Minimal PF submanifolds in Hilbert spaces with symmetries

(ヒルベルト空間内の対称性をもつ極小固有フレドホルム部分多様体)

理学研究科
数物系専攻

令和元年度

Masahiro Morimoto
(森本真弘)

Contents

Acknoledgements

I would like to express my gratitude to my supervisor Professor Yoshihiro Ohnita for helpful discussions, many invaluable suggestions and his encouragements over a long period of time. I am also grateful to Professors Takashi Sakai and Hiroshi Tamaru for useful discussions and valuable suggestions, and to Professors Mikiya Masuda and Futoshi Takahashi for their interests in my work and useful comments. Special thanks are also due to Professors Ernst Heintze and Jost-Hinrich Eschenburg for many discussions and invaluable comments, and to Professor Peter Quast for his kind hospitality during my visit to the University of Augsburg.

I was partly supported by the Grant-in-Aid for JSPS Research Fellow (No.18J14857) and by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics). I am also grateful for this support.

Introduction

In mathematics, submanifolds in Euclidean spaces are fundamental objects and have been studied by many researchers. From the view point of extrinsic differential geometry there are two important of generalizations of them. One generalization is given by submanifolds in Riemannian manifolds. Here Riemannian manifolds are always assumed to be of finite dimension throughout this thesis. The other generalization is given by submanifolds in Hilbert spaces which can be either of finite or infinite dimension. Submanifolds in Riemannian manifolds have been studied by many geometers since a long time ago. Submanifolds in Hilbert spaces were studied first by C.-L. Terng in 1980's and also by G. Thorbergsson, E. Heintze and other geometers. In both cases many interesting problems and profound results are known.

In this thesis we investigate submanifolds in Hilbert spaces which are proper Fredholm (PF), minimal, and having certain symmetries.

Submanifolds in Hilbert spaces

Strictly speaking, a submanifold in a Hilbert space is a Hilbert manifold smoothly immersed in a separable Hilbert space. Here a Hilbert manifold is defined as a smooth manifold locally modeled on a separable Hilbert space.

A motivation to study submanifolds in Hilbert spaces may come from a natural question asking differences or similarities between the finite and infinite dimensional cases. It is known that the standard theorems of differential calculus such as the inverse function theorem and the unique existence theorem for ordinary differential equations are still valid in the infinite dimensional case ([33]). A Riemannian metric g on a Hilbert manifold M is defined similarly to the finite dimensional case and a pair (M, g) is called a Riemannian Hilbert manifold. The Levi-Civita connection and geodesics for a Riemannian Hilbert manifold are defined just as in the finite dimensional case. However the Hopf-Rinow theorem does not hold in infinite dimensions ([1], [2]). A Hilbert manifold equipped with a smooth group structure is called a Hilbert Lie group, where subgroups and homomorphisms are defined similarly to the finite dimensional case. However despite the finite dimensional affirmative result the Hilbert's 5th problem is not true in infinite dimensions; an unusual example of a Hilbert manifold with a non-differentiable topological group structure appears in connection with affine Kac-Moody algebras. Also in this thesis a critical difference between finite and infinite dimensional submanifolds will become clear (Section 4.1).

A systematic study of submanifolds in Hilbert spaces was initiated by Terng [50]. The second fundamental form, the shape operator and the normal connection of a submanifold in a Hilbert space are defined just as in the finite dimensional case. However the spectral theory of the shape operator are complicated in general and infinite dimensional differential topology ([48]) and Morse theory ([40], [47]) can not be applied easily to submanifolds in Hilbert spaces without further restrictions. From this viewpoint she introduced a suitable class of submanifolds in Hilbert spaces, namely *proper Fredholm submanifolds* (shortly, PF submanifolds). Roughly speaking a PF submanifold is a submanifold in a Hilbert space where the shape operators are compact operators and the distance functions satisfy the Palais-Smale condition ([40], [47]). In [50] Terng gave examples of PF submanifolds which are orbits of the gauge transformations and studied isoparametric PF submanifolds. After that, the relation between isoparametric PF submanifolds and affine Kac-Moody algebras was discussed by her with coworkers ([50], [21]) and then a profound result on homogeneous property of isoparametric PF submanifolds was shown by Heintze and Liu [19]. These lead us to the concept of affine Kac-Moody symmetric spaces, which are the infinite dimensional analogue to finite dimensional Riemannian symmetric spaces ([16], [43], [10]).

In the study of submanifolds in Hilbert spaces an important tool is given by a certain Riemannian submersion $\Phi_K : V_{\mathfrak{g}} \to G/K$ which is called the *parallel transport map*. Here G/K is a compact normal homogeneous space and $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$ the Hilbert space of all L^2-paths with values in the Lie algebra $\mathfrak{g}$ of G (see Section 1.3). It is known ([52]) that if N is a closed submanifold of G/K then its inverse image $\Phi_K^{-1}(N)$ is a PF submanfold of $V_{\mathfrak{g}}$. Thus via the parallel transport map we can obtain many examples of PF submanifolds. Moreover the parallel transport map is also known as a tool to linearlize geometrical problems in G/K: we can reduce a non-linear problem in G/K into a linear problem in the Hilbert space $V_{\mathfrak{g}}$ (e.g. [52], [11], [9]). From these perspectives it is important to study the geometrical relation between N and $\Phi_K^{-1}(N)$. For example, it was shown ([29], [31], [20]) that if N is minimal then the PF submanifold $\Phi_K^{-1}(N)$ is also minimal. Here, for precise definitions of minimal PF submanifolds, see Section 1.4 and references therein.

Minimal submanifolds with symmetries: the Riemannian case

In Riemannian manifolds (always assumed to be finite dimensional) there are several kinds of minimal submanifolds with certain symmetries.

An *austere submanifold* is a minimal submanifold of a Riemannian manifold which has a local symmetry. More precisely a submanifold M immersed in a Riemannian manifold $\bar{M}$ is called *austere* if for each normal vector ξ the set of eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1). This notion was originally introduced by Harvey and Lawson [15] in the study of calibrated geometry. Except for the case of surfaces the austere condition is much stronger than the minimal one. It is an interesting problem to classify austere submanifolds under suitable conditions (e.g. [4], [7], [27], [26], [28]).

In [26] Ikawa Sakai and Tasaki introduced a certain kind of austere submanifold which has a global symmetry, which they call a *weakly reflective submanifold*. A submanifold M immersed in a Riemannian manifold $\bar{M}$ is called *weakly reflective* if for each normal vector ξ at each $p \in M$ there exists an isometry ν_ξ of $\bar{M}$ which satisfies

$$\nu_\xi(p) = p, \quad d\nu_\xi(\xi) = -\xi, \quad \nu_\xi(M) = M.$$

Here we call ν_ξ a reflection with respect to ξ. A reflective submanifold ([34]), defined as a connected component of the fixed point set of an involutive isometry on $\bar{M}$, is an example of a weakly reflective submanifold. Another example is a singular orbit of a cohomogeneity one action, which was essentially shown to be weakly reflective by Podestà [44]. It is an interesting problem to study submanifold geometry of orbits under isometric actions of Lie groups and to determine their weakly reflective orbits (e.g. [26], [39], [8]).

Recently Taketomi [49] introduced a generalized concept of weakly reflective submanifolds, namely *arid submanifolds*. A submanifold M immersed in a Riemannian manifold $\bar{M}$ is called *arid* if for each nonzero normal vector ξ at each $p \in M$ there exists an isometry φ_ξ of $\bar{M}$ which satisfies

$$\varphi_\xi(p) = p, \quad d\varphi_\xi(\xi) \neq \xi, \quad \varphi_\xi(M) = M.$$

Here we call φ_ξ an isometry with respect to ξ. From this definition we have:

$$\begin{array}{ccccccc} & & & & \text{austere} & & \\ \text{reflective} & \Rightarrow & \text{weakly reflective} & \begin{array}{c}\nearrow\\ \searrow\end{array} & & \begin{array}{c}\searrow\\ \nearrow\end{array} & \text{minimal.} \\ & & & & \text{arid} & & \end{array}$$

In [49] he gave an example of an arid submanifold which is *not* an austere submanifold (therefore *not* a weakly reflective submanifold). Also he showed that any isolated orbit of a proper isometric action is an arid submanifold. It is a problem for a given proper isometric action to determine its arid orbits which are not isolated.

Note that those submanifolds are defined and studied only in the finite dimensional Riemannian case.

The purpose and main results

The purpose of this thesis is:

(i) to define reflective PF submanifolds, weakly reflective PF submanifolds, austere PF submanifolds and arid PF submanifolds in Hilbert spaces,

(ii) to study the geometrical relations between a submanifold N of compact normal homogeneous space G/K and the PF submanifold $\Phi_K^{-1}(N)$ of the Hilbert space $V_\mathfrak{g}$, where $\Phi_K : V_\mathfrak{g} \to G/K$ denotes the parallel transport map, and

(iii) to obtain many examples of minimal PF submanifolds with symmetries.

In fact we define such PF submanifolds with symmetries just as in the finite dimensional case (Section 1.5). Then we show the following geometrical relations, which are main results of this thesis:

Theorem A. *Let G/K be a compact normal homogeneous space. Then each fiber of the parallel transport map $\Phi_K : V_{\mathfrak{g}} \to G/K$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Theorem B. *Let G/K be an irreducible Riemannian symmetric space of compact type. If N is a weakly reflective submanifold of G/K then $\Phi_K^{-1}(N)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Theorem C. *Let N be a closed submanifold of the l-dimensional sphere G/K, where $(G, K) = (SO(l + 1), SO(l))$. Then the the following are equivalent:*

(i) *N is an austere submanifold of G/K,*
(ii) *$\Phi_K^{-1}(N)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Theorem D. *Let G/K be an irreducible Riemannian symmetric space of compact type. If N is an arid submanifold of G/K then $\Phi_K^{-1}(N)$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

Note that these theorems except for Theorem C are proved under somewhat weaker assumptions; for details see Theorems 3.1.2, 3.2.1, 3.3.1, 3.4.1 and their corollaries. Applying these theorems to examples of weakly reflective submanifolds, austere submanifolds and arid submanifolds in G/K we obtain many examples of weakly reflective PF submanifolds, austere PF submanifolds and arid PF submanifolds respectively in the Hilbert space $V_{\mathfrak{g}}$ (cf. Section 3).

Notice that all known examples of such minimal submanifolds in G/K are homogeneous, that is, orbits of isometric actions by certain Lie groups. Then it follows that so obtained minimal PF submanifolds are also homogeneous. Note also that except for rare cases so obtained PF submanifolds are *not* totally geodesic (and thus not reflective). More precisely the following theorem will be shown (Corollary 2.3.2):

Theorem E. *Let G/K be a compact normal homogeneous space and N a connected closed submanifold of G/K through $eK \in G/K$. Denote by $\mathfrak{c}(\mathfrak{g})$ the center of the Lie algebra $\mathfrak{g}$ of G. Then the following are equivalent:*

(i) *$\Phi_K^{-1}(N)$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$,*
(ii) *N is a totally geodesic submanifold of G/K such that $T_{eK}^{\perp}N \subset \mathfrak{c}(\mathfrak{g})$.*

As a consequence we obtain the following corollary (Theorem 4.1.2):

Corollary. *In infinite dimensional Hilbert spaces there exist many homogeneous minimal PF submanifolds which are not totally geodesic.*

This corollary shows a critical difference between finite and infinite dimensional submanifolds because it is known that in finite dimensional Euclidean spaces all homogeneous minimal submanifolds are totally geodesic ([46]).

The organization of this thesis

In Section 1 we review the fundamental facts and results related to PF submanifolds in Hilbert spaces. In Section 1.1 we review definitions and the fundamental properties of PF submanfiolds and PF actions. In Sections 1.2 and 1.3 we prepare the setting of $P(G, H)$-actions and the parallel transport map. In Section 1.4 we review definitions and the fundamental properties for minimal PF submanifolds. In Section 1.5 we define reflective PF submanifolds, weakly reflective PF submanifolds, austere PF submanifolds and arid PF submanifolds in Hilbert spaces, and study their fundamental properties.

In Section 2 we study submanifold geometry of PF submanifolds obtained through the parallel transport map. In Sections 2.1 and 2.2 we give explicit formulas for the second fundamental forms and the shape operators of so obtained PF submanifolds. Using these formulas we study their totally geodesic property and principal curvatures in Sections 2.3 and 2.4 respectively. All these results are interesting themselves and also will be the foundations for the later sections.

In Section 3 we study the symmetric properties for PF submanifolds. In Section 3.1 we define the canonical reflection of the Hilbert space $V_{\mathfrak{g}}$ and show that each fiber of the parallel transport map is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$. Motivated by this result, in Sections 3.2, 3.3 and 3.4 we study the weakly reflective property, austere property and arid property for PF submanifolds obtained through the parallel transport map. Moreover we show many examples of minimal PF submanifolds with such symmetries.

In Section 4 we focus on homogeneous minimal PF submanifolds. In Section 4.1 we show the critical difference between finite and infinite dimensional cases and in Sections 4.2 and 4.3 we mention the related problems.

1 Fundamental facts

1.1 PF submanifolds and PF actions

In this subsection we review the definitions and the fundamental properties for PF submanifolds and PF actions.

Let V be a separable Hilbert space over $\mathbb{R}$ and M a Hilbert manifold immersed in V. Suppose that for each $p \in M$ the tangent space T_pM is a closed subspace of $T_pV \cong V$. We denote by $T^{\perp}M$ the normal bundle of M. The end point map $Y : T^{\perp}M \to V$ is defined by $Y(\xi) := p + \xi$ for $\xi \in T_p^{\perp}M$.

Definition (Terng [50])**.** M is called a *proper Fredholm* (PF) submanifold of V if it satisfies the following two conditions:

- (i) the end point map Y is Fredholm, that is, the differential $dY_{(p,\xi)}$ at each $(p, \xi) \in T^{\perp}M$ is a Fredholm operator,
- (ii) the restriction of Y to a normal disc bundle of any finite radius is proper.

Note that if V is of finite dimension then the condition (i) is automatically satisfied.

The following proposition gives the fundamental properties of PF submanifolds in Hilbert spaces. For a proof, see [50, Propositions 2.7 and 2.16].

Proposition 1.1.1 (Terng [50])**.** *Let M be a PF submanifold of V.*

- (i) *For each $p \in M$ and each $\xi \in T_p^{\perp}M$ the shape operator $A_\xi : T_pM \to T_pM$ is a self-adjoint compact operator.*
- (ii) *For each $u \in V$ the function $f : M \to \mathbb{R}$ defined by $f(p) := \|p - u\|^2$ satisfies the Palais-Smale condition* ([40], [47]): *For any sequence $\{x_n\}$ in M satisfying $\sup_n |f(x_n)| < \infty$ and $\|df_{x_n}\| \to 0$ it follows that $\{x_n\}$ has a convergent subsequence.*

From the first condition the spectral theory of the shape operator is somewhat simplified and we can deal with principal curvatures of PF submanifolds. However note that the shape operator is not of trace class in general and thus there is no natural definition of mean curvatures of PF submanifolds (cf. Section 1.4). The second condition implies that we can apply infinite dimensional Morse theory to PF submanifolds. In this thesis we mainly use the fist condition.

Let $\mathcal{G}$ be a Hilbert Lie group, that is, a Hilbert manifold with a smooth group structure. Suppose that $\mathcal{G}$ acts on a separable Hilbert space V. The $\mathcal{G}$-action is called *proper* if the map $\mathcal{G} \times V \to V$, $(g, u) \mapsto (g \cdot u, u)$ is proper, and is called *Fredholm* if for each $u \in V$ the map $\mathcal{G} \to V$, $g \mapsto g \cdot u$ is Fredhlom. Note that if $\mathcal{G}$ and V are of finite dimension then the $\mathcal{G}$-action on V is automatically Fredholm. For a proof of the next proposition, see [41, Theorem 7.1.6].

Proposition 1.1.2 (Palais-Terng [41])**.** *Suppose that $\mathcal{G}$ is an infinite dimensional Hilbert Lie group acting on a separable Hilbert space V. If the action is isometric, proper and Fredholm then every orbit is a PF submanifold of V.*

An important example of a PF action is the $P(G,H)$-*action*, which we review in the next subsection.

1.2 The $P(G,H)$-action

In this subsection we prepare the setting of $P(G,H)$-actions.

Let G be a (finite dimensional) connected compact Lie group with Lie algebra $\mathfrak{g}$. Choose an $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ and equip the corresponding bi-invariant Riemannian metric with G. For simplicity of notation we regard G as a subgroup of a general linear group. We set

$$\mathcal{G} := H^1([0,1],G), \qquad V_{\mathfrak{g}} := H^0([0,1],\mathfrak{g}),$$

where $H^1([0,1],G)$ denotes the Hilbert Lie group of all Sobolev H^1-paths in G parametrized on $[0,1]$ and $H^0([0,1],\mathfrak{g})$ the Hilbert space of all Sobolev H^0-paths (i.e. L^2-paths) in $\mathfrak{g}$ parametrized on $[0,1]$. Note that from the Sobolev embedding theorem any Sobolev H^1-path is continuous. The Lie algebra $\mathrm{Lie}\,\mathcal{G}$ of $\mathcal{G}$ is the Hilbert sapce $H^1([0,1],\mathfrak{g})$ of all Sobolev H^1-paths in $\mathfrak{g}$. The exponential map $\exp^{\mathcal{G}}$ of $\mathcal{G}$ is given by

$$(\exp^{\mathcal{G}} Z)(t) := \exp^{G}(Z(t)), \quad Z \in \mathrm{Lie}\,\mathcal{G},\ t \in [0,1],$$

where $\exp^G$ denotes the exponential map of G. We denote by $\hat{}$ the map which associates to each $a \in G$ (resp. $x \in \mathfrak{g}$) the constant path $\hat{a} \in \mathcal{G}$ (resp. $\hat{x} \in V_{\mathfrak{g}}$) which values at a (resp. x).

Define the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ via the gauge transformations:

$$g * u := gug^{-1} - g'g^{-1}, \quad g \in \mathcal{G},\ u \in V_{\mathfrak{g}}, \tag{1.2.1}$$

where g' denotes the weak derivative of g with respect to the parameter on $[0,1]$. The differential $d(g*)_{\hat{0}}$ of the transformation $g* : V \to V$, $u \mapsto g * u$ at $\hat{0} \in V_{\mathfrak{g}}$ is given by

$$d(g*)_{\hat{0}}(X) = gXg^{-1}, \quad X \in T_{\hat{0}}V_{\mathfrak{g}} \cong V_{\mathfrak{g}}.$$

Thus the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ is isometric. Also it can be easily seen that the $\mathcal{G}$-action on $V_{\mathfrak{g}}$ is transitive ([51, p. 132]). Moreover we know ([50, p. 89]):

Proposition 1.2.1 (Terng [50])**.** *The $\mathcal{G}$-action on $V_{\mathfrak{g}}$ is proper and Fredholm.*

Let H be a closed subgroup of $G \times G$ with Lie algebra $\mathfrak{h}$. Define a Lie subgroup $P(G,H)$ of $\mathcal{G}$ by

$$P(G,H) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in H\}$$

with Lie algebra

$$\mathrm{Lie}\,P(G,H) = \{Z \in H^1([0,1],\mathfrak{g}) \mid (Z(0), Z(1)) \in \mathfrak{h}\}.$$

The induced action of $P(G,H)$ on $V_{\mathfrak{g}}$ is called the $P(G,H)$-*action* ([51]). Note that $P(G,H)$ is the inverse image of H under the Lie group homomorphism $\Psi : \mathcal{G} \to G \times G$ which is defined by

$$\Psi(g) := (g(0), g(1)), \quad g \in \mathcal{G},$$

with differential

$$(d\Psi)(Z) = (Z(0), Z(1)), \quad Z \in \operatorname{Lie} \mathcal{G}. \tag{1.2.2}$$

This together with Proposition 1.2.1 shows that the $P(G,H)$-action is also proper and Fredholm ([51, p. 132]). Thus by Proposition 1.1.2 every orbit of the $P(G,H)$-action is a PF submanifold of $V_{\mathfrak{g}}$.

Since

$$\left.\frac{d}{ds}\right|_{s=0} \left(\exp^{\mathcal{G}} sZ\right) * \hat{0} = -Z', \quad Z \in \operatorname{Lie} P(G,H), \tag{1.2.3}$$

the tangent space of the orbit $P(G,H) * \hat{0}$ through $\hat{0} \in V_{\mathfrak{g}}$ is written by

$$T_{\hat{0}}(P(G,H) * \hat{0}) = \{-Z' \in H^0([0,1], \mathfrak{g}) \mid Z \in \operatorname{Lie} P(G,H)\}. \tag{1.2.4}$$

We know that if $H = \{e\} \times G$ then $P(G,H) = P(G, \{e\} \times G)$ acts on $V_{\mathfrak{g}}$ transitively and freely ([52, Corollary 4.2]). Similarly $P(G, G \times \{e\})$ acts on $V_{\mathfrak{g}}$ transitively and freely.

The structure of the $P(G,H)$-action can be understood by the parallel transport map, which we review in the next subsection.

1.3 The parallel transport map

In this subsection we prepare the setting of the parallel transport map.

As in the last subsection we denote by G a connected compact Lie group with a bi-invariant Riemannian metric. Define a map $E : V_{\mathfrak{g}} \to P(G, \{e\} \times G)$, $u \mapsto E_u$ by the unique solution to the linear ordinary differential equation

$$\begin{cases} E_u^{-1} E_u' = u, \\ E_u(0) = e. \end{cases}$$

Note that the inverse map $\bar{E} : P(G, \{e\} \times G) \to V_{\mathfrak{g}}$ of E is given by

$$\bar{E}(g) = g^{-1} * \hat{0}, \quad g \in P(G, \{e\} \times G).$$

From (1.2.3) the differential $d\bar{E} : T_{\hat{e}} P(G, \{e\} \times G) \to T_{\hat{0}} V_{\mathfrak{g}}$ at $\hat{e}$ is given by

$$(d\bar{E})_{\hat{e}}(Z) = Z'.$$

Thus the differential of $(dE)_{\hat{0}} : T_{\hat{0}} V_{\mathfrak{g}} \to T_{\hat{e}} P(G, \{e\} \times G)$ at $\hat{0}$ is given by

$$(dE)_{\hat{0}}(X) = \int_0^t X(t) dt, \quad X \in T_{\hat{0}} V_{\mathfrak{g}} \cong V_{\mathfrak{g}}. \tag{1.3.1}$$

From the direct computations we have ([51, Proposition 1.1]):

$$E_{g*u} = E_u g^{-1}, \quad g \in P(G, \{e\} \times G),\ u \in V_{\mathfrak{g}}.$$

Thus defining the right invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on $P(G, \{e\} \times G)$ by

$$\langle Z, W \rangle := \langle Z', W' \rangle_{L^2}, \quad Z, W \in T_{\hat{e}} P(G, \{e\} \times G),$$

the map $E : V_{\mathfrak{g}} \to P(G, \{e\} \times G)$ becomes an isometry ([29, Proposition 3.2]).

Definition (Terng [51])**.** The *parallel transport map* $\Phi : V_{\mathfrak{g}} \to G$ is defined by

$$\Phi(u) := E_u(1), \quad u \in V_{\mathfrak{g}}.$$

Let H be a closed subgroup of $G \times G$. Then H acts on G isometrically by

$$(b_1, b_2) \cdot a := b_1 a b_2^{-1}, \quad a \in G,\ (b_1, b_2) \in H. \tag{1.3.2}$$

It follows ([51, Proposition 1.1]) that for each $g \in P(G, H)$ and $u \in V_{\mathfrak{g}}$

$$\Phi(g * u) = (g(0), g(1)) \cdot \Phi(u) \tag{1.3.3}$$

and that for each $u \in V_{\mathfrak{g}}$

$$P(G, H) * u = \Phi^{-1}(H \cdot \Phi(u)). \tag{1.3.4}$$

Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathcal{G} & \supset & P(G,H) & \curvearrowright & V_{\mathfrak{g}} & \supset & P(G,H) * u & = & \Phi^{-1}(H \cdot \Phi(u)) \\
\Psi \downarrow & & \Psi \downarrow & & \Phi \downarrow & & \Phi \downarrow & & \\
G \times G & \supset & H & \curvearrowright & G & \supset & H \cdot \Phi(u). & &
\end{array}$$

From (1.2.2) and (1.3.1) the differential $(d\Phi)_{\hat{0}} : T_{\hat{0}} V_{\mathfrak{g}} \to \mathfrak{g}$ of the parallel transport map Φ at $\hat{0} \in V_{\mathfrak{g}}$ is given by

$$(d\Phi)_{\hat{0}}(X) = \int_0^1 X(t) dt, \quad X \in T_{\hat{0}} V_{\mathfrak{g}} \cong V_{\mathfrak{g}}.$$

From this we obtain the orthogonal direct sum decomposition

$$T_{\hat{0}} V_{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathrm{Ker}(d\Phi)_{\hat{0}}, \quad X = \left(\int_0^1 X(t) dt \right) \oplus \left(X - \int_0^1 X(t) dt \right). \tag{1.3.5}$$

Moreover the following properties holds ([52, p. 686], [52, Lemma 5.1]):

Proposition 1.3.1 (Terng-Thorbergsson [52])**.**

(i) *Φ is a Riemannian submersion.*

(ii) *Any two fibers of Φ are congruent under the isometries on $V_{\mathfrak{g}}$.*

(iii) *$P(G, \{e\} \times \{e\})$ acts on each fiber of Φ transitively and freely.*

(iv) Φ *is a principal* $P(G, \{e\} \times \{e\})$*-bundle.*

(v) *If* N *is a closed submanifold of* G *then* $\Phi^{-1}(N)$ *is a PF submanifold of* $V_{\mathfrak{g}}$.

Let K be a closed subgroup of G with Lie algebra $\mathfrak{k}$. Denote by $\mathfrak{g} = \mathfrak{k}+\mathfrak{m}$ the orthogonal direct sum decomposition. Restricting the $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ to $\mathfrak{m}$ we define the induced G-invariant Riemannian metric on the homogeneous space G/K. Thus G/K is a compact normal homogeneous space. We denote by $\pi : G \to G/K$ the natural projection, which is a Riemannian submersion with totally geodesic fiber. For each $x \in \mathfrak{g}$, $x_{\mathfrak{k}}$ and $x_{\mathfrak{m}}$ denote the $\mathfrak{k}$- and $\mathfrak{m}$-components, respectively.

Definition (Terng-Thorbergsson [52])**.** The *parallel transport map* Φ_K *over* G/K is defined by

$$\Phi_K := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K. \tag{1.3.6}$$

Note that if $K = \{e\}$ then $\Phi_K = \Phi$. Note also that Φ_K has the similar properties as in Proposition 1.3.1.

In the rest of this subsection, we mention several facts which will be used later. By (1.3.3) the following diagram commutes for each $g \in P(G, G \times \{e\})$:

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ G & \xrightarrow{(g(0),e)} & G. \end{array} \tag{1.3.7}$$

Let G, K be as above. For $a \in G$ we denote by l_a the left translation by a and L_a the isometry on G/K defined by $L_a(bK) := abK$ for $b \in G$. Then the diagram

$$\begin{array}{ccc} G & \xrightarrow{l_a} & G \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ G/K & \xrightarrow{L_a} & G/K \end{array} \tag{1.3.8}$$

commutes. Combining (1.3.7) with (1.3.8), the following diagram commutes for $g \in P(G, G \times \{e\})$ and $a := g(0)$:

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{g*} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi_K}\downarrow & & {\scriptstyle\Phi_K}\downarrow \\ G/K & \xrightarrow{L_a} & G/K. \end{array} \tag{1.3.9}$$

Let G, H be as above. For each $a \in G$, set $H^a := (a, e)^{-1}H(a, e)$. We have $H \cdot a = l_a(H^a \cdot e)$. Then it follows from (1.3.3) (ii) and (1.3.7) that for $g \in P(G, G \times \{e\})$, $u := g * \hat{0}$ and $a := \Phi(u) = g(0)$,

$$P(G, H) * u = g * (P(G, H^a) * \hat{0}). \tag{1.3.10}$$

1.4 Minimality for PF submanifolds

In this subsection we review the fundamental facts on minimality for PF submanifolds. Since there is no infinite dimensional version of Lebesgue measure on Hilbert spaces, the volume of PF submanifolds with respect to the induced metric is not meaningful and thus the minimality of PF submanifolds can not be defined via the local variation of the volume. Moreover the shape operator of a PF submanifold is a self-adjoint compact operator which is not of trace class and thus there is no natural definition of mean curvatures for PF submanifolds in general.

Fortunately for a suitable class of PF submanifolds the trace of the shape operator can be defined and thus the minimality is defined so that the trace vanishes. At present there are three kinds of definitions for the trace of the shape operator (mean curvature) and respectively there are three kinds of definitions for minimal PF submanifolds:

Definition (King-Terng [29], Heintze-Liu-Olmos [20], Koike [31])**.**

Let M be a PF submanifold of a separable Hilbert space V and $\xi \in T^{\perp}M$. We denote by

$$\mu_1 \leq \mu_2 \leq \cdots < 0 < \cdots \leq \lambda_2 \leq \lambda_1.$$

the eigenvalues repeated with multiplicities of the shape operator A_ξ. Also denote by $\{\kappa_k\}_{k=1}^{\infty}$ the set of all distinct non-zero eigenvalues of A_ξ arranged so that

$$|\kappa_k| > |\kappa_{k+1}| \quad \text{or} \quad \kappa_k = -\kappa_{k+1} \geq 0,$$

and by m_k the multiplicity of κ_k.

We say that A_ξ is *ζ-regularizable* ([29]) if $\sum_k \lambda_k^s + \sum_k |\mu_k|^s < \infty$ for all $s > 1$ and

$$\operatorname{tr}_\zeta A_\xi := \lim_{s \searrow 1} \left(\sum_k \lambda_k^s - \sum_k |\mu_k|^s \right)$$

exists. Then we call $\operatorname{tr}_\zeta A_\xi$ the *ζ-regularized mean curvature* in the direction of ξ. M is called *ζ-regularizable* if A_ξ is ζ-regularizable for all $\xi \in T^{\perp}M$. If M is ζ-regularizable and $\operatorname{tr}_\zeta A_\xi$ vanishes for all $\xi \in T^{\perp}M$ we say that M is *ζ-minimal.*

We say that A_ξ is *regularizable* ([20]) if $\operatorname{tr} A_\xi^2 < \infty$ and

$$\operatorname{tr}_r A_\xi := \sum_{k=1}^{\infty} (\lambda_k + \mu_k)$$

converges, where we regard λ_k or μ_k as zero if there are less than k positive or negative eigenvalues, respectively. Then we call $\operatorname{tr}_r A_\xi$ the *regularized mean curvature* in the direction of ξ. M is called *regularizable* if A_ξ is regularizable for all $\xi \in T^{\perp}M$. If M is regularizable and $\operatorname{tr}_r A_\xi$ vanishes for all $\xi \in T^{\perp}M$ we say that M is *r-minimal.*

We say that A_ξ is *formally regularizable* (shortly *f-regularizable*) if

$$\operatorname{tr}_f A_\xi := \sum_{k=1}^{\infty} m_k \kappa_k$$

converges. Then we call $\operatorname{tr}_f A_\xi$ the *f-regularized mean curvature* in the direction of ξ. M is called *f-regularizable* if A_ξ is regularizable for all $\xi \in T^\perp M$. If M is f-regularizable and $\operatorname{tr}_f A_\xi$ vanishes for all $\xi \in T^\perp M$ we say that M is f-minimal ([31]).

For proofs of the following facts, see [29, Theorem 4.12], [20, Lemma 5.2].

Proposition 1.4.1 (King-Terng [29], Heintze-Liu-Olmos [20]).
Let G be a connected compact Lie group with a bi-invariant Riemannian metric, Φ the parallel transport map and N a closed submanifold of G. Then

(i) *the PF submanifold $\Phi^{-1}(N)$ is both ζ-regularizable and regularizable,*
(ii) *for each $X \in T^\perp \Phi^{-1}(N)$ the following mean curvatures coincide:*
   (a) *The ζ-regularized mean curvature of $\Phi^{-1}(N)$ in the direction of X,*
   (b) *The regularized mean curvature of $\Phi^{-1}(N)$ in the direction of X,*
   (c) *the mean curvature of N in the direction of $d\Phi(X) \in T^\perp N$.*
(iii) *the following are equivalent:*
   (a) *$\Phi^{-1}(N)$ is ζ-minimal,* (b) *$\Phi^{-1}(N)$ is r-minimal,* (c) *N is minimal.*

Note that the parallel transport map $\Phi_K : V_{\mathfrak{g}} \to G/K$ satisfies the same properties as in Proposition 1.4.1.

1.5 Symmetric properties for PF submanifolds

In this subsection we define and study reflectivity, weak reflectivity, austerity and aridity for PF submanifolds in Hilbert spaces. They are defined just as in the finite dimensional case:

Definition. Let M be a PF submanifold of a separable Hilbert space V. M is called *reflective* if it is a connected component of the fixed point set of an involutive isometry of V. M is called *totally geodesic* if its second fundamental form is identically zero. M is called *weakly reflective* if for each $p \in M$ and each $\xi \in T_p^\perp M$ there exists an isometry ν_ξ of V which satisfies

$$\nu_\xi(p) = p, \quad (d\nu_\xi)_p \xi = -\xi, \quad \nu_\xi(M) = M.$$

Here we call such an isometry ν_ξ a *reflection* with respect to ξ. M is called *austere* if for each $\xi \in T^\perp M$ the set of eigenvalues with their multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1). M is called *arid* if for each $p \in M$ and each $\xi \in T_p^\perp M \backslash \{0\}$ there exists an isometry φ_ξ of V which satisfies

$$\varphi_\xi(p) = p, \quad d\varphi_\xi(\xi) \neq \xi, \quad \varphi_\xi(M) = M.$$

Reflective PF submanifolds in Hilbert spaces are characterized as follows:

Proposition 1.5.1. *Let M be a connected PF submanifold of V. Then the following are equivalent:*
(i) *M is a reflective PF submanifold of V,*
(ii) *M is a totally geodesic closed PF submanifold of V,*
(iii) *M is a closed affine subspace of V.*

Proof. We can assume without loss of generality that M is through $0 \in V$. "(i) $\Rightarrow$ (iii)": Suppose that $\sigma : V \to V$ is an involutive isometry such that the connected component of the fixed point set of σ coincides with M. Then σ is a linear orthogonal transformation of V and its fixed point set is a closed linear subspace of V. This shows (iii). "(iii) $\Rightarrow$ (i)": Since M is a closed subspace of V there is the orthogonal direct sum decomposition $V = M \oplus M^{\perp}$. Defining $\sigma : V \to V$ by $\sigma(x \oplus y) := x \oplus (-y)$ for $x \oplus y \in M \oplus M^{\perp} = V$ it follows that M is a reflective PF submanifold of V. "(iii) $\Rightarrow$ (ii)": Since M is a closed subspace of V then it is invariant under the linear orthogonal transformation $V \to V$, $v \mapsto -v$. Thus the second fundamental form α of M satisfies $-\alpha(-x, -y) = \alpha(x, y)$ for $x, y \in TM$, which shows α is identically zero and thus M is totally geodesic. "(ii) $\Rightarrow$ (iii)": Since M is totally geodesic it follows that the geodesic in V along each $x \in TM$ lies in M ([33, Chapter XIV, Corollary 1.4]). Since the geodesic of V is a straight line $t \mapsto tx$ our claim follows. $\square$

Similarly to the finite dimensional case we have the following relation:

$$\text{reflective PF} \;\Rightarrow\; \text{weakly reflective PF} \begin{array}{l} \nearrow \ \text{austere PF} \\ \searrow \ \text{arid PF.} \end{array}$$

Moreover the following properties hold:

Proposition 1.5.2. *Let M be a PF submanifold of V.*
(i) (a) *Suppose M is regularizable. If M is austere, then it is r-minimal.*
(b) *Suppose M is ζ-regularizable. If M is austere, then it is both ζ-minimal and r-minimal.*
(ii) (a) *Suppose that M is ζ-regularizable and that for each $p \in M$ the map $T_p^{\perp}M \to \mathbb{R}$, $\xi \mapsto \operatorname{tr}_\zeta A_\xi$ is linear. If M is arid, then it is ζ-minimal.*
(b) *Suppose that M is regularizable and that for each $p \in M$ the map $T_p^{\perp}M \to \mathbb{R}$, $\xi \mapsto \operatorname{tr}_r A_\xi$ is linear. If M is arid, then it is r-minimal.*

Proof. (i): (a) is clear. (b): Since M is austere, the series $\sum_k(\lambda_k + \mu_k)$ converges absolutely and thus $\operatorname{tr}_\zeta A_\xi = \operatorname{tr}_r A_\xi$ holds for any $\xi \in T^{\perp}M$ (see [20, Remark in Section 4]). Thus (b) follows. (ii): Choose a basis $\{e_1, \cdots, e_r\}$ of $T_p^{\perp}M$. From the assumption the mean curvature vector H_p of M at $p \in M$ is well-defined by

$$H_p := \sum_{i=1}^{r} \operatorname{tr}_\zeta(A_{e_i})e_i.$$

Note that H_p is invariant under any isometry φ of V satisfying $\varphi(p) = p$ and $\varphi(M) = M$ because the eigenvalues of A_{e_i} and $A_{d\varphi(e_i)}$ coincide. Thus if M is arid, then H_p must be zero. This shows that $\mathrm{tr}_\zeta(A_{e_i}) = 0$ for any $i \in \{1, \cdots, r\}$ and hence we obtain (a). By similar arguments (b) also follows. $\square$

Since PF submanifolds obtained through the parallel transport map satisfy all assumptions in Proposition 1.5.2 (see [29, Theorem 4.12], [20, Lemma 5.2]) we obtain the following corollary:

Corollary 1.5.3. *Let G/K be a compact normal homogeneous space, $\Phi_K : V_{\mathfrak{g}} \to G/K$ the parallel transport map and N a closed submanifold of G/K.*

(i) *If the PF submanifold $\Phi_K^{-1}(N)$ is austere then it is both ζ-minimal and r-minimal.*

(ii) *If the PF submanifold $\Phi_K^{-1}(N)$ is arid then it is both ζ-minimal and r-minimal.*

In other cases it is not clear that austere PF submanifolds and arid PF submanifolds are ζ-minimal or r-minimal. However it will not interfere our purpose because we will give attention to PF submanifolds obtained through the parallel transport map, where austerity or aridity implies both ζ-minimality and r-minimality due to the above corollary.

Later we show that each fiber of the parallel transport map $\Phi_K : V_{\mathfrak{g}} \to G/K$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$ (Section 3.1). After that we will see many examples of weakly reflective PF submanifolds, austere PF submanifolds and arid PF submanifolds in Hilbert spaces (Sections 3.2, 3.3 and 3.4).

2 Submanifold geometries via the parallel transport map

2.1 Second fundamental forms and shape operators I

In this subsection we study the second fundamental form and the shape operator of a PF submanifold obtained through the parallel transport map.

Let G be a connected compact Lie group equipped with a bi-invariant Riemannian metric and $\Phi : V_{\mathfrak{g}} \to G$ the parallel transport map. We write $F := \Phi^{-1}(e)$ for the fiber of Φ at $e \in G$. Denote by $\iota : F \to V_{\mathfrak{g}}$ the inclusion map and regard F as a submanifold of $V_{\mathfrak{g}}$. Recall that the subgroup $P(G, \{e\} \times \{e\})$ acts on F transitively and freely. Thus by (1.2.4) we have the following expression of the tangent space:

$$T_{\hat{0}}F = \{-Q' \in V_{\mathfrak{g}} \mid Q \in H^1([0,1], \mathfrak{g}),\ Q(0) = Q(1) = 0\}. \tag{2.1.1}$$

Note that by (1.3.5) we have $T_{\hat{0}}^{\perp}F = \hat{\mathfrak{g}}$. Define a map $\mathcal{E} : T_{\hat{0}}V_{\mathfrak{g}} \to \Gamma(\iota^* TV_{\mathfrak{g}})$ by

$$\mathcal{E}(X)_{g*\hat{0}} := gXg^{-1}, \quad X \in T_{\hat{0}}V_{\mathfrak{g}},\ g \in P(G, \{e\} \times \{e\}). \tag{2.1.2}$$

Thus $\mathcal{E}(X)$ is the $P(G, \{e\} \times \{e\})$-equivariant vector filed along F.

Lemma 2.1.1. *The Levi-Civita connection ∇^{TF}, the second fundamental form α^F, the shape operator A^F, and the normal connection $\nabla^{T^\perp F}$ of F satisfy the following. For $-Q', -R' \in T_{\hat{0}}F$, $\hat{\xi} \in T^\perp_{\hat{0}}F$,*

$$
\begin{aligned}
&\text{(i)} && \nabla^{TF}_{-Q'}\mathcal{E}(-R') = [Q, -R'] - \int_0^1 [Q, -R'](t)dt, \\
&\text{(ii)} && \alpha^F(-Q', -R') = \int_0^1 [Q, -R'](t)dt, \\
&\text{(iii)} && A^F_{\hat{\xi}}(-Q') = -[Q, \hat{\xi}] + \left[\int_0^1 Q(t)dt, \xi\right], \\
&\text{(iv)} && \nabla^{T^\perp F}_{-Q'}\mathcal{E}(\hat{\xi}) = \left[\int_0^1 Q(t)dt, \xi\right].
\end{aligned}
$$

Proof. Since $V_{\mathfrak{g}}$ is flat, it follows from (2.1.2) that

$$
\begin{aligned}
\nabla^{\iota^* TV_{\mathfrak{g}}}_{-Q'}\mathcal{E}(-R') &= \frac{d}{ds}\bigg|_{s=0} \mathcal{E}(-R')_{(\exp sQ)*\hat{0}} = [Q, -R'], \\
\nabla^{\iota^* TV_{\mathfrak{g}}}_{-Q'}\mathcal{E}(\hat{\xi}) &= \frac{d}{ds}\bigg|_{s=0} \mathcal{E}(\hat{\xi})_{(\exp sQ)*\hat{0}} = [Q, \hat{\xi}].
\end{aligned}
$$

By (1.3.5) our claim follows. □

The following theorem gives the Lie algebraic formulas for the second fundamental form and the shape operator of a PF submanifold obtained through the parallel transport map Φ.

Theorem 2.1.2. *Let N be a closed submanifold of G through $e \in G$. Denote respectively by α^N and A^N the second fundamental form and the shape operator of N, and by $\alpha^{\Phi^{-1}(N)}$ and $A^{\Phi^{-1}(N)}$ those of $\Phi^{-1}(N)$. For $X, Y \in T_{\hat{0}}\Phi^{-1}(N)$, $\hat{\xi} \in T^\perp_{\hat{0}}\Phi^{-1}(N) (\subset \hat{\mathfrak{g}})$,*

$$
\begin{aligned}
\text{(i)}\quad \alpha^{\Phi^{-1}(N)}(X, Y) &= \alpha^N\left(\int_0^1 X(t)dt,\ \int_0^1 Y(t)dt\right) \\
&\quad + \frac{1}{2}\left[\int_0^1 X(t)dt, \int_0^1 Y(t)dt\right]^\perp - \left(\int_0^1 \left[\int_0^t X(s)ds, Y(t)\right] dt\right)^\perp, \\
\text{(ii)}\quad A^{\Phi^{-1}(N)}_{\hat{\xi}}(X)(t) &= A^N_\xi\left(\int_0^1 X(t)dt\right) \\
&\quad - \frac{1}{2}\left[\int_0^1 X(t)dt, \xi\right]^\top + \left[\int_0^t X(s)ds, \xi\right] - \left[\int_0^1 \int_0^t X(s)dsdt, \xi\right]^\perp,
\end{aligned}
$$

where $\top$ and $\perp$ denote the projections of $\mathfrak{g}$ onto T_eN and $T_e^\perp N$, respectively.

Proof. (i) Recall that Φ is a Riemannian submersion with decomposition (1.3.5). We use superscripts h and v to denote the projections of $T_{\hat{0}}V_{\mathfrak{g}}$ onto $\hat{\mathfrak{g}}$ and $T_{\hat{0}}F$, respectively. Set $\bar{N} := \Phi^{-1}(N)$. Then

$$\begin{aligned}
\alpha^{\bar{N}}(X,Y) &= \alpha^{\bar{N}}(X^h,Y^h) + \alpha^{\bar{N}}(X^h,Y^v) + \alpha^{\bar{N}}(X^v,Y^h) + \alpha^{\bar{N}}(X^v,Y^v) \\
&= \alpha^N(d\Phi(X), d\Phi(Y)) \\
&\quad + (\nabla^{T^\perp F}_{Y^v}\mathcal{E}(X^h))_{T^\perp_{\hat{0}}\bar{N}} + (\nabla^{T^\perp F}_{X^v}\mathcal{E}(Y^h))_{T^\perp_{\hat{0}}\bar{N}} + \alpha^F(X^v,Y^v)_{T^\perp_{\hat{0}}\bar{N}}.
\end{aligned}$$

Define $Q, R \in H^1([0,1],\mathfrak{g})$ by

$$\begin{cases} X^v = -Q', \\ Q(0) = Q(1) = 0, \end{cases} \qquad \begin{cases} Y^v = -R', \\ Y(0) = Y(1) = 0. \end{cases}$$

Explicitly Q and R are

$$Q = tX^h - \int_0^t X(s)ds, \qquad R = tY^h - \int_0^t Y(s)ds.$$

By Lemma 2.4.4 we have

$$\begin{aligned}
&\alpha^{\bar{N}}(X,Y) - \alpha^N(d\Phi(X), d\Phi(Y)) \\
&= \left[\int_0^1 R(t)dt, X^h\right]^\perp + \left[\int_0^1 Q(t)dt, Y^h\right]^\perp + \left(\int_0^1 [Q, -R'](t)dt\right)^\perp.
\end{aligned}$$

Let us calculate each term above.

$$\begin{aligned}
\left[\int_0^1 R(t)dt, X^h\right] &= \frac{1}{2}[Y^h, X^h] - \left[\int_0^1\int_0^t Y(s)dsdt, X^h\right], \\
\left[\int_0^1 Q(t)dt, Y^h\right] &= \frac{1}{2}[X^h, Y^h] - \left[\int_0^1\int_0^t X(s)dsdt, Y^h\right].
\end{aligned}$$

For the third term, note that integrating by parts we have

$$\int_0^1 tY(t)dt = \left[t\int_0^t Y(s)ds\right]_0^1 - \int_0^1\int_0^t Y(s)dsdt = Y^h - \int_0^1\int_0^t Y(s)dsdt.$$

Using this we have

$$\begin{aligned}
\int_0^1 [Q, -R'](t)dt &= \int_0^1 \left[tX^h - \int_0^t X(s)ds, Y(t) - Y^h\right] dt \\
&= \left[X^h, \int_0^1 tY(t)dt\right] - \frac{1}{2}[X^h, Y^h] \\
&\quad - \int_0^1 \left[\int_0^t X(s)ds, Y(t)\right] dt + \left[\int_0^1\int_0^t X(s)dsdt, Y^h\right] \\
&= \frac{1}{2}\left[X^h, Y^h\right] - \left[X^h, \int_0^1\int_0^t Y(s)dsdt\right] \\
&\quad - \int_0^1 \left[\int_0^t X(s)ds, Y(t)\right] dt + \left[\int_0^1\int_0^t X(s)dsdt, Y^h\right].
\end{aligned}$$

From these calculations we obtain (i).

(ii) By (i) and $\mathrm{Ad}(G)$-invariance of the inner product of $\mathfrak{g}$, we have

$$\begin{aligned}\langle A^{\bar{N}}_{\hat{\xi}}(X), Y\rangle_{L^2} &= \langle \alpha^{\bar{N}}(X,Y), \hat{\xi}\rangle_{L^2} \\ &= \left\langle A^N_\xi(X^h) - \frac{1}{2}[X^h,\xi] + \left[\int_0^t X(s)ds, \xi\right], Y\right\rangle_{L^2}.\end{aligned}$$

This proves (ii). $\square$

Let N be a closed submanifold of G through $e \in G$. Since Φ is a Riemannian submersion we have the orthogonal direct sum decomposition:

$$T_{\hat{0}}\Phi^{-1}(N) \cong T_{\hat{0}}F \oplus T_eN, \quad X = -Q' \oplus v,$$

where Q is as in (2.1.1). With respect to this decomposition, Theorem 2.1.2 (ii) can be described as follows:

Corollary 2.1.3. *Let N be a closed submanifold of G through $e \in G$. For $\xi \in T^\perp_eN$, $y \in T_eN$, $-Q' \in T_{\hat{0}}F$,*

$$\text{(i)} \qquad A^{\Phi^{-1}(N)}_{\hat{\xi}}(\hat{v})(t) = A^N_\xi(v) + \left(t - \frac{1}{2}\right)[v,\xi],$$

$$\text{(ii)} \qquad A^{\Phi^{-1}(N)}_{\hat{\xi}}(-Q') = -[Q,\hat{\xi}] + \left[\int_0^1 Q(t)dt, \xi\right]^\perp.$$

Finally we show the second fundamental form and the shape operator of a $P(G,H)$-orbit. Recall the Lie algebraic expression of the tangent space (1.2.4).

Theorem 2.1.4. *Let G be a connected compact Lie group with a bi-invariant Riemannian metric and H be a closed subgroup of $G \times G$. Then the second fundamental form $\alpha^{P(G,H)*\hat{0}}$ and the shape operator $A^{P(G,H)*\hat{0}}$ of the orbit $P(G,H)*\hat{0}$ through $\hat{0} \in V_\mathfrak{g}$ are given as follows. For $-Z', -W' \in T_{\hat{0}}(P(G,H)*\hat{0})$, $\hat{\xi} \in T^\perp_{\hat{0}}(P(G,H)*\hat{0})$,*

$$\text{(i)} \qquad \alpha^{P(G,H)*\hat{0}}(-Z',-W') = \int_0^1 \{[Z,-W'](t)\}^\perp dt,$$

$$\text{(ii)} \qquad A^{P(G,H)*\hat{0}}_{\hat{\xi}}(-Z') = -[Z,\hat{\xi}] + \left[\int_0^1 Z(t)dt, \xi\right]^\perp,$$

where $\top$ and $\perp$ denote the projections of $\mathfrak{g}$ onto $T_e(H\cdot e)$ and $T^\perp_e(H\cdot e)$, respectively.

To prove this theorem we show the following proposition which describes the second fundamental form and the shape operator of the H-orbit through $e \in G$. Note that the tangent space of the orbit $H \cdot e$ is

$$T_e(H\cdot e) = \{x - y \in \mathfrak{g} \mid (x,y) \in \mathfrak{h}\}.$$

Proposition 2.1.5. *Let G, H be as in Theorem 2.1.4. Then the second fundamental form $\alpha^{H\cdot e}$ and the shape operator $A^{H\cdot e}$ of the orbit $H\cdot e$ through $e\in G$ are given as follows. For $x-y, z-w\in T_e(H\cdot e)$, $\xi\in T_e^{\perp}(H\cdot e)$,*

$$\text{(i)}\qquad \alpha^{H\cdot e}(x-y,z-w) = -\frac{1}{2}[x-y,\ z+w]^{\perp} = -\frac{1}{2}\left([x,w]-[y,z]\right)^{\perp},$$

$$\text{(ii)}\qquad A_{\xi}^{H\cdot e}(x-y) = -\frac{1}{2}[x+y,\xi]^{\top}.$$

Proof. (i) Denote by ΔG the diagonal of $G\times G$. Set $(G_1,K_1):=(H,H\cap\Delta G)$ and $(G_2,K_2):=(G\times G,\Delta G)$ so that the diffeomorphism $H\cdot e\cong G_1/K_1$ and the isometry $G\cong G_2/K_2$ hold. For $i=1,2$ we denote by $\pi_i: G_i\to G_i/K_i$ the projection and by $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{m}_i$ the orthogonal direct sum decomposition associated to the pair (G_i,K_i). Also denote by $i: G_1\to G_2$ the inclusion and by $f: G_1/K_1\to G_2/K_2$ the induced map. Then we have the commutative diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\ i\ } & G_2 \\ {\scriptstyle\pi_1}\big\downarrow & & {\scriptstyle\pi_2}\big\downarrow \\ G_1/K_1 & \xrightarrow{\ f\ } & G_2/K_2. \end{array}$$

Note that $di(\mathfrak{m}_1)\not\subset\mathfrak{m}_2$ in general. For each $X\in\mathfrak{g}_2$ we denote by X^* the basic vector field on G_2/K_2, that is,

$$X^*_{aK_2} := \left.\frac{d}{dt}\right|_{t=0}(\exp tX)\cdot aK_2, \qquad aK_2\in G_2/K_2.$$

Note that if $X\in di(\mathfrak{m}_1)$ then X^* restricted to $f(G_1/K_1)$ is tangent to $f(G_1/K_1)$. Then for $(x,y),(z,w)\in\mathfrak{m}_1$ we have

$$\begin{aligned}
\alpha^f((x,y),(z,w)) &= \left(\nabla^{T(G_2/K_2)}_{di(x,y)^*} di(z,w)^*\right)^{\perp}_{eK_2}\\
&= -[di(x,y)_{\mathfrak{m}_2}, di(z,w)_{\mathfrak{k}_2}]^{\perp}\\
&= -\left[\frac{1}{2}(x-y,-(x-y)),\frac{1}{2}(z+w,z+w)\right]^{\perp}\\
&= -\frac{1}{4}([x-y,z+w],-[x-y,z+w])^{\perp}.
\end{aligned}$$

Thus by the identification $(\mathfrak{g}\times\mathfrak{g})/\Delta\mathfrak{g}\cong\mathfrak{g}$, $(x,y)\mapsto x-y$ we obtain

$$\begin{aligned}
\alpha^{H\cdot e}(x-y,z-w) &= -\frac{1}{4}([x-y,z+w]+[x-y,z+w])^{\perp}\\
&= -\frac{1}{2}[x-y,z+w]^{\perp}\\
&= -\frac{1}{2}([x,z]+[x,w]-[y,z]-[y,w])^{\perp}.
\end{aligned}$$

Since $(x,y),(z,w) \in \mathfrak{h}$ we have $([x,z],[y,w]) \in \mathfrak{h}$, which shows $[x,z]-[y,w]$ is tangent to $H \cdot e$. Hence (i) follows. (ii) For each $z-w \in T_e(H \cdot e)$ we have

$$\begin{aligned}\langle A^{H\cdot e}_{\xi}(x-y), z-w\rangle &= \langle \alpha^{H\cdot e}(z-w,x-y),\xi\rangle \\ &= -\frac{1}{2}\langle [z-w,x+y],\xi\rangle \\ &= -\frac{1}{2}\langle [x+y,\xi], z-w\rangle.\end{aligned}$$

Thus the desired formula follows. □

Proof of Theorem 2.1.4. Set $N := H \cdot e$ and $\bar{N} := P(G,H) * \hat{0}$ so that $\bar{N} = \Phi^{-1}(N)$. By Theorem 2.1.2 (i), Proposition 2.1.5 (i) and the fact that α^N is a symmetric bilinear form, we have

$$\begin{aligned}&\alpha^{\bar{N}}(-Z',-W') \\ &= \alpha^N(W(0)-W(1), Z(0)-Z(1)) + \frac{1}{2}\,[Z(0)-Z(1), W(0)-W(1)]^{\perp} \\ &\quad - \left(\int_0^1 [Z(0)-Z(t), -W'(t)]\,dt\right)^{\perp} \\ &= -\frac{1}{2}[W(0)-W(1), Z(0)+Z(1)]^{\perp} + \frac{1}{2}\,[Z(0)-Z(1), W(0)-W(1)]^{\perp} \\ &\quad - \left(\left[Z(0), \int_0^1 -W'(t)dt\right] - \int_0^1 [Z,-W']\,(t)dt\right)^{\perp} \\ &= \left(\int_0^1 [Z,-W']\,(t)dt\right)^{\perp}.\end{aligned}$$

This proves (i). (ii) follows from Theorem 2.1.2 (ii) and Proposition 2.1.5 (ii). □

2.2 Second fundamental forms and shape operators II

In this subsection we study the second fundamental form and the shape operator of a PF submanifold obtained through the parallel transport map over a compact normal homogeneous space.

Let G/K be a compact normal homogeneous space with projection $\pi : G \to G/K$. Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the orthogonal direct sum decomposition. Suppose that a closed submanifold N of G/K through eK is given. Since π is a Riemannian submersion we have the orthogonal direct sum decomposition

$$T_e\pi^{-1}(N) = \mathfrak{k} \oplus T_{eK}N, \quad v = v_{\mathfrak{k}} \oplus v_{\mathfrak{m}}.$$

Denote respectively by α^N and A^N the second fundamental form and the shape operator of N, and by $\alpha^{\pi^{-1}(N)}$, $A^{\pi^{-1}(N)}$ those of $\pi^{-1}(N)$.

Proposition 2.2.1. *For $v, w \in T_e\pi^{-1}(N) \subset \mathfrak{g}$, $\xi \in T_{eK}^{\perp}N \cong T_e^{\perp}\pi^{-1}(N) \subset \mathfrak{m}$,*

$$\text{(i)} \qquad \alpha^{\pi^{-1}(N)}(v,w) = \alpha^N(v_{\mathfrak{m}}, w_{\mathfrak{m}}) - \frac{1}{2}[v_{\mathfrak{k}}, w_{\mathfrak{m}}]^{\perp} + \frac{1}{2}[v_{\mathfrak{m}}, w_{\mathfrak{k}}]^{\perp},$$

$$\text{(ii)} \qquad A_{\xi}^{\pi^{-1}(N)}(v) = A_{\xi}^N(v_{\mathfrak{m}}) - \frac{1}{2}[v_{\mathfrak{m}}, \xi]_{\mathfrak{k}} + \frac{1}{2}[v_{\mathfrak{k}}, \xi]^{\top}.$$

where $\top$ and $\perp$ denote the projections of $\mathfrak{g}$ onto $T_e\pi^{-1}(N)$ and $T_e^{\perp}\pi^{-1}(N)$, respectively.

Proof. (i) By identification $T_eG \cong \mathfrak{g}$ we regard v and w as left invariant vector fields on G. On the other hand we denote respectively by $\widetilde{v_{\mathfrak{m}}}$ and $\widetilde{w_{\mathfrak{m}}}$ the *right* invariant vector fields on G such that $\widetilde{v_{\mathfrak{m}}}(e) = v_{\mathfrak{m}}(e)$ and $\widetilde{w_{\mathfrak{m}}}(e) = w_{\mathfrak{m}}(e)$. Then $\widetilde{v_{\mathfrak{m}}}|_{\pi^{-1}(eK)}$ and $\widetilde{w_{\mathfrak{m}}}|_{\pi^{-1}(eK)}$ are the horizontal lift of $d\pi(v(e))$ and $d\pi(w(e))$. Since $\pi : G \to G/K$ is a Riemannian submersion with totally geodesic fiber $\pi^{-1}(eK) = K$ we have

$$\begin{aligned}
\alpha^{\pi^{-1}(N)}(v(e), w(e)) &= \alpha^{\pi^{-1}(N)}(v_{\mathfrak{m}}(e), w_{\mathfrak{m}}(e)) + \alpha^{\pi^{-1}(N)}(v_{\mathfrak{m}}(e), w_{\mathfrak{k}}(e)) \\
&\quad + \alpha^{\pi^{-1}(N)}(v_{\mathfrak{k}}(e), w_{\mathfrak{m}}(e)) + \alpha^{\pi^{-1}(N)}(v_{\mathfrak{k}}(e), w_{\mathfrak{k}}(e)) \\
&= \alpha^N(v_{\mathfrak{m}}(e), w_{\mathfrak{m}}(e)) + \left(\nabla^{T^{\perp}\pi^{-1}(eK)}_{w_{\mathfrak{k}}(e)}(\widetilde{v_{\mathfrak{m}}}|_{\pi^{-1}(eK)})\right)^{\perp} \\
&\quad + \left(\nabla^{T^{\perp}\pi^{-1}(eK)}_{v_{\mathfrak{k}}(e)}(\widetilde{w_{\mathfrak{m}}}|_{\pi^{-1}(eK)})\right)^{\perp} + \alpha^{\pi^{-1}(eK)}(v_{\mathfrak{k}}(e), w_{\mathfrak{k}}(e)) \\
&= \alpha^N(v_{\mathfrak{m}}(e), w_{\mathfrak{m}}(e)) + (\nabla^{TG}_{w_{\mathfrak{k}}(e)}\widetilde{v_{\mathfrak{m}}})^{\perp} + (\nabla^{TG}_{v_{\mathfrak{k}}(e)}\widetilde{w_{\mathfrak{m}}})^{\perp} + 0 \\
&= \alpha^N(v_{\mathfrak{m}}(e), w_{\mathfrak{m}}(e)) + \frac{1}{2}[\widetilde{w_{\mathfrak{k}}}, \widetilde{v_{\mathfrak{m}}}](e)^{\perp} + \frac{1}{2}[\widetilde{v_{\mathfrak{k}}}, \widetilde{w_{\mathfrak{m}}}](e)^{\perp} \\
&= \alpha^N(v_{\mathfrak{m}}(e), w_{\mathfrak{m}}(e)) - \frac{1}{2}[w_{\mathfrak{k}}, v_{\mathfrak{m}}](e)^{\perp} - \frac{1}{2}[v_{\mathfrak{k}}, w_{\mathfrak{m}}](e)^{\perp} \\
&= \alpha^N(v_{\mathfrak{m}}(e), w_{\mathfrak{m}}(e)) - \frac{1}{2}[v_{\mathfrak{k}}, w_{\mathfrak{m}}](e)^{\perp} + \frac{1}{2}[v_{\mathfrak{m}}, w_{\mathfrak{k}}](e)^{\perp}.
\end{aligned}$$

Thus we obtain the desired formula.

(ii) Using (i) we have

$$\begin{aligned}
\langle A_{\xi}^{\pi^{-1}(N)}(v), w\rangle &= \langle \alpha^{\pi^{-1}(N)}(w, v), \xi\rangle \\
&= \langle \alpha^N(w_{\mathfrak{m}}, v_{\mathfrak{m}}), \xi\rangle - \frac{1}{2}\langle [w_{\mathfrak{k}}, v_{\mathfrak{m}}], \xi\rangle + \frac{1}{2}\langle [w_{\mathfrak{m}}, v_{\mathfrak{k}}], \xi\rangle \\
&= \langle A_{\xi}^N(v_{\mathfrak{m}}), w_{\mathfrak{m}}\rangle - \frac{1}{2}\langle [v_{\mathfrak{m}}, \xi], w_{\mathfrak{k}}\rangle + \frac{1}{2}\langle [v_{\mathfrak{k}}, \xi], w_{\mathfrak{m}}\rangle \\
&= \langle A_{\xi}^N(v_{\mathfrak{m}}), w\rangle - \frac{1}{2}\langle [v_{\mathfrak{m}}, \xi]_{\mathfrak{k}}, w\rangle + \frac{1}{2}\langle [v_{\mathfrak{k}}, \xi], w\rangle \\
&= \left\langle A_{\xi}^N(v_{\mathfrak{m}}) - \frac{1}{2}[v_{\mathfrak{m}}, \xi]_{\mathfrak{k}} + \frac{1}{2}[v_{\mathfrak{k}}, \xi], w\right\rangle.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
A_{\xi}^{\pi^{-1}(N)}(v) &= A_{\xi}^N(v_{\mathfrak{m}}) - \frac{1}{2}[v_{\mathfrak{m}}, \xi]_{\mathfrak{k}}^{\top} + \frac{1}{2}[v_{\mathfrak{k}}, \xi]^{\top} \\
&= A_{\xi}^N(v_{\mathfrak{m}}) - \frac{1}{2}[v_{\mathfrak{m}}, \xi]_{\mathfrak{k}} + \frac{1}{2}[v_{\mathfrak{k}}, \xi]^{\top}.
\end{aligned}$$

This proves (ii). $\square$

Combining Theorem 2.1.2 with Proposition 2.2.1 we easily obtain the following relational formulas:

Corollary 2.2.2. *Let G/K be a compact normal homogeneous space and N a closed submanifold of G/K through $eK \in G/K$. Denote respectively by α^N and A^N the second fundamental form and the shape operator of N, and by $\alpha^{\Phi_K^{-1}(N)}$ and $A^{\Phi_K^{-1}(N)}$ those of $\Phi_K^{-1}(N)$. For $X, Y \in T_{\hat{0}}\Phi_K^{-1}(N)$, $\hat{\xi} \in T_{\hat{0}}^{\perp}\Phi_K^{-1}(N)(\subset \hat{\mathfrak{g}})$,*

$$
\begin{aligned}
\text{(i)}\quad & \alpha^{\Phi_K^{-1}(N)}(X,Y) = \alpha^N\left(\int_0^1 X(t)_{\mathfrak{m}}dt,\ \int_0^1 Y(t)_{\mathfrak{m}}dt\right) \\
& \quad - \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{k}}dt, \int_0^1 Y(t)_{\mathfrak{m}}dt\right]^{\perp} + \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{m}}dt, \int_0^1 Y(t)_{\mathfrak{k}}dt\right]^{\perp} \\
& \quad + \frac{1}{2}\left[\int_0^1 X(t)dt, \int_0^1 Y(t)dt\right]^{\perp} - \left(\int_0^1 \left[\int_0^t X(s)ds, Y(t)\right] dt\right)^{\perp}, \\
\text{(ii)}\quad & A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(X)(t) = A_{\xi}^N\left(\int_0^1 X(t)_{\mathfrak{m}}dt\right) \\
& \quad - \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{m}}dt, \xi\right]_{\mathfrak{k}} + \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{k}}dt, \xi\right]^{\top} \\
& \quad - \frac{1}{2}\left[\int_0^1 X(t)dt, \xi\right]^{\top} + \left[\int_0^t X(s)ds, \xi\right] - \left[\int_0^1 \int_0^t X(s)dsdt, \xi\right]^{\perp},
\end{aligned}
$$

where $\top$ and $\perp$ denote the projections of $\mathfrak{g}$ onto T_eN and $T_e^{\perp}N$, respectively.

For later use we give a simple expression of the shape operator $A_{\hat{\xi}}^{\Phi_K^{-1}(N)}$ as follows. Denote by $F_K := \Phi_K^{-1}(eK) = \Phi^{-1}(K)$ the fiber of Φ_K at $eK \in G/K$. Since

$$F_K = \Phi^{-1}((\{e\} \times K) \cdot e) = P(G, \{e\} \times K) * \hat{0},$$

from (1.2.4) we have

$$T_{\hat{0}}F_K = \{-Z' \mid Z \in H^1([0,1], \mathfrak{g}),\ Z(0) = 0,\ Z(1) \in \mathfrak{k}\}. \tag{2.2.1}$$

In particular we have

$$T_{\hat{0}}F = \{-Q' \mid Q \in H^1([0,1], \mathfrak{g}),\ Q(0) = Q(1) = 0\}. \tag{2.2.2}$$

Since Φ and Φ_K are Riemannian submersions we have the orthogonal direct sum decompositions

$$T_{\hat{0}}\Phi_K^{-1}(N) \cong T_{\hat{0}}F_K \oplus T_{eK}N \cong T_{\hat{0}}F \oplus \mathfrak{k} \oplus T_{eK}N.$$

We fix a normal vector ξ of N at eK. Then its horizontal lift at $\hat{0} \in \Phi_K^{-1}(N)$ is given by the constant path $\hat{\xi}$.

Proposition 2.2.3. *Suppose* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. *For* $-Q' \in T_{\hat{0}}F$, $x \in \mathfrak{k}$, $y \in T_{eK}N$

$$\text{(i)} \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(-Q') = -[Q, \hat{\xi}] + \left[\int_0^1 Q(t)dt, \xi\right]^{\perp},$$

$$\text{(ii)} \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x) = -\frac{1}{2}[x, \xi]^{\perp} + t[x, \xi],$$

$$\text{(iii)} \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y) = A_{\xi}^{N}(y) - (1-t)[y, \xi],$$

where $\perp$ *denote the projection from* $\mathfrak{g}$ *onto* $T_{eK}^{\perp}N (\subset \mathfrak{m})$.

Proof. The formula (i) follows from Corollary 2.1.3 (ii). Also from Corollary 2.1.3 (i) we have the following formula: for $v \in T_e\pi^{-1}(N)$

$$A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(v) = A_{\xi}^{\pi^{-1}(N)}(v) + \left(t - \frac{1}{2}\right)[v, \xi].$$

This together with Proposition 2.2.1 (ii) and the assumption $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ we obtain the formulas (ii) and (iii). □

Corollary 2.2.4. *Suppose* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. *For* $-Z' \in T_{\hat{0}}F_K$

$$A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(-Z') = -[Z, \hat{\xi}] + \left[\int_0^1 Z(t)dt, \xi\right]^{\perp}.$$

Proof. Set $Q := Z - tZ(1)$ and $x := -Z(1)$ so that $-Z' = -Q' + x$. By Proposition 2.2.3 (i) and (ii) the desired formula follows. □

2.3 The totally geodesic property

In this subsection we study the totally geodesic property of a PF submanifold obtained through the parallel transport map. As a consequence we will see that so obtained PF submanifolds are *not* totally geodesic except for rare cases. This leads us to a remarkable property of homogeneous minimal PF submanifolds in Hilbert spaces (Section 4.1).

Let G be a connected compact Lie group with a bi-invariant Riemannian metric. Denote by $\mathfrak{g}^{ss} = [\mathfrak{g}, \mathfrak{g}]$ the semisimple part and $\mathfrak{c}(\mathfrak{g})$ the center of $\mathfrak{g}$. We know the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{c}(\mathfrak{g})$. We write G^{ss} for the connected subgroup of G generated by $\mathfrak{g}^{ss}$.

Theorem 2.3.1. *Let G be a connected compact Lie group with bi-invariant Riemannian metric, $\Phi : V_{\mathfrak{g}} \to G$ the parallel transport map and N a closed connected submanifold of G through $e \in G$. Then the following are equivalent:*

(i) $\Phi^{-1}(N)$ *is a totally geodesic PF submanifold of* $V_{\mathfrak{g}}$.
(ii) N *is a closed subgroup of* G *such that* $\mathfrak{g}^{ss} \subset T_eN$.
(iii) N *is a closed subgroup of* G *such that* $T_e^{\perp}N \subset \mathfrak{c}(\mathfrak{g})$.
(iv) N *is a closed subgroup of* G *which contains* G^{ss}.

Proof. Equivalence of (ii), (iii) and (iv) is clear. (iii) $\Rightarrow$ (i): Since N is totally geodesic and $T_e^{\perp}N \subset \mathfrak{c}(\mathfrak{g})$, it follows from Theorem 2.1.2 (ii) that $\Phi^{-1}(N)$ is totally geodesic at $\hat{0} \in V_{\mathfrak{g}}$. Since N is a closed subgroup of G, we have $\Phi^{-1}(N) = \Phi^{-1}((\{e\} \times N) \cdot e) = P(G, e \times N) * \hat{0}$ and in particular $\Phi^{-1}(N)$ is homogeneous. Thus $\Phi^{-1}(N)$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$. (i) $\Rightarrow$ (iii): Let $\xi \in T_e^{\perp}N$ and $x \in \mathfrak{g}$. Since Φ is a Riemannian submersion, N is totally geodesic. Thus by Corollary 2.1.3 (i) we have

$$0 = A_{\hat{\xi}}^{\Phi^{-1}(N)}(\hat{x})(t) = (t - \tfrac{1}{2})[x, \xi]$$

for all $t \in [0, 1]$. This shows $[x, \xi] = 0$ and thus we obtain $T_e^{\perp}N \subset \mathfrak{c}(\mathfrak{g})$, which is equivalent to $\mathfrak{g}^{ss} \subset T_eN$. Then T_eN is a Lie subalgebra of $\mathfrak{g}$ because $\mathfrak{g}^{ss} = [\mathfrak{g}, \mathfrak{g}]$. Since N is connected and totally geodesic, N is identical to a connected Lie subgroup of G generated by T_eN. Hence N is a closed subgroup of G and (iii) follows. □

Corollary 2.3.2. *Let G/K be a compact normal homogeneous space, $\Phi_K : V_{\mathfrak{g}} \to G/K$ the parallel transport map and N a connected closed submanifold of G/K through $eK \in G/K$. Then the following are equivalent:*

(i) *$\Phi_K^{-1}(N)$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$,*
(ii) *N is a totally geodesic submanifold of G/K such that $T_{eK}^{\perp}N \subset \mathfrak{c}(\mathfrak{g})$.*

Corollary 2.3.3. *Let G/K, Φ_K be as above.*

(i) *If G is abelian then $\Phi_K^{-1}(N)$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$ for any closed connected totally geodesic submanifold N of G/K.*
(ii) *If G is semisimple then for a closed connected submanifold N of G the following are equivalent:* (a) *$\Phi_K^{-1}(N)$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$.* (b) *$N = G/K$.* (c) *$\Phi_K^{-1}(N) = V_{\mathfrak{g}}$.*

For fibers of the parallel transport map, we have the following corollaries.

Corollary 2.3.4. *Let G/K, Φ_K be as above.Then the following are equivalent.*

(i) *The fiber of Φ_K at eK is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$.*
(ii) *Each fiber of Φ_K is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$.*
(iii) *$\mathfrak{m} \subset \mathfrak{c}(\mathfrak{g})$.*

Corollary 2.3.5. *Let G be a connected compact Lie group with a bi-invariant Riemannain metric and $\Phi : V_{\mathfrak{g}} \to G$ the parallel transport map. Then the following are equivalent:*

(i) *The fiber of Φ at $e \in G$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$.*
(ii) *Each fiber of Φ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$.*
(iii) *G is a torus.*

Remark 2.3.6. Recall that $\Phi : V_{\mathfrak{g}} \to G$ is a principal $P(G, \{e\} \times \{e\})$-bundle which is not trivial in general. Corollary 2.3.5 shows that Φ is a Hilbert space bundle if and only if G is a torus. In this case it can be checked that Φ is the trivial bundle. This agrees with Kuiper's theorem ([3, p. 67]), stating that any Hilbert space bundle must be trivial.

2.4 Principal curvatures

In this subsection we calculate the principal curvatures of PF submanifolds obtained through the parallel transport map. For technical reasons, here we will restrict our attention to PF submanifolds obtained from *curvature adapted* submanifolds in compact *symmetric* spaces. Although such a subject has been studied by Koike [31] there are some inaccuracies in his eigenspace decomposition and so here we give the corrected formula with another elementary proof by using the formulas for shape operators given in Section 2.2.

Recall that a submanifold M immersed in a Riemannian manifold $\bar{M}$ is called *curvature adapted* if for each $p \in M$ and each $\xi \in T_p^\perp M$ the Jacobi operator $R_\xi := \bar{R}(\cdot, \xi)\xi : T_p\bar{M} \to T_p\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$, satisfies

$$R_\xi(T_pM) \subset T_pM \quad \text{and} \quad A_\xi^M \circ R_\xi|_{T_pM} = R_\xi|_{T_pM} \circ A_\xi^M,$$

where A_ξ^M denotes the shape operator of M in the direction of ξ.

Let G be a connected compact Lie group and K a closed subgroup of G. Suppose that K is symmetric, that is, there exists an involutive automorphism θ of G such that $G_0^\theta \subset K \subset G^\theta$, where G^θ is the fixed point subgroup of θ and G_0^θ the identity component. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively and by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the direct sum decomposition into the ± 1-eigenspaces of $d\theta$. We fix an inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ which is invariant under both $Ad(G)$ and θ. Then the above direct sum decomposition is orthogonal with respect to this inner product $\langle \cdot, \cdot \rangle$. We equip the corresponding bi-invariant Riemannian metric with G and a normal homogeneous metric with G/K. Then G/K is a compact symmetric space and the natural projection $\pi : G \to G/K$ is a Riemannian submersion with totally geodesic fiber. We denote by $\Phi_K : V_\mathfrak{g} \to G/K$ the parallel transport map.

Let N be a curvature adapted closed submanifold of G/K. Note that in order to calculate the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$ we can assume without loss of generality that N contains eK and moreover it suffices to consider normal vectors only at $\hat{0} \in \Phi_K^{-1}(N)$ because of the commutativity (1.3.9). Thus in the rest of this subsection we fix $\xi \in T_{eK}^\perp N$ and calculate the principal curvatures of $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^\perp \Phi_K^{-1}(N)$. Note that in this case the Jacobi operator is given by $R_\xi = -\operatorname{ad}(\xi)^2 : \mathfrak{m} \to \mathfrak{m}$.

Denote by $\{\sqrt{-1}\,\nu\}$ the set of all distinct eigenvalues of the skew adjoint operator $\operatorname{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$. Consider the complexification $\operatorname{ad}(\xi) : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C}$ and the eigenspace decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}_0^\mathbb{C} + \sum_{\nu \neq 0} \mathfrak{g}_\nu,$$

$$\begin{aligned} \mathfrak{g}_0 &:= \{x \in \mathfrak{g} \mid \operatorname{ad}(\xi)(x) = 0\}, \\ \mathfrak{g}_\nu &:= \{z \in \mathfrak{g}^\mathbb{C} \mid \operatorname{ad}(\xi)(z) = \sqrt{-1}\,\nu z\}. \end{aligned}$$

Since $\bar{\mathfrak{g}}_\nu = \mathfrak{g}_{-\nu}$ we can write

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} + \sum_{\nu>0} (\mathfrak{g}_\nu + \mathfrak{g}_{-\nu})$$

and thus we obtain

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\nu>0} (\mathfrak{g}_\nu + \mathfrak{g}_{-\nu})_{\mathbb{R}},$$

$$(\mathfrak{g}_\nu + \mathfrak{g}_{-\nu})_{\mathbb{R}} = \{x \in \mathfrak{g} \mid \mathrm{ad}(\xi)^2(x) = -\nu^2 x\},$$

which is nothing but the eigenspace decomposition with respect to $\mathrm{ad}(\xi)^2 : \mathfrak{g} \to \mathfrak{g}$. Since $\mathrm{ad}(\xi)^2$ commutes with involution θ we have the simultaneous eigenspace decomposition

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\nu>0} \mathfrak{k}_\nu, \qquad \mathfrak{m} = \mathfrak{m}_0 + \sum_{\nu>0} \mathfrak{m}_\nu, \tag{2.4.1}$$

$$\begin{aligned} \mathfrak{k}_0 &:= \{x \in \mathfrak{k} \mid \mathrm{ad}(\xi)(x) = 0\}, \\ \mathfrak{m}_0 &:= \{y \in \mathfrak{m} \mid \mathrm{ad}(\xi)(y) = 0\}, \\ \mathfrak{k}_\nu &:= \{x \in \mathfrak{k} \mid \mathrm{ad}(\xi)^2(x) = -\nu^2 x\}, \\ \mathfrak{m}_\nu &:= \{y \in \mathfrak{m} \mid \mathrm{ad}(\xi)^2(y) = -\nu^2 y\}. \end{aligned}$$

By similar arguments as in [36, p. 60], for each $\nu > 0$ we can take bases $\{x_1^\nu, \cdots, x_{m(\nu)}^\nu\}$ of $\mathfrak{k}_\nu$ and $\{y_1^\nu, \cdots, y_{m(\nu)}^\nu\}$ of $\mathfrak{m}_\nu$ where $m(\nu) := \dim \mathfrak{k}_\nu = \dim \mathfrak{m}_\nu$ such that

$$[\xi, x_k^\nu] = -\nu y_k^\nu, \qquad [\xi, y_k^\nu] = \nu x_k^\nu. \tag{2.4.2}$$

Thus a linear isomorphism $\varphi_\nu : \mathfrak{k}_\nu \to \mathfrak{m}_\nu$ is defined by

$$\varphi_\nu(x) := -\frac{1}{\nu}[\xi, x]. \tag{2.4.3}$$

Let $\{\lambda\}$ denote the set of all distinct eigenvalues of the shape operator A_ξ^N. Set

$$S_\lambda := \mathrm{Ker}(A_\xi^N - \lambda\, \mathrm{id}).$$

Since N is curvature adapted, for each $\nu \geq 0$ we have the decomposition

$$\begin{array}{ccccc} \mathfrak{m} & = & T_{eK}N & \oplus & T_{eK}^\perp N \\ \cup & & \cup & & \cup \\ \mathfrak{m}_\nu & = & \mathfrak{m}_\nu \cap T_{eK}N & \oplus & \mathfrak{m}_\nu \cap T_{eK}^\perp N \\ & & \| & & \\ & & \sum_\lambda (\mathfrak{m}_\nu \cap S_\lambda). & & \end{array}$$

For each $\nu \geq 0$ we set

$$m(\nu, \lambda) := \dim(\mathfrak{m}_\nu \cap S_\lambda), \qquad m(\nu, \perp) := \dim(\mathfrak{m}_\nu \cap T_{eK}^\perp N).$$

For each $\nu \geq 0$ and λ, choose

$$\begin{aligned}
&\{y_1^{(\nu,\lambda)}, \cdots, y_{m(\nu,\lambda)}^{(\nu,\lambda)}\} : \text{ a basis of } \mathfrak{m}_\nu \cap S_\lambda, \\
&\{y_1^{(\nu,\perp)}, \cdots, y_{m(\nu,\perp)}^{(\nu,\perp)}\} : \text{ a basis of } \mathfrak{m}_\nu \cap T_{eK}^\perp N.
\end{aligned}$$

Then for each $\nu \geq 0$ we obtain

$$\{y_1^{(\nu,\lambda)}, \cdots, y_{m(\nu,\lambda)}^{(\nu,\lambda)}\}_\lambda \cup \{y_1^{(\nu,\perp)}, \cdots, y_{m(\nu,\perp)}^{(\nu,\perp)}\} : \text{ a basis of } \mathfrak{m}_\nu.$$

Thus for each $\nu > 0$, via an isomorphism (2.4.3) we obtain

$$\{x_1^{(\nu,\lambda)}, \cdots, x_{m(\nu,\lambda)}^{(\nu,\lambda)}\}_\lambda \cup \{x_1^{(\nu,\perp)}, \cdots, x_{m(\nu,\perp)}^{(\nu,\perp)}\} : \text{ a basis of } \mathfrak{k}_\nu.$$

For $\nu = 0$ we choose and denote by

$$\{x_1^0, \cdots, x_{\dim \mathfrak{k}_0}^0\} : \text{ a basis of } \mathfrak{k}_0.$$

Note that these satisfy

$$\begin{array}{ll}
[\xi, x_i^0] = 0, & [\xi, y_j^{(0,\lambda)}] = [\xi, y_l^{(0,\perp)}] = 0, \\
[\xi, x_k^{(\nu,\lambda)}] = -\nu\, y_k^{(\nu,\lambda)}, & [\xi, y_k^{(\nu,\lambda)}] = \nu\, x_k^{(\nu,\lambda)}, \\
[\xi, x_r^{(\nu,\perp)}] = -\nu\, y_r^{(\nu,\perp)}, & [\xi, y_r^{(\nu,\perp)}] = \nu\, x_r^{(\nu,\perp)}.
\end{array}$$

Set $V(\mathfrak{g}) := V_\mathfrak{g} = H^0([0,1], \mathfrak{g})$. We decompose

$$V(\mathfrak{g}) = \sum_{\nu \geq 0} V(\mathfrak{k}_\nu) + \sum_{\nu \geq 0} \big(V(\mathfrak{m}_\nu \cap T_{eK}N) + V(\mathfrak{m}_\nu \cap T_{eK}^\perp N)\big)$$

and equip a basis with each term above. Recall that there are well-known three kinds of orthonormal bases in $H^0([0,1], \mathbb{R})$:

$$\{1,\ \sqrt{2}\cos 2n\pi t,\ \sqrt{2}\sin 2n\pi t\}_{n=1}^\infty\ , \tag{2.4.4}$$

$$\{1,\ \sqrt{2}\cos n\pi t\}_{n=1}^\infty\ , \tag{2.4.5}$$

$$\{\sqrt{2}\sin n\pi t\}_{n=1}^\infty\ . \tag{2.4.6}$$

For $\nu = 0$ we consider the following bases:

$$\begin{array}{lcccc}
\text{a basis of } V(\mathfrak{k}_0) & : & & & \{x_i^0 \sin n\pi t\}_{i,n}\ , \\
\text{a basis of } V(\mathfrak{m}_0 \cap T_{eK}N) & : & \{y_j^{(0,\lambda)}\}_{\lambda,j} & \cup & \{y_j^{(0,\lambda)} \cos n\pi t\}_{\lambda,j,n}\ , \\
\text{a basis of } V(\mathfrak{m}_0 \cap T_{eK}^\perp N) & : & \{y_l^{(0,\perp)}\}_l & \cup & \{y_l^{(0,\perp)} \cos n\pi t\}_{l,n}\ .
\end{array}$$

For each $\nu > 0$ we consider the following bases:

$$\begin{array}{lcccc}
\text{a basis of } V(\mathfrak{k}_\nu) & : & \{x_k^{(\nu,\lambda)} \sin n\pi t\}_{\lambda,k,n} & \cup & \{x_r^{(\nu,\perp)} \sin n\pi t\}_{r,n}\ , \\
\text{a basis of } V(\mathfrak{m}_\nu \cap T_{eK}N) & : & \{y_k^{(\nu,\lambda)}\}_{\lambda,k} & \cup & \{y_k^{(\nu,\lambda)} \cos n\pi t\}_{\lambda,n,k}\ , \\
\text{a basis of } V(\mathfrak{m}_\nu \cap T_{eK}^\perp N) & : & \{y_r^{(\nu,\perp)}\}_r & \cup & \{y_r^{(\nu,\perp)} \cos n\pi t\}_{n,r}\ .
\end{array}$$

Clearly all these bases form a basis of $V(\mathfrak{g}) = V_{\mathfrak{g}}$. Identifying $T_{\hat{0}}V_{\mathfrak{g}} \cong V_{\mathfrak{g}}$ and considering the orthogonal direct sum decomposition

$$T_{\hat{0}}V_{\mathfrak{g}} = T_{\hat{0}}\Phi_K^{-1}(N) \oplus T_{eK}^{\perp}N, \qquad X = \left(X - \int_0^1 X(t)^{\perp}dt\right) \oplus \int_0^1 X(t)^{\perp}dt$$

we consequently obtain the following basis of $T_{\hat{0}}\Phi_K^{-1}(N)$:

$$\begin{aligned}
&\{x_i^0 \sin n\pi t\}_{i,\,n} \cup \{y_j^{(0,\lambda)}\}_{\lambda,\,j} \cup \{y_j^{(0,\lambda)} \cos n\pi t\}_{\lambda,\,j,\,n} \cup \{y_r^{(0,\perp)} \cos n\pi t\}_{r,\,n} \\
&\cup \bigcup_{\nu>0} \left(\{x_k^{(\nu,\lambda)} \sin n\pi t\}_{\lambda,\,k,\,n} \cup \{y_k^{(\nu,\lambda)}\}_{\lambda,\,k} \cup \{y_k^{(\nu,\lambda)} \cos n\pi t\}_{\lambda,\,k,\,n}\right) \\
&\cup \bigcup_{\nu>0} \left(\{x_r^{(\nu,\perp)} \sin n\pi t\}_{r,n} \cup \{y_r^{(\nu,\perp)} \cos n\pi t\}_{r,\,n}\right).
\end{aligned}$$

Lemma 2.4.1.

$$\begin{aligned}
&\text{(i)} && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x_i^0 \sin n\pi t) = 0, \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_j^{(0,\lambda)}) = \lambda y_j^{(0,\lambda)}, \\
&\text{(ii)} && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_j^{(0,\lambda)} \cos n\pi t) = A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_l^{(0,\perp)} \cos n\pi t) = 0, \\
&\text{(iii)} && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x_r^{(\nu,\perp)} \sin n\pi t) = -\frac{\nu}{n\pi} y_r^{(\nu,\perp)} \cos n\pi t, \\
& && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_r^{(\nu,\perp)} \cos n\pi t) = -\frac{\nu}{n\pi} x_r^{(\nu,\perp)} \sin n\pi t, \\
&\text{(iv)} && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_k^{(\nu,\lambda)}) = \lambda y_k^{(\nu,\lambda)} + \frac{2\nu}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (x_k^{(\nu,\lambda)} \sin n\pi t), \\
&\text{(v)} && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x_k^{(\nu,\lambda)} \sin n\pi t) = -\frac{\nu}{n\pi} y_k^{(\nu,\lambda)} (-1 + \cos n\pi t), \\
&\text{(vi)} && A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_k^{(\nu,\lambda)} \cos n\pi t) = -\frac{\nu}{n\pi} x_k^{(\nu,\lambda)} \sin n\pi t.
\end{aligned}$$

Proof. (i) and (ii): The second equality of (i) follows from Proposition 2.2.3 (iii). Let Q be

$$\frac{1}{n\pi} x_i^0 \cos n\pi t, \quad -\frac{1}{n\pi} y_j^{(0,\lambda)} \sin n\pi t \quad \text{or} \quad -\frac{1}{n\pi} y_l^{(0,\perp)} \sin n\pi t.$$

By Proposition 2.2.3 (i), we obtain the first formula of (i) and formulas in (ii). (iii): Set

$$Z_1 := \frac{1}{n\pi} x_r^{(\nu,\perp)} (-1 + \cos n\pi t), \quad Z_2 := -\frac{1}{n\pi} y_r^{(\nu,\perp)} \sin n\pi t.$$

Then we have

$$[\xi, Z_1] = -\frac{\nu}{n\pi} y_r^{(\nu,\perp)} (-1 + \cos n\pi t), \quad [\xi, Z_2] = -\frac{\nu}{n\pi} x_r^{(\nu,\perp)} \sin n\pi t.$$

Thus we have

$$\left[\xi, \int_0^1 Z_1(t)dt\right]^{\perp} = \frac{\nu}{n\pi} y_r^{(\nu,\perp)}, \quad \left[\xi, \int_0^1 Z_2(t)dt\right]^{\perp} = 0.$$

Hence by Corollary 2.2.4 the desired equalities follow. (iv): By Proposition 2.2.3 (iii), we have

$$A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_k^{(\nu,\lambda)}) = \lambda y_k^{(\nu,\lambda)} + (1-t)\,\nu x_k^{(\nu,\lambda)}.$$

Since $\int_0^1 (1-t)\sin n\pi t\,dt = (n\pi)^{-1}$ the Fourier expansion with respect to a basis (2.4.6) of a function $f : [0,1] \to \mathbb{R}$, $t \mapsto 1-t$ is given by

$$f = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(\sin n\pi t).$$

This shows the desired equality. (v) and (vi): Set

$$Z_1 := \frac{1}{n\pi} x_k^{(\nu,\lambda)}(-1+\cos n\pi t), \quad Z_2 := -\frac{1}{n\pi} y_k^{(\nu,\lambda)} \sin n\pi t.$$

Then we have

$$[\hat{\xi}, Z_1] = -\frac{\nu}{n\pi} y_k^{(\nu,\lambda)}(-1+\cos n\pi t), \quad [\hat{\xi}, Z_2] = -\frac{\nu}{n\pi} x_k^{(\nu,\lambda)} \sin n\pi t,$$

Thus we have

$$\left[\xi, \int_0^1 Z_1(t)dt\right]^{\perp} = \left[\xi, \int_0^1 Z_2(t)dt\right]^{\perp} = 0.$$

Hence by Corollary 2.2.4 we obtain the desired formulas. □

We come now to the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$:

Theorem 2.4.2. *Let G/K be a compact symmetric space, $\Phi_K : V_{\mathfrak{g}} \to G/K$ the parallel transport map, N a curvature adapted closed submanifold of G/K through $eK \in G/K$, and $\xi \in T_{eK}^{\perp}N \subset \mathfrak{m}$. Denote by $\{\sqrt{-1}\,\nu\}$ the set of all distinct eigenvalues of $\operatorname{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$ and by $\{\lambda\}$ the set of all distinct eigenvalues of the shape operator A_ξ^N. For each $\nu > 0$, each λ and each $m \in \mathbb{Z}$ we set*

$$\mu = \mu(\nu, \lambda, m) := \frac{\nu}{\arctan\frac{\nu}{\lambda} + m\pi},$$

where we set $\arctan(\nu/\lambda) := \pi/2$ if $\lambda = 0$. Then the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^{\perp}\Phi^{-1}(N)$ are given by

$$\left\{0,\ \lambda,\ \frac{\nu}{n\pi},\ \mu(\nu,\lambda,m)\right\}_{\lambda,\ \nu>0,\ n\in\mathbb{Z}\setminus\{0\},\ m\in\mathbb{Z}}.$$

The eigenfunctions and the multiplicities are given in the following table.

Remark 2.4.3. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$ and Δ^+ the set of positive roots satisfying $\alpha(\xi) \geq 0$ for each $\alpha \in \Delta^+$. Then for each $\nu > 0$ there exists $\alpha \in \Delta^+$ such that $\nu = \alpha(\xi)$ and thus the above eigenvalues coincide with those given by Koike [31, Theorem 3.3]. However note that the eigenspace decomposition [31, p. 73, line 3] does not hold in general.

eigenvalue	basis of eigenfunctions	multiplicity
0	$\{x_i^0 \sin n\pi t, y_j^{(0,\lambda)} \cos n\pi t, y_l^{(0,\perp)} \cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1}, \lambda, i, j, l}$	∞
λ	$\{y_j^{(0,\lambda)}\}_j$	$m(0,\lambda)$
$\dfrac{\nu}{n\pi}$	$\{x_r^{(\nu,\perp)} \sin n\pi t - y_r^{(\nu,\perp)} \cos n\pi t\}_r$	$m(\nu,\perp)$
$\mu(\nu,\lambda,m)$	$\left\{\sum_{n\in\mathbb{Z}} \frac{\nu}{n\pi\mu+\nu}(x_k^{(\nu,\lambda)} \sin n\pi t + y_k^{(\nu,\lambda)} \cos n\pi t)\right\}_k$	$m(\nu,\lambda)$

Proof of Theorem 2.4.2. By Lemma 2.4.1 (i) - (iii) it follows that 0, λ and $\frac{\nu}{n\pi}$ are eigenvalues of $A_{\hat{\xi}}^{\Phi_K^{-1}(N)}$ with eigenfunctions described above. Let W denote a subspace of $T_{\hat{0}}\Phi_K^{-1}(N)$ spanned by all such eigenfunctions and consider its orthogonal complements $W^\perp$ in $T_{\hat{0}}\Phi_K^{-1}(N)$. We know that one basis of $W^\perp$ is given by

$$\bigcup_{\nu>0} \left(\{y_k^{(\nu,\lambda)}\}_{\lambda, k} \cup \{x_k^{(\nu,\lambda)} \sin n\pi t,\ y_k^{(\nu,\lambda)} \cos n\pi t\}_{\lambda,\ k,\ n\in\mathbb{Z}_{\geq 1}}\right).$$

In particular Lemma 2.4.1 (iv) - (vi) show that for each $\nu > 0$, λ and k, a subspace of $T_{\hat{0}}\Phi_K^{-1}(N)$ spanned by

$$\{y_k^{(\nu,\lambda)}\} \cup \{x_k^{(\nu,\lambda)} \sin n\pi t,\ y_k^{(\nu,\lambda)} \cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1}}$$

is invariant under $A_{\hat{\xi}}^{\Phi_K^{-1}(N)}$. We denote such a subspace by $W^\perp_{(\lambda,\nu,k)}$. Suppose that for constants $a_n, b_n, c \in \mathbb{R}$

$$\varphi := c y_k^{(\nu,\lambda)} + \sum_{n=1}^{\infty}\{a_n(x_k^{(\nu,\lambda)} \sin n\pi t) + b_n(y_k^{(\nu,\lambda)} \cos n\pi t)\} \quad \in W^\perp_{(\lambda,\nu,k)}$$

is a (nonzero) eigenfunction of $A_{\hat{\xi}}^{\Phi_K(N)}$ for some eigenvalue μ. By Lemma 2.4.1 (iv) - (vi) we have

$$\begin{aligned} A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(\varphi) &= \left(c\lambda + \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\right) y_k^{(\nu,\lambda)} \\ &\quad + \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{2c-b_n}{n}(x_k^{(\nu,\lambda)} \sin n\pi t) - \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}(y_k^{(\nu,\lambda)} \cos n\pi t). \end{aligned}$$

Comparing with

$$\mu\varphi = \mu c y_k^{(\nu,\lambda)} + \sum_{n=1}^{\infty}\{\mu a_n(x_k^{(\nu,\lambda)} \sin n\pi t) + \mu b_n(y_k^{(\nu,\lambda)} \cos n\pi t)\}$$

we obtain a system of equations

$$c\lambda + \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n} = c\mu, \tag{2.4.7}$$

$$\frac{\nu}{\pi}\frac{2c - b_n}{n} = \mu a_n, \tag{2.4.8}$$

$$-\frac{\nu}{\pi}\frac{a_n}{n} = \mu b_n. \tag{2.4.9}$$

Summing (2.4.9) with respect to $n \in \mathbb{Z}_{\geq 1}$ we have

$$-\frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n} = \mu\sum_{n=1}^{\infty} b_n.$$

Applying this to (2.4.7) we obtain

$$c\lambda - \mu\sum_{n=1}^{\infty} b_n = c\mu. \tag{2.4.10}$$

On the other hand, multiplying (2.4.9) by μ we have

$$-\frac{\nu}{n\pi}\mu a_n = \mu^2 b_n.$$

Applying (2.4.8) to this we obtain

$$-2c\left(\frac{\nu}{n\pi}\right)^2 = b_n\left(\mu^2 - \left(\frac{\nu}{n\pi}\right)^2\right)$$

for *all* $n \in \mathbb{Z}_{\geq 1}$. Note that this implies $c \neq 0$, $\mu \neq 0$ and $\mu^2 - (\frac{\nu}{n\pi})^2 \neq 0$ for all $n \in \mathbb{Z}$. Therefore without loss of generality we can (and will) assume $c = 1$ from now on, and

$$b_n = \frac{-2r^2}{n^2 - r^2}, \quad \text{where} \quad r := \frac{\nu}{\pi\mu}. \tag{2.4.11}$$

From (2.4.10), (2.4.11) and the standard formula (see [12, 1.449 - 4]):

$$\sum_{n=1}^{\infty}\frac{1}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\cot(\pi a), \quad a \in \mathbb{R}\backslash\mathbb{Z}$$

we have

$$\lambda - \mu = \mu\sum_{n=1}^{\infty} b_n = -\mu + \nu\cot(\nu/\mu).$$

Thus

$$\mu = \frac{\nu}{\arctan(\nu/\lambda) + m\pi}, \quad m \in \mathbb{Z}.$$

By (2.4.11) we have

$$
\begin{aligned}
b_n &= \frac{-2\nu^2}{(n\pi\mu)^2 - \nu^2} = \frac{-2(\arctan(\nu/\lambda) + m\pi)^2}{(n\pi)^2 - (\arctan(\nu/\lambda) + m\pi)^2} \\
&= (\arctan(\nu/\lambda) + m\pi) \\
&\quad \times \left(\frac{1}{n\pi + (\arctan(\nu/\lambda) + m\pi)} - \frac{1}{n\pi - (\arctan(\nu/\lambda) + m\pi)} \right).
\end{aligned}
$$

By (2.4.9) we have

$$
\begin{aligned}
a_n &= -n\pi b_n \mu\nu^{-1} = \frac{2n\pi(\arctan(\nu/\lambda) + m\pi)}{(n\pi)^2 - (\arctan(\nu/\lambda) + m\pi)^2} \\
&= (\arctan(\nu/\lambda) + m\pi) \\
&\quad \times \left(\frac{1}{n\pi + (\arctan(\nu/\lambda) + m\pi)} + \frac{1}{n\pi - (\arctan(\nu/\lambda) + m\pi)} \right).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\varphi &= y_k^{(\nu,\lambda)} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\arctan(\nu/\lambda) + m\pi}{n\pi + (\arctan(\nu/\lambda) + m\pi)} \{(x_k^{(\nu,\lambda)} \sin n\pi t) + (y_k^{(\nu,\lambda)} \cos n\pi t)\} \\
&= \sum_{n\in\mathbb{Z}} \frac{\nu}{n\pi\mu + \nu} \{(x_k^{(\nu,\lambda)} \sin n\pi t) + (y_k^{(\nu,\lambda)} \cos n\pi t)\}.
\end{aligned}
$$

This proves the theorem. □

Considering the case that $N = \{eK\}$ we obtain:

Corollary 2.4.4. *Let G/K be a compact symmetric space. Take $\xi \in T_{eK}(G/K) = \mathfrak{m}$. Denote by $\{\sqrt{-1}\,\nu\}$ the set of all distinct eigenvalues of $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$. Then the principal curvatures of the fiber $\Phi_K^{-1}(eK)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^{\perp}\Phi_K^{-1}(eK)$ are given by*

$$
\left\{0,\ \frac{\nu}{n\pi}\right\}_{\nu>0,\ n\in\mathbb{Z}\setminus\{0\}}.
$$

Denoting by $\{x_i^0\}_i$ a basis of $\mathfrak{k}_0$, by $\{y_j^0\}_j$ a basis of $\mathfrak{m}_0$ and by $\{x_k^\nu\}_k$, $\{y_k^\nu\}_k$ the bases defined by (2.4.2), the eigenfunctions and the multiplicities are given in the following table.

eigenvalue	basis of eigenfunctions	multiplicity
0	$\{x_i^0 \sin n\pi t, y_j^0 \cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1},\lambda,i,j}$	∞
$\dfrac{\nu}{n\pi}$	$\{x_k^\nu \sin n\pi t - y_k^\nu \cos n\pi t\}_k$	$m(\nu)$

The principal curvatures of $\pi^{-1}(N)$ are described as follows:

Proposition 2.4.5. *With notation as in Theorem 2.4.2, for each $\nu > 0$ and each λ we set*

$$\kappa_+ = \kappa_+(\nu, \lambda) := \frac{1}{2}(\lambda + \sqrt{\lambda^2 + \nu^2}),$$
$$\kappa_- = \kappa_-(\nu, \lambda) := \frac{1}{2}(\lambda - \sqrt{\lambda^2 + \nu^2}).$$

Then the principal curvatures of $\pi^{-1}(N)$ in the direction of $\xi \in T_e^\perp \pi^{-1}(N) \cong T_{eK}^\perp N$ are given by

$$\{0,\ \lambda,\ \kappa_+(\nu, \lambda),\ \kappa_-(\nu, \lambda)\}_{\lambda,\ \nu>0}\,.$$

The eigenfunctions and the multiplicities are given in the following table.

eigenvalue	basis of eigenfunctions	multiplicity
0	$\{x_i^0\}_i,\ \{x_r^{(\nu,\perp)}\}_{r,\nu}$	$\dim \mathfrak{k}_0 + \dim T_{eK}^\perp N$
λ	$\{y_j^{(0,\lambda)}\}_j$	$m(0, \lambda)$
$\kappa_+(\nu, \lambda)$	$\{\nu x_k^{(\nu,\lambda)} + 2\kappa_+ y_k^{(\nu,\lambda)}\}_k$	$m(\nu, \lambda)$
$\kappa_-(\nu, \lambda)$	$\{\nu x_k^{(\nu,\lambda)} + 2\kappa_- y_k^{(\nu,\lambda)}\}_k$	$m(\nu, \lambda)$

Proof. From Proposition 2.2.1 (ii) we have

$$A_\xi^{\pi^{-1}(N)}(x) = \frac{1}{2}[x, \xi]^\top, \quad x \in \mathfrak{k},$$
$$A_\xi^{\pi^{-1}(N)}(y) = A_\xi^N(y) - \frac{1}{2}[y, \xi]_{\mathfrak{k}}, \quad y \in T_{eK}N.$$

From this we have

$$A_\xi^{\pi^{-1}(N)}(x_k^0) = 0, \quad A_\xi^{\pi^{-1}(N)}(x_k^{(\nu,\lambda)}) = \frac{\nu}{2} y_k^{(\nu,\lambda)}, \quad A_\xi^{\pi^{-1}(N)}(x_k^{(\nu,\perp)}) = 0$$
$$A_\xi^{\pi^{-1}(N)}(y_k^{(\nu,\lambda)}) = \frac{\nu}{2} x_k^{(\nu,\lambda)} + \lambda y_k^{(\nu,\lambda)}.$$

Suppose that

$$\varphi := a x_k^{(\nu,\lambda)} + b y_k^{(\nu,\lambda)}, \quad a, b \in \mathbb{R}$$

is an eigenfunction of $A_\xi^{\pi^{-1}(N)}$ with eigenvalue κ. Then we have

$$A_\xi^{\pi^{-1}(N)}(\varphi) := a A_\xi^{\pi^{-1}(N)}(x_k^{(\nu,\lambda)}) + b A_\xi^{\pi^{-1}(N)}(y_k^{(\nu,\lambda)})$$
$$= b\frac{\nu}{2} x_k^{(\nu,\lambda)} + \left(a\frac{\nu}{2} + b\lambda\right) y_k^{(\lambda,\nu)},$$
$$\kappa\varphi = \kappa a x_k^{(\lambda,\nu)} + \kappa b y_k^{(\lambda,\nu)}.$$

Comparing these we obtain a system of equations

$$\begin{aligned} b\frac{\nu}{2} &= \kappa a, \\ a\frac{\nu}{2} + b\lambda &= \kappa b. \end{aligned}$$

If $a = 0$ then $b = 0$, which contradicts the fact that φ is an eigenfunction. Thus $a \neq 0$ and hence we can assume without loss of generality that $a = 1$. Then we have

$$\begin{aligned} b\nu &= 2\kappa, \\ \nu + 2b\lambda &= 2\kappa b. \end{aligned}$$

From this we easily obtain

$$b = \frac{\lambda \pm \sqrt{\lambda^2 + \nu^2}}{\nu}, \qquad \kappa = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 + \nu^2}).$$

and

$$\varphi = x_k^{(\nu,\lambda)} + \frac{\lambda \pm \sqrt{\lambda^2 + \nu^2}}{\nu} y_k^{(\nu,\lambda)}.$$

Thus our claim follows. □

3 Symmetric properties via the parallel transport map

3.1 The canonical reflection of the path space

In this subsection we focus on the intrinsic symmetry of the parallel transport map and show that each fiber of the parallel transport map is a weakly reflective PF submanifold.

Let G be a connected compact Lie group with a bi-invariant Riemannian metric. As in Subsection 1.2 we denote by $\mathcal{G} := H^1([0,1], G)$ the Hilbert Lie group of all Sobolev H^1-paths in G parametrized on $[0,1]$ and $V_{\mathfrak{g}} := H^0([0,1], \mathfrak{g})$ the Hilbert space of all Sobolev H^0-paths in $\mathfrak{g}$ parametrized on $[0,1]$. We write $_{\#}$ for the map which associates to each $g \in \mathcal{G}$ (resp. $u \in V_{\mathfrak{g}}$) the *inverse path* $g_{\#} \in \mathcal{G}$ (resp. $u_{\#} \in V_{\mathfrak{g}}$):

$$g_{\#}(t) := g(1-t), \qquad u_{\#}(t) := u(1-t).$$

Definition. The *canonical reflection* $\mathfrak{r}$ of $V_{\mathfrak{g}}$ is the involutive linear orthogonal transformation of $V_{\mathfrak{g}}$ defined by

$$\mathfrak{r}(u) := -u_{\#}, \quad u \in V_{\mathfrak{g}}.$$

Since $(g_\#)' = -(g')_\#$ for each $g \in \mathcal{G}$ we have

$$\mathfrak{r}(g * \hat{0}) = g_\# * \hat{0}, \quad g \in \mathcal{G}.$$

Thus by (1.3.3) we obtain the commutative diagram

$$\begin{array}{ccc} V_\mathfrak{g} & \xrightarrow{\mathfrak{r}} & V_\mathfrak{g} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ G & \xrightarrow{\mathfrak{i}} & G, \end{array} \tag{3.1.1}$$

where Φ denotes the parallel transport map and $\mathfrak{i}$ the isometry of G defined by $\mathfrak{i}(a) = a^{-1}$ for $a \in G$. Also it follows that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{G} & \supset & P(G, \{e\} \times G) & \curvearrowright & V_\mathfrak{g} \\ \# \downarrow & & \# \downarrow & & \mathfrak{r} \downarrow \\ \mathcal{G} & \supset & P(G, G \times \{e\}) & \curvearrowright & V_\mathfrak{g}. \end{array} \tag{3.1.2}$$

We can easily see that for each $g \in P(G, \{e\} \times G)$,

$$g_\# g(1)^{-1} \in P(G, \{e\} \times G) \quad \text{and} \quad ((g_\#)g(1)^{-1}) * \hat{0} = g_\# * \hat{0}.$$

Hence via the isometry $E : V_\mathfrak{g} \to P(G, \{e\} \times G)$ (cf. Section 1.3) $\mathfrak{r}$ induces an involutive isometry $\tilde{\mathfrak{r}}$ of $P(G, \{e\} \times G)$:

$$\tilde{\mathfrak{r}}(g) = g(1)^{-1} g_\#, \quad g \in P(G, \{e\} \times G).$$

The reflective submanifold associated to $\mathfrak{r}$ is described as follows.

Proposition 3.1.1. *Let W denote the fixed point set of $\mathfrak{r}$. Then*
- (i) *W is a closed linear subspace of $V_\mathfrak{g}$,*
- (ii) *W is isomorphic to the Hilbert space $H^0([0, 1/2], \mathfrak{g})$,*
- (iii) *W is contained in the fiber of Φ at $e \in G$.*

Proof. (i) follows from linearity of $\mathfrak{r}$. (ii) is clear by the expression $W = \{u \in V_\mathfrak{g} \mid \forall t \in [0, 1],\ u(t) = -u(1-t)\}$. (iii) follows from commutativity (3.1.1). □

The following theorem gives a certain class of weakly reflective PF submanifolds in the Hilbert space $V_\mathfrak{g}$.

Theorem 3.1.2. *Let G be a connected compact Lie group with a bi-invariant Riemannian metric and H a closed subgroup of $G \times G$. Suppose that the orbit $H \cdot e$ through $e \in G$ satisfies the condition $(H \cdot e)^{-1} = H \cdot e$. Then*
- (i) *$H \cdot e$ is a totally geodesic weakly reflective submanifold of G,*
- (ii) *$P(G, H) * \hat{0}$ is a weakly reflective PF submanifold of $V_\mathfrak{g}$.*

Proof. (i) It is easy to see that the condition $(H \cdot e)^{-1} = H \cdot e$ implies the totally geodesic property of $H \cdot e$. Also it can be easily seen that $\mathfrak{i}$ is a reflection of $H \cdot e$ with respect to any normal vector at $e \in G$. Thus by homogeneity $H \cdot e$ is a weakly reflective submanifold of G. (ii) By commutativity (3.1.1) the canonical reflection $\mathfrak{r}$ is a reflection of $P(G, H) * \hat{0}$ with respect to any normal vector at $\hat{0}$. Since $P(G, H) * \hat{0}$ is homogeneous, our claim follows. $\square$

An example of H satisfying the condition $(H \cdot e)^{-1} = H \cdot e$ is that $H = \{e\} \times K$ or $K \times \{e\}$, where K is a closed subgroup of G. The following is another example such that $H \cdot e$ is *not* a subgroup of G.

Example 3.1.3. For each automorphism σ of G, define a closed subgroup of $G \times G$ by $G(\sigma) := \{(a, \sigma(a)) \mid a \in G\}$. Then $G(\sigma)$ acts on G by (1.3.2), which is called the *Conlon's σ-action* ([5]). From now on we suppose that $\sigma^2 = \mathrm{id}$. Note that $G(\sigma) \cdot e$ is nothing but the Cartan immersion ([23, p. 347]) $G/K \to G$, $aK \mapsto a\sigma(a)^{-1}$, where K is the fixed point set of σ. It easily follows that $H := G(\sigma)$ satisfies $(H \cdot e)^{-1} = H \cdot e$. Thus by Theorem 3.1.2, $G(\sigma) \cdot e$ is a totally geodesic weakly reflective submanifold of G, and $P(G, G(\sigma)) * \hat{0}$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$. Note that $G(\sigma) \cdot e$ is not a subgroup of G in general. On the other hand $P(G, G(\sigma)) * \hat{0}$ is not totally geodesic in most cases by Theorem 2.3.1.

It was essentially proved ([29, Theorem 4.11], [20, Corollary 6.3]) that each fiber of the parallel transport map is an austere PF submanifold of $V_{\mathfrak{g}}$. The following corollary asserts that the fibers have higher symmetry.

Corollary 3.1.4. *Let G/K be a compact normal homogeneous space. Then each fiber of the parallel transport map $\Phi_K : V_{\mathfrak{g}} \to G/K$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Proof. By Theorem 3.1.2 the fiber of Φ_K at eK is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$. Since any two fibers of Φ_K are congruent under the isometry on $V_{\mathfrak{g}}$, each fiber of Φ_K is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$. $\square$

Corollary 3.1.5. *Let G be a connected compact Lie group with a bi-invariant Riemannian metric. Then each fiber of the parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

3.2 The weakly reflective property

In this subsection, motivated by the last subsection, we study the weakly reflective property for PF submanifolds obtained through the parallel transport map. The main theorem is the following:

Theorem 3.2.1. *Let G be a connected compact semisimple Lie group with a bi-invariant Riemannian metric induced from a negative multiple of the Killing form and K a symmetric subgroup of G such that the pair (G, K) effective. If N is a weakly reflective submanifold of the symmetric space G/K then*

(i) $\pi^{-1}(N)$ *is a weakly reflective submanifold of* G, *and*
(ii) $\Phi_K^{-1}(N)$ *is a weakly reflective PF submanifold of* $V_{\mathfrak{g}}$.

From this theorem we obtain the following corollary:

Corollary 3.2.2. *Let* M *be an irreducible Riemannian symmetric space of compact type. Denote by* G *the identity component of the group of isometries of* M *and by* K *the isotropy subgroup of* G *at a fixed* $p \in M$. *If* N *is a weakly reflective submanifold of* $M = G/K$ *then* $\Phi_K^{-1}(N)$ *is a weakly reflective PF submanifold of* $V_{\mathfrak{g}}$.

To prove Theorem 3.2.1 we need the following lemma:

Lemma 3.2.3. *Let* $\mathcal{M}$ *and* $\mathcal{B}$ *be Riemannian Hilbert manifolds,* $\phi : \mathcal{M} \to \mathcal{B}$ *a Riemannian submersion and* N *a closed submanifold of* $\mathcal{B}$. *Fix* $p \in \phi^{-1}(N)$ *and* $X \in T_p^{\perp}\phi^{-1}(N)$. *Suppose that* $\nu_{\mathcal{M}}$ *is an isometry of* $\mathcal{M}$ *fixing* p, *that* $\nu_{\mathcal{B}}$ *is an isometry of* $\mathcal{B}$ *fixing* $\phi(p)$ *and that the diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\nu_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi}\big\downarrow \\ \mathcal{B} & \xrightarrow{\nu_{\mathcal{B}}} & \mathcal{B} \end{array}$$

commutes. Then the following are equivalent:
(i) $\nu_{\mathcal{M}}$ *satisfies* $\nu_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ *and* $d\nu_{\mathcal{M}}(X) = -X$,
(ii) $\nu_{\mathcal{B}}$ *satisfies* $\nu_{\mathcal{B}}(N) = N$ *and* $d\nu_{\mathcal{B}}(d\phi(X)) = -d\phi(X)$.

Proof. It is easily seen that the condition $\nu_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ is equivalent to the condition $\nu_{\mathcal{B}}(N) = N$. Then by commutativity of the diagram

$$\begin{array}{ccc} T_p^{\perp}\phi^{-1}(N) & \xrightarrow{d\nu_{\mathcal{M}}} & T_p^{\perp}\phi^{-1}(N) \\ {\scriptstyle d\phi}\big\downarrow & & {\scriptstyle d\phi}\big\downarrow \\ T_{\phi(p)}^{\perp}N & \xrightarrow{d\nu_{\mathcal{B}}} & T_{\phi(p)}^{\perp}N, \end{array}$$

the condition $d\nu_{\mathcal{M}}(X) = -X$ is equivalent to the condition $d\nu_{\mathcal{B}}(d\phi(X)) = -d\phi(X)$. This proves the lemma. $\square$

Proof of Theorem 3.2.1. (i) Take $a \in \pi^{-1}(N)$ and $w \in T_a^{\perp}\pi^{-1}(N)$. Set $\eta := d\pi(w) \in T_{aK}^{\perp}N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in T_{eK}^{\perp}N'$. Denote by $v \in T_e^{\perp}\pi^{-1}(N')$ the horizontal lift of ξ. By commutativity of (1.3.7) (ii) we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the existence of a reflection ν_w with respect to w it suffices to construct a reflection ν_v with respect to v. Let ν_η be a reflection with respect to η. Then a reflection ν_ξ with respect to ξ is defined by $\nu_\xi := L_a^{-1} \circ \nu_\eta \circ L_a$. Now we define ν_v as follows. Denote by $I(G/K)$ the group of isometries of G/K. From the assumption the map $L : G \to I(G/K)$, $a \mapsto L_a$ is a Lie group isomorphism onto the identity component $I_0(G/K)$ ([23, Theorem 4.1 in Chapter V]). Since $I_0(G/K)$ is a normal subgroup of $I(G/K)$ an automorphism $\nu_v : G \to G$, $b \mapsto \nu_v(b)$ is defined by

$$L_{\nu_v(b)} := \nu_\xi \circ L_b \circ \nu_\xi^{-1}. \tag{3.2.1}$$

Since the bi-invariant Riemannian metric on G is induced from the Killing form of $\mathfrak{g}$ the automorphism ν_v is an isometry of G. Moreover since

$$\nu_\xi \circ \pi(b) = \nu_\xi(bK) = \nu_\xi \circ L_b(eK) = \nu_\xi \circ L_b \circ \nu_\xi^{-1}(eK) \quad \text{and}$$
$$\pi \circ \nu_v(b) = \nu_v(b)K = L_{\nu_v(b)}(eK) = \nu_\xi \circ L_b \circ \nu_\xi^{-1}(eK)$$

hold for all $b \in G$ it follows from Lemma 3.2.3 that ν_v is a reflection with respect to v. This proves (i).

(ii) Take $u \in \Phi_K^{-1}(N)$ and $X \in T_u^\perp \Phi_K^{-1}(N)$. Take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$ and $\eta := d\Phi_K(X) \in T_{aK}^\perp N$. Define N', ξ, v as in the above (i). Denote by $\hat{\xi} \in (T_{\hat{0}}^\perp \Phi_K^{-1}(N')) \backslash \{0\}$ the horizontal lift of ξ with respect to the Riemannian submersion $\Phi_K : V_\mathfrak{g} \to G/K$. By commutativity of (1.3.9) we have $g * \Phi_K^{-1}(N') = \Phi_K^{-1}(N)$ and $d(g*)\hat{\xi} = X$. Thus in order to show the existence of a reflection ν_X with respect to X it suffices to construct a reflection $\nu_{\hat{\xi}}$ with respect to $\hat{\xi}$. By the same way as in (i) we can define a reflection ν_v with respect to $v \in (T_e^\perp \pi^{-1}(N')) \backslash \{0\}$. Moreover we define a linear orthogonal transformation $\nu_{\hat{\xi}}$ of $V_\mathfrak{g}$ by

$$\nu_{\hat{\xi}}(u) := d\nu_v \circ u, \quad u \in V_\mathfrak{g}. \tag{3.2.2}$$

Since ν_v is an automorphism of G we have $\nu_{\hat{\xi}}(g * \hat{0}) = (\nu_v \circ g) * \hat{0}$ for all $g \in \mathcal{G}$. This together with (1.3.3) implies that the following diagram commutes:

$$\begin{array}{ccc} V_\mathfrak{g} & \xrightarrow{\nu_{\hat{\xi}}} & V_\mathfrak{g} \\ {\scriptstyle\Phi}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow \\ G & \xrightarrow{\nu_v} & G. \end{array}$$

Thus by Lemma 3.2.3 $\nu_{\hat{\xi}}$ is an isometry with respect to $\hat{\xi}$ and (ii) follows. □

Remark 3.2.4. Even if N is reflective in Theorem 3.2.1, $\Phi_K^{-1}(N)$ can not be reflective due to Corollary 2.3.3 (ii). In this case there exists one more reflective submanifold $N^\perp$ of G/K corresponding to N ([35, p. 328]) and thus a pair of two weakly reflective PF submanifolds appears in the Hilbert space $V_\mathfrak{g}$.

Remark 3.2.5. Let G/K be a compact normal homogeneous space and $\Phi_K : V_\mathfrak{g} \to G/K$ the parallel transport map. The fact that each fiber of Φ_K is a weakly reflective PF submanifold of $V_\mathfrak{g}$ also follows from Theorem 3.2.1 if (G, K) satisfies the assumptions in Theorem 3.2.1. The advantage of Corollary 3.1.4 is that it does not require such assumptions. It is also noted that under such assumptions each of the fibers has at least two different weakly reflective structures.

Example 3.2.6. Set $\bar{M} := S^{2n-1}(\sqrt{2}) \subset \mathbb{R}^{2n}$ and

$$M := S^{n-1}(1) \times S^{n-1}(1) \subset \bar{M}.$$

Ikawa, Sakai and Tasaki [26, Example 2.3] showed that M is a weakly reflective submanifold of $\bar{M}$. Set $(G, K) := (SO(2n), SO(2n-1))$ so that $G/K = \bar{M}$. Then by Theorem 3.2.1 the inverse image $\Phi_K^{-1}(M)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Example 3.2.7. Ikawa, Sakai and Tasaki ([26, Theorem 4]) classified weakly reflective submanifolds of the standard sphere given as orbits of s-representations of irreducible Riemannian symmetric pairs. Applying Theorem 3.2.1 to their result we obtain weakly reflective PF submanifolds as follows. Let (U, L) be a compact Riemannian symmetric pair. Suppose that L is connected. Denote by $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ the canonical decomposition and $\mathrm{Ad} : L \to SO(\mathfrak{p})$ the isotropy representation. If an orbit $\mathrm{Ad}(L) \cdot x$ through $x \in \mathfrak{p}$ is a weakly reflective submanifold of the hypersphere $S(\|x\|)$ in $\mathfrak{p}$, then the orbit $P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_x) * \hat{0}$ is a weakly reflective PF submanifold of the Hilbert space $V_{\mathfrak{o}(\mathfrak{p})}$.

Example 3.2.8. Enoyoshi ([8, Proposition 4]) gave an example of a weakly reflective submanifold in the symmetric space $SO(7)/SO(3) \times SO(4)$ by the action of the exceptional Lie group G_2. Applying Theorem 3.2.1 to her result an orbit $P(SO(7), G_2 \times (SO(3) \times SO(4))) * \hat{0}$ is a weakly reflective PF submanifold of the Hilbert space $V_{\mathfrak{o}(7)}$.

In general it is not clear that conversely the weakly reflective property of $\Phi_K^{-1}(N)$ implies the weakly reflective property of N or not. However the next theorem shows that under suitable assumptions the weakly reflective property of $\Phi_K^{-1}(N)$ is equivalent to that of N. To explain this we now introduce some terminologies on weakly reflective submanifolds. Let M be a submanifold immersed in a finite dimensional Riemannian manifold $\bar{M}$. Denote by $I(\bar{M})$ the group of isometries of $\bar{M}$. For a closed subgroup $\mathfrak{G}$ of $I(\bar{M})$ we say that M is $\mathfrak{G}$*-weakly reflective* if for each $p \in M$ and each $\xi \in T_p^{\perp} M$ there exists a reflection ν_ξ with respect to ξ satisfying $\nu_\xi \in \mathfrak{G}_p$, where $\mathfrak{G}_p$ denotes the isotropy subgroup of $\mathfrak{G}$ at p. If $\mathfrak{G} = I(\bar{M})$ then "$\mathfrak{G}$-weakly reflective" is nothing but "weakly reflective". The same concepts and relation are also valid for PF submanifolds in Hilbert spaces.

In the rest of this subsection we denote by G a connected compact Lie group equipped with a bi-invariant Riemannian metric, K a closed subgroup of G and G/K the compact normal homogeneous space. Recall that the Hilbert Lie group $\mathcal{G} := H^1([0,1], G)$ acts on $V_{\mathfrak{g}}$ via the gauge transformations (1.2.1). We denote by $\mathcal{G}_u$ the isotropy subgroup of $\mathcal{G}$ at $u \in V_{\mathfrak{g}}$. If $u = \hat{0}$ then $\mathcal{G}_{\hat{0}}$ is the set of constant paths $\hat{G} := \{\hat{b} \in G \mid b \in G\}$. Thus if $u = g * \hat{0}$ for some $g \in \mathcal{G}$ then $\mathcal{G}_u = g \hat{G} g^{-1}$. Recall also that $G \times G$ acts on G by the formula (1.3.2). We denote by $(G \times G)_a = (a, e) \Delta G (a, e)^{-1}$ the isotropy subgroup of $G \times G$ at $a \in G$, where $\Delta G := \{(b, b) \mid b \in G\}$. Finally we recall the G-action on G/K

defined by $b \cdot (aK) := (ba)K$ for $a, b \in G$. We denote by $G_{aK} = aKa^{-1}$ the isotropy subgroup of G at $aK \in G/K$.

Theorem 3.2.9.

(i) *Let N be a closed submanifold of a compact Lie group G equipped with a bi-invariant Riemannian metric. Then the following are equivalent:*

   (a) *N is a $(G \times G)$-weakly reflective submanifold of G.*
   (b) *$\Phi^{-1}(N)$ is a $\mathcal{G}$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

(ii) *Let N be a closed submanifold of a compact normal homogeneous space G/K. Then the following are equivalent:*

   (a) *N is a G-weakly reflective submanifold of G/K.*
   (b) *$\pi^{-1}(N)$ is a $(G \times K)$-weakly reflective submanifold of G.*
   (c) *$\Phi_K^{-1}(N)$ is a $P(G, G \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Proof. (i) "(a) $\Rightarrow$ (b)": Take $u \in \Phi^{-1}(N)$ and $X \in T_u^{\perp}\Phi^{-1}(N)$. Take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$, $\eta := d\Phi(X) \in T_a^{\perp}N$, $N' := a^{-1}N$ and $\xi := dL_a^{-1}(\eta) \in T_e^{\perp}N'$. Denote by $\hat{\xi} \in T_{\hat{0}}^{\perp}\Phi^{-1}(N')$ the horizontal lift of ξ. By commutativity of (1.3.7) we have $g * (\Phi^{-1}(N')) = \Phi^{-1}(N)$ and $(dg*)\hat{\xi} = X$. Thus in order to show the existence of a reflection ν_X with respect to X satisfying $\nu_X \in \mathcal{G}_u$ it suffices to construct a reflection $\nu_{\hat{\xi}}$ with respect to $\hat{\xi}$ satisfying $\nu_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$. Let ν_η be a reflection with respect to η which is given by $\nu_\eta(c) = b'cb^{-1}$ for some $(b', b) \in (G \times G)_a$. Then a reflection ν_ξ with respect to ξ is defined by $\nu_\xi := (a, e)^{-1} \circ \nu_\eta \circ (a, e)$, that is, $\nu_\xi(c) := bcb^{-1}$ for $c \in G$. Now we define a linear orthogonal transformation $\nu_{\hat{\xi}}$ of $V_{\mathfrak{g}}$ by

$$\nu_{\hat{\xi}}(u) := d\nu_\xi \circ u = bub^{-1} = \hat{b} * u, \quad u \in V_{\mathfrak{g}}.$$

Clearly $\nu_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$. Moreover by (1.3.3) the following diagram commutes:

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\nu_{\hat{\xi}}} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ G & \xrightarrow{\nu_\xi} & G. \end{array}$$

From Lemma 3.2.3 $\nu_{\hat{\xi}}$ is a refection with respect to $\hat{\xi}$ and (b) follows.

(i) "(b) $\Rightarrow$ (a)": Take $a \in N$ and $\eta \in T_a^{\perp}N$. Fix $u \in \Phi^{-1}(a)$. Denote by $X \in (T_u^{\perp}\Phi^{-1}(N))$ the horizontal lift of η. Take $g \in P(G, G \times \{e\})$ so that $g * \hat{0} = u$ and define N', ξ, $\hat{\xi}$ as in the above (i). Let ν_X be a reflection with respect to X satisfying $\nu_X \in \mathcal{G}_u$. Then an isometry $\nu_{\hat{\xi}}$ with respect to $\hat{\xi} \in (T_{\hat{0}}^{\perp}\Phi^{-1}(N'))$ is defined by $\nu_{\hat{\xi}} := (g*)^{-1} \circ \nu_X \circ (g*)$. By definition $\nu_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$ and thus there exists $b \in G$ such that $\nu_{\hat{\xi}}(u) = bub^{-1}$. Hence defining an isometry ν_ξ of G by $\nu_\xi(c) := bcb^{-1}$ for $c \in G$ it follows from Lemma 3.2.3 that ν_ξ is a reflection with respect to ξ satisfying $\nu_\xi \in (G \times G)_e$. Therefore an isometry ν_η with respect to η is defined by $\nu_\eta := l_a \circ \nu_\xi \circ l_a^{-1}$ so that $\nu_\eta \in (G \times G)_a$. This shows (a).

(ii) "(a) $\Rightarrow$ (b)": Take $a \in \pi^{-1}(N)$ and $w \in (T_a^{\perp}\pi^{-1}(N))$. Set $\eta := d\pi(w) \in (T_{aK}^{\perp}N)$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in (T_{eK}^{\perp}N')$. Denote by $v \in (T_e^{\perp}\pi^{-1}(N'))$ the horizontal lift of ξ. By commutativity of (1.3.8) we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the existence of an isometry ν_w with respect to w satisfying $\nu_w \in (G\times K)_a = (a,e)\Delta K(a,e)^{-1}$ it suffices to construct an isometry ν_v with respect to v satisfying $\nu_v \in (G\times K)_e = \Delta K$. Let ν_η be a reflection with respect to η which is given by $\nu_\eta(cK) = (bc)K$ for some $b \in G_{aK}$. Then there exists $k \in K$ such that $b = aka^{-1}$. Thus an isometry ν_ξ with respect to ξ is defined by $\nu_\xi := L_a^{-1} \circ \nu_\xi \circ L_a$, that is, $\nu_\xi = L_k$. Define an isometry ν_v of G by

$$\nu_v(c) := kck^{-1}, \quad c \in G.$$

Clearly $\nu_v \in (G \times K)_e$. Moreover the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\nu_v} & G \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ G/K & \xrightarrow{\nu_\xi} & G/K. \end{array}$$

Thus by Lemma 3.2.3 ν_v is a reflection with respect to v and (b) follows.

(ii) "(b) $\Rightarrow$ (a)": Take $aK \in N$ and $\eta \in T_{aK}^{\perp}N$. Denote by $w \in T_a^{\perp}\pi^{-1}(N)$ the horizontal lift of η. Define N', ξ, v as above. Let ν_w be a reflection with respect to w satisfying $\nu_w \in (G \times K)_a$. Then a reflection with respect to v is defined by $\nu_v := l_a^{-1} \circ \nu_w \circ l_a$ so that $\nu_v \in (G \times K)_e$. Then there exists $k \in K$ such that $\nu_v(c) = kck^{-1}$. Thus defining an isometry ν_ξ of G/K by $\nu_\xi := L_k$ it follows from Lemma 3.2.3 that ν_ξ is a reflection with respect to ξ satisfying $\nu_\xi \in G_{eK}$. Hence an isometry ν_η with respect to η is defined by $\nu_\eta := l_a \circ \nu_\xi \circ l_a^{-1}$ so that $\nu_\eta \in G_{aK}$. This shows (b).

Using the fact $gP(G, G \times K)_{\hat{0}}g^{-1} = P(G, G \times K)_{g*\hat{0}}$ for $g \in P(G, G \times \{e\})$ the equivalence of (b) and (c) of (ii) follows by the similar arguments to (i). $\square$

Corollary 3.2.10. *Let G, G/K be as in Theorem 3.2.9.*

(i) *Let H be a closed subgroup of $G \times G$. Then the following are equivalent:*
   - (a) *an orbit $H \cdot a$ through $a \in G$ is an H-weakly reflective submanifold of G,*
   - (b) *an orbit $P(G,H)*u$ through $u \in \Phi(a)$ is a $P(G,H)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

(ii) *Let K' be a closed subgroup of G. Then the following are equivalent:*
   - (a) *an orbit $K' \cdot aK$ through $aK \in G/K$ is a K'-weakly reflective submanifold of G/K,*
   - (b) *an orbit $(K' \times K) \cdot a$ through $a \in G$ is a $(K' \times K)$-weakly reflective submanifold of G,*
   - (c) *an orbit $P(G, K' \times K) * u$ through $u \in \Phi^{-1}(a)$ is a $P(G, K' \times K)$-weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Proof. From (1.3.10) we can assume without loss of generality that $a = e$. Moreover by homogeneity it suffices to consider normal vectors only at $e \in G$ or $\hat{0} \in V_{\mathfrak{g}}$. By similar arguments as in Theorem 3.2.9 (i) our claim follows. (ii) Similarly we can reduce the case $a = e$ and the assertion follows by similar arguments as in Theorem 3.2.9 (ii). $\square$

Example 3.2.11. It was proved ([26, p. 442], [44]) that any singular orbit of a cohomogeneity one action is weakly reflective. In this case each reflection is given by the action of the isotropy subgroup. Thus by Corollary 3.2.10 we have the following examples.

(i) Let G be a connected compact Lie group with a bi-invariant Riemannian metric and H a closed subgroup of $G \times G$. Suppose that the H-action is of cohomogeneity 1. If an orbit $H \cdot a$ through $a \in G$ is singular, then $H \cdot a$ is a weakly reflective submanifold of G, and the orbit $P(G, H) * u$ through $u \in \Phi^{-1}(a)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

(ii) Let G/K be a compact normal homogeneous space and K' a closed subgroup of G. Suppose that the K'-action is of cohomogeneity 1. If an orbit $K' \cdot aK$ through $aK \in G/K$ is singular, then orbits $K' \cdot aK$ and $(K' \times K) \cdot a$ are weakly reflective submanifolds of G/K and G, respectively. Moreover the orbit $P(G, K' \times K) * u$ through $u \in \Phi^{-1}(a)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Example 3.2.12. Let G be a connected compact semisimple Lie group. Let $K = K_1$ and $K' = K_2$ be connected symmetric subgroups of G with involutions θ_1 and θ_2, respectively. Suppose that $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Ohno ([39, Theorem 5]) gave a sufficient condition for orbits $(K_2 \times K_1) \cdot a$ and $K_2 \cdot aK_1$ to be weakly reflective submanifolds of G and G/K_1, respectively. By Corollary 3.2.10, in this case the orbits $P(G, K_2 \times K_1) * u$ through $u \in \Phi^{-1}(a)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Compared to Corollary 3.2.10 (i) the following proposition covers only H-orbits through $e \in G$. However the reflection ν_ξ with respect to each normal vector ξ at $e \in G$ need *not* belong to the isotropy subgroup H_e at $e \in G$.

Theorem 3.2.13. *Let G be a connected compact Lie group with a bi-invariant Riemannian metric and H be a closed subgroup of $G \times G$. Suppose that the orbit $H \cdot e$ through $e \in G$ is an weakly reflective submanifold of G such that for each $\xi \in T_e^{\perp}(H \cdot e)$ a reflection ν_ξ with respect to ξ is an automorphism of G. Then the orbit $P(G, H) * \hat{0}$ through $\hat{0} \in V_{\mathfrak{g}}$ is an weakly reflective PF submanifold of $V_{\mathfrak{g}}$.*

Proof. Let ν_ξ be a reflection with respect to $\xi \in T_e^{\perp}(H \cdot e)$ which is an automorphism of G. Then an isometry $\nu_{\hat{\xi}}$ with respect to $\hat{\xi} \in T_{\hat{0}}^{\perp} P(G, H) * \hat{0}$ is defined similarly to (3.2.2). By homogeneity of $P(G, H) * \hat{0}$ our claim follows. $\square$

Example 3.2.14. Let G, K_1, K_2 be as in Example 3.2.12. Ohno ([39, Theorem 4]) also gave another sufficient condition for an orbit $N := (K_2 \times K_1) \cdot a$ to be a weakly reflective submanifold of G. In this case $\nu_a := l_a \circ \theta_1 \circ l_a^{-1}$ was shown to be a reflection of N with respect to any normal vector at $a \in G$. Applying Theorem 3.2.13 to his result we can see that $\Phi^{-1}(N) = P(G, K_2 \times K_1) * u$ ($u \in \Phi^{-1}(a)$) is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$ as follows. Choose $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Then $a = \Phi(u) = g(0)$. Set $N' := a^{-1}N = ((a^{-1}K_2 a) \times K_1) \cdot e$. Then θ_1 is a reflection of N' with respect to any normal vector at $e \in N'$. Since θ_1 is an automorphism of G, it follows from Theorem 3.2.13 that $\Phi^{-1}(N')$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$. By commutativity of (1.3.7) we have $g * \Phi^{-1}(N') = \Phi^{-1}(N)$. Thus $\Phi^{-1}(N)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

3.3 The austere property

In this subsection we study the austere property of PF submanifolds obtained through the parallel transport map Φ_K. Notice that even if N is an austere curvature adapted submanifold of a compact symmetric space G/K, it is not clear in general whether the inverse image $\Phi_K^{-1}(N)$ is austere or not; according to Theorem 2.4.2 it is not clear whether the set of eigenvalues with *multiplicities* of the shape operator is invariant under the multiplication by (-1) or not. From this reason here we will restrict our attention further to the case that G/K is a *sphere* and show that in this case N is austere if and only if $\Phi_K^{-1}(N)$ is austere (Theorem 3.3.1).

Let $S^l(r) = G/K$ denote the l-dimensional sphere of radius $r > 0$, where $l \in \mathbb{Z}_{\geq 1}$ and $(G, K) = (SO(l+1), SO(l))$. Note that in this case any submanifold of G/K is automatically curvature adapted. Let N be a closed submanifold of G/K. Suppose $eK \in N$ and fix $\xi \in T^{\perp}_{eK}N$. Then for $v \in T_{eK}N$ and $\eta \in T^{\perp}_{eK}N$ the Jacobi operator R_ξ satisfies

$$R_\xi(v) := \frac{\|\xi\|^2}{r^2} v, \qquad R_\xi(\eta) := \frac{1}{r^2}\{\|\xi\|^2 \eta - \langle \eta, \xi \rangle \xi\}.$$

Thus in this case the eigenspace decomposition (2.4.1) of $\mathfrak{m}$ is given by

$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_\nu, \qquad \text{where} \quad \nu := \|\xi\|/r,$$

$$\begin{aligned} \mathfrak{m}_0 &= \mathbb{R}\xi \subset T^{\perp}_{eK}N, \\ \mathfrak{m}_\nu &= T_{eK}N \oplus \{\eta \in T^{\perp}_{eK}N \mid \eta \perp \xi\}. \end{aligned}$$

In particular we have

$$\begin{aligned} &\mathfrak{m}_0 \cap T_{eK}N = \{0\}, && \mathfrak{m}_0 \cap T^{\perp}_{eK}N = \mathbb{R}\xi, \\ &\mathfrak{m}_\nu \cap T_{eK}N = T_{eK}N, && \mathfrak{m}_\nu \cap T^{\perp}_{eK}N = \{\eta \in T^{\perp}_{eK}N \mid \eta \perp \xi\}. \end{aligned}$$

Hence by Proposition 2.4.5 the principal curvatures of $\pi^{-1}(N)$ in the direction of $\xi \in T^{\perp}_e \pi^{-1}(N) \cong T^{\perp}_{eK}N$ are given by

$$\{0, \ \kappa_+(\|\xi\|/r, \lambda), \ \kappa_-(\|\xi\|/r, \lambda)\}_\lambda . \tag{3.3.1}$$

Further by Theorem 2.4.2 the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^{\perp}\Phi^{-1}(N)$ are given by

$$\left\{0,\ \frac{\|\xi\|}{rn\pi},\ \mu(\|\xi\|/r, \lambda, m)\right\}_{\lambda,\ n\in\mathbb{Z}\setminus\{0\},\ m\in\mathbb{Z}}. \tag{3.3.2}$$

Notice that in this case the multiplicities of

$$\lambda, \quad \kappa_+(\|\xi\|/r, \lambda), \quad \kappa_-(\|\xi\|/r, \lambda), \quad \mu(\|\xi\|/r, \lambda, m)$$

are the same for each λ.

Theorem 3.3.1. *Let N be a closed submanifold of the l-dimensional sphere $S^l(r) = G/K$ of radius $r > 0$, where $l \in \mathbb{Z}_{\geq 1}$ and $(G, K) = (SO(l+1), SO(l))$. Then the the following are equivalent:*

(i) *N is an austere submanifold of G/K,*
(ii) *$\pi^{-1}(N)$ is an austere submanifold of G,*
(iii) *$\Phi_K^{-1}(N)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Proof. "(i) $\Rightarrow$ (ii)": Take $a \in \pi^{-1}(N)$ and $w \in T_a^{\perp}\pi^{-1}(N)$. Set $\eta := d\pi(w) \in T_{aK}^{\perp}N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in T_{eK}^{\perp}N'$. Denote by $v \in T_e^{\perp}\pi^{-1}(N')$ the horizontal lift of ξ. By commutativity of (1.3.7) (ii), we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the austerity of $A_w^{\pi^{-1}(N)}$ it suffices to show that of $A_v^{\pi^{-1}(N')}$. For each eigenvalue λ of $A_\xi^{N'}$ it follows from the austerity of $A_\xi^{N'}$ that $-\lambda$ is also an eigenvalue of $A_\xi^{N'}$ and

$$(-1) \times \kappa_+(\|\xi\|/r, \lambda) = \kappa_-(\|\xi\|/r, -\lambda),$$
$$(-1) \times \kappa_-(\|\xi\|/r, \lambda) = \kappa_+(\|\xi\|/r, -\lambda).$$

Note that these identities still hold even if the multiplicities are taking account of. This shows that the set (3.3.1) with multiplicities is invariant under the multiplication by (-1) and (ii) follows.

"(ii) $\Rightarrow$ (i)": Take $aK \in N$ and $\eta \in T_{aK}^{\perp}N$. Denote by $w \in T_a^{\perp}\pi^{-1}(N)$ the horizontal lift of η. Defining N', ξ, v by the above way it suffices to show the austerity of $A_\xi^{N'}$. Let λ be an eigenvalue of $A_\xi^{N'}$. Since the set (3.3.1) is invariant under the multiplication by (-1) there exist eigenvalues λ' and λ'' of $A_\xi^{N'}$ such that

$$(-1) \times \kappa_+(\|\xi\|/r, \lambda) = \kappa_-(\|\xi\|/r, \lambda'),$$
$$(-1) \times \kappa_-(\|\xi\|/r, \lambda) = \kappa_+(\|\xi\|/r, \lambda'').$$

Note that the function $\mathbb{R} \to \mathbb{R}_{>0}$, $x \mapsto \kappa_+(\|\xi\|/r, x)$ is monotonically increasing. Thus the relation

$$\kappa_-(\|\xi\|/r, x) = -\kappa_+(\|\xi\|/r, -x)$$

shows that also the function $\mathbb{R} \to \mathbb{R}_{<0}$, $x \mapsto \kappa_-(\|\xi\|/r, x)$ is monotonically increasing. From these we obtain $\lambda' = \lambda'' = -\lambda$. Note that this identity still holds even if the multiplicities are taken account of. This shows (i).

"(i) $\Rightarrow$ (iii)": Take $u \in \Phi_K^{-1}(N)$ and $X \in T_u^\perp \Phi_K^{-1}(N)$. Also take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := g(0) = \Phi(u)$, $\eta := d\Phi_K(X) \in T_{aK}^\perp N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in T_{eK}^\perp N'$. Denote by $\hat{\xi} \in T_{\hat{0}}^\perp \Phi_K^{-1}(N')$ the horizontal lift of ξ. By commutativity of (1.3.9) we have $g * (\Phi_K^{-1}(N')) = \Phi_K^{-1}(N)$ and $dg * (\hat{\xi}) = X$. Thus in order to show the austerity of $A_X^{\Phi_K^{-1}(N)}$ it suffices to show that of $A_{\hat{\xi}}^{\Phi_K^{-1}(N')}$. For each λ it follows from the austerity of $A_\xi^{N'}$ that $-\lambda$ is also an eigenvalue of $A_\xi^{N'}$ and

$$(-1) \times \mu(\|\xi\|/r, \lambda, m) = \mu(\|\xi\|/r, -\lambda, -m).$$

Note that this identity still hold even if the multiplicities are taken account of. This shows that the set

$$\{\mu(\|\xi\|/r, \lambda, m)\}_{\lambda,\ m \in \mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). This together with (3.3.2) shows the austerity of $A_{\hat{\xi}}^{\Phi_K^{-1}(N')}$ and (iii) follows.

"(iii) $\Rightarrow$ (i)": Take $aK \in N$ and $\eta \in T_{aK}^\perp N$. Choose $u \in \Phi_K^{-1}(a)$. Denote by $X \in T_u^\perp \Phi_K^{-1}(N)$ the horizontal lift of η. Defining N', ξ, $\hat{\xi}$ by the above way it suffices to show the austerity of $A_\xi^{N'}$. From (3.3.2) and the assumption the set

$$\{\mu(\|\xi\|/r, \lambda, m)\}_{\lambda,\ m \in \mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). Thus for each eigenvalue λ of $A_\xi^{N'}$ and each $m \in \mathbb{Z}$ there exists an eigenvalue λ' of $A_\xi^{N'}$ and $m' \in \mathbb{Z}$ such that

$$(-1) \times \mu(\|\xi\|/r, \lambda, m) = \mu(\|\xi\|/r, \lambda', m').$$

That is,

$$-\arctan \frac{\|\xi\|}{r\lambda} - m\pi = \arctan \frac{\|\xi\|}{r\lambda'} + m'\pi.$$

Since $-\pi/2 < \arctan x < \pi/2$, the above equality shows $m' = -m$ and

$$\lambda' = -\lambda.$$

Note that this identity still holds even if the multiplicities are taking account of. This shows (i). $\square$

Example 3.3.2. Ikawa, Sakai and Tasaki ([26, Theorem 5.1]) classified austere submanifolds of the standard sphere given as orbits of s-representations of irreducible Riemannian symmetric pairs. Applying Theorem 3.3.1 to their result we obtain austere PF submanifolds as follows. Let (U, L) be a compact

Riemannian symmetric pair, where L is connected. Denote by $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ the canonical decomposition and by $\mathrm{Ad} : L \to SO(\mathfrak{p})$ the isotropy representation. If an orbit $\mathrm{Ad}(L) \cdot x$ through $x \in \mathfrak{p}$ is an austere submanifold of the hypersphere $S(\|x\|)$ in $\mathfrak{p}$ then the orbit $P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_x) * \hat{0}$ is an austere PF submanifold of the Hilbert space $V_{\mathfrak{o}(\mathfrak{p})}$.

3.4 The arid property

In this subsection we study the arid property of PF submanifolds obtained through the parallel transport map Φ_K. We see that all theorems on weakly reflective submanifolds in Section 3.2 are formulated to the arid case. Although the essential idea of the proof does not change, for convenience we give complete proofs here. Together with the result in Section 3.3 we show examples of arid PF submanifolds which are *not* austere (therefore *not* weakly reflective) PF submanifolds.

The following theorem can be thought of an analogue of Theorem 3.2.1.

Theorem 3.4.1. *Let G be a connected compact semisimple Lie group equipped with a bi-invariant Riemannian metric induced from a negative multiple of the Killing form and K a symmetric subgroup of G such that the pair (G, K) effective. If N is an arid submanifold of the symmetric space G/K then*

(i) *$\pi^{-1}(N)$ is an arid submanifold of G, and*
(ii) *$\Phi_K^{-1}(N)$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

From this theorem we obtain the following corollary:

Corollary 3.4.2. *Let M be an irreducible Riemannian symmetric space of compact type. Denote by G the identity component of the group of isometries of M and by K the isotropy subgroup of G at a fixed $p \in M$. If N is an arid submanifold of $M = G/K$ then $\Phi_K^{-1}(N)$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

To prove Theorem 3.4.1 we need the following lemma:

Lemma 3.4.3. *Let $\mathcal{M}$ and $\mathcal{B}$ be Riemannian Hilbert manifolds, $\phi : \mathcal{M} \to \mathcal{B}$ a Riemannian submersion and N a closed submanifold of $\mathcal{B}$. Fix $p \in \phi^{-1}(N)$ and $X \in (T_p^{\perp} \phi^{-1}(N)) \backslash \{0\}$. Suppose that $\varphi_{\mathcal{M}}$ is an isometry of $\mathcal{M}$ fixing p, that $\varphi_{\mathcal{B}}$ is an isometry of $\mathcal{B}$ fixing $\phi(p)$ and that the diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \phi}\downarrow \\ \mathcal{B} & \xrightarrow{\varphi_{\mathcal{B}}} & \mathcal{B} \end{array}$$

commutes. Then the following are equivalent:

(i) *$\varphi_{\mathcal{M}}$ satisfies $\varphi_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ and $d\varphi_{\mathcal{M}}(X) \neq X$.*
(ii) *$\varphi_{\mathcal{B}}$ satisfies $\varphi_{\mathcal{B}}(N) = N$ and $d\varphi_{\mathcal{B}}(d\phi(X)) \neq d\phi(X)$.*

Proof. It is easy to see that the condition $\varphi_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ is equivalent to the condition $\varphi_{\mathcal{B}}(N) = N$. Then by commutativity of the diagram

$$\begin{array}{ccc} T_p^{\perp}\phi^{-1}(N) & \xrightarrow{d\varphi_{\mathcal{M}}} & T_p^{\perp}\phi^{-1}(N) \\ {\scriptstyle d\phi}\downarrow & & {\scriptstyle d\phi}\downarrow \\ T_{\phi(p)}^{\perp}N & \xrightarrow{d\varphi_{\mathcal{B}}} & T_{\phi(p)}^{\perp}N, \end{array}$$

the condition $d\varphi_{\mathcal{M}}(X) \neq X$ is equivalent to the condition $d\varphi_{\mathcal{B}}(d\phi(X)) \neq d\phi(X)$. This shows the lemma. □

Proof of Theorem 3.4.1. (i) Let $a \in \pi^{-1}(N)$ and $w \in (T_a^{\perp}\pi^{-1}(N))\backslash\{0\}$. Set $\eta := d\pi(w) \in T_{aK}^{\perp}N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in (T_{eK}^{\perp}N')\backslash\{0\}$. Denote by $v \in (T_e^{\perp}\pi^{-1}(N'))\backslash\{0\}$ the horizontal lift of ξ. From commutativity of (1.3.8) we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the existence of an isometry φ_w with respect to w it suffices to construct an isometry φ_v with respect to v. Let φ_η be an isometry with respect to η. Then an isometry φ_ξ with respect to ξ is defined by $\varphi_\xi := L_a^{-1} \circ \varphi_\eta \circ L_a$. Now we define φ_v as follows. Denote by $I(G/K)$ the group of isometries of G/K. By the assumption the map $L : G \to I(G/K)$, $a \mapsto L_a$ is a Lie group isomorphism onto the identity component $I_0(G/K)$ ([23, Theorem 4.1 in Chapter V]). Since $I_0(G/K)$ is a normal subgroup of $I(G/K)$ an automorphism $\varphi_v : G \to G$, $b \mapsto \varphi_v(b)$ is defined by

$$L_{\varphi_v(b)} := \varphi_\xi \circ L_b \circ \varphi_\xi^{-1}. \tag{3.4.1}$$

Note that φ_v is an isometry of G since the bi-invariant Riemannian metric on G is induced from the Killing form of $\mathfrak{g}$. Moreover since

$$\begin{aligned} &\varphi_\xi \circ \pi(b) = \varphi_\xi(bK) = \varphi_\xi \circ L_b(eK) = \varphi_\xi \circ L_b \circ \varphi_\xi^{-1}(eK) \quad \text{and} \\ &\pi \circ \varphi_v(b) = \varphi_v(b)K = L_{\varphi_v(b)}(eK) = \varphi_\xi \circ L_b \circ \varphi_\xi^{-1}(eK) \end{aligned}$$

hold for all $b \in G$ it follows from Lemma 3.4.3 that φ_v is an isometry with respect to v. This shows (i).

(ii) Let $u \in \Phi_K^{-1}(N)$ and $X \in (T_u^{\perp}\Phi_K^{-1}(N))\backslash\{0\}$. Take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$ and $\eta := d\Phi_K(X) \in (T_{aK}^{\perp}N)\backslash\{0\}$. Define N', ξ, v as in the above (i). Denote by $\hat{\xi} \in (T_{\hat{0}}^{\perp}\Phi_K^{-1}(N'))\backslash\{0\}$ the horizontal lift of ξ with respect to the Riemannian submersion $\Phi_K : V_{\mathfrak{g}} \to G/K$. From commutativity of (1.3.9) we have $g * \Phi_K^{-1}(N') = \Phi_K^{-1}(N)$ and $d(g*)\hat{\xi} = X$. Thus in order to show the existence of an isometry φ_X with respect to X it suffices to construct an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi}$. By the similar way as in (i) an isometry φ_v with respect to $v \in (T_e^{\perp}\pi^{-1}(N'))\backslash\{0\}$ can be defined. Moreover define a linear orthogonal transformation $\varphi_{\hat{\xi}}$ of $V_{\mathfrak{g}}$ by

$$\varphi_{\hat{\xi}}(u) := d\varphi_v \circ u, \quad u \in V_{\mathfrak{g}}. \tag{3.4.2}$$

Since φ_v is an automorphism of G the identity $\varphi_{\hat{\xi}}(g * \hat{0}) = (\varphi_v \circ g) * \hat{0}$ holds for all $g \in \mathcal{G}$. This together with (1.3.3) shows that the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\varphi_{\hat{\xi}}} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ G & \xrightarrow{\varphi_v} & G \end{array}$$

commutes. Hence by Lemma 3.4.3 $\varphi_{\hat{\xi}}$ is an isometry with respect to $\hat{\xi}$ and we obtain (ii). $\square$

Example 3.4.4. Let $m, n \in \mathbb{Z}_{\geq 2}$. Set $\bar{M} := S^{mn-1}(\sqrt{m}) \subset \mathbb{R}^{mn}$ and

$$M := \underbrace{S^{n-1}(1) \times \cdots \times S^{n-1}(1)}_{m \text{ times}} \subset \bar{M}.$$

Ikawa, Sakai and Tasaki [26, Example 2.3] showed that if $m = 2$ then M is a weakly reflective submanifold of $\bar{M}$. Taketomi [49, Proposition 3.1] showed that if $m \geq 3$ then M is an arid submanifold of $\bar{M}$ and is not an austere submanifold (therefore not a weakly reflective submanifold) of $\bar{M}$.

Set $G := SO(mn)$ and $K := SO(mn-1)$ so that $\bar{M} = G/K$. If $m = 2$ then $\Phi_K^{-1}(M)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$ by [37, Theorem 8] and is not a totally geodesic PF submanifold of $V_{\mathfrak{g}}$ by [37, Theorem 3]. If $m \geq 3$ then $\Phi_K^{-1}(M)$ is an arid PF submanifold of $V_{\mathfrak{g}}$ by Theorem 3.4.1 and is not an austere PF submanifold (therefore not a weakly reflective PF submanifold) of $V_{\mathfrak{g}}$ by Theorem 3.3.1.

In general it is not clear that conversely the arid property of $\Phi_K^{-1}(N)$ implies the arid property of N or not. However the next theorem shows that under suitable assumptions the arid property of $\Phi_K^{-1}(N)$ is equivalent to that of N. To explain this we now introduce some terminologies which are used in a context slightly wider than [49]. Let M be a submanifold immersed in a finite dimensional Riemannian manifold $\bar{M}$. Denote by $I(\bar{M})$ the group of isometries of $\bar{M}$. For a closed subgroup $\mathfrak{G}$ of $I(\bar{M})$ we say that M is $\mathfrak{G}$-*arid* if for each $p \in M$ and each $\xi \in T_p^{\perp} M \backslash \{0\}$ there exists an isometry φ_ξ with respect to ξ satisfying $\varphi_\xi \in \mathfrak{G}_p$, where $\mathfrak{G}_p$ denotes the isotropy subgroup of $\mathfrak{G}$ at p. Clearly $\mathfrak{G}$-weakly reflective submanifols are $\mathfrak{G}$-arid submanifolds. If $\mathfrak{G} = I(\bar{M})$ then "$\mathfrak{G}$-arid" is nothing but "arid" The same concepts and relation are also valid for PF submanifolds in Hilbert spaces.

The following theorem can be thought of an analogue of Theorem 3.2.9.

Theorem 3.4.5.
(i) *Let N be a closed submanifold of a compact Lie group G equipped with a bi-invariant Riemannian metric. Then the following are equivalent:*
  (a) *N is a $(G \times G)$-arid submanifold of G.*
  (b) *$\Phi^{-1}(N)$ is a $\mathcal{G}$-arid PF submanifold of $V_{\mathfrak{g}}$.*

(ii) *Let N be a closed submanifold of a compact normal homogeneous space G/K. Then the following are equivalent:*

(a) *N is a G-arid submanifold of G/K.*
(b) *$\pi^{-1}(N)$ is a $(G \times K)$-arid submanifold of G.*
(c) *$\Phi_K^{-1}(N)$ is a $P(G, G \times K)$-arid PF submanifold of $V_{\mathfrak{g}}$.*

Proof. (i) "(a) $\Rightarrow$ (b)": Let $u \in \Phi^{-1}(N)$ and $X \in (T_u^{\perp}\Phi^{-1}(N))\backslash\{0\}$. Take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$, $\eta := d\Phi(X) \in (T_a^{\perp}N)\backslash\{0\}$, $N' := a^{-1}N$ and $\xi := dL_a^{-1}(\eta) \in (T_e^{\perp}N')\backslash\{0\}$. Denote by $\hat{\xi} \in (T_{\hat{0}}^{\perp}\Phi^{-1}(N'))\backslash\{0\}$ the horizontal lift of ξ. From commutativity of (1.3.7) we have $g * (\Phi^{-1}(N')) = \Phi^{-1}(N)$ and $(dg*)\hat{\xi} = X$. Thus in order to show the existence of an isometry φ_X with respect to X satisfying $\varphi_X \in \mathcal{G}_u$ it suffices to construct an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi}$ satisfying $\varphi_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$. Let φ_η be an isometry with respect to η which is given by $\varphi_\eta(c) = b'cb^{-1}$ for some $(b', b) \in (G \times G)_a$. Then an isometry φ_ξ with respect to ξ is defined by $\varphi_\xi := (a, e)^{-1} \circ \varphi_\eta \circ (a, e)$, that is, $\varphi_\xi(c) := bcb^{-1}$ for $c \in G$. Define a linear orthogonal transformation $\varphi_{\hat{\xi}}$ of $V_{\mathfrak{g}}$ by

$$\varphi_{\hat{\xi}}(u) := d\varphi_\xi \circ u = bub^{-1} = \hat{b} * u, \quad u \in V_{\mathfrak{g}}.$$

Note that $\varphi_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$. Moreover by (1.3.3) the diagram

$$\begin{array}{ccc} V_{\mathfrak{g}} & \xrightarrow{\varphi_{\hat{\xi}}} & V_{\mathfrak{g}} \\ {\scriptstyle\Phi}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow \\ G & \xrightarrow{\varphi_\xi} & G \end{array}$$

commutes. Thus by Lemma 3.4.3 $\varphi_{\hat{\xi}}$ is an isometry with respect to $\hat{\xi}$ and we obtain (b).

(i) "(b) $\Rightarrow$ (a)": Let $a \in N$ and $\eta \in (T_a^{\perp}N)\backslash\{0\}$. Fix $u \in \Phi^{-1}(a)$. Denote by $X \in (T_u^{\perp}\Phi^{-1}(N))\backslash\{0\}$ the horizontal lift of η. Take $g \in P(G, G \times \{e\})$ so that $g * \hat{0} = u$ and define N', ξ, $\hat{\xi}$ as in the above (i). Let φ_X be an isometry with respect to X satisfying $\varphi_X \in \mathcal{G}_u$. Then an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi} \in (T_{\hat{0}}^{\perp}\Phi^{-1}(N'))\backslash\{0\}$ is defined by $\varphi_{\hat{\xi}} := (g*)^{-1} \circ \varphi_X \circ (g*)$. By definition $\varphi_\xi \in \mathcal{G}_{\hat{0}}$ and thus there exists $b \in G$ such that $\varphi_{\hat{\xi}}(u) = bub^{-1}$. Thus defining an isometry φ_ξ of G by $\varphi_\xi(c) := bcb^{-1}$ for $c \in G$ it follows from Lemma 3.4.3 that φ_ξ is an isometry with respect to ξ satisfying $\varphi_\xi \in (G \times G)_e$. Hence an isometry φ_η with respect to η is defined by $\varphi_\eta := l_a \circ \varphi_\xi \circ l_a^{-1}$ so that $\varphi_\eta \in (G \times G)_a$. Therefore we obtain (a).

(ii) "(a) $\Rightarrow$ (b)": Let $a \in \pi^{-1}(N)$ and $w \in (T_a^{\perp}\pi^{-1}(N))\backslash\{0\}$. Set $\eta := d\pi(w) \in (T_{aK}^{\perp}N)\backslash\{0\}$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in (T_{eK}^{\perp}N')\backslash\{0\}$. Denote by $v \in (T_e^{\perp}\pi^{-1}(N'))\backslash\{0\}$ the horizontal lift of ξ. From commutativity of (1.3.8) we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the existence of an isometry φ_w with respect to w satisfying

$\varphi_w \in (G \times K)_a = (a,e)\Delta K(a,e)^{-1}$ it suffices to construct an isometry φ_v with respect to v satisfying $\varphi_v \in (G \times K)_e = \Delta K$. Let φ_η be an isometry with respect to η which is given by $\varphi_\eta(cK) = (bc)K$ for some $b \in G_{aK}$. Then there exists $k \in K$ satisfying $b = aka^{-1}$. Thus an isometry φ_ξ with respect to ξ is defined by $\varphi_\xi := L_a^{-1} \circ \varphi_\xi \circ L_a$, that is, $\varphi_\xi = L_k$. Define an isometry φ_v of G by

$$\varphi_v(c) := kck^{-1}, \quad c \in G.$$

Note that $\varphi_v \in (G \times K)_e$. Moreover the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi_v} & G \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ G/K & \xrightarrow{\varphi_\xi} & G/K \end{array}$$

commutes. From Lemma 3.4.3 φ_v is an isometry with respect to v and we obtain (b).

(ii) "(b) $\Rightarrow$ (a)": Let $aK \in N$ and $\eta \in (T^\perp_{aK}N)\backslash\{0\}$. Denote by $w \in (T^\perp_a \pi^{-1}(N))\backslash\{0\}$ the horizontal lift of η. Define N', ξ, v as above. Let φ_w be an isometry with respect to w satisfying $\varphi_w \in (G \times K)_a$. Then an isometry with respect to v is defined by $\varphi_v := l_a^{-1} \circ \varphi_w \circ l_a$ so that $\varphi_v \in (G \times K)_e$. Thus there exists $k \in K$ such that $\varphi_v(c) = kck^{-1}$. Hence defining an isometry φ_ξ of G/K by $\varphi_\xi := L_k$ it follows from Lemma 3.4.3 that φ_ξ is an isometry with respect to ξ satisfying $\varphi_\xi \in G_{eK}$. Therefore an isometry φ_η with respect to η is defined by $\varphi_\eta := l_a \circ \varphi_\xi \circ l_a^{-1}$ so that $\varphi_\eta \in G_{aK}$. This proves (b).

Using the fact $gP(G, G \times K)_{\hat{0}}g^{-1} = P(G, G \times K)_{g*\hat{0}}$ for $g \in P(G, G \times \{e\})$ the equivalence of (b) and (c) of (ii) follows by similar arguments to (i). □

Corollary 3.4.6. *Let G, G/K be as in Theorem 3.4.5.*

(i) *Let H be a closed subgroup of $G \times G$. Then the following are equivalent:*
   (a) *an orbit $H \cdot a$ through $a \in G$ is an H-arid submanifold of G,*
   (b) *an orbit $P(G, H) * u$ through $u \in \Phi(a)$ is a $P(G, H)$-arid PF submanifold of $V_\mathfrak{g}$.*

(ii) *Let K' be a closed subgroup of G. Then the following are equivalent:*
   (a) *an orbit $K' \cdot aK$ through $aK \in G/K$ is a K'-arid submanifold of G/K,*
   (b) *an orbit $(K' \times K) \cdot a$ through $a \in G$ is a $(K' \times K)$-arid submanifold of G,*
   (c) *an orbit $P(G, K' \times K) * u$ through $u \in \Phi^{-1}(a)$ is a $P(G, K' \times K)$-arid PF submanifold of $V_\mathfrak{g}$.*

Proof. (i) By (1.3.10) we can assume without loss of generality that $a = e$. Moreover by homogeneity it suffices to consider normal vectors only at $e \in G$ or $\hat{0} \in V_\mathfrak{g}$. Thus by similar arguments as in Theorem 3.4.5 (i) the assertion follows. (ii) We can similarly reduce the case $a = e$ and by similar arguments as in Theorem 3.4.5 (ii) our claim follows. □

Example 3.4.7. Let (U, L) be a compact Riemannian symmetric pair where L connected. Denote by $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ the canonical decomposition, by $\mathrm{Ad} : L \to SO(\mathfrak{p})$ the isotropy representation and by S the standard sphere in $\mathfrak{p}$.

Let us first show that there are examples of $\mathrm{Ad}(L)$-orbits which are $\mathrm{Ad}(L)$-arid submanifolds in S. By Taketomi's result ([49, Proposition 4.4]) an orbit $N := \mathrm{Ad}(L)w$ through $w \in S$ is an $\mathrm{Ad}(L)$-arid submanifold of S if and only if N is an isolated orbit of the $\mathrm{Ad}(L)$-action on S. One can find such isolated orbits by considering the fundamental Weyl Chamber. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by F the fundamental system of the restricted root system with respect to $\mathfrak{a}$. The fundamental Weyl chamber is defined by

$$C := \{w \in \mathfrak{a} \mid \forall \alpha \in F,\ \alpha(w) > 0\}$$

with closure

$$\bar{C} := \{w \in \mathfrak{a} \mid \forall \alpha \in F,\ \alpha(w) \geq 0\}.$$

It is known ([24, Lemma 1.2]) that $\bar{C}$ is decomposed by

$$\bar{C} = \coprod_{\Delta:\ \text{subset of } F} C^{\Delta} \qquad : \text{disjoint union},$$

$$C^{\Delta} := \{w \in \mathfrak{a} \mid \forall \alpha \in \Delta,\ \alpha(w) > 0 \ \text{ and } \ \forall \beta \in F \backslash \Delta,\ \beta(w) = 0\}.$$

If a subset Δ consists of only one element then $\dim C^{\Delta} = 1$ and thus the intersection $S \cap C^{\Delta}$ consists of only one point, which implies that in this case the orbit $\mathrm{Ad}(L)w$ through $w \in C^{\Delta}$ is isolated. In this way we can obtain examples of $\mathrm{Ad}(L)$-arid submanifolds in the standard sphere S. Notice that from the classification result of austere $\mathrm{Ad}(L)$-orbits ([26, Theorem 5.1]), in particular we can choose $\mathrm{Ad}(L)$-arid orbits which are *not* austere.

Applying Corollary 3.4.6 (ii) to such examples we obtain the orbit

$$P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_w) * \hat{0}$$

which is an $P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_w)$-arid PF submanifolds in the Hilbert space $V_{\mathfrak{o}(\mathfrak{p})}$. Moreover by Theorem 3.3.1 such an arid PF submanifold is *not* an austere (therefore *not* a weakly reflective) PF submanifold in $V_{\mathfrak{o}(\mathfrak{p})}$.

The following theorem can be thought of an analogue of Theorem 3.2.13.

Theorem 3.4.8. *Let G be a connected compact Lie group with a bi-invariant Riemannian metric and H be a closed subgroup of $G \times G$. Suppose that the orbit $H \cdot e$ through $e \in G$ is an arid submanifold of G such that for each $\xi \in (T_e^{\perp}(H \cdot e))\backslash\{0\}$ an isometry φ_ξ with respect to ξ is an automorphism of G. Then the orbit $P(G, H) * \hat{0}$ through $\hat{0} \in V_{\mathfrak{g}}$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

Proof. Let φ_ξ be an isometry with respect to $\xi \in (T_e^{\perp}(H \cdot e))\backslash\{0\}$ which is an automorphism of G. Then similarly to (3.4.2) we can define an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi} \in (T_{\hat{0}}^{\perp} P(G, H) * \hat{0})\backslash\{0\}$. By homogeneity of $P(G, H) * \hat{0}$ the assertion follows. $\square$

Now we see an example of an arid submanifold $H \cdot e$ satisfying the condition in Proposition 3.4.8. Although the following $H \cdot e$ can be shown to be arid by applying Theorem 3.4.1 (i) to Taketomi's example [49, Proposition 3.1], here we give a direct proof in order to see an isometry with respect to each normal vector explicitly.

Example 3.4.9. Set $G := SO(9)$. Denote by E the 3×3 unit matrix. Define $Q \in G$ by

$$Q := \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2}E & \sqrt{3}E & E \\ \sqrt{2}E & -\sqrt{3}E & E \\ \sqrt{2}E & 0 & -2E \end{bmatrix}.$$

Set $K := Q(\{1\} \times SO(8))Q^{-1}$, $K' := SO(3) \times SO(3) \times SO(3)$ and $H := K' \times K$. Then the tangent space of the orbit $H \cdot e$ is given by

$$T_e(H \cdot e) = \mathfrak{k}' + \mathfrak{k} = \mathfrak{k}' + Q(0 \oplus \mathfrak{o}(8))Q^{-1} = Q(Q^{-1}\mathfrak{k}'Q + (0 \oplus \mathfrak{o}(8)))Q^{-1}.$$

Then it follows from the direct computations that each $X \in Q^{-1}T_e^{\perp}(H \cdot e)Q$ is written by

$$X = \begin{bmatrix} 0 & S & T \\ -S & 0 & 0 \\ -T & 0 & 0 \end{bmatrix},$$

$$S := \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T := \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

The calculation of QXQ^{-1} shows that each $Y \in T_e^{\perp}(H \cdot e)$ is written by

$$Y = \begin{bmatrix} 0 & U & V \\ -U & 0 & W \\ -V & -W & 0 \end{bmatrix},$$

$$U := \begin{bmatrix} -2x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V := \begin{bmatrix} -x-y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W := \begin{bmatrix} x-y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $x, y \in \mathbb{R}$. For each $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ we define $P_{i,j} \in O(9)$ by

$$P_{1,2} := \begin{bmatrix} 0 & E & 0 \\ E & 0 & 0 \\ 0 & 0 & E \end{bmatrix}, \quad P_{1,3} := \begin{bmatrix} 0 & 0 & E \\ 0 & E & 0 \\ E & 0 & 0 \end{bmatrix}, \quad P_{2,3} := \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & E \\ 0 & E & 0 \end{bmatrix}$$

and define an automorphism φ_{ij} of G by

$$\varphi_{ij}(A) := P_{ij}AP_{ij}^{-1}, \quad A \in G.$$

Then for each $\xi \in T_e^{\perp}(H \cdot e)$ there exists (i, j) such that φ_{ij} is an isometry with respect to ξ. Thus $H \cdot e$ is an arid submanifold of G. Since φ_{ij} is an automorphism of G it follows from Proposition 3.4.8 that the orbit $P(G, H) * \hat{0}$ is an arid PF submanifold of $V_{\mathfrak{g}}$. Notice that we can not apply Corollary 3.4.6 (i) to this example since φ_{ij} is *not* an inner automorphism of G and thus not belong to the isotropy subgroup H_e at $e \in G$.

4 Homogeneous minimal PF submanifolds in Hilbert spaces

4.1 A critical difference between finite and infinite dimensions

In the finite dimensional Euclidean spaces the following fact is known:

Theorem 4.1.1 (Di Scala [46])**.** *In finite dimensional Euclidean spaces any homogeneous minimal submanifolds must be totally geodesic.*

On the other hand the following fact follows from our results.

Theorem 4.1.2. *In infinite dimensional Hilbert spaces there exist many homogeneous minimal PF submanifolds which are not totally geodesic.*

In fact, in Section 3 we have seen many examples of minimal PF submanifolds which are orbits of the $P(G, H)$-actions. Moreover from Corollary 2.3.2 such minimal PF submanifolds are *not* totally geodesic, which shows Theorem 4.1.2. Comparing with Theorem 4.1.1, above Theorem 4.1.2 shows a critical difference between finite and infinite dimensional cases

There is a natural question:

Question 4.1.3. Can any homogeneous minimal PF submanifolds be described by an orbit of a $P(G, H)$-action ?

In the rest of this thesis, aside from this question we propose and discuss more concrete problems on homogeneous minimal PF submanifolds.

4.2 A problem related to hyperpolar $P(G, H)$-actions

Recall that an isometric action of a compact Lie group on a Riemannian manifold M is called *polar* if there exists a closed connected submanifold S of M which meets every orbit orthogonally. If S is flat in the induced metric then such an action is called *hyperpolar* ([21]). Similarly an isometric PF action of a Hilbert Lie group on a Hilbert space V is called *hyperpolar* ([21]) if there exists a closed affine subspace S of V which meets every orbit orthogonally. It was shown ([50]) that the $P(G, H)$-action is hyperpolar if the H-action on G is hyperpolar. Hyperpolar actions on irreducible Riemannian symmetric spaces of compact type were classified by Kollross [32].

Notice that almost all examples of minimal PF submanifolds we have seen in Section 3 are orbits of hyperpolar $P(G, H)$-actions. Conversely it is also interesting to consider the following problem:

Problem 4.2.1. Determine minimal orbits in a hyperpolar $P(G, H)$-action and classify the symmetric properties they have.

In order to study the above problem it is noted ([29], [20]) that a $P(G,H)$-orbit is minimal if and only if the H-orbit is minimal. This shows that the determination of minimal $P(G,H)$-orbits can be reduced to a finite dimensional problem. However the classification of symmetric properties seems not easy. As we have seen in Section 3.3 the austere property via the parallel transport map is not clear except for the spherical case. Moreover even if we classify all austere orbits in H- or $P(G,H)$-actions it seems very difficult in general to classify all weakly reflective orbits; it is very hard to assert one austere orbit is *not* weakly reflective.

In connection with such a classification problem the author is now giving attention to the structure of weakly reflective submanifolds. Let M be a weakly reflective submanifold of a Riemannian manifold $\bar{M}$. Denote by $I(\bar{M})$ the group of isometries of $\bar{M}$ and by $I_0(\bar{M})$ its identity component. There are at least two kinds of weakly reflective submanifolds:

(a) for each $p \in M$ and each $\xi \in T_p^\perp M$ there exists $\nu_\xi \in I_0(\bar{M})$ (: *identity component*) such that $\nu_\xi(p) = p$, $d\nu_\xi(\xi) = -\xi$ and $\nu_\xi(M) = M$.

(b) for each $p \in M$ there exists an *involutive* isometry $\nu_p \in I(\bar{M})$ which is *independent* of the choice of $\xi \in T_p^\perp M$ such that $\nu_p(p) = p$, $d\nu_p(\xi) = -\xi$ and $\nu_p(M) = M$.

For example, consider an isometric action of cohomogeneoity one on $\bar{M}$. It is known ([44], [26]) that in this case any singular orbit M is a weakly reflective submanifold of $\bar{M}$. More precisely if M has one codimension in $\bar{M}$ then M satisfies the condition (b) because the identity component of each isotropy subgroup acts transitively on each normal space. On the other hand, if the codimension of M is equal or greater than two then it can be seen that M satisfies the condition (a). Note that there exist examples of weakly reflective submanifolds which satisfy both conditions (a) and (b): a weakly reflective submanifold M reviewed in Example 3.4.4 (the case $m = 2$) satisfies (b) for all $n \in \mathbb{Z}_{\geq 2}$ and in particular if n is even then it also satisfies (a). Although examples by Ohno [39] and by Kimura-Mashimo [28] satisfy the condition (a), Enoyoshi's example ([8]) does not satisfy (a) but (b). It should be also noted ([37]) that under suitable assumptions if M satisfies the condition (b) then its inverse image under the parallel transport map is a weakly reflective PF submanifold satisfying the condition (b). It can be a problem to consider whether weakly reflective submanifolds which does not satisfy both conditions (a) and (b) exist or not. Also for arid submanifolds similar types may work and might be useful to study their structure.

4.3 A problem related to affine Kac-Moody symmetric spaces

In this subsection we mention a problem related to affine Kac-Moody symmetric spaces.

Following [16] (see also [17]) we first review briefly the fundamental facts on affine Kac-Moody symmetric spaces. Let $\mathfrak{g}$ be a simple Lie algebra with Killing form $(\cdot,\cdot)_0$ and $\sigma : \mathfrak{g} \to \mathfrak{g}$ an automorphism of finite order. The twisted Lie algebra $L(\mathfrak{g},\sigma)$ is defined by

$$L(\mathfrak{g},\sigma) := \{u : \mathbb{R} \to \mathfrak{g} : \ C^\infty \text{ map} \mid \forall t \in \mathbb{R},\ u(t+2\pi) = \sigma(u(t))\}$$

with pointwise bracket $[u,v]_0(t) := [u(t),v(t)]$. For each $\lambda \in \mathbb{R}\backslash\{0\}$, define an alternating linear 2-form ω_λ on $L(\mathfrak{g},\sigma)$ by

$$\omega_\lambda(u,v) := \lambda \int_0^1 (u'(t),v(t))_0 dt.$$

The *affine Kac-Moody algebra* $\hat{L}(\mathfrak{g},\sigma)$ is defined by

$$\hat{L}(\mathfrak{g},\sigma) := L(\mathfrak{g},\sigma) + \mathbb{R}c + \mathbb{R}d$$

with bracket

$$[u,v] := [u,v]_0 + \omega_\lambda(u,v)c, \quad [d,u] := u', \quad [c,x] := 0$$
$$u,v \in L(\mathfrak{g},\sigma),\ x \in \hat{L}(\mathfrak{g},\sigma),$$

with derived algebra $L(\mathfrak{g},\sigma) + \mathbb{R}c$ and center $\mathbb{R}c$. Note that the isomorphism class of $\hat{L}(\mathfrak{g},\sigma)$ does not depend on λ.

Let G be a simply connected compact simple Lie group with Lie algebra $\mathfrak{g}$ and $\sigma : G \to G$ an automorphism of finite order. The twisted loop group

$$L(G,\sigma) := \{g : \mathbb{R} \to G : \ C^\infty\text{map} \mid \forall t \in \mathbb{R},\ g(t+2\pi) = \sigma(g(t))\}$$

with pointwise multiplication is a Frechet Lie group with Lie algebra $L(\mathfrak{g},\sigma)$. Then we define a Frechet Lie group $\tilde{L}(G,\sigma)$ whose Lie algebra is $\tilde{L}(\mathfrak{g},\sigma) := L(\mathfrak{g},\sigma) + \mathbb{R}c$. Consider the 2-form ω_λ defined above. The induced left invariant closed 2-form on $L(G,\sigma)$ is still denoted by ω_λ. We can choose λ so that $\frac{1}{2\pi}\omega_\lambda$ defines an integral cohomology class in $H^2(L(G,\sigma),\mathbb{Z})$. Then there corresponds a principal S^1-bundle P over $L(G,\sigma)$ with a connection whose curvature form coincides with ω. Then $\tilde{L}(G,\sigma)$ is defined as a group of all bundle automorphisms on P preserving the connection. Note that $\tilde{L}(G,\sigma)$ extends $L(G,\sigma)$ by S^1. Note also that we can (and will) choose λ so that $\tilde{L}(G,\sigma)$ is simply connected. Finally we define the *Kac-Moody group* $\hat{L}(G,\sigma)$ whose Lie algebra is $\hat{L}(\mathfrak{g},\sigma)$. This is a Frechet Lie group defined by the semi-direct product

$$\hat{L}(G,\sigma) := S^1 \ltimes \tilde{L}(G,\sigma).$$

where the S^1-action on $\tilde{L}(G,\sigma)$ is induced from the natural $\mathbb{R}$-action on $L(G,\sigma)$ shifting the parameter on loops.

Define a non-degenerate symmetric bi-linear form of index 1 on $\hat{L}(\mathfrak{g},\sigma)$ by

$$\langle u + r_1 c + s_1 d,\ v + r_2 c + s_2 d\rangle := \int_0^{2\pi} \langle u(t), v(t)\rangle_0 dt + r_1 s_2 + r_2 s_1$$
$$u, v \in L(\mathfrak{g},\sigma),\ r_i, s_j \in \mathbb{R},$$

where $\langle\cdot,\cdot\rangle_0$ denotes the negative multiple of the Killing form. By left translation we can define a metric on $\hat{L}(G,\sigma)$ so that $\hat{L}(G,\sigma)$ is a Lorentz manifold. Since the metric is bi-invariant the map $g \mapsto g^{-1}$ is an involutive isometry and thus we can think of $\hat{L}(G,\sigma)$ as a symmetric space.

Roughly speaking, an affine Kac-Moody symmetric space is by definition either an affine Kac-Moody group $\hat{G} := \hat{L}(G,\sigma)$ with above metric or the quotient $\hat{G}/\hat{K}$, where $\hat{K}$ is the fixed point set of an involution $\hat{\theta}$ on $\hat{G}$. More precisely here we consider an involution of the *second* kind. There are two kinds of involutions on Kac-Moody group $\hat{G}$: the first kind involution inducing the identity on $S^1 \subset \tilde{L}(G,\sigma)$ and the second one inducing the reflection on this S^1. We consider only the second one so that the extension $L(G,\sigma)$ to $\hat{L}(G,\sigma)$ is not canceled in the quotient. Also we do not mention the manifold structures of $\hat{G}$ and $\hat{G}/\hat{K}$. Actually they are *tame Frechet manifolds* ([14]), where the inverse function theorem is valid. Moreover the unique existence theorem of Levi-Civita connection for $\hat{G}$ and $\hat{G}/\hat{K}$ can be proved. For details of such manifold structures and geometry of $\hat{G}$ and $\hat{G}/\hat{K}$, see Popescu [43]. For classification of involutions of affine Kac-Moody algebras, see Groß [13] and Heintze-Groß [22]. For complex Kac-Moody groups and the duality of affine Kac-Moody symmetric spaces, see Freyn [10].

One of the most striking similarities between finite dimensional Riemannian symmetric spaces and affine Kac-Moody symmetric spaces comes from their isotropy representations. In the finite dimensional case it is known that the isotropy representation of a symmetric space is polar and the converse essentially holds: any irreducible polar representation is orbit equivalent to the isotropy representation of a symmetric space (Dadok [6]). Also principal orbits of the isotropy representation are isoparametric in the sense of Terng [50], and under suitable assumptions conversely any irreducible isoparametric submanifolds in Euclidean spaces are orbits of polar representation (Thorbergsson [53]). In the case of affine Kac-Moody symmetric spaces similar properties hold. For an affine Kac-Moody symmetric space, there corresponds a hyperpolar PF action on a Hilbert space (Kollross [32], Terng [51], Heintze-Groß [22]). A principal orbit of the hyperpolar PF action is isoparametric, and the converse also holds under a suitable assumption (Heintze-Liu [19]).

We now propose the following problem:

Problem 4.3.1. Find similar properties between minimal orbits in the isotropy representation of Riemannian symmetric spaces and minimal orbits in the isotropy representation of affine Kac-Moody symmetric spaces.

For example, in the finite dimensional case it was shown that there exists a unique minimal orbit in each strata of the stratification of orbit types

(Hirohashi-Song-Takagi-Tasaki [25]). It is interesting to ask whether similar result also holds for affine Kac-Moody symmetric spaces. In the finite dimensional case austere orbits and weakly reflective orbits of the isotropy representation were classified by Ikawa, Sakai abd Tasaki [26]. It may be also interesting to study similar classification problem for affine Kac-Moody symmetric spaces. These problem would also relate with the problem mentioned in the last subsection.

References

[1] C. J. Atkin, *The Hopf-Rinow theorem is false in infinite dimensions*, Bull. London Math. Soc. **7** (1975), no. 3, 261-266.

[2] C. J. Atkin, *Geodesic and metric completeness in infinite dimensions*, Hokkaido Math. J. **26** (1997), no. 1, 1-61.

[3] B. Booss, D. D. Bleecker, *Topology and Analysis. The Atiyah-Singer index formula and gauge-theoretic physics*, Translated from the German by Bleecker and A. Mader. Universitext. Springer-Verlag, New York, 1985.

[4] R. L. Bryant, *Some remarks on the geometry of austere manifolds*, Bol. Soc. Brasil. Mat. (N.S.) **21** (1991), no. 2, 133-157.

[5] L. Conlon, *The topology of certain spaces of paths on a compact symmetric space*, Trans. Amer. Math. Soc. **112** (1964) 228-248.

[6] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), no. 1, 125-137.

[7] M. Dajczer, L. A. Florit, *A class of austere submanifolds*, Illinois J. Math. **45** (2001), no. 3, 735-755.

[8] K. Enoyoshi, *Principal curvatures of homogeneous hypersurfaces in a Grassmann manifold* $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\,\mathbb{O})$ *by the* G_2*-action*, to appear in Tokyo J. Math.

[9] H. Ewert, *A splitting theorem for equifocal submanifolds in simply connected compact symmetric spaces*, Proc. Amer. Math. Soc. **126** (1998), no. 8, 2443–2452.

[10] W. Freyn, *Affine Kac-Moody symmetric spaces*, arXiv:1109.2837 (2011).

[11] C. Gorodski, G. Thorbergsson, *Variationally complete actions on compact symmetric spaces* J. Differential Geom. **62** (2002), no. 1, 39-48.

[12] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007.

[13] C. Groß, *s-representations for involutions on affine Kac-Moody algebras are polar*, Manuscripta Math. **103** (2000), no. 3, 339-350.

[14] R. S. Hamilton, *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65-222.

[15] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math., **148** (1982), 47-157.

[16] E. Heintze, *Toward symmetric spaces of affine Kac-Moody type*, Int. J. Geom. Methods Mod. Phys. **3** (2006), no. 5-6, 881-898.

[17] E. Heintze, *Theory of Kac-Moody Symmetric Spaces*, Noted by Nobutaka Boumuki and Kaname Hashimoto, lectures at OCAMI DG workshop, Oct.1-5, 2007, OCAMI Preprint Series 2008, 08-4.

[18] E. Heintze, X. Liu, *A splitting theorem for isoparametric submanifolds in Hilbert space*, J. Differential Geom. **45 (**1997), no. 2, 319-335.

[19] E. Heintze, X. Liu, *Homogeneity of infinite-dimensional isoparametric submanifolds*, Ann. of Math. (2) **149** (1999), no. 1, 149-181.

[20] E. Heintze, X. Liu, C. Olmos, *Isoparametric submanifolds and a Chevalley-type restriction theorem*, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math., **36**, Amer. Math. Soc., Providence, RI, 2006.

[21] E. Heintze, R. Palais, C.-L. Terng, G. Thorbergsson, *Hyperpolar actions on symmetric spaces*, Geometry, topology, & physics, 214-245, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.

[22] E. Heintze, C. Groß, *Finite order automorphisms and real forms of affine Kac-Moody algebras in the smooth and algebraic category*, Mem. Amer. Math. Soc. **219** (2012)

[23] S. Helgason, *Differential Geometry, Lie groups, and Symmetric Spaces*, Corrected reprint of the 1978 original. Graduate Studies in Mathematics, **34**. American Mathematical Society, Providence, RI, 2001.

[24] D. Hirohashi, T. Kanno, H. Tasaki, *Area-minimizing of the cone over symmetric R-spaces*, Tsukuba J. Math. **24** (2000), no. 1, 171-188.

[25] D. Hirohashi, H. Tasaki, H. Song, R. Takagi, *Minimal orbits of the isotropy groups of symmetric spaces of compact type*, Differential Geom. Appl. **13** (2000), no. 2, 167-177.

[26] O. Ikawa, T. Sakai, H. Tasaki, *Weakly reflective submanifolds and austere submanifolds*. J. Math. Soc. Japan **61** (2009), no. 2, 437-481.

[27] M. Ionel, T. Ivey, *Austere submanifolds of dimension four: examples and maximal types*, Illinois J. Math. **54** (2010), no. 2, 713-746.

[28] T. Kimura, K. Mashimo, *Classification of Cartan embeddings which are austere submanifolds*, a preprint.

[29] C. King, C.-L. Terng, *Minimal submanifolds in path space*, Global analysis in modern mathematics, 253-281, Publish or Perish, Houston, TX, 1993.

[30] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*. Vol. I and Vol. II. Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[31] N. Koike, *On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space*, Japan. J. Math. (N.S.) **28** (2002), no. 1, 61-80.

[32] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 571-612.

[33] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Math-

ematics, 191. Springer-Verlag, New York, 1999.
[34] Dominic S. P. Leung, *The reflection principle for minimal submanifolds of Riemannian symmetric spaces*, J. Differential Geom. **8** (1973), 153-160.
[35] Dominic S. P. Leung, *On the classification of reflective submanifolds of Riemannian symmetric spaces*, Indiana Univ. Math. J. **24** (1974/75), 327-339.
[36] O. Loos, *Symmetric Spaces. II: Compact Spaces and Classification.* W. A. Benjamin, Inc., New York-Amsterdam 1969.
[37] M. Morimoto, *On weakly reflective PF submanifolds in Hilbert spaces*, to appear in Tokyo J. Math.
[38] M. Morimoto, *Austere and arid properties for PF submanifolds in Hilbert spaces*, arXiv:1912.08037, submitted.
[39] S. Ohno, *A sufficient condition for orbits of Hermann actions to be weakly reflective*, Tokyo J. Math. **39** (2016), no. 2, 537-564.
[40] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299-340.
[41] R. S. Palais, C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math., vol 1353, Springer-Verlag, Berlin and New York, 1988.
[42] U. Pinkall, G. Thorbergsson, *Examples of infinite dimensional isoparametric submanifolds*, Math. Z. **205** (1990), no. 2, 279-286.
[43] B. Popescu, *Infinite dimensional symmetric spaces*, Thesis, University of Augsburg (2005).
[44] F. Podestà, *Some remarks on austere submanifolds*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 157-160.
[45] A. Pressley, G. Segal, *Loop Groups*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.
[46] A. J. D. Scala, *Minimal homogeneous submanifolds in Euclidean spaces*, Ann. Global Anal. Geom. **21** (2002), no. 1, 15-18.
[47] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) **80** (1964), 382-396.
[48] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861-866.
[49] Y. Taketomi, *On a Riemannian submanifold whose slice representation has no nonzero fixed points*, Hiroshima Math. J. **48** (2018), no. 1, 1-20.
[50] C.-L. Terng, *Proper Fredholm submanifolds of Hilbert space.* J. Differential Geom. **29** (1989), no. 1, 9-47.
[51] C.-L. Terng, *Polar actions on Hilbert space.* J. Geom. Anal. **5** (1995), no. 1, 129-150.
[52] C.-L. Terng, G. Thorbergsson, *Submanifold geometry in symmetric spaces.* J. Differential Geom. **42** (1995), no. 3, 665-718.
[53] G. Thorbergsson, *Isoparametric foliations and their buildings*, Ann. of Math. (2) **133** (1991), no. 2, 429-446.

Masahiro Morimoto
Department of Mathematics, Graduate School of Science,
Osaka City University.
3-3-138 Sugimoto, Sumyoshi-ku, Osaka 558-8585 JAPAN.
email: mmasahiro0408@gmail.com
webpage: https://sites.google.com/view/mmorimoto-e