), the pair $(A, \leq)$ is right-strongly quasi-hereditary if and only if the projective dimension of H_j/H_{j+1} as a right A-module is at most one for any $0 \leq j \leq n-1$. By Lemma 3.6, this is equivalent to that H_j is projective as a right A-module for any $0 \leq j \leq n-1$. Hence (3-4) is a right-strongly heredity chain. □

Thought this thesis, we frequently use the following basic observations.

Lemma 3.8. *Let A be an artin algebra and e an idempotent of A. Then we have the following statements.*

(1) *If AeA is projective as a right A-module, then $AeA \in \mathsf{add}\, eA$.*

(2) *If Ae is projective as a right (eAe)-module, then the functor $\mathrm{Hom}_A(eA, -) : \mathsf{mod}\, A \to \mathsf{mod}\, eAe$ preserves projective modules. In particular, $\mathrm{gldim}\, eAe \leq \mathrm{gldim}\, A$.*

(3) *If AeA is projective as a right A-module, then Ae is a projective right (eAe)-module.*

Proof. (1) Take an epimorphism $f : (eAe)^l \twoheadrightarrow Ae$ in $\mathsf{mod}(eAe)$. Composing $f \otimes_{eAe} eA : (eA)^l \twoheadrightarrow Ae \otimes_{eAe} eA$ with the multiplication map $Ae \otimes_{eAe} eA \twoheadrightarrow AeA$, we have an epimorphism $(eA)^l \twoheadrightarrow AeA$ of right A-modules.

(2) For any $P \in \mathsf{proj}\, A$, we have that $\mathrm{Hom}_A(eA, P) = Pe$ is a direct summand of a finite direct sum of copies of $\mathrm{Hom}_A(eA, A) = Ae$. Hence the assertion holds.

(3) Since AeA is a projective A-module, it follows from (1) that $AeA \in \mathsf{add}\, eA$. Hence we obtain that $Ae = AeAe = \mathrm{Hom}_A(eA, AeA)$ is projective as a right (eAe)-module. □

We end this subsection with the following observations which show that right-strongly (resp. left-strongly) quasi-hereditary algebras are closed under idempotent reductions.

Proposition 3.9. *Let A be an artin algebra with a right-strongly (resp. left-strongly) heredity chain*

$$A = H_0 > H_1 > \cdots > H_n = 0.$$

Then the following statements hold.

(1) *For $0 < i \leq n-1$, A/H_i has a right-strongly (resp. left-strongly) heredity chain*

$$A/H_i = H_0/H_i > H_1/H_i > \cdots > H_{i-1}/H_i > H_i/H_i = 0.$$

(2) *Let $e_i \in A$ be an idempotent of A such that $H_i = Ae_iA$ for $0 \leq i \leq n-1$. Then e_iAe_i has a right-strongly (resp. left-strongly) heredity chain*

$$e_iAe_i = e_iH_ie_i > e_iH_{i+1}e_i > \cdots > e_iH_ne_i = 0.$$

Proof. (1) It is enough to show that H_j/H_i is projective as a right (A/H_i)-module for $1 \leq j < i$. This is immediate since H_j is projective as a right A-module and the functor $- \otimes_A (A/H_i) : \mathsf{mod}\, A \to \mathsf{mod}\, A/H_i$ reflects projectivity.

(2) We prove that $e_iH_je_i$ is a projective right (e_iAe_i)-module. By Lemma 3.8 (3), we have that Ae_i is projective as a right (e_iAe_i)-module. It follows from Lemma 3.8 (2) that H_je_i is projective as a right (e_iAe_i)-module. Since $H_je_i = e_iH_je_i \oplus (1-e_i)H_je_i$, we have $e_iH_je_i \in \mathsf{proj}(e_iAe_i)$. □

3.2 Right rejective subcategories

In this subsection, we recall the definitions of right rejective subcategories. Using them, we characterize right-strongly (resp. left-strongly) quasi-hereditary algebras. We refer to [ASS06, Appendix] for background on category theory.

Let $\mathcal{C}$ be an additive category, and put $\mathcal{C}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)$. In the rest of this thesis, *we assume that any subcategory is full and closed under isomorphisms, direct sums and direct summands.* We denote by $\mathcal{J}_{\mathcal{C}}$ the Jacobson radical of $\mathcal{C}$, and by $\mathsf{ind}\,\mathcal{C}$ the set of isoclasses of indecomposable objects in $\mathcal{C}$. For a subcategory $\mathcal{C}'$ of $\mathcal{C}$, we denote by $[\mathcal{C}']$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object of $\mathcal{C}'$. For an ideal $\mathcal{I}$ of $\mathcal{C}$, the factor category $\mathcal{C}/\mathcal{I}$ is defined by $ob(\mathcal{C}/\mathcal{I}) := ob(\mathcal{C})$ and $(\mathcal{C}/\mathcal{I})(X,Y) := \mathcal{C}(X,Y)/\mathcal{I}(X,Y)$ for any $X, Y \in \mathcal{C}$. Recall that an additive category $\mathcal{C}$ is called *Krull–Schmidt* if any object of $\mathcal{C}$ is isomorphic to a finite direct sum of objects whose endomorphism rings are local.

Definition 3.10 (Auslander–Smalø [AS80])**.** Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. We say that $f \in \mathcal{C}(Y,X)$ is a *right $\mathcal{C}'$-approximation* of X if the following equivalent conditions are satisfied.

(i) $Y \in \mathcal{C}'$ and $\mathcal{C}(-,Y) \xrightarrow{f\circ -} \mathcal{C}(-,X) \to 0$ is exact on $\mathcal{C}'$.

(ii) $Y \in \mathcal{C}'$ and the induced morphism $\mathcal{C}(-,Y) \xrightarrow{f\circ -} [\mathcal{C}'](-,X)$ is an epimorphism on $\mathcal{C}$.

Dually, a *left $\mathcal{C}'$-approximation* is defined.

Now, we introduce the following key notions in this thesis.

Definition 3.11 (Iyama [Iya03a, 2.1(1)])**.** Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$.

(1) We call $\mathcal{C}'$ a *right* (*resp. left*) *rejective subcategory* of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a right (resp. left) adjoint with a counit ε^- (resp. unit ε^+) such that ε^-_X is a monomorphism (resp. ε^+_X is an epimorphism) for $X \in \mathcal{C}$.

(2) We call $\mathcal{C}'$ a *rejective subcategory* of $\mathcal{C}$ if $\mathcal{C}'$ is a right and left rejective subcategory of $\mathcal{C}$.

We often use the following equivalent conditions.

Proposition 3.12 (Iyama [Iya03b, Definition 1.5])**.** *Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. Then the following are equivalent:*

(i) *$\mathcal{C}'$ is a right* (*resp. left*) *rejective subcategory of $\mathcal{C}$.*

(ii) *For any $X \in \mathcal{C}$, there exists a monic right* (*resp. epic left*) *$\mathcal{C}'$-approximation $f_X \in \mathcal{C}(Y,X)$* (*resp. $f^X \in \mathcal{C}(X,Y)$*) *of X.*

Proof. (i) ⇒ (ii): If the inclusion functor $F : \mathcal{C}' \hookrightarrow \mathcal{C}$ has a right adjoint G with a counit ε^-, then $\varepsilon_X^- : G(X) \to X$ is a right $\mathcal{C}'$-approximation of $X \in \mathcal{C}$. Thus the assertion follows.

(ii) ⇒ (i): We assume that, for any $X \in \mathcal{C}$, there exists a monic right $\mathcal{C}'$-approximation of X. We construct a right adjoint functor $G : \mathcal{C} \to \mathcal{C}'$ as follows. For $X \in \mathcal{C}$, take a monic right $\mathcal{C}'$-approximation $f_X : C_X \to X$. For a morphism $\varphi \in \mathcal{C}(X, Y)$, there exists a unique morphism $C_\varphi : C_X \to C_Y$ making the following diagram commutative.

$$\begin{array}{ccc} C_X & \xrightarrow{f_X} & X \\ {\scriptstyle C_\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C_Y & \xrightarrow{f_Y} & Y. \end{array}$$

It is easy to check that $G(X) := C_X$ and $G(\varphi) := C_\varphi$ give a right adjoint functor $G : \mathcal{C} \to \mathcal{C}'$ of the inclusion functor $F : \mathcal{C}' \to \mathcal{C}$ and f gives a counit. □

Right rejective subcategories of $\mathsf{mod}\, A$ are characterized as follows.

Proposition 3.13 (Iyama [Iya03b, Proposition 1.5.2])**.** *Let A be an artin algebra and $\mathcal{C}$ a subcategory of $\mathsf{mod}\, A$. Then $\mathcal{C}$ is a right (resp. left) rejective subcategory of $\mathsf{mod}\, A$ if and only if $\mathcal{C}$ is closed under factor modules (resp. submodules).*

Proof. We show the "if" part. For $M \in \mathsf{mod}\, A$, we put $G(M) := \sum_{X \in \mathcal{C}, f \in \mathrm{Hom}_A(X,M)} f(X)$. Then $G(M)$ is a factor module of some module in $\mathcal{C}$. Thus we have $G(M) \in \mathcal{C}$. Since the natural inclusion $G(M) \hookrightarrow M$ is a monic right $\mathcal{C}$-approximation of M, the assertion holds.

We show the "only if" part. For a surjection $f : M \to N$ with $M \in \mathcal{C}$, we show that N belongs to $\mathcal{C}$. Since $\mathcal{C}$ is a right rejective subcategory of $\mathsf{mod}\, A$, there exists a monic right $\mathcal{C}$-approximation $f_N : G(N) \to N$ of N. Thus we have a morphism $g : M \to G(N)$ such that $f = f_N \circ g$. Since f is surjective, we obtain that f_N is a bijection. Hence we have $N \in \mathcal{C}$. □

Proposition 3.14 (Iyama [Iya03b, Theorem 1.6.1(1)])**.** *Let A be an artin algebra. Then there exists a bijection between factor algebras B of A and rejective subcategories $\mathcal{C}$ of $\mathsf{mod}\, A$ given by $B \mapsto \mathsf{mod}\, B$.*

Proof. This is clearly from Proposition 3.13 since a full subcategory of $\mathsf{mod}\, A$ which is closed under submodules and factor modules is precisely $\mathsf{mod}\, B$ for a factor algebra B of A. □

Example 3.15. Let A be an artin algebra.

(a) Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathsf{mod}\, A$. Then $\mathcal{T}$ is a right rejective subcategory and $\mathcal{F}$ is a left rejective subcategory of $\mathsf{mod}\, A$ by Proposition 3.13.

(b) For a classical tilting A-module T, we put $\mathcal{T} := \{Y \in \mathsf{mod}\, A \mid \mathrm{Ext}_A^1(T, Y) = 0\}$ and $\mathcal{F} := \{Y \in \mathsf{mod}\, A \mid \mathrm{Hom}_A(T, Y) = 0\}$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathsf{mod}\, A$, and therefore $\mathcal{T}$ (resp. $\mathcal{F}$) is a right (resp. left) rejective subcategory of $\mathsf{mod}\, A$.

(c) Assume that A is right-strongly quasi-hereditary and let T be a characteristic tilting module (see Definition-Theorem 4.4). Then T is a classical tilting module by [DR92, Lemma 4.1] and hence $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathsf{mod}\, A$. Since $\mathcal{T}$ coincides with the subcategory

$$\mathcal{F}(\Delta)^{\perp} := \{Y \in \mathsf{mod}\, A \mid \operatorname{Ext}^i_A(\mathcal{F}(\Delta), Y) = 0 \text{ for all } i \geq 1\}$$

(see [DR92, §4]), we have that $\mathcal{F}(\Delta)^{\perp}$ is a right rejective subcategory of $\mathsf{mod}\, A$.

Right rejective subcategories of $\mathsf{proj}\, A$ are characterized as follows.

Proposition 3.16 (Iyama [Iya03b, Theorem 3.2 (2)]). *Let A be an artin algebra and e an idempotent of A. Then $\mathsf{add}\, eA$ is a right (resp. left) rejective subcategory of $\mathsf{proj}\, A$ if and only if AeA is a projective right (resp. left) A-module. In this case, we have* $\operatorname{gldim} eAe \leq \operatorname{gldim} A$.

Proof. Assume that $\mathsf{add}\, eA$ is a right rejective subcategory of $\mathsf{proj}\, A$. Then there exists $a \in \operatorname{Hom}_A(P, A)$ with $P \in \mathsf{add}(eA)_A$ such that

$$P = \operatorname{Hom}_A(A, P) \xrightarrow{a\circ -} [\mathsf{add}\, eA](A, A) = AeA$$

is an isomorphism. Hence $AeA \cong P$ is a projective right A-module.

Conversely, we assume that AeA is a projective right A-module. By Lemma 3.8 (1), we have $AeA \in \mathsf{add}\, eA$ as a right A-module. The inclusion map $i : AeA \hookrightarrow A$ gives a right $(\mathsf{add}\, eA)$-approximation of A since

$$Ae = AeAe = \operatorname{Hom}_A(eA, AeA) \xrightarrow{i\circ -} \operatorname{Hom}_A(eA, A) = Ae$$

is an isomorphism.

In this case, Ae is projective as a right (eAe)-module by Lemma 3.8 (3). Thus it follows from Lemma 3.8 (2) that $\operatorname{gldim} eAe \leq \operatorname{gldim} A$.

By the duality $\operatorname{Hom}_A(-, A) : \mathsf{proj}\, A \to \mathsf{proj}\, A^{\mathrm{op}}$, we have that $\mathsf{add}\, eA$ is left rejective in $\mathsf{proj}\, A$ if and only if $\mathsf{add}\, Ae$ is right rejective in $\mathsf{proj}\, A^{\mathrm{op}}$. Hence the statement for left rejective subcategories follows. □

To introduce rejective chains, we need the following notion.

Definition 3.17. Let $\mathcal{C}$ be a Krull–Schmidt category.

(1) We call $\mathcal{C}$ a *semisimple* category if $\mathcal{J}_{\mathcal{C}} = 0$.

(2) A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called *cosemisimple* in $\mathcal{C}$ if the factor category $\mathcal{C}/[\mathcal{C}']$ is semisimple.

We often use the fact that $\mathcal{C}'$ is a cosemisimple subcategory of $\mathcal{C}$ if and only if $[\mathcal{C}'](-, X) = \mathcal{J}_{\mathcal{C}}(-, X)$ holds for any $X \in \mathsf{ind}\, \mathcal{C} \setminus \mathsf{ind}\, \mathcal{C}'$.

Lemma 3.18. *Let A be an artin algebra and $Ae'A \subset AeA$ idempotent ideals of A. Then the following conditions are equivalent:*

(i) $\mathsf{add}\, e'A$ *is a cosemisimple subcategory of* $\mathsf{add}\, eA$.

(ii) $J(eAe/eAe'Ae) = 0$.

(iii) $(AeA/Ae'A)J(A/Ae'A)(AeA/Ae'A) = 0$.

Proof. (i) ⇔ (ii): Let $\mathcal{C} := \mathsf{add}\, eA/[\mathsf{add}\, e'A]$. The condition (i) means $\mathcal{J}_{\mathcal{C}} = 0$. This is equivalent to (ii) since $\mathcal{J}_{\mathcal{C}}(eA, eA) = J(\mathrm{End}_{\mathcal{C}}(eA)) = J(eAe/eAe'Ae)$.

(ii) ⇔ (iii): Since $J(eAe/eAe'Ae) = eJ(A/Ae'A)e$, we have the assertion. □

Now, we introduce the following central notion in this thesis.

Definition 3.19 (Iyama [Iya03a, 2.1(2)], [Iya03b, Definition 2.2])**.** Let $\mathcal{C}$ be a Krull–Schmidt category and

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0 \tag{3-6}$$

a chain of subcategories.

(1) We call (3-6) a *rejective chain* (resp. *right rejective, left rejective*) if $\mathcal{C}_i$ is a cosemisimple rejective (resp. right rejective, left rejective) subcategory of $\mathcal{C}_{i-1}$ for $1 \leq i \leq n$.

(2) We call (3-6) a *total right* (resp. *left*) *rejective chain* if the following conditions hold for $1 \leq i \leq n$:

  (a) $\mathcal{C}_i$ is a right (resp. left) rejective subcategory of $\mathcal{C}$;

  (b) $\mathcal{C}_i$ is a cosemisimple subcategory of $\mathcal{C}_{i-1}$.

Remark 3.20. (1) Rejective chains are total right rejective chains and total left rejective chains by [Iya03b, 2.1(3)].

(2) Our total right rejective chains are called right rejective chains in [Iya04, Definition 2.6].

Example 3.21. Let A be the $\mathbf{k}$-algebra given in Example 2.10. Then

$$\mathsf{proj}\, A = \mathsf{add}\, A \supset \mathsf{add}(\varepsilon_2 + \varepsilon_3)A \supset \mathsf{add}\, \varepsilon_3 A \supset 0$$

is a total right rejective chain of $\mathsf{proj}\, A$. In fact, the conditions (a) and (b) in Definition 3.19 (2) are satisfied by Proposition 3.16 and $\varepsilon_1 J(A)\varepsilon_1 = 0 = \varepsilon_2 J(A)\varepsilon_2$ respectively.

Total right (resp. left) rejective chains are useful to study the endomorphism algebras with finite global dimension.

Proposition 3.22 (Iyama [Iya03b, Theorem 2.2.2])**.** *Let A be an artin algebra. If* $\mathsf{proj}\, A$ *has a total right* (*resp. left*) *rejective chain of length $n > 0$, then* $\mathrm{gldim}\, A \leq n$.

Now, we are ready to prove the following main result.

Theorem 3.23. *Let A be an artin algebra and let*

$$A = H_0 > H_1 > \cdots > H_n = 0 \tag{3-7}$$

be a chain of idempotent ideals of A. For $0 \le i \le n-1$, we write $H_i = Ae_iA$, where e_i is an idempotent of A. Then the following conditions are equivalent:

(i) *The chain* (3-7) *is a right-strongly* (*resp. left-strongly*) *heredity chain.*

(ii) *The following chain is a total right* (*resp. left*) *rejective chain of* $\mathsf{proj}\, A$.

$$\mathsf{proj}\, A = \mathsf{add}\, e_0A \supset \mathsf{add}\, e_1A \supset \cdots \supset \mathsf{add}\, e_nA = 0.$$

In particular, an artin algebra A is strongly (*resp. right-strongly, left-strongly*) *quasi-hereditary if and only if* $\mathsf{proj}\, A$ *has a rejective* (*resp. total right rejective, total left rejective*) *chain.*

Proof. It follows from Proposition 3.16 that $H_i \in \mathsf{proj}\, A$ if and only if $\mathsf{add}\, e_iA$ is a right rejective subcategory of $\mathsf{proj}\, A$. Thus we have that H_i satisfies the condition (a) in Definition 3.1 (1) if and only if $\mathsf{add}\, e_iA$ satisfies the condition (a) in Definition 3.19 (2). From Lemma 3.18, we have that $(H_i/H_{i+1})J(A/H_{i+1})(H_i/H_{i+1}) = 0$ holds if and only if $\mathsf{add}\, e_{i+1}A$ is a cosemisimple subcategory of $\mathsf{add}\, e_iA$. Thus we obtain that H_i and H_{i+1} satisfy the condition (b) in Definition 3.1 (1) if and only if $\mathsf{add}\, e_iA$ and $\mathsf{add}\, e_{i+1}A$ satisfy the condition (b) in Definition 3.19 (2). Hence the proof is complete. □

By combining Proposition 3.22 and Theorem 3.23, we can recover the following result which was obtained by [Rin10, Proposition in §4].

Corollary 3.24. *Let A be a right-strongly* (*resp. left-strongly*) *quasi-hereditary algebra with a right-strongly* (*resp. left-strongly*) *heredity chain of length $n > 0$. Then we have*

$$\operatorname{gldim} A \le n.$$

Proof. Since A is a right-strongly quasi-hereditary algebra, $\mathsf{proj}\, A$ has a total right rejective chain of length $n > 0$ by Theorem 3.23. Hence the assertion follows from Proposition 3.22. □

We can rephrase Theorem 3.23 as follows.

Corollary 3.25. *Let A be an artin algebra and M a right A-module. Then the following statements are equivalent.*

(i) $\operatorname{End}_A(M)$ *is a right-strongly* (*resp. left-strongly*) *quasi-hereditary algebra.*

(ii) $\mathsf{proj}\, \operatorname{End}_A(M)$ *has a total right* (*resp. left*) *rejective chain.*

(iii) $\mathsf{add}\, M$ *has a total right* (*resp. left*) *rejective chain.*

Proof. Let $B := \mathrm{End}_A(M)$.

(i) $\Leftrightarrow$ (ii): This is from Theorem 3.23.

(ii) $\Leftrightarrow$ (iii): Since the functor $\mathrm{Hom}_A(M,-)$ (resp. $\mathrm{Hom}_B(\mathrm{Hom}_A(-,M),B)$) induces an equivalence $\mathsf{add}\, M \to \mathsf{proj}\, B$, we obtain that $\mathsf{proj}\, B$ has a total right (resp. left) rejective chain if and only if $\mathsf{add}\, M$ has a total right (resp. left) rejective chain. This finishes the proof. □

Moreover we have the following well-known result.

Corollary 3.26. *Let A be an artin algebra.*

(1) (Iyama [Iya03a, Theorem 1.1]) *For any $M \in \mathsf{mod}\, A$, there exists $N \in \mathsf{mod}\, A$ such that $\mathsf{add}\, N$ contains M and has a total right rejective chain.*

(2) (Ringel [Rin10, Theorem in §5]) *There exists a right-strongly quasi-hereditary algebra B and an idempotent e of B such that $A = eBe$.*

Proof. (1) For the reader's convenience, we recall the construction. Let $M_0 := M$ and $M_{i+1} := J(\mathrm{End}_A(M_i))M_i$ inductively. We take the smallest $n > 0$ such that $M_n = 0$, and let $N := \bigoplus_{k=0}^{n-1} M_k$ and $\mathcal{C}_i := \mathsf{add}(\bigoplus_{k=i}^{n-1} M_k)$. Then

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$

is a total right rejective chain by [Iya03a, Lemma 2,2].

(2) Applying (1) to $M = A$, we obtain that $N \in \mathsf{mod}\, A$ such that $B = \mathrm{End}_A(N)$ is right-strongly quasi-hereditary by Corollary 3.25. Let $e \in B$ be the idempotent corresponding to the direct summand A of N. Then $eBe = A$ holds as desired. □

We end this subsection with characterizations of cosemisimple right (resp. left) rejective subcategories. The first one is crucial in the proof of Corollary 3.26 (1).

Proposition 3.27 (Iyama [Iya03b, 1.5.1])**.** *Let $\mathcal{C}$ be a Krull–Schmidt category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. Then $\mathcal{C}'$ is a cosemisimple right* (*resp. left*) *rejective subcategory of $\mathcal{C}$ if and only if, for any $X \in \mathsf{ind}\, \mathcal{C} \setminus \mathsf{ind}\, \mathcal{C}'$, there exists a morphism $\varphi : Y \to X$* (*resp. $\varphi : X \to Y$*) *such that $Y \in \mathcal{C}'$ and $\mathcal{C}(-,Y) \xrightarrow{\varphi\circ-} \mathcal{J}_{\mathcal{C}}(-,X)$* (*resp. $\mathcal{C}(Y,-) \xrightarrow{-\circ\varphi} \mathcal{J}_{\mathcal{C}}(X,-)$*) *is an isomorphism on $\mathcal{C}$.*

Proof. We show the "only if" part. For any $X \in \mathsf{ind}\, \mathcal{C} \setminus \mathsf{ind}\, \mathcal{C}'$, we take a morphism $\varphi : Y \to X$ such that $Y \in \mathcal{C}'$ and $\mathcal{C}(-,Y) \xrightarrow{\varphi\circ-} [\mathcal{C}'](-,X)$ is an isomorphism on $\mathcal{C}$. This gives a desired morphism since cosemisimplicity of $\mathcal{C}'$ implies that $\mathcal{J}_{\mathcal{C}}(-,X) = [\mathcal{C}'](-,X)$.

We show the "if" part. It suffices to prove that $[\mathcal{C}'](-,X) = \mathcal{J}_{\mathcal{C}}(-,X)$ for any $X \in \mathsf{ind}\, \mathcal{C} \setminus \mathsf{ind}\, \mathcal{C}'$. For any $X \in \mathsf{ind}\, \mathcal{C} \setminus \mathsf{ind}\, \mathcal{C}'$, we take a morphism $\varphi : Y \to X$ such that $Y \in \mathcal{C}'$ and $\mathcal{C}(-,Y) \xrightarrow{\varphi\circ-} \mathcal{J}_{\mathcal{C}}(-,X)$ is an isomorphism on $\mathcal{C}$. Then $\mathcal{J}_{\mathcal{C}}(-,X) \subseteq \mathrm{Im}(\varphi \circ -) \subseteq [\mathcal{C}'](-,X)$ holds. Since $X \notin \mathcal{C}'$, this clearly implies $\mathcal{J}_{\mathcal{C}}(-,X) = [\mathcal{C}'](-,X)$, and hence we have the assertion. □

The second one is a reformulation of Proposition 3.27.

Proposition 3.28 (Iyama [Iya03b, Theorem 3.2(3)]). *Let A be a basic artin algebra and e an idempotent of A. Then $\mathsf{add}\, eA$ is a cosemisimple right (resp. left) rejective subcategory of $\mathsf{proj}\, A$ if and only if $(1-e)J(A) \in \mathsf{add}\, eA$ as a right A-module (resp. $J(A)(1-e) \in \mathsf{add}\, Ae$ as a left A-module).*

Proof. Applying Proposition 3.27 to $\mathcal{C} := \mathsf{proj}\, A$ and $\mathcal{C}' := \mathsf{add}\, eA$, we have that $\mathcal{C}'$ is a cosemisimple right rejective subcategory of $\mathcal{C}$ if and only if there exists a morphism $\varphi : Y \to (1-e)A$ with $Y \in \mathcal{C}'$ such that

$$Y \cong \mathcal{C}(A, Y) \xrightarrow{\varphi\circ-} \mathcal{J}_{\mathcal{C}}(A, (1-e)A) = (1-e)J(A)$$

is an isomorphism. This means that $(1-e)J(A) \in \mathcal{C}'$ holds. □

3.3 Coreflective subcategories

In this subsection, we study a weaker notion of right (resp. left) rejective subcategories called coreflective (resp. reflective) subcategories. They appeared in the classical theory of localizations of abelian categories [Ste75]. Let us start with recalling their definitions.

Definition 3.29 (Cf. Stenström [Ste75]). Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. We call $\mathcal{C}'$ a *coreflective* (resp. *reflective*) subcategory of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a right (resp. left) adjoint.

Clearly right (resp. left) rejective subcategories are coreflective (resp. reflective). The following proposition is an analogue of Proposition 3.12.

Proposition 3.30. *Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$. Then the following conditions are equivalent:*

(i) *$\mathcal{C}'$ is a coreflective (resp. reflective) subcategory of $\mathcal{C}$.*

(ii) *For any $X \in \mathcal{C}$, there exists a right (resp. left) $\mathcal{C}'$-approximation $f_X \in \mathcal{C}(Y, X)$ (resp. $f^X \in \mathcal{C}(X, Y)$) of X such that $\mathcal{C}(-, Y) \xrightarrow{f_X\circ-} \mathcal{C}(-, X)$ (resp. $\mathcal{C}(Y, -) \xrightarrow{-\circ f^X} \mathcal{C}(X, -)$) is an isomorphism on $\mathcal{C}'$.*

We omit the proof since it is similar to Proposition 3.12.

The following proposition is an analogue of Proposition 3.16.

Proposition 3.31 (Iyama [Iya03b, Theorem 3.2 (1)]). *Let A be an artin algebra and e an idempotent of A. Then $\mathsf{add}\, eA$ is a coreflective (resp. reflective) subcategory of $\mathsf{proj}\, A$ if and only if Ae (resp. eA) is a projective right (resp. left) eAe-module.*

Proof. Assume that $\mathsf{add}\, eA$ is a coreflective subcategory of $\mathsf{proj}\, A$. Then there exists a right $(\mathsf{add}\, eA)$-approximation $a \in \mathrm{Hom}_A(P, A)$ of A such that

$$\mathrm{Hom}_A(eA, P) \xrightarrow{a\circ -} \mathrm{Hom}_A(eA, A)$$

is an isomorphism. Thus we have an isomorphism $Ae \cong Pe \in \mathsf{add}(eAe)$ of right (eAe)-modules and we obtain $Ae \in \mathsf{proj}(eAe)$.

Conversely, we assume that Ae is projective as right (eAe)-modules. Then there exists $P \in \mathsf{add}\, eA$ as a right A-module such that $Pe \cong Ae$ as right (eAe)-modules. This is induced by a morphism $a : P \to A$ since $\mathrm{Hom}_A(P, A) = \mathrm{Hom}_{eAe}(Pe, Ae)$ (see [ARS95, Proposition 2.1 (a)]). Since

$$\mathrm{Hom}_A(eA, P) \xrightarrow{a\circ -} \mathrm{Hom}_A(eA, A)$$

is an isomorphism, $\mathsf{add}\, eA$ is coreflective in $\mathsf{proj}\, A$. □

Right (resp. left) rejective subcategories are coreflective (resp. reflective) subcategories, but the converse is not true as the following example shows.

Example 3.32. Let A be the preprojective algebra of type $\mathbb{A}_3$. It is defined by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3$$

with relations $\alpha_1\beta_1, \beta_1\alpha_1 - \alpha_2\beta_2$ and $\beta_2\alpha_2$. Then $A\varepsilon_3 A$ is not projective as a right A-module, but $A\varepsilon_3$ is projective as a right $\varepsilon_3 A\varepsilon_3$-module. Thus $\mathsf{add}\, \varepsilon_3 A$ is not a right rejective subcategory of $\mathsf{proj}\, A$ by Proposition 3.16, but a coreflective subcategory of $\mathsf{proj}\, A$ by Proposition 3.31.

We introduce the following analogue of Definition 3.19.

Definition 3.33. Let $\mathcal{C}$ be an additive category. We call a chain of subcategories

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$

a *coreflective* (*resp. reflective*) *chain* if $\mathcal{C}_i$ is a cosemisimple coreflective (resp. reflective) subcategory of $\mathcal{C}_{i-1}$ for $1 \leq i \leq n$.

Clearly right (resp. left) rejective chains are coreflective (resp. reflective) chains. The converse is not true as the following example shows.

Example 3.34. Let A be the preprojective algebra of type $\mathbb{A}_2$. It is defined by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with relations $\beta\alpha$ and $\alpha\beta$. Then

$$\mathsf{proj}\, A = \mathsf{add}\, A \supset \mathsf{add}\, \varepsilon_2 A \supset 0$$

is not a right rejective chain, but a coreflective chain of $\mathsf{proj}\, A$. In fact, the conditions (a) and (b) in Definition 3.33 follow from Proposition 3.31 and $\varepsilon_1 J(A)\varepsilon_1 = 0$ respectively. However the condition (a) in Definition 3.19 dose not hold by Proposition 3.16.

We are ready to state the following main result in this thesis.

Theorem 3.35. *In Theorem 3.23, the conditions* (i) *and* (ii) *are equivalent to the following condition.*

(iii) (3-7) *is a heredity chain of A and the following chain is a coreflective* (*resp. reflective*) *chain of* $\mathsf{proj}\,A$.

$$\mathsf{proj}\,A = \mathsf{add}\,e_0A \supset \cdots \supset \mathsf{add}\,e_{n-1}A \supset \mathsf{add}\,e_nA = 0.$$

To prove Theorem 3.35, we need the following lemma.

Lemma 3.36. *Let A be an artin algebra and $I' \subset I$ idempotent ideals of A. Let e and e' be idempotents of A such that $I = AeA$ and $I' = Ae'A$. We assume that I/I' is a projective right* (*resp. left*) *(A/I')-module and $\mathrm{Tor}_2^A(A/I, A/I') = 0$* (*resp.* $\mathrm{Tor}_2^A(A/I', A/I) = 0$). *If Ae* (*resp. eA*) *is a projective right* (*resp. left*) *(eAe)-module, then I is a projective right* (*resp. left*) *A-module.*

Proof. Let $0 \to K \to P \to I \to 0$ be a projective cover of the right A-module I. Then $P \in \mathsf{add}\,eA$ as a right A-module and $K \subset PJ(A)$ hold. Applying the functor $(-)e : \mathsf{mod}\,A \to \mathsf{mod}\,eAe$, we have a short exact sequence $0 \to Ke \to Pe \to Ie \to 0$. Since $Ie = Ae$ and Pe are projective (eAe)-modules and $Ke \subset PeJ(eAe)$, we have $Ke = 0$.

On the other hand, applying the functor $- \otimes_A (A/I')$ to the short exact sequence $0 \to K \to P \to I \to 0$, we have an exact sequence

$$\mathrm{Tor}_1^A(I, A/I') \to K/KI' \to P/PI' \to I/I' \to 0,$$

where $\mathrm{Tor}_1^A(I, A/I') = \mathrm{Tor}_2^A(A/I, A/I') = 0$ holds by our assumption. Since I/I' is a projective right (A/I')-module, the sequence splits, and hence K/KI' is a direct summand of P/PI'. On the other hand, $K \subset PJ(A)$ implies that $K/KI' \subset (P/PI')J(A)$. Thus $K/KI' = 0$ holds. Consequently, $K = KI' \subset KI = 0$ holds as desired. □

We are ready to prove Theorem 3.35.

Proof of Theorem 3.35. Since (ii) ⇒ (iii) clearly holds, it suffices for us to prove that (iii) ⇒ (i). We show this claim by induction on n. If $n = 1$, then the assertion holds since $H_0 = A$ is projective as a right A-module.

For $n \geq 2$ we proceed by induction. Let e_i denote the idempotent $e_i + H_{n-1}$ of A/H_{n-1} for $0 \leq i \leq n-2$. Firstly, we claim that

$$A/H_{n-1} > \cdots > (A/H_{n-1})e_i(A/H_{n-1}) > \cdots > H_{n-1}/H_{n-1} = 0$$

is a heredity chain of A/H_{n-1} such that $\mathsf{add}\,e_i(A/H_{n-1})$ is a coreflective subcategory of $\mathsf{proj}(A/H_{n-1})$ for $0 \leq i \leq n-2$. Since $(A/H_{n-1})e_i(A/H_{n-1}) = H_i/H_{n-1}$ for $0 \leq i \leq n-2$, the above chain is a heredity chain of A/H_{n-1}. Since $Ae_i \in \mathsf{proj}(e_iAe_i)$, we have that $Ae_i \otimes_{e_iAe_i} e_i(A/H_{n-1})e_i = (A/H_{n-1})e_i$ is projective as a right $(e_i(A/H_{n-1})e_i)$-module.

Therefore it follows from Proposition 3.31 that $\mathsf{add}\, e_i(A/H_{n-1})$ is a coreflective subcategory of $\mathsf{proj}(A/H_{n-1})$ for $0 \leq i \leq n-2$.

Now, we deduce from the induction hypothesis that H_i/H_{n-1} is a projective module as a right (A/H_{n-1})-module for $0 \leq i \leq n-2$. For any $0 \leq i \leq n-1$, we obtain from the hypothesis (iii) that $\mathsf{add}\, e_iA$ is a coreflective subcategory of $\mathsf{proj}\, A$, and hence Ae_i is a projective right (e_iAe_i)-module by Proposition 3.31. Thus we have idempotent ideals H_{n-1}, H_i such that H_{n-1} is a heredity ideal of A, H_i/H_{n-1} is projective as a right (A/H_{n-1})-module and Ae_i is a projective right (e_iAe_i)-module for $0 \leq i \leq n-1$. Moreover $\mathrm{Tor}_2^A(A/H_i, A/H_{n-1}) = \mathrm{Tor}_2^{A/H_{n-1}}(A/H_i, A/H_{n-1}) = 0$ holds since H_{n-1} is a heredity ideal of A. Therefore we deduce from Lemma 3.36 that H_i is a projective right A-modules. $\square$

4 Applications

4.1 Algebras of global dimension at most two are right-strongly quasi-hereditary

The aim of this subsection is to prove the following result.

Theorem 4.1. *Let A be an artin algebra such that* $\operatorname{gldim} A \leq 2$. *Then the following statements hold.*

(1) *A is a right-strongly quasi-hereditary algebra.*

(2) (Iyama [Iya03b, Theorem 3.6]). *The category $\mathsf{proj}\, A$ has a total right rejective chain*

$$\mathsf{proj}\, A = \mathsf{add}\, e_0 A \supset \cdots \supset \mathsf{add}\, e_{n-1} A \supset \mathsf{add}\, e_n A = 0.$$

Note that for an artin algebra A of global dimension at most two, we can similarly construct a total left rejective chain

$$\mathsf{proj}\, A = \mathsf{add}\, \epsilon_0 A \supset \cdots \supset \mathsf{add}\, \epsilon_{n-1} A \supset \mathsf{add}\, \epsilon_n A = 0.$$

Hence such an algebra is left-strongly quasi-hereditary. However it is not necessarily strongly quasi-hereditary.

We need the following preparation.

Lemma 4.2 (Iyama [Iya03b, Lemma 3.6.1])**.** *Let A be an artin algebra with* $\operatorname{gldim} A = m$, *where $2 \leq m < \infty$. Then there exists simple right A-modules S and S' such that the projective dimensions of S and S' are $m-1$ and m respectively.*

Proof. Existence of S' is clear since $\operatorname{gldim} A$ is supremum of the projective dimensions of simple A-modules. Let $0 \to X \to P \to S' \to 0$ be an exact sequence with a projective A-module P. Then the projective dimension of X is precisely $m-1$. We assume that X is not simple. Then there exists a proper simple submodule L of X. Consider the short exact sequence

$$0 \to L \to X \to X/L \to 0.$$

Since the projective dimension of X is $m-1$ and $\operatorname{gldim} A = m$, the projective dimension of L is at most $m-1$. We assume that the projective dimension of L is strictly less than $m-1$. Then the projective dimension of X/L is precisely $m-1$. Therefore we obtain the assertion by replacing X by X/L and repeating this argument. □

We are ready to prove the main theorem in this subsection.

Proof of Theorem 4.1. (2) We show by induction on the number of simple modules. We may assume that A is basic. Let n be the number of simple A-modules.

Assume that $n = 1$. Since A is simple, the assertion holds.

For $n \geq 2$ we proceed by induction. If A is semisimple, then the assertion is obvious. Thus we assume that A is non-semisimple. It follows from Lemma 4.2 that there exists a simple A-module S such that the projective dimension of S is precisely one since $\operatorname{gldim} A = 1$ or $\operatorname{gldim} A = 2$. Let f be a primitive idempotent of A such that $S = f(A/J(A))$. Let $e := 1 - f$ and $A' := eAe$.

(i) We claim that $\mathsf{add}\, eA$ is a cosemisimple right subcategory of $\mathsf{proj}\, A$ and $\operatorname{gldim} A' \leq \operatorname{gldim} A \leq 2$. There exists a short exact sequence

$$0 \to fJ(A) \xrightarrow{\varphi} fA \to S \to 0.$$

Since the projective dimension of S is one, we have $fJ(A) \in \mathsf{proj}\, A$. Since fA is not an indecomposable direct summand of $fJ(A)$, we have $fJ(A) \in \mathsf{add}\, eA$ as a right A-module. It follows from Proposition 3.28 that $\mathsf{add}\, eA$ is a cosemisimple right rejective subcategory of $\mathsf{proj}\, A$. Thus A/AeA is simple. Since $\mathsf{add}\, eA$ is a right rejective subcategory of $\mathsf{proj}\, A$, it follows from Proposition 3.16 that $\operatorname{gldim} A' \leq \operatorname{gldim} A \leq 2$.

(ii) We claim that any monomorphism in $\mathsf{add}\, eA$ is monic in $\mathsf{proj}\, A$. Let $a : P_1 \to P_0$ be a monomorphism in $\mathsf{add}\, eA$. Then we have an exact sequence

$$0 \to \operatorname{Ker} a \to P_1 \xrightarrow{a} P_0 \to \operatorname{Cok} a \to 0$$

in $\mathsf{mod}\, A$. Since $\operatorname{gldim} A \leq 2$, we obtain that $P_2 := \operatorname{Ker} a \in \mathsf{proj}\, A$. Since a is a monomorphism in $\mathsf{add}\, eA$, we have $P_2 e = \operatorname{Hom}_A(eA, P_2) = 0$. This implies that P_2 is a module over a simple algebra A/AeA. Thus we obtain that P_2 is isomorphic to S^l for some $l \geq 0$. If $l > 0$, then S is projective as a right A-module. This is a contradiction since the projective dimension of S is one. Therefore we have $l = 0$ and $P_2 = 0$. Thus a is a monomorphism of A-modules, and hence the assertion follows.

(iii) We claim that any right rejective subcategory $\mathcal{C}$ of $\mathsf{add}\, eA$ is also right rejective in $\mathsf{proj}\, A$. In fact, $A = eA \oplus fA$ and $\mathsf{add}\, eA$ has a right $\mathcal{C}$-approximation which is monic in $\mathsf{add}\, eA$ and hence it is also monic in $\mathsf{proj}\, A$ by (ii). Similarly, composing a right $\mathcal{C}$-approximation of $fJ(A) \in \mathsf{add}\, eA$ and $\varphi : fJ(A) \hookrightarrow fA$, we have a right $\mathcal{C}$-approximation of fA which is monic in $\mathsf{add}\, eA$, and hence in $\mathsf{proj}\, A$ by (ii).

(iv) We complete the proof by induction on the number of simple A-modules. By induction hypothesis, $\mathsf{proj}\, A' \simeq \mathsf{add}\, eA$ has a total right rejective chain.

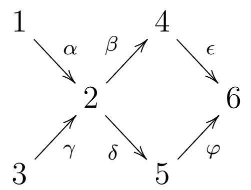
$$\mathsf{proj}\, A' \simeq \mathsf{add}\, eA \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_{n-1} \supset \mathcal{C}_n = 0.$$

Composing it with $\mathsf{proj}\, A \supset \mathsf{add}\, eA$, and applying (iii), we have a total right rejective chain of $\mathsf{proj}\, A$.

(1) The assertion follows from (2) and Theorem 3.23. □

If A is a strongly quasi-hereditary algebra, then the global dimension of A is at most two [Rin10]. The converse is not true as the following example shows.

Example 4.3. Let Q be the quiver $1 \leftarrow 2 \rightarrow 3$ whose underlying graph is the Dynkin graph $\mathbb{A}_3$ and A the Auslander algebra of $\mathbf{k}Q$. Then A is defined by the quiver



with relations $\alpha\beta$, $\gamma\delta$ and $\beta\epsilon - \delta\varphi$. The global dimension of A is two. However we can not construct a strongly heredity chain of A. This example can be explained by Theorem 4.6.

We end this subsection with describing a certain class of artin algebras which is called Ringel self-dual. We recall the following result.

Definition-Theorem 4.4 (Ringel [Rin91, Theorem 5]). *Let A be a quasi-hereditary algebra and a partially ordered set $I := \{1 < \cdots < n\}$. Then there exist the indecomposable A-modules $T(1), T(2), \dots, T(n)$ such that $T(i)$ is Ext-injective in $\mathcal{F}(\Delta)$ and the standard module $\Delta(i)$ is embedded to $T(i)$ with $T(i)/\Delta(i) \in \mathcal{F}(\Delta(j) \mid j < i)$. Let $T := \bigoplus_{i=1}^{n} T(i)$ and $R(A) := \mathrm{End}_A(T)^{\mathrm{op}}$. Then $R(A)$ is a quasi-hereditary algebra with respect to the opposite order of $\leq$. We call $R(A)$ a* Ringel dual *of A and T a characteristic tilting module.*

Let $(A, \leq_A)$ and $(B, \leq_B)$ be quasi-hereditary algebras with simple A-modules $\{S_A(i) \mid i \in I\}$ and simple B-modules $\{S_B(i') \mid i' \in I'\}$. We say that $(A, \leq_A)$ *is isomorphic to* $(B, \leq_B)$ *as a quasi-hereditary algebra* if there exists an algebra isomorphism $f : A \xrightarrow{\sim} B$ such that the induced map $\varphi : I \to I'$ is a poset isomorphism.

Let $(A, \leq)$ be a quasi-hereditary algebra and $\leq^{\mathrm{op}}$ the opposite order of $\leq$. We say that A is *Ringel self-dual* if $(A, \leq_A)$ is isomorphic to $(R(A), \leq^{\mathrm{op}})$ as a quasi-hereditary algebra.

Corollary 4.5. *Let A be a Ringel self-dual algebra. Then the following conditions are equivalent:*

(i) *A has global dimension at most two.*

(ii) *A is strongly quasi-hereditary.*

(iii) *A is right strongly quasi-hereditary.*

Proof. (ii) $\Rightarrow$ (i): This is shown in [Rin10].

(i) $\Rightarrow$ (iii): This follows from Theorem 4.1 immediately.

(iii) $\Rightarrow$ (ii): Let A be a right strongly quasi-hereditary algebra. Since the Ringel dual of a right-strongly quasi-hereditary algebra is left-strongly quasi-hereditary with the opposite order by [Rin10, Proposition A.2], A is strongly quasi-hereditary. □

4.2 Strongly quasi-hereditary Auslander algebras

In this subsection, we study strongly quasi-hereditary Auslander algebras. We start this subsection with recalling the definition of Auslander algebras. Let A be a representation-finite artin algebra and M a direct sum of all pairwise non-isomorphic indecomposable A-modules. Then the endomorphism algebra $B := \mathrm{End}_A(M)$ is called the Auslander algebra of A. It is well known that B is an Auslander algebra if and only if $\mathrm{gldim}\, B \leq 2 \leq \mathrm{dom.dim} B$ [Aus71, Theorem in § 4].

Since the global dimensions of Auslander algebras are at most two, it follows from Theorem 4.1 that each Auslander algebra is right-strongly quasi-hereditary. However it is not necessarily true that each Auslander algebra is strongly quasi-hereditary. The aim of this subsection is to provide the following characterization of Auslander algebras which are strongly quasi-hereditary. Recall that an artin algebra A is a Nakayama algebra if and only if every indecomposable A-module is uniserial (see for example [ARS95, §4.2]).

Theorem 4.6. *Let A be a representation-finite artin algebra and B the Auslander algebra of A. Then the following conditions are equivalent.*

(i) *B is strongly quasi-hereditary.*

(ii) $\mathsf{proj}\, B$ *has a rejective chain.*

(iii) *A is a Nakayama algebra.*

To prove Theorem 4.6, we need the following observation.

Lemma 4.7. *Let A be an artin algebra and B a factor algebra of A such that* $\mathsf{mod}\, B$ *is a cosemisimple subcategory of* $\mathsf{mod}\, A$*. Then the following statements hold.*

(1) *Let X be an indecomposable A-module which does not belong to* $\mathsf{mod}\, B$*. Then X is a projective-injective A-module such that $XJ(A)$ is an indecomposable B-module.*

(2) *B is a Nakayama algebra if and only if A is a Nakayama algebra.*

Proof. (1) By Proposition 3.27, there exists a morphism $\varphi : Y \to X$ of A-modules such that $Y \in \mathsf{mod}\, B$ and $\mathrm{Hom}_A(-, Y) \xrightarrow{\varphi\circ -} \mathcal{J}_{\mathsf{mod}\, A}(-, X)$ is an isomorphism on $\mathsf{mod}\, A$. Then φ is a minimal right almost split morphism of X in $\mathsf{mod}\, A$. If X is not a projective A-module, then φ is surjective and hence $X \in \mathsf{mod}\, B$, a contradiction. Therefore X is a projective A-module, and φ is an inclusion map $XJ(A) \to X$. Thus $XJ(A) = Y$ is a B-module. The dual argument shows that X is an injective A-module, and hence $XJ(A)$ is indecomposable.

(2) Since the "if" part is obvious, we prove the "only if" part. Let M be an indecomposable A-module which is either projective or injective. We show that M is a uniserial A-module. If M is a B-module, then this is clear. Assume that M is not a B-module. By (1), M is a projective-injective A-module such that $MJ(A)$ is an indecomposable B-module. Since B is a Nakayama algebra, $MJ(A)$ is uniserial. Hence M is also uniserial. □

We are ready to show the main theorem in this subsection.

Proof of Theorem 4.6. It suffices to show from Theorem 3.23 that (ii) is equivalent to (iii).

(ii) $\Rightarrow$ (iii): We show by induction on the length $l(A)$ of A as a right A-module. If $l(A) = 1$, then this is clear. For $l(A) \geq 2$ we proceed by induction. Since B is a strongly quasi-hereditary algebra, it follows from Theorem 3.23 that $\mathsf{proj}\, B \simeq \mathsf{mod}\, A$ has a rejective chain

$$\mathsf{mod}\, A \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n \supset 0.$$

Since $\mathcal{C}_1$ is a rejective subcategory of $\mathsf{mod}\, A$, there exists a two-sided ideal I of A such that $\mathcal{C}_1 = \mathsf{mod}(A/I)$ by Proposition 3.14. It follows from the induction hypothesis that A/I is a Nakayama algebra. Therefore we obtain from Lemma 4.7 (2) that A is also a Nakayama algebra.

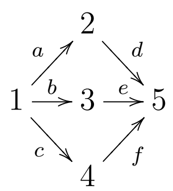
(iii) $\Rightarrow$ (ii): We show by induction on $l(A)$. If $l(A) = 1$, then the assertion holds. For $l(A) \geq 2$, we prove that $\mathsf{mod}\, A$ has a rejective chain by induction. Since A is a Nakayama algebra, there exists an indecomposable projective-injective A-module P. Let M be a direct sum of all indecomposable A-modules which are not isomorphic to P and $\mathcal{C}_1 := \mathsf{add}\, M$. Then $\mathcal{C}_1$ is closed under factor modules and submodules. It follows from Proposition 3.14 that there exists a two-sided ideal I of A such that $\mathcal{C}_1 = \mathsf{mod}(A/I)$. On the other hand, we have $\mathsf{ind}(\mathsf{mod}\, A) \setminus \mathsf{ind}(\mathcal{C}_1) = \{P\}$. Since the inclusion map $\varphi : PJ(A) \to P$ gives an isomorphism $\mathrm{Hom}_A(-, PJ(A)) \xrightarrow{\varphi\circ -} \mathcal{J}_{\mathsf{mod}\, A}(-, P)$ on $\mathsf{mod}\, A$, we obtain from Proposition 3.27 that $\mathsf{mod}\, A/I$ is a cosemisimple right rejective subcategory of $\mathsf{mod}\, A$. Since $l(A) > l(A/I)$, we obtain from the induction hypothesis that there exists a rejective chain $\mathsf{mod}(A/I) = \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_i \supset \cdots \supset \mathcal{C}_n = 0$ of $\mathsf{mod}(A/I)$. Composing with $\mathsf{mod}\, A \supset \mathsf{mod}(A/I)$, we have a rejective chain of $\mathsf{mod}\, A$

$$\mathsf{mod}\, A \supset \mathcal{C}_1 = \mathsf{mod}(A/I) \supset \cdots \supset \mathcal{C}_n \supset 0.$$

The proof is complete. □

We end this subsection with providing a strongly quasi-hereditary algebra of dominant dimension less than two, that is, it is not an Auslander algebra.

Example 4.8. Let A be the **k**-algebra defined by the quiver



with relations $ad + be + cf$. We can easily check that the global dimension of A is two and the dominant dimension of A is less than two since A dose not have a projective-injective module. On the other hand, A is a strongly quasi-hereditary algebra with respect to $\{2 < 1 < 3 < 4 < 5\}$.

4.3 Strongly quasi-hereditary ADR algebras

Let A be an artin algebra and $m > 0$ the smallest integer such that $J(A)^m = 0$. In [Aus71], Auslander studied the endomorphism algebra $\widetilde{B} := \mathrm{End}_A(\bigoplus_{k=0}^{m} A/J(A)^k)$ and proved that $\widetilde{B}$ has finite global dimension. Furthermore, Dlab and Ringel showed that $\widetilde{B}$ is a quasi-hereditary algebra [DR89b]. Hence $\widetilde{B}$ is called an *Auslander–Dlab–Ringel algebra* (for short ADR algebra). Recently, Conde introduced the notion of left ultra strongly quasi-hereditary algebras as a special class of left-strongly quasi-hereditary algebras and showed all ADR algebras are left ultra strongly quasi-hereditary algebras [Con16].

In this subsection, we study a relationship between ADR algebras and strongly quasi-hereditary algebras. Since the ADR algebra of a Nakayama algebra coincides with the Auslander algebra of the Nakayama algebra, it is strongly quasi-hereditary algebra by Theorem 4.6. However, ADR algebras are not necessarily strongly quasi-hereditary algebras.

We start with recalling the quasi-hereditary structure of an ADR algebra. Let

$$G := \bigoplus_{i\in I}\bigoplus_{j=1}^{l_i} P(i)/P(i)J(A)^j,$$

where l_i is the Loewy length of the indecomposable projective A-module $P(i)$. Then the endomorphism algebra $B := \mathrm{End}_A(G)$ is a basic algebra of $\widetilde{B}$ and called also an *ADR algebra* of A. Note that the basic algebra of a quasi-hereditary algebra is also quasi-hereditary. Since the indecomposable projective B-modules are given by $P_{i,j} := \mathrm{Hom}_A(G, P(i)/P(i)J(A)^j)$ for $i \in I$ and $1 \le j \le l_i$, we denote by

$$\Lambda := \{(i,j) \mid i \in I,\ 1 \le j \le l_i\}$$

a label of simple B-modules. For $(i,j),(k,l) \in \Lambda$, we write $(i,j) \trianglelefteq (k,l)$ if $j > l$. Then $\trianglelefteq$ gives a partial order on Λ. We call the partial order $\trianglelefteq$ the ADR order of B.

Proposition 4.9. (Conde [Con16]) *Let A be an artin algebra and B the ADR algebra of A. Then $(B, \trianglelefteq)$ is a left-strongly quasi-hereditary algebra.*

We can also prove Proposition 4.9 by the following two ways.

(1) Let A be an artin algebra and let $A = I_0 > I_1 > \cdots > I_m = 0$ be a chain of two-sided ideals such that $J(A)I_n \subset I_{n+1}$ for any n. Let $G_n := \bigoplus_{i=1}^{n} A/I_i$ for $1 \le n \le m$ and $B := \mathrm{End}_A(G_m)$. Then

$$\mathsf{proj}\, B \simeq \mathsf{add}\, G_m \supset \cdots \supset \mathsf{add}\, G_2 \supset \mathsf{add}\, G_1 \supset 0$$

is a total left rejective chain by [Iya03b, Example 2.2.3]. Since we can apply this result to the radical series of A, the ADR algebra $(B, \trianglelefteq)$ is left-strongly quasi-hereditary by Theorem 3.23.

(2) Since A is a semilocal module, B is the ADR algebra in the sense of § 4.4. Hence Proposition 4.9 follows from Theorem 4.16.

First we consider the case where A is a self-injective algebra. Then we obtain the following characterization of ADR algebras to be strongly quasi-hereditary.

Proposition 4.10. *Let A be a self-injective algebra and B the ADR algebra of A. Then the following statements are equivalent.*

(i) *B is a strongly quasi-hereditary algebra.*

(ii) *A is a Nakayama algebra.*

Proof. (i) $\Rightarrow$ (ii): Since A is a self-injective algebra, we have $A \simeq DA \in \operatorname{add} G$. By [Mül68, Theorem 2], the dominant dimension of B is at least two. On the other hand, the global dimension of B is at most two since B is a strongly quasi-hereditary algebra. Hence B is a strongly quasi-hereditary Auslander algebra. By Theorem 4.6, there exists a Nakayama algebra A' such that B is the Auslander algebra of A'. By the Morita-Tachikawa correspondence (see for example [FK11]), we have A is Morita equivalent to A'. Hence A is a Nakayama algebra.

(ii) $\Rightarrow$ (i): Since B coincides with the Auslander algebra of A, the assertion follows from Theorem 4.6. ☐

Even if A is neither self-injective nor Nakayama, the ADR algebra of A can be a strongly quasi-hereditary algebra as the following example shows.

Example 4.11. Let A be the **k**-algebra defined by the quiver

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2$$

with relations $\alpha\beta$ and α^2. Then the ADR algebra B of A is given by the quiver

$$\begin{array}{ccc} P(1) & & \\ {\scriptstyle a}\uparrow\downarrow{\scriptstyle b} & \searrow^{c} & \\ S(1) & & S(2) \end{array}$$

with relations ab and ac. We can check that B is strongly quasi-hereditary with respect to $\{S(1) > P(1) < S(2)\}$.

The following theorem is our main result in this subsection.

Theorem 4.12. *Let A be an artin algebra with Loewy length $m \geq 2$ and $G_n := \bigoplus_{i=1}^{n} A/J(A)^i$ for any $1 \leq n \leq m$. Let $B := \operatorname{End}_A(G)$ be the ADR algebra of A. Then the following statements are equivalent.*

(i) $(B, \trianglelefteq)$ *is a strongly quasi-hereditary algebra.*

(ii) *The following chain is a rejective chain of* $\mathsf{add}\, G$.

$$\mathsf{add}\, G \supset \mathsf{add}\, G_{m-1} \supset \cdots \supset \mathsf{add}\, G_1 \supset 0.$$

(iii) $\operatorname{gldim} B = 2$.

(iv) *For any* $i \in I$, $P(i)J(A) \in \mathsf{add}\, G$.

To prove Theorem 4.12, we need the following lemma.

Lemma 4.13. *Let* A *be an artin algebra. If* $P(i)J(A) \in \mathsf{add}\, G$ *for any* $i \in I$, *then* $P(i)J(A)/P(i)J(A)^j \in \mathsf{add}\, G$ *for* $1 \leq j \leq l_i$.

Proof. Since $P(i)J(A) \in \mathsf{add}\, G$, we have $P(i)J(A) \cong \bigoplus_{k,l} P(k)/P(k)J(A)^l$. For simplicity, we write $P(i)J(A) \cong P(k)/P(k)J(A)^l$. Then we have $P(i)J(A)/P(i)J(A)^j \cong (P(k)/P(k)J(A)^l)/(P(k)J(A)^j/P(k)J(A)^l) \cong P(k)/P(k)J(A)^j \in \mathsf{add}\, G$. □

Now we are ready to prove Theorem 4.12.

Proof of Theorem 4.12. (i) ⇔ (ii): The assertion follows from Corollary 3.25.

(i) ⇒ (iii): It follows from [Rin10, Proposition A.2] that the global dimension of B is at most two. It is enough to show that there exists a B-module such that its projective dimension is two. Let S be a simple A-module. Then we have the following short exact sequence.

$$0 \to \mathcal{J}_{\mathsf{mod}\, A}(G, S) \to \operatorname{Hom}_A(G, S) \to \mathsf{top}\operatorname{Hom}_A(G, S) \to 0.$$

Assume that $\mathcal{J}_{\mathsf{mod}\, A}(G, S)$ is a projective right B-module. Then there exists an A-module $Y \in \mathsf{add}\, G$ such that $\mathcal{J}_{\mathsf{mod}\, A}(G, S) \cong \operatorname{Hom}_A(G, Y)$. By $S \in \mathsf{add}\, G$, there exists a non-zero morphism $f : Y \to S$ such that $\operatorname{Hom}_A(G, f) : \operatorname{Hom}_A(G, Y) \to \operatorname{Hom}_A(G, S)$ is an injective map. Since the functor $\operatorname{Hom}_A(G, -)$ is faithful, f is an injective map. Hence f is an isomorphism. This is a contradiction since $\mathcal{J}_{\mathsf{mod}\, A}(G, S) \cong \operatorname{Hom}_A(G, S)$. Therefore we obtain the assertion.

(iii) ⇔ (iv): This follows from [Sma78, Proposition 2].

(iv) ⇒ (ii): First we show that $\mathsf{add}\, G_{m-1}$ is a cosemisimple rejective subcategory of $\mathsf{add}\, G$. For any $X \in \mathsf{ind}(\mathsf{add}\, G) \setminus \mathsf{ind}(\mathsf{add}\, G_{m-1})$, there exists an inclusion map $\varphi : XJ(A) \hookrightarrow X$ with $XJ(A) \in \mathsf{add}\, G_{m-1}$ by the condition (iv). Since X is a projective A-module such that its Loewy length coincides with the Loewy length of A, φ induces an isomorphism

$$\operatorname{Hom}_A(G, XJ(A)) \xrightarrow{\varphi \circ -} \mathcal{J}_{\mathsf{mod}\, A}(G, X).$$

It follows from Proposition 3.27 that $\mathsf{add}\, G_{m-1}$ is a cosemisimple right rejective subcategory of $\mathsf{add}\, G$. Moreover, there exists canonical surjection $\psi : X \twoheadrightarrow X/XJ(A)^{m-1}$

with $X/XJ(A)^{m-1} \in \mathsf{add}\, G_{m-1}$. Since we have $f(XJ(A)^{m-1}) \subset J(A)^m = 0$ for any $f \in \mathsf{rad}_A(X, G)$, the map ψ induces the following isomorphism.

$$\mathrm{Hom}_A(X/XJ(A)^{m-1}, G) \xrightarrow{-\circ\psi} \mathcal{J}_{\mathsf{mod}\, A}(X, G).$$

Hence we obtain that $\mathsf{add}\, G_{m-1}$ is a cosemisimple left rejective subcategory of $\mathsf{add}\, G$ by Proposition 3.27.

Next we prove that $\mathsf{add}\, G$ has a rejective chain

$$\mathsf{add}\, G \supset \mathsf{add}\, G_{m-1} \supset \cdots \supset \mathsf{add}\, G_1 \supset 0$$

by induction on m. If $m = 2$, then the assertion holds. Assume that $m > 2$. Let $X \in \mathsf{ind}(\mathsf{add}\, G_{m-1}) \setminus \mathsf{ind}(\mathsf{add}\, G_{m-2})$. Then $X = P(i)/P(i)J(A)^{m-1}$ for some $i \in I$ and we have

$$(P(i)/P(i)J(A)^{m-1})J(A/J^{m-1}(A)) \cong P(i)J(A)/P(i)J(A)^{m-1}.$$

Since $P(i)J(A) \in \mathsf{add}\, G$, we obtain $P(i)J(A)/P(i)J(A)^{m-1} \in \mathsf{add}\, G_{m-1}$ by Lemma 4.13. By induction hypothesis, $\mathsf{add}\, G_{m-1}$ has the following rejective chain.

$$\mathsf{add}\, G_{m-1} \supset \cdots \supset \mathsf{add}\, G_1 \supset 0.$$

Composing it with $\mathsf{add}\, G \supset \mathsf{add}\, G_{m-1}$, we obtain a rejective chain of $\mathsf{add}\, G$. □

Remark 4.14. Since the global dimension of any strongly quasi-hereditary algebra is at most two, (iii) ⇒ (i) of Theorem 4.12 implies that if the ADR algebra B is strongly quasi-hereditary, then B is strongly quasi-hereditary with respect to the ADR order.

4.4 ADR algebras of semilocal modules

In this subsection, we consider a generalization of ADR algebras in § 4.3. This generalization was introduced by Lin and Xi [LX93].

Definition 4.15. Let M be a right A-module.

(1) M is called a *local* module if $\mathsf{top}\, M$ is isomorphic to a simple A-module.

(2) M is called a *semilocal* module if M is a direct sum of local modules.

Note that any local module is indecomposable.

Throughout this subsection, suppose that M is a semilocal module with Loewy length $\ell\ell(M) = m$. We denote by $\widetilde{M}$ the basic module of $\oplus_{i=1}^m M/MJ(A)^i$. We call $B := \mathrm{End}_A(\widetilde{M})$ the ADR algebra of M. Note that in the case $M = A_A$, the ADR algebra of A is the original ADR algebra in § 4.3.

In [LX93], they proved that the ADR algebras of semilocal modules are quasi-hereditary. In this subsection, we sharpen this result. Namely, we prove the following theorem.

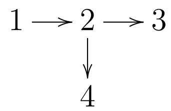
Theorem 4.16. *The ADR algebra of any semilocal module is left-strongly quasi-hereditary.*

An advantage of our theorem is to give a better upper bound on global dimension of the ADR algebras (see Remark 4.23).

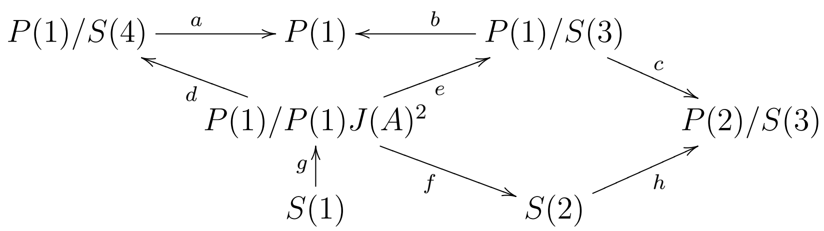
In the rest of this subsection, we give a proof of Theorem 4.16. We define the following sets of the isomorphism classes of indecomposable direct summands of $\widetilde{M}$. Let F be the set of pairwise non-isomorphic indecomposable direct summands of $\widetilde{M}$ and F_i the subset of F consisting of all modules with Loewy length $m-i$. We denote by $\mathsf{F}_{i,1}$ the subset of F_i consisting of all modules X which do not have a surjective map in $\mathcal{J}_{\mathsf{mod}\,A}(X,N)$ for all modules N in F_i. For any integer $j>1$, we inductively define the subsets $\mathsf{F}_{i,j}$ of F_i as follows: $\mathsf{F}_{i,j}$ consists of all modules $X \in \mathsf{F}_i \setminus \bigcup_{1\le k\le j-1} \mathsf{F}_{i,k}$ which do not have a surjective map in $\mathcal{J}_{\mathsf{mod}\,A}(X,N)$ for all modules $N \in \mathsf{F}_i \setminus \bigcup_{1\le k\le j-1} \mathsf{F}_{i,k}$. We set $n_i := \min\{j \mid \mathsf{F}_i = \bigcup_{1\le k\le j} \mathsf{F}_{i,k}\}$ and $n_M := \sum_{i=0}^{m-1} n_i$.

These subsets $\mathsf{F}_{i,j}$ give a partial order on the isomorphism classes of simple B-modules. In fact, $\{\mathsf{F}_{0,1} < \cdots < \mathsf{F}_{0,n_0} < \mathsf{F}_{1,1} < \cdots < \mathsf{F}_{m-1,n_{m-1}}\}$ is a partial order on the isomorphism classes of simple B-modules. If $M = A_A$, then $n_i = 1$ for all $0 \le i \le m-1$ and such a partial order coincides with the ADR order in § 4.3. However, the ADR algebra of a semilocal module is not necessarily left-strongly quasi-hereditary with respect to the ADR order.

Example 4.17. Let A be the $\mathbf{k}$-algebra defined by the quiver



and $M := P(1) \oplus P(1)/S(3) \oplus P(1)/S(4) \oplus P(2)/S(3)$. Then M is a semilocal module. The ADR algebra B of M is given by the quiver



with relations $da-eb, ec-fh$ and gf. Then $\mathsf{F}_{0,1} = \{P(1)/S(4), P(1)/S(3)\}$, $\mathsf{F}_{0,2} = \{P(1)\}$, $\mathsf{F}_{1,1} = \{P(1)/P(1)J(A)^2, P(2)/S(3)\}$, $\mathsf{F}_{2,1} = \{S(1), S(2)\}$. We can check that B is left-strongly quasi-hereditary with respect to $\{\mathsf{F}_{0,1} < \mathsf{F}_{0,2} < \mathsf{F}_{1,1} < \mathsf{F}_{2,1}\}$. However we can also check that B is not left-strongly quasi-hereditary with respect to the ADR order $\{P(1)/S(3), P(1)/S(4), P(1) < P(1)/P(1)J(A)^2, P(2)/S(3) < S(1), S(2)\}$.

We introduce a special right (resp. left) rejective chain which plays crucial role in the proof of Theorem 4.16.

Definition 4.18 (Iyama [Iya03b, Definition 2.2])**.** Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\, A$ and

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0 \tag{4-8}$$

a chain of subcategories. We call (4-8) an *A-total right* (resp. *left*) *rejective chain* if the following conditions hold for $1 \leq i \leq n$:

(a) for any $X \in \mathcal{C}_{i-1}$, there exists a monic (resp. epic) in $\mathsf{mod}\, A$ right (resp. left) $\mathcal{C}_i$-approximation of X;

(b) $\mathcal{C}_i$ is a cosemisimple subcategory of $\mathcal{C}_{i-1}$.

Remark 4.19. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\, A$. All A-total right (resp. left) rejective chains of $\mathcal{C}$ are total right (resp. left) rejective chains. If $A \in \mathcal{C}$, then the converse holds.

Recall that if $\mathsf{proj}\, A$ has a total left rejective chain, then A is left-strongly quasi-hereditary by Theorem 3.23. To prove Theorem 4.16, it is enough to show that $\mathsf{proj}\, A$ has a total left rejective chain. For $0 \leq i \leq m-1$ and $1 \leq j \leq n_i$, we set

$$\mathsf{F}_{>(i,j)} := \mathsf{F} \setminus ((\cup_{-1 \leq k \leq i-1} \mathsf{F}_k) \cup (\cup_{1 \leq l \leq j} \mathsf{F}_{i,l})),$$
$$\mathcal{C}_{i,j} := \mathsf{add} \bigoplus_{N \in \mathsf{F}_{>(i,j)}} N,$$

where $\mathsf{F}_{-1} := \emptyset$. Then we have the following result.

Proposition 4.20. *Let A be an artin algebra and M a semilocal A-module. Then* $\mathsf{add}\, \widetilde{M}$ *has the following A-total left rejective chain with length n_M.*

$$\mathsf{add}\, \widetilde{M} =: \mathcal{C}_{0,0} \supset \mathcal{C}_{0,1} \supset \cdots \supset \mathcal{C}_{0,n_0} \supset \mathcal{C}_{1,1} \supset \cdots \supset \mathcal{C}_{m-1,n_{m-1}} = 0.$$

Before proving Proposition 4.20, we give a proof of Theorem 4.16.

Proof of Theorem 4.16. By Corollary 3.25, it is enough to show that $\mathsf{add}\, \widetilde{M}$ has a total left rejective chain. Hence the assertion follows from Proposition 4.20. □

To show Proposition 4.20, we need the following lemma.

Lemma 4.21. *For any $M' \in \mathsf{F}_{0,1}$, the canonical surjection $\rho : M' \twoheadrightarrow M'/M'J(A)^{m-1}$ induces an isomorphism.*

$$\varphi : \mathrm{Hom}_A(M'/M'J(A)^{m-1}, \widetilde{M}) \xrightarrow{-\circ \rho} \mathcal{J}_{\mathsf{mod}\, A}(M', \widetilde{M}).$$

Proof. Since φ is a well-defined injective map, we show that φ is surjective. Let N be an indecomposable summand of $\widetilde{M}$ with Loewy length k and let $f : M' \to N$ be any morphism in $\mathcal{J}_{\mathsf{mod}\, A}(M', N)$. Then we show $f(M'J(A)^{m-1}) = 0$.

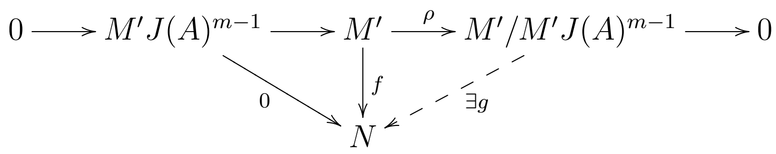
(i) Assume that $\mathsf{top}\, M' \not\cong \mathsf{top}\, N$ or $k = m$. Then we have $\mathrm{Im}\, f \subset NJ(A)$, and hence

$$f(M'J(A)^{m-1}) = f(M')J(A)^{m-1} \subset (NJ(A))J(A)^{m-1} = 0.$$

(ii) Assume that $\mathsf{top}\, M' \cong \mathsf{top}\, N$ and $k < m$. Since $m - k > 0$ holds, we obtain

$$f(M'J(A)^{m-1}) = f(M')J(A)^{m-1} \subset NJ(A)^{m-1} = (NJ(A)^k)J(A)^{m-k-1} = 0.$$

Since $f(M'J(A)^{m-1}) = 0$ holds, there exists $g : M'/M'J(A)^{m-1} \to N$ such that $f = g \circ \rho$.



Hence the assertion follows. $\square$

Now we are ready to prove Proposition 4.20.

Proof of Proposition 4.20. We show by induction on n_M. If $n_M = 1$, then this is clear. Assume that $n_M > 1$. By Proposition 3.27 and Lemma 4.21, $\mathcal{C}_{0,1}$ is a cosemisimple left rejective subcategory of $\mathsf{add}\, \widetilde{M}$. Since $N := M/(\oplus_{X \in \mathsf{F}_{0,1}} X) \oplus (\oplus_{X \in \mathsf{F}_{0,1}} XJ(A)^{m-1})$ is a semilocal module satisfying $\widetilde{N} = \widetilde{M}/\oplus_{X \in \mathsf{F}_{0,1}} X$ and $n_N < n_M$, we obtain that

$$\mathsf{add}\, \widetilde{N} = \mathcal{C}_{0,1} \supset \cdots \supset \mathcal{C}_{0,n_0} \supset \mathcal{C}_{1,1} \supset \cdots \supset \mathcal{C}_{m-1,n_{m-1}} = 0$$

is an A-total left rejective chain by induction hypothesis. By composing $\mathcal{C}_{0,0} \supset \mathcal{C}_{0,1}$ and it, we have the desired A-total left rejective chain. $\square$

As an application, we give an upper bound on global dimension on ADR algebras.

Corollary 4.22. *Let A be an artin algebra and M a semilocal A-module. Then we have*

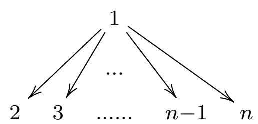
$$\operatorname{gldim} \operatorname{End}_A(\widetilde{M}) \leq n_M.$$

Proof. By Proposition 4.20, $\mathsf{proj}\, B \simeq \mathsf{add}\, \widetilde{M}$ has a total left rejective chain with length n_M. Hence the assertion follows from Proposition 3.22. $\square$

Remark 4.23. In [LX93], they showed that the ADR algebra of a semilocal module M is quasi-hereditary. This implies $\operatorname{gldim} \operatorname{End}_A(\widetilde{M}) \leq 2(n_M - 1)$ by [DR89c, Statement 9]. By Corollary 4.22, we obtain better upper bound on global dimension of ADR algebras.

The following example shows us that the bound on the global dimension in Corollary 4.22 are tightly.

Example 4.24. Let $n \geq 2$. Let A be the $\mathbf{k}$-algebra defined by the quiver



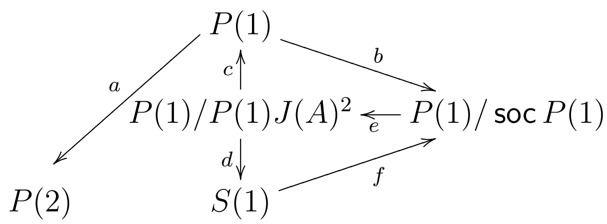
and M a direct sum of all factor modules of $P(1)$. Clearly M is semilocal. Then $\operatorname{gldim} B = n - 1 = n_M - 1$ for $n \geq 3$. In the case $n = 2$, $\operatorname{gldim} B = 2 = n_M$.

By Remark 4.14, if B is the ADR algebra of A, then its strongly quasi-hereditary structure can be realized by the ADR order. However, for any semilocal module, the assertion does not necessarily hold as the following example shows.

Example 4.25. Let A be the $\mathbf{k}$-algebra defined by the quiver

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2$$

with relations $\alpha\beta$ and α^3. Then $M := P(1) \oplus P(1)/\operatorname{soc} P(1) \oplus P(2)$ is a semilocal module. The ADR algebra B of M is given by the quiver



with relations eca, fed and $cb - df$. Then B is not strongly quasi-hereditary with respect to $\{\mathsf{F}_{0,1} < \mathsf{F}_{1,1} < \mathsf{F}_{1,2} < \mathsf{F}_{2,1}\}$. However B is strongly quasi-hereditary with respect to $\{P(1) < P(1)/P(1)J(A)^2 < P(1)/\operatorname{soc} P(1) < P(2), S(1)\}$.

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