Brane transitions in a circle and Exceptional groups

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Abstract

It is known that multiple M2-branes on various backgrounds are described by the $\mathcal{N} = 4$ super Chern-Simons theories. After applying the localization technique, these theories reduce to matrix models which we call the super Chern-Simons matrix models. And the super Chern Simons matrix models without rank differences are described with the Fredholm determinant for the quantum curves. At the same time, those matrix models are expressed by the brane configurations in a circle in the IIB string theory. In this thesis, from the correspondences between the brane configurations and the quantum curves, we explain the new brane transitions and the hidden structure behind the brane transitions, which were found in my doctoral course.

From the correspondences between the brane configurations and the quantum curves, we can regard the Weyl groups which are the symmetries of the curves, as the brane transitions. Then after separating the Hanany-Witten transitions, the new brane transitions were found (called local rule).

Moreover, by the duality cascades with the Hanany-Witten transitions, the affine Weyl group appears in the space of the brane configurations naturally. We found that the fundamental domain (that the Hanany-Witten transitions do not cause the duality cascades) in the Weyl chamber is nothing but the affine Weyl chamber. The affine Weyl group takes any brane configuration into the fundamental domain by the translations in the space of the brane configurations, which realize the duality cascades. At the same time, it is found that the fundamental domain is a convex polytope that can fill the space of the brane configurations.

Also, in the correspondences between the quantum curves and the brane configurations, the Weyl groups of the curves interpreted as the brane transitions are smaller than the original symmetries of the curves by $\mathbb{Z}_2$ folding. However, by introducing the super determinant operator interpreted as the insertion of the Fayet-Iliopoulos parameters, we can regard the full Weyl group as the brane transitions.

For the brane configurations relating to the super Chern-Simons matrix models in the case that the symmetries of the quantum curves are not fully known yet, we cannot construct the affine Weyl group. However, from the property of the Hanany-Witten transitions, the fundamental domains of the duality cascades are at least polytopes whose planes facing each other are parallel.

This dissertation is mainly based on the joint research with Professor Sanefumi Moriyama, Tomoki Nakanishi and Kazunobu Matsumura [1,2].

Contents

1 Introduction

The goal of physics is to explain our mother nature. However, due to its complexity, we often construct simple solvable models to capture its essence (such as ideal gas). Particularly, in particle physics, we have been searching for an ultimate theory unifying all interactions. At present, M theory and string theory are expected to be one of the best candidates. We would like to find the essence of these theories from the correspondence to the integrable systems. The integrable systems are solvable by the high symmetries, such as soliton equations and Painlevé equations.

Here, string theory is a ten-dimensional theory consisting of strings (one-dimensional objects) and branes (higher-dimensional objects), naturally contain not only electromagnetic, weak and strong interactions but also gravity and perturbatively five types of such a theory are known. And in the low energy limit, there are the supergravity theories corresponding to the string theories. Furthermore, the supergravity theory is composed up to eleven dimension. M theory is given as the theory containing it in the low energy limit and string theory by compactifying one dimension. Due to the T and S dualities among string theories, all string theories are derived from M theory, therefore the M theory is expected as an ultimate theory unifying all interactions.

Although the M theory and the string theories were mysterious for a long time, various interesting aspects become clearer by now. As explained below, ABJM theory and its generalizations describe multiple M2-branes. Since they have a large number of supersymmetries, we expect them to have a simple and systematic structure such as the integrable systems. Indeed, in previous works, the partition function and some supersymmetric correlation functions of these theories were reduced to matrix models [3]; the non-perturbative instanton effects were computed exactly [4–7] and related to the free energy of topological string theories [8, 9]; the relation between the ABJM theory and the Painlevé equation was observed [10].

Besides, in my doctoral course, with my collaborators, we have added many interesting structures. We clarified the group-theoretical structure and the multi-covering structure of the quantum A-period [11]; we revealed the correspondence between the ABJM matrix model and the two-dimensional Toda lattice (and mKP) hierarchy [12, 13]; we revealed the correspondence between the duality cascades and the affine Weyl groups [2]. In my doctoral thesis, I mainly focus on the last topic of the correspondence between the duality cascades and the affine Weyl groups in the main text. I also add appendices to cover various other structures and to introduce our works in my doctoral course.

The ABJM theory is the $\mathcal{N} = 6$ superconformal Chern-Simons theory with the two bifundamental matters and the gauge group $U(N_1)_k \times U(N_2)_{-k}$ [14–16] where the subscripts denote the Chern-Simons levels with $k > 0$. It is known that this theory describes the worldvolume theory of the multiple M2-branes with $\min(N_1, N_2)$ M2-branes and $|N_1 - N_2|$ flactional M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$. In the IIB string theory, this situation is described by a brane configuration with 5-branes (an NS5-brane and a $(1,k)$5-brane) aligned in a circle and D3-branes located at the two intervals between the 5-branes.

The ABJM theory is reduced to the matrix model by applying the localization technique on S^3 [3,17]. This is called ABJM matrix model. Also, we can apply the localization technique to the vacuum expectation values of the BPS Wilson loops [18] besides the partition function. It was also found that the dualities between the vacuum expectation values of the half BPS Wilson loops and the partition function [19, 20]. The two-point functions were constructed as the generalizations of the one-point functions of the half BPS Wilson loops [21]. Furthermore, it was found that the correlation functions satisfy many relations deeply concerning with the integrable systems [12, 22, 23] and correspond to the soliton models [13]. In particular, the one- and two-point functions correspond to the mKP and two-dimensional Toda lattice (2DTL) equations respectively. The reduction from the two-point functions to the one-point functions agrees with the reduction from the 2DTL equations to the mKP equations in the correspondence.

In the ABJM matrix model, by the Fermi gas formalism [8, 24], when we regard the gauge rank $\min(N_1, N_2)$ as the number of free fermions, we can consider the grand canonical partition function by introducing the fugacity z. Also, in the 't Hooft expansion, for the partition function of the ABJM matrix model, it was found that the perturbative corrections are summed up to the Airy function and reproduce the degree of freedom $N^{3/2}$ [25,26]. Furthermore, This expansion allows us to reveal the non-perturbative effects, worldsheet instantons [27] and membrane instantons [28].

In the case with equal ranks $N_1 = N_2 = N$, the grand canonical partition function is described by the Fredholm determinant with the spectral operator $\widehat{H}$,

$$\Xi_k(z) = \sum_{N=0}^{\infty} z^N Z_k(N) = \det\left(1 + z\widehat{H}^{-1}\right), \tag{1.1}$$

where the spectral operator takes the form as $\widehat{H} = \widehat{\mathcal{Q}}\widehat{\mathcal{P}}$ with the hyperbolic cosine operators

$$\widehat{\mathcal{Q}} = \widehat{Q}^{\frac{1}{2}} + \widehat{Q}^{-\frac{1}{2}}, \quad \widehat{\mathcal{P}} = \widehat{P}^{\frac{1}{2}} + \widehat{P}^{-\frac{1}{2}}, \quad (\widehat{Q} = e^{\widehat{q}}, \quad \widehat{P} = e^{\widehat{p}}), \tag{1.2}$$

and z denotes the fugacity which relates to the chemical potential μ as $z = e^{\mu}$. In this canonical quantization, the Chern-Simons level k plays the role of the Plank constant as $\hbar = 2\pi k$ and the above operators $\widehat{q}$ and $\widehat{p}$ satisfy the commutation relation $[\widehat{q}, \widehat{p}] = i\hbar$. If we consider the WKB expansion for the grand canonical partition function, the surface of the phase space becomes the Fermi surface. From the area of this Fermi surface, it was found not only that the ABJM matrix model reproduces the degree of freedom of the N multiple M2-branes $N^{3/2}$ in the large N limit but also that the partition function is the Airy function [8]. Furthermore from the shape of the Fermi surface, it was found that the Fermi surface relates to the dual toric diagram associated with the local $\mathbb{P}^1 \times \mathbb{P}^1$. Actually, this spectral operator is derived by quantizing the mirror curve of the local $\mathbb{P}^1 \times \mathbb{P}^1$ [29], therefore we also call this operator the quantum curve.

In the non-perturbative effects, both the worldsheet instantons and the membrane instantons are divergent at some values of k. But, by considering the bound states of them, the divergences are canceled among the instantons [6]. Also, if we redefine the fugacity properly, the contributions from the bound states are included in the worldsheet instantons and we can take a simpler view that the cancelations for the divergences of the instantons appear between the worldsheet instantons and the membrane instantons. Also, from the geometric viewpoint, we can reexamine this discussion for the instantons as follows. First, we can compute the two periods by integrating along two cycles, A-period and B-period, for the algebraic curve of genus one. As mentioned above, the ABJM matrix model is described by the quantum curve associated with the algebraic curve of genus one. Thus, we can construct the quantum corrected periods and compute them [7, 30]. As the result, the quantum A- and B-period give the redefinition of the chemical potential and the derivatives of the free energy respectively. The B-period is well analyzed and it was found that it is determined by the BPS indices and has the group-theoretical structure and the multi-covering structure [31–34]. Here, the group-theoretical structure means that the B-period is written in terms of the characters for the symmetries of the curve. And the multi-covering structure means that the instanton effects of high degree include those of lower degrees. And in our works, we found that there is the same structures for the quantum A-period [11, 34].

There are generalizations of the ABJM theory which keep $\mathcal{N} = 4$ while increasing the number of the 5-branes [35–39]. The matrix models obtained from the theories by the localization technique, are the super Chern-Simons matrix models with the gauge group $\prod_{i=1}^{r} U(N_i)$ and the Chern-Simons levels given by

$$k_a = \frac{k}{2}(s_a - s_{a-1}), \quad s_a = \pm 1, \quad (a = 1, 2, \cdots, r), \tag{1.3}$$

where the models are of the circular type with the identification $s_0 = s_r$. Then, the brane configurations are constructed with the 5-branes that an NS5-brane and a $(1,k)$5-brane are respectively labeled by $s_a = +1$ and $s_a = -1$ and they are arranged in the sequence of s_a. For example, in the case that s_a are arranged as

$$\{s_1, s_2, \cdots\} = \{\underbrace{+1, \cdots, +1}_{p_1} \underbrace{-1, \cdots, -1}_{q_1} \underbrace{+1, \cdots, +1}_{p_2} \underbrace{-1, \cdots, -1}_{q_1}, \cdots\}, \tag{1.4}$$

the corresponding brane configuration is

$$\langle \underbrace{\bullet \bullet \cdots \bullet}_{p_1} \underbrace{\circ \circ \cdots \circ}_{q_1} \underbrace{\bullet \bullet \cdots \bullet}_{p_2} \underbrace{\circ \circ \cdots \circ}_{q_2} \cdots \rangle, \tag{1.5}$$

where $\bullet$ and $\circ$ respectively correspond to an NS5-brane and a $(1,k)$5-brane, also the grand canonical partition function of the corresponding matrix model is described with the quantum curve taking the form as

$$\widehat{H} = \cdots \widehat{\mathcal{Q}}^{q_2} \widehat{\mathcal{P}}^{p_2} \widehat{\mathcal{Q}}^{q_1} \widehat{\mathcal{P}}^{p_1}, \tag{1.6}$$

we call this model $(p_1, q_1, p_2, q_2, \cdots)$ model.

In the case with the rank differences, there are the above super Chern-Simons matrix models generalized from the ABJM matrix model. However, the grand canonical partition functions in those models have not been fully described by the Fredholm determinant, and the quantum curves corresponding to those matrix models are not clear (see [40] for recent progress). To overcome the difficulties, the correspondences between the quantum curves and the brane configurations with the rank differences were constructed by the Hanany-Witten transitions [41],

$$\cdots K \circ L \circ M \cdots = \cdots K \circ K - L + M \circ M \cdots, \tag{1.7}$$

$$\cdots K \bullet L \bullet M \cdots = \cdots K \bullet K - L + M \bullet M \cdots, \tag{1.8}$$

$$\cdots K \bullet L \circ M \cdots = \cdots K \circ K - L + M + k \bullet M \cdots, \tag{1.9}$$

$$\cdots K \circ L \bullet M \cdots = \cdots K \bullet K - L + M + k \circ M \cdots, \tag{1.10}$$

in addition to the correspondences between the quantum curves and the brane configurations without the rank differences. Here, $\bullet$ and $\circ$ respectively denote an NS5-brane and a $(1,k)$5-brane, also K, L and M denote D3-branes in each interval of 5-branes. These transitions can be stated as the conservation of the RR-charge for an NS5-brane,

$$q_{\mathrm{RR}} = -\frac{1}{2}\Big((\#\mathrm{D5})|_{\mathrm{L}} - (\#\mathrm{D5})|_{\mathrm{R}}\Big) + (\#\mathrm{D3})|_{\mathrm{L}} - (\#\mathrm{D3})|_{\mathrm{R}}, \tag{1.11}$$

and that for a $(1,k)$5-brane,

$$q_{\mathrm{RR}} = -\frac{k}{2}\Big((\#\mathrm{NS5})|_{\mathrm{L}} - (\#\mathrm{NS5})|_{\mathrm{R}}\Big) + (\#\mathrm{D3})|_{\mathrm{L}} - (\#\mathrm{D3})|_{\mathrm{R}}, \tag{1.12}$$

where $(\#\mathrm{D5})|_{\mathrm{L/R}}$ and $(\#\mathrm{NS5})|_{\mathrm{L/R}}$ are respectively the numbers of the D5-branes and the NS5-branes on the left/right sides of an original 5-brane (NS5 and $(1,k)$5-brane), and $(\#\mathrm{D3})|_{\mathrm{L\ or\ R}}$ denotes the number of the D3-branes ending to the original 5-brane from the left/right.

In this thesis, we consider the curves whose parameters are transformed by the known groups. Namely, for certain brane configurations, the corresponding quantum curves constructed in (1.6) has nice classical counterparts known as the del Pezzo curves. The transformations of the parameters of these curves form the Weyl group of exceptional Lie algebras. In particular, the A_1, D_5 and E_7 curves correspond respectively to the brane configurations with (one NS5-brane, one $(1,k)$5-brane), (two NS5-branes, two $(1,k)$5-branes) and (two NS5-branes, four $(1,k)$5-branes). And if we regard the Weyl group symmetries of the curves as the brane transitions, it is found that there are the new brane transitions after separating the Hanany-Witten transitions [1, 42].

For the brane configurations in a circle, the Hanany-Witten transitions may reduce the number of the D3-branes at the interval of the 5-branes. The number of the D3-branes corresponds to the rank of the gauge group in the field theory and if we continue to apply this process to reduce the rank, finally we arrive at the situation of either the lower ranks do not appear or a rank becomes negative. The negative rank is regarded as the anti-D3-branes in the string theory, so when a negative rank appears, it is interpreted as the supersymmetry is broken. The process reducing the ranks is known as the duality cascade [43–46]. For the supersymmetric brane configurations in a circle, we can ask following questions,

- Is the process of the duality cascades always completed regardless of the initial brane configuration?
- Is the endpoint of the duality cascades unique regardless of the flow path of the duality cascades?

- After applying the duality cascades, if the brane configurations appearing in the duality cascades are all identified, what the fundamental domain of duality cascades is, more concretely, whether it is finite and whether it is the connected one?

As in the following paragraph, we can answer these questions with the help of the affine Weyl groups.

When we consider the correspondences between the supersymmetric brane configurations and the quantum curves, we need to choose the lowest rank as the reference and Weyl groups act to the brane configurations so that the reference rank is invariant. However, through the Hanany-Witten transitions, we may encounter one of the ranks lower than the reference rank. In this case, we need to choose the lowest rank as a new reference rank again. And we can continue this process until no lower ranks appear. This series of the duality transformations is nothing but the duality cascade [46]. And, from the symmetries of the quantum curves we investigate carefully the fundamental domain of the duality cascades that the lower ranks do not appear by the Hanany-Witten transitions. The symmetries of the quantum curves are the Weyl groups and regarded as the brane transitions. Then since the reference rank is fixed, of course, the duality cascade changing the reference rank is not included there. However, after we identify the fundamental domain of the duality cascades, by considering the reflections about the boundary planes of the region, we can extend the Weyl groups to the affine Weyl groups and it is found that the extensions agree with the duality cascades. Then, it is also found that the overall rank plays a role of the eigenvalue of the grading operator of the affine Weyl groups. Furthermore, the fundamental domain divided by the Weyl group is no other than the affine Weyl chamber, therefore it is found that the endpoint of the duality cascades is unique and the fundamental domain is a convex polytope that can fill the space of the brane configurations.

Also, when we relate the brane configurations in a circle to the quantum curve, only part of the Weyl group surviving in the $\mathbb{Z}_2$ folding remains as symmetries and we shall regard them as brane transitions. Concretely, the B_3 Weyl group for the D_5 curve and F_4 Weyl group for E_7 curve are regarded as brane transitions. In this thesis, we show that the full Weyl group of the quantum curve can be regarded as brane transitions by the deformation with the FI parameters. Then, from the full Weyl group of the curve, the brane transitions interpreted as "halves" Hanany-Witten transitions appear.

In section 2, we discuss the correspondences between the quantum curves and the cyclic brane configurations. And we propose the local rule as the new brane transitions after separating the Hanany-Witten transitions. In section 3, we derive the affine Weyl groups from duality cascades. This is a main topic in this thesis. In section 4, we discuss advanced topics for the case when the symmetries regarded as the brane transitions is not known and the case with the deformation with the FI parameters.

2 Quantum curves and brane configurations

In this section, we explain several topics. First, we explain that the super Chern-Simons matrix models without the rank differences are represented with the Fredholm determinant of the quantum curves briefly. Afterwards, we consider the generalizations to the case with the rank differences. Then we can embed the space of the brane configurations into the space of the parameters of the quantum curves. And at the same time, the Weyl group symmetries of the curve are folded to the smaller symmetries. Since the Weyl group acting to the subspace of the brane configurations in the parameter space of the curves includes the transitions interpreted as the Hanany-Witten transitions, by regarding the Weyl groups as the brane transitions, we can propose the new brane transitions after separating the Hanany-Witten transitions [1,42].

2.1 Super Chern-Simons matrix models and quantum curves

In this subsection, we explain the relation between the super Chern-Simons matrix models and quantum curves (see appendix A for detail calculations).

The $\mathcal{N}=4$ superconformal Chern-Simons theories describe the brane configurations with 5-branes in a circle in the IIB string theory. By the localization technique, they reduce to the super Chern-Simons matrix models. In the matrix model with the gauge group $\prod_{a=1}^{r}U(N)_{k_a}$, the partition function labeled by $p_1+q_1+p_2+q_2+\cdots=r$ corresponding to the brane configuration in (1.5), is denoted by

$$Z_k^{(p_1,q_1,\cdots)}(N)=\int\prod_{a=1}^{r}\frac{D^Nx^{(a)}}{N!}\prod_{a=1}^{r}\Delta^{(N)}(x^{(a-1)},x^{(a)}),\tag{2.1}$$

with

$$D^Nx^{(a)}=\prod_{\ell=1}^{N}\frac{dx_\ell^{(a)}}{2\pi}e^{\frac{is_a}{4\pi k}(x_\ell^{(a)})^2},\tag{2.2}$$

$$\Delta^{(N)}(x^{(a-1)},x^{(a)})=\frac{\prod_{m<m'}^{N}2k\sinh\frac{x_m^{(a-1)}-x_{m'}^{(a-1)}}{2k}\prod_{n<n'}^{N}2k\sinh\frac{x_n^{(a)}-x_{n'}^{(a)}}{2k}}{\prod_{m=1}^{N}\prod_{n=1}^{N}2k\cosh\frac{x_m^{(a-1)}-x_n^{(a)}}{2k}}.\tag{2.3}$$

The partition function in the matrix model is also shown in the case with the rank differences. However, since the relation between the matrix model and the quantum curve is not fully known and it is enough to consider the case without the rank differences for our discussions for the quantum curves below, we give only the partition function without the rank differences here.

And we proceed with the analysis by using the Fermi gas formalism [8, 24]. In the formalism, by regarding the rank of the gauge group N as the number of free fermions, the grand canonical partition function is represented as

$$\Xi_k^{(p_1,q_1,\cdots)}(z) = \sum_{N=0}^{\infty} Z_k^{(p_1,q_1,\cdots)}(N) z^N, \tag{2.4}$$

with the fugacity z. By introducing the operators $\widehat{q}$ and $\widehat{p}$ satisfying the commutation relation $[\widehat{q}, \widehat{p}] = i\hbar$, this grand canonical partition function (without the rank differences) is expressed by the Fredholm determinant of the spectral operator $\widehat{H}^{-1}$ as

$$\Xi_k^{(p_1,q_1,\cdots)}(z) = \det\Big(1 + z\widehat{H}^{-1}_{(p_1,q_1,\cdots)}\Big), \tag{2.5}$$

where the spectral operator is represented in terms of the hyperbolic cosine operators (1.2) as

$$\widehat{H}^{-1}_{(p_1,q_1,\cdots)} = \widehat{\mathcal{P}}^{-p_1}\widehat{\mathcal{Q}}^{-q_1}\cdots. \tag{2.6}$$

Since the matrix models describing the brane configurations are realized by the spectral operators, we would like to analyze the brane configuration in the viewpoint of the spectral operators. Therefore, we consider the quantum curves obtained from the del Pezzo curves through the quantization. When we choose the parameters of the quantum curves specially, the spectral operators in the case without the rank differences are reproduced. And in this section, we reveal that the symmetries of the parameters of the quantum curves contain the transitions interpreted as the Hanany-Witten transitions by identifying the parameters of the quantum curves and the rank differences. Furthermore, by regarding the symmetries of the curves as the brane transitions, we propose the new brane transitions after separating the Hanany-Witten transitions [1, 42].

2.2 Symmetries of curves

To investigate the correspondences between quantum curves and brane configurations, the quantum curves need to be written in terms of the product of the hyperbolic cosine operators. Therefore, in this subsection, we give the quantum curves associated with the A_1, D_5 and E_7 del Pezzo curves [1, 29, 42, 47, 48]. These quantum curves are constructed by replacing the variables of the algebraic curves with the operators $\widehat{q}$ and $\widehat{p}$ satisfying the commutation relation $[\widehat{q}, \widehat{p}] = i\hbar$. In this quantization, the operator order is important because the operators must be ordered so that the Hamiltonian becomes the same function before and after the quantization.

When the mirror curves of the A_1, D_5 and E_7 del Pezzo curves are denoted $H_{(A_1)}$, $H_{(D_5)}$ and $H_{(E_7)}$ respectively and they are the functions of exponentials like $H = H(e^x, e^y)$, these curves are quantized as $\widehat{H} = \widehat{H}(\widehat{Q}, \widehat{\mathcal{P}})$ by introducing the q-order[1],

$$[\widehat{Q}^{\alpha_1}\widehat{P}^{\beta_1}\widehat{Q}^{\alpha_2}\widehat{P}^{\beta_2}\cdots]_q = q^{-\frac{\alpha\beta}{2}}\widehat{Q}^{\alpha}\widehat{P}^{\beta}, \quad (q = e^{i\hbar}). \tag{2.7}$$

The A_1, D_5 and E_7 quantum curves associated with the del Pezzo curves $\widehat{H}^{(1)}$, $\widehat{H}^{(5)}$ and $\widehat{H}^{(7)}$ are respectively given as

$$\frac{\widehat{H}_{(A_1)}}{\alpha} =[\widehat{Q}^{\frac{1}{2}}\widehat{P}^{\frac{1}{2}} + \widehat{Q}^{\frac{1}{2}}\widehat{P}^{-\frac{1}{2}} + \widehat{Q}^{-\frac{1}{2}}\widehat{P}^{\frac{1}{2}} + m\widehat{Q}^{-\frac{1}{2}}\widehat{P}^{-\frac{1}{2}}]_q, \tag{2.8}$$

$$\begin{aligned}\frac{\widehat{H}_{(D_5)}}{\alpha} =&[e_3e_4\widehat{Q}^{-1}\widehat{P} + (e_3 + e_4)\widehat{P} + \widehat{Q}\widehat{P}\\ &+ e_3e_4\Big(\frac{e_5}{h_2} + \frac{e_6}{h_2}\Big)\widehat{Q}^{-1} + \frac{E}{\alpha} + \Big(\frac{1}{e_1} + \frac{1}{e_2}\Big)\widehat{Q}\\ &+ \frac{1}{e_1e_2}\frac{h_1^2}{e_7e_8}\widehat{Q}^{-1}\widehat{P}^{-1} + \frac{1}{e_1e_2}\Big(\frac{h_1}{e_7} + \frac{h_1}{e_8}\Big)\widehat{P}^{-1} + \frac{1}{e_1e_2}\widehat{Q}\widehat{P}^{-1}]_q,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\frac{\widehat{H}_{(E_7)}}{\alpha} =&[\widehat{Q}^2\widehat{P}\\ &+ \widehat{Q}\Big(F_1^{(+)}\widehat{P} + H_1^{(-)}\Big)\\ &+ F_2^{(+)}\widehat{P} + \frac{E}{\alpha} + H_2^{(-)}\widehat{P}^{-1}\\ &+ F_4^{(+)}\widehat{Q}^{-1}\Big(\widehat{P} + g_1\Big)\Big(\widehat{P} + g_2\Big)\Big(F_1^{(-)}\widehat{P} + g_1g_2H_1^{(+)}\Big)\widehat{P}^{-2}\\ &+ F_4^{(+)}\widehat{Q}^{-2}\Big(\widehat{P} + g_1\Big)\Big(\widehat{P} + g_1\Big)\Big(\widehat{P} + g_2\Big)\Big(\widehat{P} + g_2\Big)\widehat{P}^{-3}]_q,\end{aligned} \tag{2.10}$$

with

$$F_n^{(\pm)} := \sum_{i_1<\cdots<i_n}\Big(f_{i_1}\cdots f_{i_n}\Big)^{\pm 1}, \quad H_n^{(\pm)} := \sum_{i_1<\cdots<i_n}\Big(h_{i_1}\cdots h_{i_n}\Big)^{\pm 1}, \tag{2.11}$$

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[1] The q-order is understood from the Baker-Cambell-Hausdorff formula. For example, if we quantize e^{ax+by} by $x \to \widehat{q}$ and $y \to \widehat{p}$ as $e^{ax+by} \to e^{a\widehat{q}+b\widehat{p}}$, then we do not need to consider the operator order since the operaters $\widehat{q}$ and $\widehat{p}$ are commutative in $e^{a\widehat{q}+b\widehat{p}}$. However, if we quantize e^{ax+by} by $e^x \to \widehat{Q}$ and $e^y \to \widehat{P}$, we must consider the operator order so that the operators $\widehat{Q}$ and $\widehat{P}$ are not commutative.

where m is the parameter of the A_1 curve, $e_1, \cdots, e_8, h_1, h_2$ are the parameters of the D_5 curve and $f_1, f_2, f_3, f_4, g_1, g_2, h_1, h_2, h_3, h_4$ are the parameters of the E_7 curve. And α and E are the overall factor and the constant respectively. Also, in the D_5 curve the ten parameters $e_1, \cdots, e_8, h_1, h_2$ satisfy the Vieta's formula $(h_1 h_2)^2 = \prod_{i=1}^{8} e_i$, while in the E_7 curve the parameters satisfy the Vieta's formula $F_4^{(+)}(g_1 g_2)^2 H_4^{(+)} = 1$. By calculating the q-order, the quantum curves are clearly represented as

$$\begin{aligned}
\frac{\widehat{H}_{(A_1)}}{\alpha} =& q^{-\frac{1}{8}} \widehat{Q}^{\frac{1}{2}} \widehat{P}^{\frac{1}{2}} + q^{\frac{1}{8}} \widehat{Q}^{\frac{1}{2}} \widehat{P}^{-\frac{1}{2}} + q^{\frac{1}{8}} \widehat{Q}^{-\frac{1}{2}} \widehat{P}^{\frac{1}{2}} + q^{-\frac{1}{8}} m \widehat{Q}^{-\frac{1}{2}} \widehat{P}^{-\frac{1}{2}}, && (2.12)\\
\frac{\widehat{H}_{(D_5)}}{\alpha} =& q^{\frac{1}{2}} e_3 e_4 \widehat{Q}^{-1} \widehat{P} + (e_3 + e_4) \widehat{P} + q^{-\frac{1}{2}} \widehat{Q} \widehat{P} \\
&+ e_3 e_4 \Big(\frac{e_5}{h_2} + \frac{e_6}{h_2} \Big) \widehat{Q}^{-1} + \frac{E}{\alpha} + \Big(\frac{1}{e_1} + \frac{1}{e_2} \Big) \widehat{Q} \\
&+ q^{-\frac{1}{2}} \frac{1}{e_1 e_2} \frac{h_1^2}{e_7 e_8} \widehat{Q}^{-1} \widehat{P}^{-1} + \frac{1}{e_1 e_2} \Big(\frac{h_1}{e_7} + \frac{h_1}{e_8} \Big) \widehat{P}^{-1} + q^{\frac{1}{2}} \frac{1}{e_1 e_2} \widehat{Q} \widehat{P}^{-1}, && (2.13)\\
\frac{\widehat{H}_{(E_7)}}{\alpha} =& q^{-1} \widehat{Q}^2 \widehat{P} \\
&+ q^{-\frac{1}{2}} \widehat{Q} \Big(F_1^{(+)} \widehat{P} + q^{\frac{1}{2}} H_1^{(-)} \Big) \\
&+ F_2^{(+)} \widehat{P} + \frac{E}{\alpha} + H_2^{(-)} \widehat{P}^{-1} \\
&+ q^{\frac{1}{2}} F_4^{(+)} \widehat{Q}^{-1} \Big(\widehat{P} + q^{-\frac{1}{2}} g_1 \Big) \Big(\widehat{P} + q^{-\frac{1}{2}} g_2 \Big) \Big(F_1^{(-)} \widehat{P} + q^{-\frac{1}{2}} g_1 g_2 H_1^{(+)} \Big) \widehat{P}^{-2} \\
&+ q F_4^{(+)} \widehat{Q}^{-2} \Big(\widehat{P} + q^{-\frac{3}{2}} g_1 \Big) \Big(\widehat{P} + q^{-\frac{1}{2}} g_1 \Big) \Big(\widehat{P} + q^{-\frac{3}{2}} g_2 \Big) \Big(\widehat{P} + q^{-\frac{1}{2}} g_2 \Big) \widehat{P}^{-3}, && (2.14)
\end{aligned}$$

where we introduce the q-number $[n]_q$ for the sum of q,

$$[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \tag{2.15}$$

By applying the similarity transitions to the quantum curves $\widehat{H}$, we adjust the coefficients of each term and transform $\widehat{H}$ to make the asymptotic values easier to see,

$$\begin{aligned}
\frac{\widehat{H}_{(A_1)}}{\alpha} =& \widehat{Q}^{\frac{1}{2}} \widehat{P}^{\frac{1}{2}} + \widehat{Q}^{\frac{1}{2}} \widehat{P}^{-\frac{1}{2}} + \widehat{Q}^{-\frac{1}{2}} \widehat{P}^{\frac{1}{2}} + q^{-\frac{1}{2}} m \widehat{Q}^{-\frac{1}{2}} \widehat{P}^{-\frac{1}{2}}, && (2.16)\\
\frac{\widehat{H}_{(D_5)}}{\alpha} =& q^{-\frac{1}{2}} \widehat{Q}^{-1} \big(\widehat{Q} + q^{\frac{1}{2}} e_3 \big) \big(\widehat{Q} + q^{\frac{1}{2}} e_4 \big) \widehat{P} \\
&+ e_3 e_4 \Big(\frac{e_5}{h_2} + \frac{e_6}{h_2} \Big) \widehat{Q}^{-1} + \frac{E}{\alpha} + \Big(\frac{1}{e_1} + \frac{1}{e_2} \Big) \widehat{Q} \\
&+ q^{\frac{1}{2}} \frac{1}{e_1 e_2} \widehat{Q}^{-1} \Big(\widehat{Q} + q^{-\frac{1}{2}} \frac{h_1}{e_7} \Big) \Big(\widehat{Q} + q^{-\frac{1}{2}} \frac{h_1}{e_8} \Big) \widehat{P}^{-1}, && (2.17)
\end{aligned}$$

$$
\begin{aligned}
\frac{\widehat{H}_{(E_7)}}{\alpha} =& q^{-1}\widehat{Q}^2\Big(\widehat{P}+q^{\frac{3}{2}}g_1\Big)\Big(\widehat{P}+q^{\frac{1}{2}}g_1\Big)\widehat{P}^{-1} \\
&+q^{-\frac{1}{2}}\widehat{Q}\Big(\widehat{P}+q^{\frac{1}{2}}g_1\Big)\Big(F_1^{(+)}\widehat{P}+q^{\frac{1}{2}}H_1^{(-)}\Big)\widehat{P}^{-1} \\
&+F_2^{(+)}\widehat{P}+\frac{E}{\alpha}+H_2^{(-)}\widehat{P}^{-1} \\
&+q^{\frac{1}{2}}F_4^{(+)}\widehat{Q}^{-1}\Big(\widehat{P}+q^{-\frac{1}{2}}g_2\Big)\Big(F_1^{(-)}\widehat{P}+q^{-\frac{1}{2}}g_1g_2H_1^{(+)}\Big)\widehat{P}^{-1} \\
&+qF_4^{(+)}\widehat{Q}^{-2}\Big(\widehat{P}+q^{-\frac{3}{2}}g_2\Big)\Big(\widehat{P}+q^{-\frac{1}{2}}g_2\Big)\widehat{P}^{-1},
\end{aligned}
\tag{2.18}
$$

where we give the similarity transitions applying the E_7 curve in appendix B.

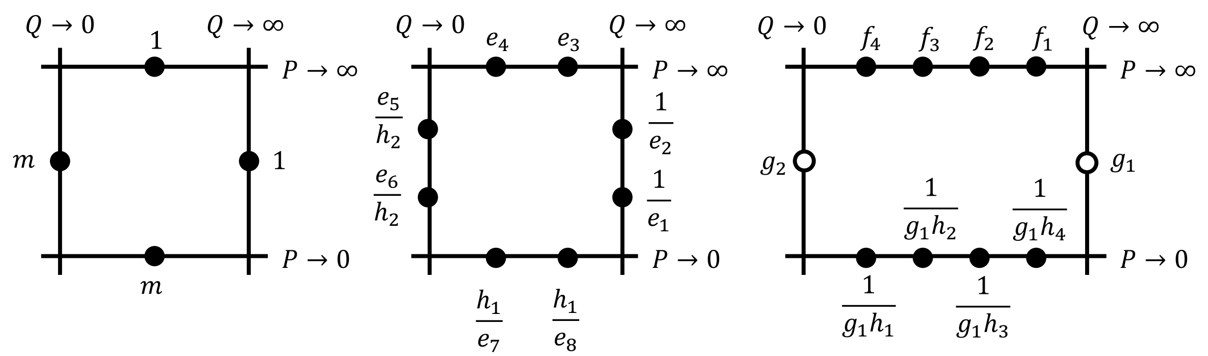

Figure 1: Asymptotic values of the A_1, D_5 and E_7 curves in the limits $Q \to \infty$, $P \to \infty$, $Q \to 0$ and $P \to 0$. The symbols $\bullet$ and $\circ$ denote an asymptotic value and a doubly degenerated asymptotic value, respectively. (Left) The four asymptotic values of the A_1 curve are parametrized with one parameter m. (Center) The eight asymptotic values of the D_5 curve are parametrized with ten parameters. (Right) The ten asymptotic values of the E_7 curve are parametrized with ten parameters.

In the A_1 curve, the asymptotic values in the limits $Q \to \infty$, $P \to \infty$, $Q \to 0$ and $P \to 0$ are given as 1, 1, m and m respectively (see figure 1). Here we omitted the minus signs of the asymptotic values since it does not affect our discussion. These asymptotic values are transformed as

$$
s_1: \quad m \leftrightarrow 1/m, \tag{2.19}
$$

so that it preserves the A_1 curve. This transition is given by the similarity transition $(Q,P) \leftrightarrow (P^{-1},Q)$ which means the clockwide rotation of the asymptotic values in the figure 1. As in the A_1 case, in the D_5 curve the asymptotic values in the limits $Q \to \infty$, $P \to \infty$, $Q \to 0$ and $P \to 0$ are given as $\{1/e_1, 1/e_2\}$, $\{e_3, e_4\}$, $\{e_5/h_2, e_6/h_2\}$ and $\{h_1/e_7, h_1/e_8\}$

respectively (see figure 1) and they are transformed while preserving the D_5 curve as

$$
\begin{aligned}
s_5: &\quad 1/e_1 \leftrightarrow 1/e_2, \\
s_2: &\quad e_3 \leftrightarrow e_4, \\
s_0: &\quad e_5/h_2 \leftrightarrow e_6/h_2, \\
s_1: &\quad h_1/e_7 \leftrightarrow h_1/e_8, \\
s_4: &\quad 1/e_1 \leftrightarrow e_5/h_2, \\
s_3: &\quad e_3 \leftrightarrow h_1/e_7,
\end{aligned}
\tag{2.20}
$$

where s_0 is described in terms of other transitions $s_{i\neq 0}$ as $s_0 = s_4 s_3 s_2 s_5 s_4 s_3 s_1 s_3 s_4 s_5 s_2 s_3 s_4$ because the parameters are not independent. And in the E_7 curve, the asymptotic values in the limits $Q \to \infty$, $P \to \infty$, $Q \to 0$ and $P \to 0$ are expected as g_1, $\{f_1, f_2, f_3, f_4\}$, g_2 and $\{(g_1 h_1)^{-1}, (g_1 h_2)^{-1}, (g_1 h_3)^{-1}, (g_1 h_4)^{-1}\}$ where g_1 and g_2 are doubly degenerate (see figure 1), which are resoled by quantization. These asymptotic values are transformed while preserving the E_7 curve as

$$
\begin{aligned}
s_2: &\quad f_1 \leftrightarrow f_2, \\
s_1: &\quad f_2 \leftrightarrow f_3, \\
s_0: &\quad f_3 \leftrightarrow f_4, \\
s_5: &\quad (g_1 h_1)^{-1} \leftrightarrow (g_1 h_2)^{-1}, \\
s_6: &\quad (g_1 h_2)^{-1} \leftrightarrow (g_1 h_3)^{-1}, \\
s_7: &\quad (g_1 h_3)^{-1} \leftrightarrow (g_1 h_4)^{-1}, \\
s_4: &\quad g_1 \leftrightarrow g_2, \\
s_3: &\quad f_1 \leftrightarrow (g_1 h_1)^{-1},
\end{aligned}
\tag{2.21}
$$

where since s_0 can be described in terms of other transitions $s_{i\neq 0}$ as in the D_5 case, s_0 is auxiliary transition in this sense.

In the A_1 case, the transition s_1 acts to one parameter m and is a generator of the Weyl group $W(A_1)$ on the one-dimensional parameter space $C_{\mathrm{P}}^{A_1} = \{m\}$. On the other hand, in the D_5 and E_7 cases, the numbers of the transitions s_i and the parameters do not match. This problem is solved by considering one constraint (Vieta's formula), fixing two degrees of freedom derived from the similarity transitions $\widehat{Q} \to A\widehat{Q}$ and $\widehat{P} \to B\widehat{P}$ and fixing extra degrees of freedom created by the differences between the numbers of the asymptotic values and parameters[2]. For the parameters of the D_5 curve, we choose the parameters as

$$e_2 = e_4 = e_6 = e_8 = 1, \quad e_7 = \frac{(h_1h_2)^2}{e_1e_3e_5}, \tag{2.22}$$

and in the case of the E_7 curve as

$$f_4 = g_2 = 1, \quad h_4 = \frac{1}{f_1f_2f_3g_1^2h_1h_2h_3}. \tag{2.23}$$

By these parameter fixings, in both cases of the D_5 and E_7 curves the numbers of the transitions s_i and the parameters match. In the D_5 case, transitions s_i act to the five-dimensional parameter space $C_{\mathrm{P}}^{D_5} = \{(h_1, h_2, e_1, e_3, e_5)\}$ as

$$\begin{aligned}
s_5 :\quad & (h_1,h_2,e_1,e_3,e_5) \mapsto \Big(h_1, \frac{h_2}{e_1}, \frac{1}{e_1}, e_3, e_5\Big),\\
s_2 :\quad & (h_1,h_2,e_1,e_3,e_5) \mapsto \Big(\frac{h_1}{e_3}, h_2, e_1, \frac{1}{e_3}, e_5\Big),\\
s_0 :\quad & (h_1,h_2,e_1,e_3,e_5) \mapsto \Big(h_1, \frac{h_2}{e_5}, e_1, e_3, \frac{1}{e_5}\Big),\\
s_1 :\quad & (h_1,h_2,e_1,e_3,e_5) \mapsto \Big(\frac{e_1e_3e_5}{h_1h_2^2}, h_2, e_1, e_3, e_5\Big),\\
s_4 :\quad & (h_1,h_2,e_1,e_3,e_5) \mapsto \Big(\frac{h_1h_2}{e_1e_5}, h_2, \frac{h_2}{e_5}, e_3, \frac{h_2}{e_1}\Big),\\
s_3 :\quad & (h_1,h_2,e_1,e_3,e_5) \mapsto \Big(h_1, \frac{e_1e_5}{h_1h_2}, e_1, \frac{e_1e_3e_5}{h_1h_2^2}, e_5\Big),
\end{aligned} \tag{2.24}$$

and generate the Weyl group $W(D_5)$. Similarly, in the E_7 case the transitions s_i act to the seven-dimensional parameter space $C_{\mathrm{P}}^{E_7} = \{(f_1, f_2, f_3, g_1, h_1, h_2, h_3)\}$ as

$$\begin{aligned}
s_2 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto (f_2,f_1,f_3,g_1,h_1,h_2,h_3),\\
s_1 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto (f_1,f_3,f_2,g_1,h_1,h_2,h_3),\\
s_0 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto \Big(\frac{f_1}{f_3}, \frac{f_2}{f_3}, \frac{1}{f_3}, g_1, f_3h_1, f_3h_2, f_3h_3\Big),\\
s_5 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto (f_1,f_2,f_3,g_1,h_2,h_1,h_3),\\
s_6 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto (f_1,f_2,f_3,g_1,h_1,h_3,h_2),\\
s_7 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto \Big(f_1,f_2,f_3,g_1,h_1,h_2, \frac{1}{f_1f_2f_3g_1^2h_1h_2h_3}\Big),\\
s_4 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto \Big(f_1,f_2,f_3, \frac{1}{g_1}, h_1,h_2,h_3\Big),\\
s_3 :\quad & (f_1,f_2,f_3,g_1,h_1,h_2,h_3) \mapsto \Big(\frac{1}{g_1f_1}, f_2, f_3, \frac{1}{f_1h_1}, h_1, (f_1g_1h_1)h_2, (f_1g_1h_1)h_3\Big),
\end{aligned} \tag{2.25}$$

and generate the Weyl group $W(E_7)$.

[2]if there are n asymptotic values and $m(\leq n)$ parameters of the curve, we can fix the $(n-m)$ parameters.

2.3 Case without rank differences

In this subsection, we consider the quantum curves associated with the super Chern-Simons matrix models having the gauge groups, $U(N)\times U(N)$, $U(N)\times U(N)\times U(N)\times U(N)$ and $\underbrace{U(N)\times U(N)\times\cdots U(N)}_{\text{six}}$. Then, these curves are respectively represented as the combinations of the hyperbolic cosine operators $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{P}}$ in the order of the Chern-Simons levels (1.4). Among them, the combinations of one $\widehat{\mathcal{Q}}$ and one $\widehat{\mathcal{P}}$,

$$\widehat{\mathcal{Q}}\widehat{\mathcal{P}},\quad \widehat{\mathcal{P}}\widehat{\mathcal{Q}}, \tag{2.26}$$

namely the quantum curve of $(1,1)$ model $\widehat{H}_{(1,1)}$[3] is included in the A_1 quantum curve. Similarly, the combinations of two $\widehat{\mathcal{Q}}$ and two $\widehat{\mathcal{P}}$,

$$\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}},\quad \widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}},\quad \widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}},\quad \widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}},\quad \widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}},\quad \widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}, \tag{2.27}$$

are included in the D_5 quantum curve. These operators are those in the $(2,2)$ model and the $(1,1,1,1)$ model. Finally, we consider the combinations of four $\widehat{\mathcal{Q}}$ and two $\widehat{\mathcal{P}}$ included in the E_7 quantum curve. Since the E_7 curve have the following structure,

$$\frac{\widehat{H}_{(E_7)}}{\alpha} = q^{-1}\widehat{Q}^2\Big(\widehat{P}+q^{\frac{3}{2}}g_1\Big)\Big(\widehat{P}+q^{\frac{1}{2}}g_1\Big)\widehat{P}^{-1}+\cdots+qF_4^{(+)}\widehat{Q}^{-2}\Big(\widehat{P}+q^{-\frac{3}{2}}g_2\Big)\Big(\widehat{P}+q^{-\frac{1}{2}}g_2\Big)\widehat{P}^{-1}, \tag{2.28}$$

the curves included in this curve must have the structure: $\cdots\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\cdots$. Thus, only parts of the combinations,

$$\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}},\quad \widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}},\quad \widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}, \tag{2.29}$$

are $\widehat{H}_{(2,1,2,1)}$[4] included in the E_7 quantum curve.

There are the brane configurations corresponding to above curves constructed by the combinations of the hyperbolic cosine operators $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{P}}$ and at the same time we can derive asymptotic values for each curve (see table 1). From table 1, it is found that we make pairs in the asymptotic values facing each other (compare table 1 and figure 1). Also, we can also understand these pairings from

$$\widehat{\mathcal{P}}\widehat{Q}^{\frac{n}{2}} = \widehat{Q}^{\frac{n}{2}}\Big(q^{-\frac{n}{4}}\widehat{P}^{\frac{1}{2}}+q^{\frac{n}{4}}\widehat{P}^{-\frac{1}{2}}\Big),$$

[3]The operators in (2.26) are equivalent under the similarity tansitions.

[4]The $(2,1,2,1)$ model, the $(1,1,2,1,1)$ and the $(1,2,1,2)$ model are equivalent under the similarity transformations.

types	1	1	m	m	brane configurations
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	1	1	$q^{\frac{1}{2}}$	$q^{\frac{1}{2}}$	$\langle 0\bullet 0\circ\rangle$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	1	1	$q^{-\frac{1}{2}}$	$q^{-\frac{1}{2}}$	$\langle 0\circ 0\bullet\rangle$

types	$\{e_1^{-1},e_2^{-1}\}$	$\{e_3,e_4\}$	$\{h_2^{-1}e_5,h_2^{-1}e_6\}$	$\{h_1e_7^{-1},h_1e_8^{-1}\}$	brane configurations
$\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}$	$\{q^{-\frac{1}{2}},q^{-\frac{1}{2}}\}$	$\{q^{-\frac{1}{2}},q^{-\frac{1}{2}}\}$	$\{q^{\frac{1}{2}},q^{\frac{1}{2}}\}$	$\{q^{\frac{1}{2}},q^{\frac{1}{2}}\}$	$\langle 0\bullet 0\bullet 0\circ 0\circ\rangle$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\{q^{-\frac{1}{2}},1\}$	$\{q^{-\frac{1}{2}},1\}$	$\{1,q^{\frac{1}{2}}\}$	$\{1,q^{\frac{1}{2}}\}$	$\langle 0\bullet 0\circ 0\bullet 0\circ\rangle$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\{1,1\}$	$\{q^{-\frac{1}{2}},q^{\frac{1}{2}}\}$	$\{1,1\}$	$\{q^{-\frac{1}{2}},q^{\frac{1}{2}}\}$	$\langle 0\circ 0\bullet 0\bullet 0\circ\rangle$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\{q^{-\frac{1}{2}},q^{\frac{1}{2}}\}$	$\{1,1\}$	$\{q^{-\frac{1}{2}},q^{\frac{1}{2}}\}$	$\{1,1\}$	$\langle 0\bullet 0\circ 0\circ 0\bullet\rangle$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\{1,q^{\frac{1}{2}}\}$	$\{1,q^{\frac{1}{2}}\}$	$\{q^{-\frac{1}{2}},1\}$	$\{q^{-\frac{1}{2}},1\}$	$\langle 0\circ 0\bullet 0\circ 0\bullet\rangle$
$\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}$	$\{q^{\frac{1}{2}},q^{\frac{1}{2}}\}$	$\{q^{\frac{1}{2}},q^{\frac{1}{2}}\}$	$\{q^{-\frac{1}{2}},q^{-\frac{1}{2}}\}$	$\{q^{-\frac{1}{2}},q^{-\frac{1}{2}}\}$	$\langle 0\circ 0\circ 0\bullet 0\bullet\rangle$

types	$\{f_i\}_{i=1,\cdots,4}$	g_1	g_2	$\{(g_1h_i)^{-1}\}_{i=1,\cdots,4}$	brane configurations
$\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}$	$\{q^{-\frac{1}{2}},q^{-\frac{1}{2}},1,1\}$	$q^{-\frac{1}{2}}$	$q^{\frac{1}{2}}$	$\{1,1,q^{\frac{1}{2}},q^{\frac{1}{2}}\}$	$\langle 0\bullet 0\circ 0\circ 0\bullet 0\circ 0\circ\rangle$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\{q^{-\frac{1}{2}},1,1,q^{\frac{1}{2}}\}$	1	1	$\{q^{-\frac{1}{2}},1,1,q^{\frac{1}{2}}\}$	$\langle 0\circ 0\bullet 0\circ 0\circ 0\bullet 0\circ\rangle$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2$	$\{1,1,q^{\frac{1}{2}},q^{\frac{1}{2}}\}$	$q^{\frac{1}{2}}$	$q^{-\frac{1}{2}}$	$\{q^{-\frac{1}{2}},q^{-\frac{1}{2}},1,1\}$	$\langle 0\circ 0\circ 0\bullet 0\circ 0\circ 0\bullet\rangle$

Table 1: Asymptotic values of A_1, D_5 and E_7 quantum curves and corresponding brane configurations. The asymptotic values are arranged in the order of those in the limits $Q\to\infty$, $P\to\infty$, $Q\to 0$ and $P\to 0$.

$$\widehat{P}^{\frac{n}{2}}\widehat{\mathcal{Q}} = \left(q^{-\frac{n}{4}}\widehat{Q}^{\frac{1}{2}} + q^{\frac{n}{4}}\widehat{Q}^{-\frac{1}{2}}\right)\widehat{P}^{\frac{n}{2}}. \tag{2.30}$$

In this subsection without the rank differences, we do not need to consider these pairings, however in the next subsection, namely in the case with the rank differences, these parameter pairings are needed for the correspondences between the parameters of the curves and the rank differences which are the numbers of the D3-branes stretching in each interval of 5-branes.

Originally there is only one parameter in the A_1 curve, thus the number of the parameters describing the asymptotic values does not decrease due to the pairing between the asymptotic values. However, in the cases of the D_5 and E_7 curves, the number of the parameters decreases by pairing the asymptotic values. In the D_5 case, since there are four pairs of the asymptotic values, the curves in D_5 curve type are labeled with the three parameters. And in the E_7 case, the curves are distinguished by only four parameters due to the pairings. Therefore by reducing the parameters as

$$(h_1,e_3,h_2,e_1,e_5) = \left(\frac{m_2m_3}{m_1}, m_2m_3, \frac{m_1m_3}{m_2}, \frac{m_3}{m_2}, \frac{m_3}{m_2}\right),$$

$$(f_1, f_2, f_3, g_1, h_1, h_2, h_3) = \Big(f_1, f_2, f_3, g_1, \sqrt{\frac{f_1}{f_2 f_3 g_1}}, \sqrt{\frac{f_2}{f_3 f_1 g_1}}, \sqrt{\frac{f_3}{f_1 f_2 g_1}}\Big), \tag{2.31}$$

and by labeling formally the hyperbolic cosine operators, we correspond the labeled quantum curves to the points of subspace of the parameter space, $C_{\mathrm{P}}^{D_5} \supset C_{\mathrm{B}}^{(2,2)} = \{(m_1, m_2, m_3)\}$[5] in the case of the D_5 curve type and $C_{\mathrm{P}}^{E_7} \cap C_{\mathrm{B}}^{(2,4)} = \{(f_1, f_2, f_3, g_1)\}$ in the case of the E_7 curve type (see table 2 and table 3).

types	quantum curves	$(h_1, e_3,$	$h_2, e_1, e_5)$	(m_1, m_2, m_3)
$\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(q, 1,$	$q^{-1}, 1, 1)$	$(q^{-1}, 1, 1)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(q, q^{\frac{1}{2}},$	$q^{-1}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(1, q^{-\frac{1}{2}},$	$q^{-1}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{-\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}$	$(q, q^{\frac{1}{2}},$	$1, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{-\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}$	$(1, q^{-\frac{1}{2}},$	$1, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(q^{-1}, q,$	$1, 1, 1)$	$(1, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(q, q^{-1},$	$1, 1, 1)$	$(1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}$	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(1, 1,$	$q^{-1}, q^{-1}, q^{-1})$	$(1, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}$	$(1, 1,$	$q, q, q)$	$(1, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(1, q^{\frac{1}{2}},$	$1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(q^{-1}, q^{-\frac{1}{2}},$	$1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$	$(q^{\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_1$	$(1, q^{\frac{1}{2}},$	$q, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2$	$(q^{-1}, q^{-\frac{1}{2}},$	$q, q^{\frac{1}{2}}, q^{\frac{1}{2}})$	$(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$
$\widehat{\mathcal{P}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1$	$(q^{-1}, 1,$	$q, 1, 1)$	$(q, 1, 1)$

Table 2: Point configurations for the D_5 quantum curves. In the parameter space $C_P^{D_5} = \{(h_1, h_2, e_1, e_3, e_5)\}$, the parameters h_1 and e_3 are decided from the asymptotic values of the limits $P \to 0, \infty$, also the parameters h_2, e_1 and e_5 are decided from the asumptotic values of the limits $Q \to 0, \infty$. And the point configurations corresponding to the brane configurations give the three-dimensional subspace $C_{\mathrm{B}}^{(2,2)} = \{(m_1, m_2, m_3)\}$ in the parameter space $C_{\mathrm{P}}^{D_5}$ under the constraint (2.31).

[5] $C_{\mathrm{B}}^{(p,q)}$ denotes the space of the brane configurations described by p NS5-branes and q $(1,k)$5-branes. In the case with the equal ranks, the models are distinguished by the order of the 5-branes. For example, $(2,2)$ model and $(1,1,1,1)$ model are not equivalent under the similarity transitions thus are distinguished. However, in the case with the difference ranks, the models cannot be distinguished due to the Hanany-Witten transitions. Thus, we distinguish the spaces of the brane configurations with the rank differences by the number of the NS5-branes and the number of the $(1,k)$5-branes as $C_{\mathrm{B}}^{(p,q)}$.

types	quantum curves	$(f_1, f_2, f_3,$	$g_1,$	$h_1, h_2, h_3)$
$\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1,$	$q^{-1},$	$q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(q^{\frac{1}{2}}, 1, q^{\frac{1}{2}},$	$q^{-1},$	$q^{\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(1, q^{\frac{1}{2}}, q^{\frac{1}{2}},$	$q^{-1},$	$1, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}},$	$q^{-1},$	$q, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(q^{-\frac{1}{2}}, 1, q^{-\frac{1}{2}},$	$q^{-1},$	$q^{\frac{1}{2}}, q, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, 1,$	$q^{-1},$	$q^{\frac{1}{2}}, q^{\frac{1}{2}}, q)$
$\widehat{\mathcal{Q}}\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\widehat{\mathcal{Q}}$	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(q, q^{\frac{1}{2}}, q^{\frac{1}{2}},$	$1,$	$1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(q^{\frac{1}{2}}, q, q^{\frac{1}{2}},$	$1,$	$q^{-\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_3$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, q,$	$1,$	$q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(q^{\frac{1}{2}}, 1, q^{-\frac{1}{2}},$	$1,$	$q^{\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(1, q^{\frac{1}{2}}, q^{-\frac{1}{2}},$	$1,$	$1, q^{\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_4$	$(q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-1},$	$1,$	$q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1,$	$1,$	$q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_3$	$(1, q^{-\frac{1}{2}}, q^{\frac{1}{2}},$	$1,$	$1, q^{-\frac{1}{2}}, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_4$	$(q^{-\frac{1}{2}}, q^{-1}, q^{-\frac{1}{2}},$	$1,$	$q^{\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, 1,$	$1,$	$q^{-\frac{1}{2}}, q^{\frac{1}{2}}, 1)$
	$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_3$	$(q^{-\frac{1}{2}}, 1, q^{\frac{1}{2}},$	$1,$	$q^{-\frac{1}{2}}, 1, q^{\frac{1}{2}})$
	$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_4$	$(q^{-1}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}},$	$1,$	$1, q^{\frac{1}{2}}, q^{\frac{1}{2}})$
$\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2\widehat{\mathcal{P}}\widehat{\mathcal{Q}}^2$	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1,$	$q,$	$q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-1})$
	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1$	$(q^{\frac{1}{2}}, 1, q^{\frac{1}{2}},$	$q,$	$q^{-\frac{1}{2}}, q^{-1}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2$	$(1, q^{\frac{1}{2}}, q^{\frac{1}{2}},$	$q,$	$q^{-1}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1$	$(1, q^{\frac{1}{2}}, q^{\frac{1}{2}},$	$q,$	$1, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2$	$(q^{\frac{1}{2}}, 1, q^{\frac{1}{2}},$	$q,$	$q^{-\frac{1}{2}}, 1, q^{-\frac{1}{2}})$
	$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3$	$(q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1,$	$q,$	$q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, 1)$

Table 3: Point configurations for the E_7 quantum curves. The labeled quantum curves are corresponded to the point of $C_{\mathrm{B}}^{(2,4)} \cap C_{\mathrm{P}}^{E_7} = \{(f_1, f_2, f_3, g_1)\}$, since h_i are written as $h_i = \sqrt{\frac{f_i}{f_j f_k g_1}}$ (subscripts i, j, k are different).

2.4 Case with rank differences

In the previous subsection, we have discussed the correspondences between the quantum curves and the brane configurations without the rank differences. In this subsection, through the Hanany-Witten transitions [41], we formulate the correspondences in the case with the rank differences.

When we consider the case with the rank differences, we correspond the rank differences to the parameter differences of the curves. Then, since the parameters are labeled and distinguished, the quantum curves need to be labeled and we need to decide a reference. In the case of the A_1 curve, if we consider $\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}$ as a reference and the parameter m of this curve is denoted $m_{(s)}$, then the difference from $m_{(s)}$ for another curve $\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_1$ is denoted $\delta m = m/m_{(s)}$. On the other hand, since we can consider the Hanany-Witten transitions (1.10) for brane configurations, we can align the 5-branes of the brane configuration corresponding to each curve with same sequence as a reference curve (we call this sequence standard sequence) and read the difference of the number of the D3-branes in the interval of 5-branes from the reference rank. We identify the parameter difference δm and the rank difference M as

$$\delta m = \frac{m}{m_{(s)}} = e^{-2\pi i M}, \tag{2.32}$$

and summarize the comparison between the rank differences and the brane configurations in table 4.

quantum curves	M	brane configurations
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}$	0	$\langle 0 \overset{\rm i}{\bullet} 0 \overset{1}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_1$	k	$\langle 0 \overset{1}{\circ} 0 \overset{\rm i}{\bullet} \rangle = \langle 0 \overset{\rm i}{\bullet} k \overset{1}{\circ} \rangle$

Table 4: Comparison between the rank differences and the brane configurations for the $(1,1)$ model. Here we omit the overall rank N.

As in the A_1 case, in the cases of the D_5 and E_7 curves, we choose $\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{P}}_{\rm ii}$ and $\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}$ as references respectively. And we identify the parameter differences and the rank differences as follows,

$$\begin{aligned}(\delta m_1, \delta m_2, \delta m_3) &= \left(\frac{m_1}{m_{1(s)}}, \frac{m_2}{m_{2(s)}}, \frac{m_3}{m_{3(s)}}\right) = (e^{2\pi i M_1}, e^{2\pi i M_2}, e^{2\pi i M_3}),\\ (\delta f_1, \delta f_2, \delta f_3, \delta g_1) &= \left(\frac{f_1}{f_{1(s)}}, \frac{f_2}{f_{2(s)}}, \frac{f_3}{f_{3(s)}}, \frac{g_1}{g_{1(s)}}\right) = (e^{2\pi i F_1}, e^{2\pi i F_2}, e^{2\pi i F_3}, e^{2\pi i G_1}).\end{aligned} \tag{2.33}$$

In the E_7 case, since the asymptotic values in the limit $Q \to 0, \infty$ are paired, we decide to place the operator $\widehat{\mathcal{P}}_{\mathrm{i}}$ on the left and $\widehat{\mathcal{P}}_{\mathrm{ii}}$ on the right. Namely, when we correspond the brane configurations to the labeled E_7 curves, the Hanany-Witten transition between the NS5-branes is restricted.

quantum curves	(M_1, M_2, M_3)	brane configurations
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(0,0,0)$	$\langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{1}{\circ} 0 \overset{2}{\circ} \rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(\frac{k}{2},\frac{k}{2},0)$	$\langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{1}{\circ} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{2}{\circ} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} k \overset{1}{\circ} 0 \overset{2}{\circ} \rangle$
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(\frac{k}{2},0,-\frac{k}{2})$	$\langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{2}{\circ} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{1}{\circ} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} k \overset{1}{\circ} k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}$	$(\frac{k}{2},0,\frac{k}{2})$	$\langle 0 \overset{\mathrm{i}}{\bullet} 0 \overset{1}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{2}{\circ} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} k \overset{\mathrm{i}}{\bullet} k \overset{1}{\circ} 0 \overset{2}{\circ} \rangle$
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}$	$(\frac{k}{2},-\frac{k}{2},0)$	$\langle 0 \overset{\mathrm{i}}{\bullet} 0 \overset{2}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{1}{\circ} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} k \overset{\mathrm{i}}{\bullet} k \overset{1}{\circ} k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(k,\frac{k}{2},\frac{k}{2})$	$\langle 0 \overset{1}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{2}{\circ} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} k \overset{\mathrm{i}}{\bullet} 2k \overset{1}{\circ} 0 \overset{2}{\circ} \rangle$
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(k,-\frac{k}{2},-\frac{k}{2})$	$\langle 0 \overset{2}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{1}{\circ} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} k \overset{\mathrm{i}}{\bullet} 2k \overset{1}{\circ} 2k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}$	$(k,\frac{k}{2},-\frac{k}{2})$	$\langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{1}{\circ} 0 \overset{2}{\circ} 0 \overset{\mathrm{i}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} 2k \overset{1}{\circ} k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}$	$(k,-\frac{k}{2},\frac{k}{2})$	$\langle 0 \overset{\mathrm{i}}{\bullet} 0 \overset{1}{\circ} 0 \overset{2}{\circ} 0 \overset{\mathrm{ii}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 2k \overset{\mathrm{i}}{\bullet} 2k \overset{1}{\circ} k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1$	$(\frac{3k}{2},\frac{k}{2},0)$	$\langle 0 \overset{1}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{2}{\circ} 0 \overset{\mathrm{i}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} k \overset{\mathrm{i}}{\bullet} 3k \overset{1}{\circ} k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2$	$(\frac{3k}{2},0,-\frac{k}{2})$	$\langle 0 \overset{2}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{1}{\circ} 0 \overset{\mathrm{i}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} k \overset{\mathrm{i}}{\bullet} 3k \overset{1}{\circ} 2k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_1$	$(\frac{3k}{2},0,\frac{k}{2})$	$\langle 0 \overset{1}{\circ} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{2}{\circ} 0 \overset{\mathrm{ii}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 2k \overset{\mathrm{i}}{\bullet} 3k \overset{1}{\circ} k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{Q}}_2$	$(\frac{3k}{2},-\frac{k}{2},0)$	$\langle 0 \overset{2}{\circ} 0 \overset{\mathrm{i}}{\bullet} 0 \overset{1}{\circ} 0 \overset{\mathrm{ii}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 2k \overset{\mathrm{i}}{\bullet} 3k \overset{1}{\circ} 2k \overset{2}{\circ} \rangle$
$\widehat{\mathcal{P}}_{\mathrm{i}}\widehat{\mathcal{P}}_{\mathrm{ii}}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1$	$(2k,0,0)$	$\langle 0 \overset{1}{\circ} 0 \overset{2}{\circ} 0 \overset{\mathrm{ii}}{\bullet} 0 \overset{\mathrm{i}}{\bullet} \rangle = \langle 0 \overset{\mathrm{ii}}{\bullet} 2k \overset{\mathrm{i}}{\bullet} 4k \overset{1}{\circ} 2k \overset{2}{\circ} \rangle$

Table 5: Comparison between the rank differences and the brane configurations for the $(2,2)$ models and the $(1,1,1,1)$ models. Here we omit the overall rank N.

It is found that the correspondences between the point configurations and the brane configurations from table 4, 5 and 6. If the overall rank is denoted N, we identify the point configurations with the brane configurations for the A_1 curve, the D_5 curve and the E_7 curve respectively as

$$\langle N_1 \overset{\mathrm{i}}{\bullet} N_2 \overset{1}{\circ} \rangle = \langle N \overset{\mathrm{i}}{\bullet} N + M \overset{1}{\circ} \rangle, \tag{2.34}$$

$$\langle N_1 \overset{\mathrm{ii}}{\bullet} N_2 \overset{\mathrm{i}}{\bullet} N_3 \overset{1}{\circ} N_4 \overset{2}{\circ} \rangle = \langle N \overset{\mathrm{ii}}{\bullet} N + M_1 - M_2 + M_3 \overset{\mathrm{i}}{\bullet} N + 2M_1 \overset{1}{\circ} N + M_1 - M_2 - M_3 \overset{2}{\circ} \rangle, \tag{2.35}$$

and

$$\langle N_1 \overset{\mathrm{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\mathrm{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ} \rangle$$

quantum curves	(F_1,F_2,F_3,G_1)	brane configurations
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}$	$(\frac{k}{2},\frac{k}{2},0,-k)$	$\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm i}{\bullet}0\overset{3}{\circ}0\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}$	$(\frac{k}{2},0,\frac{k}{2},-k)$	$\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}0\overset{2}{\circ}0\overset{4}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}k\overset{\rm i}{\bullet}k\overset{3}{\circ}0\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}$	$(0,\frac{k}{2},\frac{k}{2},-k)$	$\langle 0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}0\overset{1}{\circ}0\overset{4}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}k\overset{2}{\circ}k\overset{\rm i}{\bullet}k\overset{3}{\circ}0\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}$	$(0,-\frac{k}{2},-\frac{k}{2},-k)$	$\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{2}{\circ}0\overset{3}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}k\overset{\rm i}{\bullet}k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}$	$(-\frac{k}{2},0,-\frac{k}{2},-k)$	$\langle 0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{1}{\circ}0\overset{3}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}k\overset{2}{\circ}k\overset{\rm i}{\bullet}k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm ii}$	$(-\frac{k}{2},-\frac{k}{2},0,-k)$	$\langle 0\overset{\rm ii}{\bullet}0\overset{3}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}0\overset{1}{\circ}k\overset{2}{\circ}2k\overset{\rm i}{\bullet}2k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_1$	$(k,\frac{k}{2},\frac{k}{2},0)$	$\langle 0\overset{1}{\circ}0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}0\overset{4}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm i}{\bullet}k\overset{3}{\circ}0\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_2$	$(\frac{k}{2},k,\frac{k}{2},0)$	$\langle 0\overset{2}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}0\overset{4}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}k\overset{2}{\circ}0\overset{\rm i}{\bullet}k\overset{3}{\circ}0\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_3$	$(\frac{k}{2},\frac{k}{2},k,0)$	$\langle 0\overset{3}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm i}{\bullet}0\overset{4}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}k\overset{2}{\circ}k\overset{\rm i}{\bullet}2k\overset{3}{\circ}0\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_1$	$(\frac{k}{2},0,-\frac{k}{2},0)$	$\langle 0\overset{1}{\circ}0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{3}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm i}{\bullet}k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_2$	$(0,\frac{k}{2},-\frac{k}{2},0)$	$\langle 0\overset{2}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{3}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}k\overset{2}{\circ}0\overset{\rm i}{\bullet}k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_4$	$(-\frac{k}{2},-\frac{k}{2},-k,0)$	$\langle 0\overset{4}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm i}{\bullet}0\overset{3}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}k\overset{2}{\circ}k\overset{\rm i}{\bullet}2k\overset{3}{\circ}2k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_1$	$(\frac{k}{2},-\frac{k}{2},0,0)$	$\langle 0\overset{1}{\circ}0\overset{\rm ii}{\bullet}0\overset{3}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{2}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}0\overset{2}{\circ}k\overset{\rm i}{\bullet}2k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_3$	$(0,-\frac{k}{2},\frac{k}{2},0)$	$\langle 0\overset{3}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{2}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}k\overset{2}{\circ}2k\overset{\rm i}{\bullet}3k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_4$	$(-\frac{k}{2},-k,-\frac{k}{2},0)$	$\langle 0\overset{4}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}0\overset{2}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}0\overset{2}{\circ}2k\overset{\rm i}{\bullet}3k\overset{3}{\circ}2k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_2$	$(-\frac{k}{2},\frac{k}{2},0,0)$	$\langle 0\overset{2}{\circ}0\overset{\rm ii}{\bullet}0\overset{3}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{1}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}2k\overset{2}{\circ}k\overset{\rm i}{\bullet}2k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_3$	$(-\frac{k}{2},0,\frac{k}{2},0)$	$\langle 0\overset{3}{\circ}0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}0\overset{1}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}2k\overset{2}{\circ}2k\overset{\rm i}{\bullet}3k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_4$	$(-k,-\frac{k}{2},-\frac{k}{2},0)$	$\langle 0\overset{4}{\circ}0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}0\overset{1}{\circ}\rangle=\langle 0\overset{\rm ii}{\bullet}k\overset{1}{\circ}2k\overset{2}{\circ}2k\overset{\rm i}{\bullet}3k\overset{3}{\circ}2k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1$	$(\frac{k}{2},\frac{k}{2},0,k)$	$\langle 0\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm ii}{\bullet}0\overset{3}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}\rangle=\langle 0\overset{\rm ii}{\bullet}2k\overset{1}{\circ}k\overset{2}{\circ}0\overset{\rm i}{\bullet}2k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1$	$(\frac{k}{2},0,\frac{k}{2},k)$	$\langle 0\overset{1}{\circ}0\overset{3}{\circ}0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}\rangle=\langle 0\overset{\rm ii}{\bullet}2k\overset{1}{\circ}k\overset{2}{\circ}k\overset{\rm i}{\bullet}3k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2$	$(0,\frac{k}{2},\frac{k}{2},k)$	$\langle 0\overset{2}{\circ}0\overset{3}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{4}{\circ}0\overset{\rm i}{\bullet}\rangle=\langle 0\overset{\rm ii}{\bullet}2k\overset{1}{\circ}2k\overset{2}{\circ}k\overset{\rm i}{\bullet}3k\overset{3}{\circ}k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_2\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_1$	$(0,-\frac{k}{2},-\frac{k}{2},k)$	$\langle 0\overset{1}{\circ}0\overset{4}{\circ}0\overset{\rm ii}{\bullet}0\overset{2}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}\rangle=\langle 0\overset{\rm ii}{\bullet}2k\overset{1}{\circ}k\overset{2}{\circ}k\overset{\rm i}{\bullet}3k\overset{3}{\circ}2k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_3\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_2$	$(-\frac{k}{2},0,-\frac{k}{2},k)$	$\langle 0\overset{2}{\circ}0\overset{4}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{3}{\circ}0\overset{\rm i}{\bullet}\rangle=\langle 0\overset{\rm ii}{\bullet}2k\overset{1}{\circ}2k\overset{2}{\circ}k\overset{\rm i}{\bullet}3k\overset{3}{\circ}2k\overset{4}{\circ}\rangle$
$\widehat{\mathcal{P}}_{\rm i}\widehat{\mathcal{Q}}_2\widehat{\mathcal{Q}}_1\widehat{\mathcal{P}}_{\rm ii}\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3$	$(-\frac{k}{2},-\frac{k}{2},0,k)$	$\langle 0\overset{3}{\circ}0\overset{4}{\circ}0\overset{\rm ii}{\bullet}0\overset{1}{\circ}0\overset{2}{\circ}0\overset{\rm i}{\bullet}\rangle=\langle 0\overset{\rm ii}{\bullet}2k\overset{1}{\circ}2k\overset{2}{\circ}2k\overset{\rm i}{\bullet}4k\overset{3}{\circ}2k\overset{4}{\circ}\rangle$

Table 6: Comparison between the rank differences and the brane configurations for the $(2,4)$ models. Here we omit the overall rank N.

$$=\langle N \overset{\text{ii}}{\bullet} N+G_1+k \overset{1}{\circ} N+\frac{1}{2}(-3F_1+F_2+F_3+G_1)+k \overset{2}{\circ} N-F_1-F_2+F_3+k$$
$$\overset{\text{i}}{\bullet} N-F_1-F_2+F_3+G_1+2k \overset{3}{\circ} N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k\overset{4}{\circ}\rangle. \tag{2.36}$$

Then, in the case of the E_7 curve, the extra constraint appears

$$N_1+N_5=N_2+N_4. \tag{2.37}$$

This constraint means that the sums of the numbers of the D3-branes stretching in both sides of NS5-branes are equal and is called a balanced condition [1].

2.5 Brane transitions from Weyl group

In this subsection, we review for the local brane transitions (local rule) [1, 42]. These transitions are the brane transitions proposed from the viewpoint of the quantum curves and derived by regarding the Weyl reflections for the parameters as the brane transitions under identifications between the point configurations and the brane configurations given in the previous subsection.

In the case of the A_1 curve, the Weyl group $W(A_1)$ is generated by

$$s_1 : M \mapsto k-M. \tag{2.38}$$

For the brane configurations, this transition is expressed as

$$s_1 : \langle N_1 \overset{\text{i}}{\bullet} N_2 \overset{1}{\circ}\rangle \mapsto \langle N_1 \overset{1}{\circ} N_2 \overset{\text{i}}{\bullet}\rangle. \tag{2.39}$$

In this case, there is no reason to interpret the symmetry of the curve as the brane transition, however the Weyl groups of the D_5 and E_7 curves include the Hanany-Witten transitions. Therefore, other transitions after separating the Hanany-Witten transitions are also interpreted as the brane transitions.

For the D_5 and the E_7 curves, the Weyl groups are folded due to the pairings of the parameters ($\mathbb{Z}_2$ folding). The D_5 and E_7 Dynkin diagrams are folded as the B_3 and F_4 Dynkin diagrams respectively.

In the case of the D_5 curve, under the identification between the point configurations and the brane configurations, since the asymptotic values are paired, the parameter transitions are also paired (see figure 2). Therefore the symmetries of the curve are expressed as

$$s_1s_2 : (M_1, M_2, M_3) \mapsto (M_1, -M_3, -M_2),$$

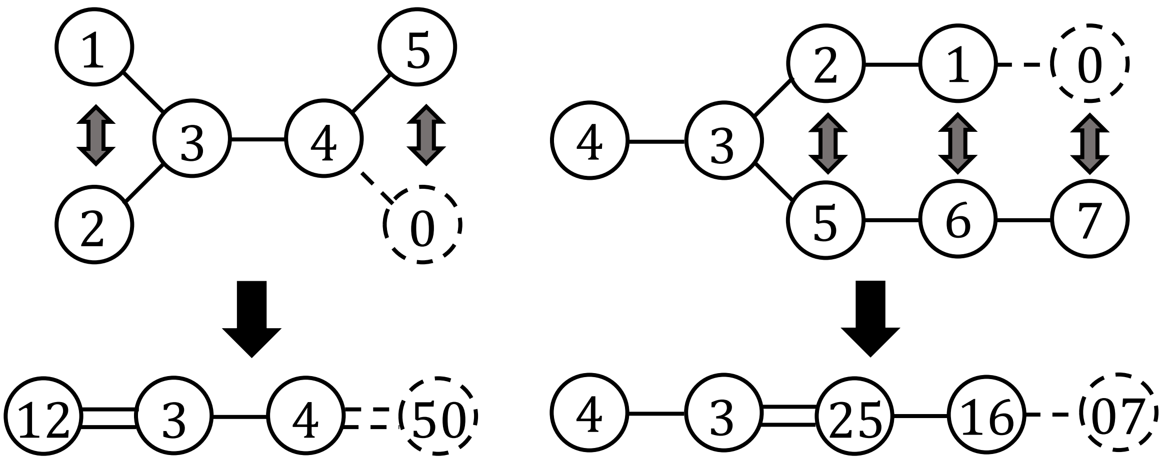

Figure 2: $\mathbb{Z}_2$ folding. Here, s_0 is the auxiliary transition.

$$
\begin{aligned}
s_3 &: (M_1, M_2, M_3) \mapsto (-M_2 - M_3 + k, \frac{1}{2}(-M_1 + M_2 - M_3 + k), \frac{1}{2}(-M_1 - M_2 + M_3 + k)),\\
s_4 &: (M_1, M_2, M_3) \mapsto (-M_2 + M_3 + k, \frac{1}{2}(-M_1 + M_2 + M_3 + k), \frac{1}{2}(M_1 + M_2 + M_3 - k)),\\
s_5 s_0 &: (M_1, M_2, M_3) \mapsto (M_1, M_3, M_2),
\end{aligned}
\tag{2.40}
$$

and generate the Weyl group $W(B_3)$[6]. These transitions are expressed for the brane configurations as

$$
\begin{aligned}
s_1 s_2 &: \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{\text{i}}{\bullet} N_3 \overset{1}{\circ} N_4 \overset{2}{\circ} \rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{\text{i}}{\bullet} N_3 \overset{1}{\circ} N_3 - N_4 + N_1 \overset{2}{\circ} \rangle,\\
s_3 &: \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3' \overset{\text{i}}{\bullet} N_4 \overset{2}{\circ} \rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_3' \overset{1}{\circ} N_2 \overset{\text{i}}{\bullet} N_4 \overset{2}{\circ} \rangle,\\
s_4 &: \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3' \overset{\text{i}}{\bullet} N_4 \overset{2}{\circ} \rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_4 \overset{\text{i}}{\bullet} N_3' \overset{2}{\circ} \rangle,\\
s_5 s_0 &: \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{\text{i}}{\bullet} N_3 \overset{1}{\circ} N_4 \overset{2}{\circ} \rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_1 - N_2 + N_3 \overset{\text{i}}{\bullet} N_3 \overset{1}{\circ} N_4 \overset{2}{\circ} \rangle,
\end{aligned}
\tag{2.41}
$$

with $N_3' = N_2 - N_3 + N_4 + k$. Here, the transitions $s_1 s_2$ and $s_5 s_0$ are interpreted as the Hanany-Witten transitions between $(1,k)$5-branes and between the NS5-branes respectively. Moreover, after separating the Hanany-Witten transitions, if s_3 and s_4 are also interpreted as the brane transitions, they are respectively interpreted as the exchanges between the number of the D3-branes stretching both sides of $(1,k)$5-brane surrounded by NS5-branes and NS5-brane surrounded by $(1,k)$5-branes.

As in the D_5 case, the symmetries of the E_7 curve are also folded to ones of the Weyl

[6] At this time, we cannot determine whether the symmetries of the curve generate the Weyl group $W(B_3)$ or $W(C_3)$. However, from the discussion about the affine Weyl group in the next section, the group is decided as $W(B_3)$.

group $W(F_4)$ as

$$
\begin{aligned}
s_4 &: (F_1, F_2, F_3, G_1) \mapsto (F_1, F_2, F_3, -G_1),\\
s_3 &: (F_1, F_2, F_3, G_1) \mapsto (\frac{1}{2}(-F_1 + F_2 + F_3 - G_1), F_2, F_3, \frac{1}{2}(-3F_1 + F_2 + F_3 + G_1)),\\
s_2 s_5 &: (F_1, F_2, F_3, G_1) \mapsto (F_2, F_1, F_3, G_1),\\
s_1 s_6 &: (F_1, F_2, F_3, G_1) \mapsto (F_1, F_3, F_2, G_1),\\
s_0 s_7 &: (F_1, F_2, F_3, G_1) \mapsto (F_1 - F_3, F_2 - F_3, -F_3, G_1),
\end{aligned} \tag{2.42}
$$

(see also figure 2). And for the brane configurations, these transitions are written as

$$
\begin{aligned}
s_4 :&\langle N_1 \overset{1}{\circ} N_2' \overset{\text{ii}}{\bullet} N_3 \overset{2}{\circ} N_4 \overset{3}{\circ} N_5' \overset{\text{i}}{\bullet} N_6 \overset{4}{\circ}\rangle \mapsto \langle N_1 \overset{1}{\circ} N_3 \overset{\text{ii}}{\bullet} N_2' \overset{2}{\circ} N_4 \overset{3}{\circ} N_6 \overset{\text{i}}{\bullet} N_5' \overset{4}{\circ}\rangle,\\
s_3 :&\langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{\text{i}}{\bullet} N_4' \overset{2}{\circ} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_3 \overset{1}{\circ} N_2 \overset{\text{i}}{\bullet} N_4' \overset{2}{\circ} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle,\\
s_2 s_5 :&\langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\text{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_2 - N_3 + N_4 \overset{2}{\circ} N_4 \overset{\text{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle,\\
s_1 s_6 :&\langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\text{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4'' \overset{\text{i}}{\bullet} N_5'' \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle,\\
s_0 s_7 :&\langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\text{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle \mapsto \langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\text{i}}{\bullet} N_5 \overset{3}{\circ} N_5 - N_6 + N_1 \overset{4}{\circ}\rangle,
\end{aligned} \tag{2.43}
$$

with $N_2' = N_1 - N_2 + N_3 + k$, $N_4' = N_3 - N_4 + N_5 + k$, $N_4'' = N_3 - N_4' + N_6 + k$, $N_5' = N_4 - N_5 + N_6 + k$ and $N_5'' = N_3 - N_4 + N_6 + k$. $s_2 s_5$, $s_1 s_6$ and $s_0 s_7$ are interpreted as the brane transitions between $(1,k)$5-branes. And s_4 and s_3 means the exchanges the numbers of the D3-branes as in the D_5 curve.

2.6 Local rule

Since the Weyl groups contain the transitions interpreted as Hanany-Witten transitions, if we regard the Weyl groups as the brane transitions, the remaining transitions after separating the Hanany-Witten transitions are the new transitions. Then, we have proposed the local rule for the brane transitions,

$$
\cdots \circ N \bullet N' \circ \cdots = \cdots \circ N' \bullet N \circ \cdots, \quad \cdots \bullet N \circ N' \bullet \cdots = \cdots \bullet N \circ N' \bullet \cdots. \tag{2.44}
$$

These local transitions mean the brane transitions without referring to whole configurations.

When we discuss the brane transitions from the quantum curves, we use the fact that the grand canonical partition functions are written by the Fredholm determinant. Therefore, we cannot discuss about the reference rank in our analysis. For the reason, we do not know if the local rule holds separately for each rank.

3 Duality cascade and affine Weyl group

For the ABJM matrix model, namely for the A_1 curve, in the duality cascade the number of the D3-branes is decreased by the Hanany-Witten transitions as discussed in [46]. In this section, we reveal that the regions where the duality cascade does not occur (fundamental domains) are parallelotopes in the subspaces of the brane configurations in the parameter spaces of the curves $C_{\mathrm{B}} \cap C_{\mathrm{P}}$. Then the affine Weyl groups are found by considering the fundamental domains in addition to the symmetries of curves and the transitions to replace reference rank N mean both the duality cascades and the translations in the space $C_{\mathrm{B}} \cap C_{\mathrm{P}}$. These results mean finiteness of the process of duality cascades and the uniqueness of the endpoint of the duality cascades [2].

3.1 Duality cascade

The Hanany-Witten transitions may decrease the number of D3-branes at the interval of the 5-branes by exchanging the 5-branes. And we can continue to apply the Hanany-Witten transitions until the negative ranks appear or the lower ranks do not appear. This series of the dualities are called a duality cascade. If a rank becomes negative, then the supersymmetries are broken. And we call the region where the duality cascades do not occur in the case if the overall rank is large enough, fundamental domain of the duality cascades.

For example, in the ABJM matrix model, namely $(1,1)$ model consisting of one $(1,k)$5-brane, one NS5-brane and D3-branes perpendicular to the 5-branes which are located at each interval of the 5-branes in a circle, when the reference rank and the rank difference are denoted N and M respectively, the Hanany-Witten transition makes the following brane configurations dual,

$$\langle N \bullet N + M \circ \rangle = \langle N \circ N + k - M \bullet \rangle. \tag{3.1}$$

Here since we decide N as a reference, M is a positive number from the inequality $N \leq N + M$. In the right hand side of the above equality, if the rank difference M is less than the Chern-Simons level k, the duality cascade does not occur. But, if M is greater than k, we reconsider the lowest rank $N + k - M$ as a new reference and duality cascade occurs as

$$\langle N \circ N + k - M \bullet \rangle = \langle N' \bullet N' - k + M \circ \rangle = \langle N' \circ N' + 2k - M \bullet \rangle, \tag{3.2}$$

with $N' = N + k - M$. And if $2k - M$ is negative, we continue to apply the Hanany-Witten transitions until the rank lower than the reference rank does not appear or the negative rank appears. Under this duality cascade, the reference rank N and the rank difference M continue to be redefined as

$$(N, M) \to (N + k - M, -k + M) \to (N + 3k - 2M, -2k + M) \to \cdots. \tag{3.3}$$

In the ABJM matrix model, if the overall rank N is large enough, we find that the fundamental domain is clearly $0 \leq M \leq k$ as we see also below [46]. However, more generally, it is not clear that the endpoint of the duality cascades is uniquely determined, and any brane configuration in a circle reduces to the fundamental domain of the duality cascades. We reveal them for the brane configurations corresponding to the D_5 and E_7 curves in [2] and explain them in the following sections.

3.2 Fundamental domain

The Hanany-Witten transitions may reduce each rank. And if the rank becomes lower than the reference rank, the duality cascade occurs [43–46]. In this subsection, we discuss the fundamental domain where we assume N_1 as the reference rank.

In the case of the A_1 curve, we consider the Hanany-Witten transitions for the brane configuration $\langle N_1 \overset{\text{i}}{\bullet} N_2 \overset{1}{\circ} \rangle = \langle N \overset{\text{i}}{\bullet} N + M \overset{1}{\circ} \rangle$. The brane configurations obtained through the Hanany-Witten transitions while preserving the reference rank are

$$\langle N \overset{\text{i}}{\bullet} N + M \overset{1}{\circ} \rangle, \quad \langle N \overset{1}{\circ} N + k - M \overset{\text{i}}{\bullet} \rangle. \tag{3.4}$$

Since the fundamental domain of the duality cascades is obtained from the condition $N_1 \leq N_2$, the region is

$$0 \leq M \leq k. \tag{3.5}$$

This region is a line segment in the one-dimensional space of the brane configurations $\langle N \overset{\text{i}}{\bullet} N + M \overset{1}{\circ} \rangle$, $C_{\text{B}}^{(1,1)} = \{M\}$. The boundaries of this fundamental domain are $M = 0, k$, thus these correspond to the brane configurations $\langle N \overset{\text{i}}{\bullet} N \overset{1}{\circ} \rangle$ and $\langle N \overset{\text{i}}{\bullet} N + k \overset{1}{\circ} \rangle$ respectively. These are just those without the rank difference in table 4.

In the case of the D_5 curve, we discuss the Hanany-Witten transitions for the brane configuration $\langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{\text{i}}{\bullet} N_3 \overset{1}{\circ} N_4 \overset{2}{\circ} \rangle = \langle N \overset{\text{ii}}{\bullet} N + M_1 - M_2 + M_3 \overset{\text{i}}{\bullet} N + 2M_1 \overset{1}{\circ} N + M_1 - M_2 - M_3 \overset{2}{\circ} \rangle$.

As in the A_1 case, the fundamental domain is obtained from the conditions that the each rank is greater than the reference rank $N_1 \leq N_2$, $N_1 \leq N_3$ and $N_1 \leq N_4$ as

$$\begin{gathered} 0 \leq M_1 \leq 2k, \quad -\frac{k}{2} \leq M_2 \leq \frac{k}{2}, \quad -\frac{k}{2} \leq M_3 \leq \frac{k}{2}, \\ 0 \leq M_1 + M_2 \pm M_3 \leq 2k, \quad 0 \leq M_1 - M_2 \pm M_3 \leq 2k. \end{gathered} \tag{3.6}$$

Here, above region contains extra inequality $0 \leq M_1 \leq 2k$, thus the fundamental domain for the D_5 curve can be described with less conditions by omitting this inequality. The fact that the inequality $0 \leq M_1 \leq 2k$ is extra is understood by looking at the fundamental domain in the case of, for example, $M_2 = 0$ or $M_3 = 0$. Therefore, it is found that the fundamental domain becomes as

$$\begin{gathered} -\frac{k}{2} \leq M_2 \leq \frac{k}{2}, \quad -\frac{k}{2} \leq M_3 \leq \frac{k}{2}, \\ 0 \leq M_1 + M_2 \pm M_3 \leq 2k, \quad 0 \leq M_1 - M_2 \pm M_3 \leq 2k. \end{gathered} \tag{3.7}$$

Then, this fundamental domain is a rhombic dodecahedron in the three-dimensional subspace of the brane configurations in the parameter space of the D_5 curve $C_{\mathrm{B}}^{(2,2)} = \{(M_1, M_2, M_3)\}$ (see figure 3). The vertices of this fundamental domain just correspond to the brane configurations without the reference ranks in table 5. Indeed, for example, the brane configuration $\langle N \overset{\text{ii}}{\bullet} N \overset{\text{i}}{\bullet} N \overset{1}{\circ} N \overset{2}{\circ} \rangle$ in the first line of table 5 is the vertex $(0, 0, 0)$ described by the intersection of the three planes $N_1 = N = N + M_1 - M_2 + M_3 = 0$, $N_2 = N = N + M_1$ and $N + M_1 - M_2 - M_3$. Similarly, the other brane configurations also correspond to the vertices of the fundamental domain.

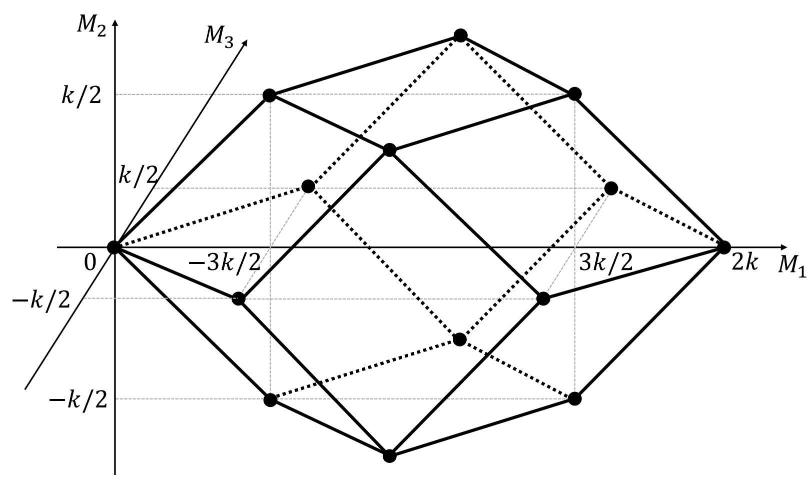

Figure 3: Fundamental domain for the D_5 curve.

In the case of the E_7 curve, we consider the Hanany-Witten transitions for the brane configuration $\langle N_1 \overset{\text{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\text{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ} \rangle = \langle N \overset{\text{ii}}{\bullet} N + G_1 + k \overset{1}{\circ} N + \frac{1}{2}(-3F_1 + F_2 + F_3 + G_1) + k \overset{2}{\circ} N - F_1 - F_2 + F_3 + k \overset{\text{i}}{\bullet} N - F_1 - F_2 + F_3 + G_1 + 2k \overset{3}{\circ} N + \frac{1}{2}(-F_1 - F_2 - F_3 + G_1) + k \overset{4}{\circ} \rangle$. The conditions $N_1 \leq N_2$, $N_1 \leq N_3$, $N_1 \leq N_4$, $N_1 \leq N_5$ and $N_1 \leq N_6$ give the fundamental domain of the duality cascades for the E_7 curve as

$$\begin{aligned} -k \leq F_1 + F_2 - F_3 \leq k, \quad -k \leq -F_1 + F_2 + F_3 \leq k, \quad -k \leq F_1 - F_2 + F_3 \leq k, \\ -2k \leq F_1 + F_2 - F_3 \pm G_1 \leq 2k, \quad -2k \leq -F_1 + F_2 + F_3 \pm G_1 \leq 2k, \\ -2k \leq F_1 - F_2 + F_3 \pm G_1 \leq 2k, \quad -2k \leq F_1 + F_2 - 3F_3 \pm G_1 \leq 2k, \\ -2k \leq -3F_1 + F_2 + F_3 \pm G_1 \leq 2k, \quad -2k \leq F_1 - 3F_2 + F_3 \pm G_1 \leq 2k, \\ -k \leq G_1 \leq k, \quad -2k \leq \sum_{i=1}^{3} F_i \pm G_1 \leq 2k. \end{aligned} \tag{3.8}$$

Here, due to $-k \leq G_1 \leq k$, it is found that above inequalities contain some extra inequalities, thus the fundamental domain is described with less conditions as

$$\begin{aligned} -k \leq F_1 + F_2 - F_3 \leq k, \quad -k \leq -F_1 + F_2 + F_3 \leq k, \quad -k \leq F_1 - F_2 + F_3 \leq k, \\ -2k \leq F_1 + F_2 - 3F_3 \pm G_1 \leq 2k, \quad -2k \leq -3F_1 + F_2 + F_3 \pm G_1 \leq 2k, \\ -2k \leq F_1 - 3F_2 + F_3 \pm G_1 \leq 2k, \quad -k \leq G_1 \leq k, \quad -2k \leq \sum_{i=1}^{3} F_i \pm G_1 \leq 2k. \end{aligned} \tag{3.9}$$

This fundamental domain is an icositetrachoron in the four-dimensional space of the brane configurations $C_{\mathrm{P}}^{E_7} \cap C_{\mathrm{B}}^{(2,4)} = \{(F_1, F_2, F_3, G_1)\}$. The vertices of the fundamental domain are just those without the rank differences in table 6. Also, the coordinate transformation

$$M_1 = F_1 - F_2 - F_3, \quad M_2 = -F_1 + F_2 - F_3, \quad M_3 = -F_1 - F_2 + F_3,$$

leads the vertices to the permutations of $(\pm k, \pm k, 0, 0)$ and it is clearer that the fundamental domain of the duality cascades is the icositetrachoron.

3.3 Affine Weyl group

In this subsection, we discuss how the Weyl groups which are symmetries of the curves act to the spaces of the brane configurations. And if we add to Weyl groups the reflection with respect to boundary planes of the fundamental domain of the duality cascades, we get the affine Weyl groups [7]. Then, the fundamental domain of the duality cascades is

[7] For affine Weyl groups, for example, see [49].

divided to the affine Weyl chambers and we also find that it can fill the spaces of the brane configurations.

The Weyl reflection for the hyperplane perpendicular to the simple root α acts to a vector v as follows,

$$s_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha, \tag{3.10}$$

where we identify a vector v in the subspace of the brane configurations in the parameter space of the curves $C_{\mathrm{B}} \cap C_{\mathrm{P}}$ as a wight vector.

In the case of the A_1 curve, the A_1 Weyl group $W(A_1)$ acts to a one-dimensional vector $v = M$ as

$$s_1 : M \mapsto -M, \tag{3.11}$$

where we redefine the parameter M as $M \to M + \frac{k}{2}$ for simplicity. Then, from $M - s_1(M) = 2M$ it is found that the metric, the simple root and the Cartan matrix are

$$g = 2, \quad \alpha_1 = 1, \quad A_{11}^{(A_1)} = 2, \tag{3.12}$$

where we decide the metric g so that the length of the root vector α_1 becomes $\sqrt{2}$. Also, the overall rank N is invariant under actions of the Weyl group.

The fundamental domain in the A_1 case has two boundary planes, thus we can consider the extra reflections about those planes in addition to the Weyl reflection. These reflections generate the affine Weyl group $W(\widehat{A}_1)$. The extra root and the Cartan matrix are respectively,

$$\tilde{\alpha}_0 = -1, \quad (A_{ij}^{(\widehat{A}_1)}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \tag{3.13}$$

and we give the Dynkin diagram in figure 4. Therefore, the affine Weyl group $W(\widehat{A}_1)$ is

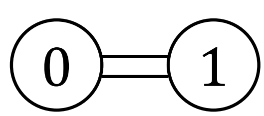

Figure 4: Affine A_1 Dynkin diagram.

obtained by considering the fundamental domain in addition to the symmetry of the A_1 curve. Moreover, by dividing the fundamental domain with the Weyl group or by dividing the space of the brane configurations $C_{\mathrm{B}}^{(1,1)}$ with the affine Weyl group, the affine Weyl chamber is obtained as

$$0 \leq M \leq \frac{k}{2}. \tag{3.14}$$

The affine Weyl group allows us to translate a point in the direction perpendicular to the boundary planes. we can fill the one-dimensional space $C_{\mathrm{B}}^{(1,1)} = C_{\mathrm{P}}^{A_1}$ by reflecting the affine Weyl chamber or translating the fundamental domain of the duality cascades with the affine Weyl group.

The reflection associated with the simple root vector $\tilde{\alpha}_0$ acts to $C_{\mathrm{B}}^{(1,1)}$ as

$$\tilde{s}_0 : M \mapsto k - M, \tag{3.15}$$

and cannot preserve the overall rank N unlike the reflections of the Weyl group $W(A_1)$. Concretely, from the fact that $\tilde{s}_0$ is a reflection, namely $\tilde{s}_0^2 = \mathrm{id}$, it is found that $\tilde{s}_0$ transforms N as

$$\tilde{s}_0 : N \mapsto N + 2aM - ak, \tag{3.16}$$

with the arbitrary parameter a. Besides, when we identify the overall rank N as the eigenvalue of the additional grading operator in the affine Lie algebra and at same time the Chern-Simons level as the level of the weight vector v, N is transited with the highest root $\theta = \alpha_1$ as

$$\tilde{s}_0 : N \mapsto N - (\theta, v) + k, \tag{3.17}$$

therefore an arbitrary parameter a is decided as $a = -1$.

In the case of the D_5 curve, with the correspondence between the parameters of the D_5 curve and the brane configuration, the transitions for the parameters of the curve acts to the three-dimensional subspace of the brane configurations in the parameter space $C_{\mathrm{B}}^{(2,2)} \subset C_{\mathrm{P}}^{D_5}$ as

$$\begin{aligned}
s_{12} := s_1 s_2 &: (M_1, M_2, M_3) \mapsto (M_1, -M_3, -M_2),\\
s_3 &: (M_1, M_2, M_3) \mapsto \left(-M_2 - M_3, \frac{1}{2}(-M_1 + M_2 - M_3), \frac{1}{2}(-M_1 - M_2 + M_3)\right),\\
s_4 &: (M_1, M_2, M_3) \mapsto \left(-M_2 + M_3, \frac{1}{2}(-M_1 + M_2 + M_3), \frac{1}{2}(M_1 + M_2 + M_3)\right),\\
s_{50} := s_5 s_0 &: (M_1, M_2, M_3) \mapsto (M_1, M_3, M_2),
\end{aligned} \tag{3.18}$$

where we redefine $M_1 \to M_1 + k$ for simplicity. These transitions generate the Weyl group $W(B_3) = \{s_{12}, s_3, s_4; s_{50}\}$. And these transitions are the Weyl reflections for a three-dimensional vector

$$v = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}. \tag{3.19}$$

By representing the reflections s_{12}, s_3, s_4, s_{50} for the vector v with the corresponding simple root vectors $\alpha_{12}, \alpha_3, \alpha_4, \alpha_{50}$ as

$$\begin{cases} v - s_{12}v = 2(M_2 + M_3)\alpha_{12}, \\ v - s_3 v = -(M_1 + M_2 + M_3)\alpha_3, \\ v - s_4 v = (M_1 + M_2 - M_3)\alpha_4, \\ v - s_{50}v = -2(M_2 - M_3)\alpha_{50}, \end{cases} \tag{3.20}$$

we find that the simple root vectors and the Cartan matrix are respectively given with the metric $(g_{ij}) = \mathrm{diag}(1, 2, 2)$ as

$$\alpha_{12} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} -1 \\ -1/2 \\ -1/2 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 \\ 1/2 \\ -1/2 \end{pmatrix}, \quad (A_{ij}^{(B_3)}) = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{3.21}$$

where we omit α_{50} because s_{50} is auxiliary transition described by the others $s_{i \neq 50}$. Here we decide the degree of freedom of overall of the metric so that the lengths of the root vectors become 1 or $\sqrt{2}$. If we consider the reflection about the boundary plane of the fundamental domain as the extra reflection $\tilde{s}_0$ of the affine Weyl group $W(\widehat{B}_3)$, the extra root vector $\tilde{\alpha}_0$ and the Cartan matrix are given by[8]

$$\tilde{\alpha}_0 = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad (A_{ij}^{(\widehat{B}_3)}) = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \tag{3.22}$$

and the Dynkin diagram is given in figure 5. Also, when we divide the fundamental domain

[8]If we change the definitions of the metric and the simple roots, we can construct the C_3 Cartan matrix. Actually, the Weyl groups $W(B_3)$ and $W(C_3)$ are indistinguishable. However, if we choose the Weyl group $W(C_3)$, we cannot construct the affine Weyl group with the reflection about the boundary plane of the fundamental domain.

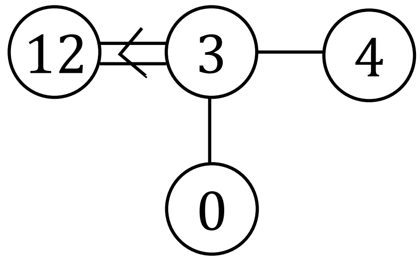

Figure 5: Affine B_3 Dynkin diagram.

by the hyperplanes parpendicular to the root vectors α_i, the affine Weyl chamber is given as

$$0 \le M_2 + M_3, \quad M_1 + M_2 + M_3 \le 0, \quad 0 \le M_1 + M_2 - M_3, \quad -k \le M_1 - M_2 + M_3, \tag{3.23}$$

where this chamber shares the surface surrounded by the three points

$$(M_1, M_2, M_3) = \left(0, \frac{k}{2}, \frac{k}{2}\right), \ \left(-\frac{k}{2}, \frac{k}{2}, 0\right), \ \left(-\frac{k}{2}, \frac{k}{4}, -\frac{k}{4}\right), \tag{3.24}$$

with the fundamental domain (see figure 6). As in the A_1 case, this fundamental affine

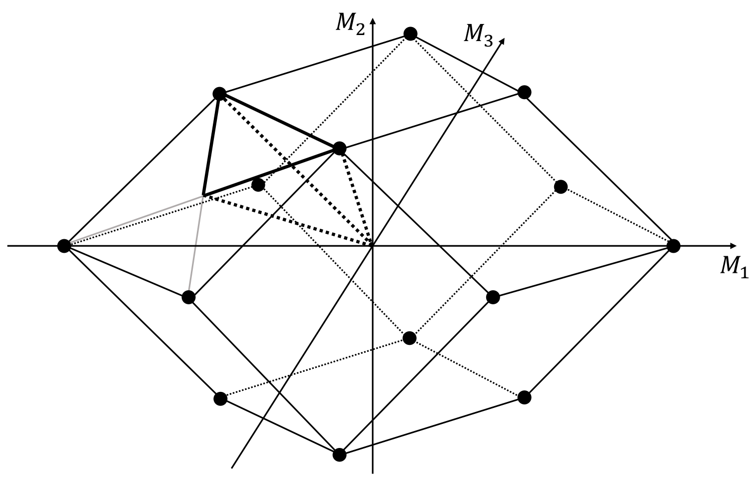

Figure 6: Affine B_3 Weyl chamber. The region is the part obtaned by dividing the fundamental domain by the planes consisting of the dotted lines. Then dotted lines intersect the surface of fundamental domain at (3.24).

Weyl chamber (or the fundamental domain) can fill the three-dimensional subspace of the brane configurations $C_{\mathrm{B}}^{(2,2)} \subset C_{\mathrm{P}}^{D_5}$ with actions of the affine Weyl group, and we identify the overall rank N and the Chern-Simons level as the eigenvalue of the additional grading operator and the level of the weight vector respectively. Actually, the reflection $\tilde{s}_0$ acts to (M_1, M_2, M_3) and N as

$$\begin{aligned}\tilde{s}_0 : (M_1, M_2, M_3) \mapsto & (M_2 - M_3 - k, \frac{1}{2}(M_1 + M_2 + M_3 + k), \frac{1}{2}(-M_1 + M_2 + M_3 - k)), \\ \tilde{s}_0 : N \mapsto & N + M_1 - M_2 + M_3 + k.\end{aligned} \tag{3.25}$$

This result for N is reproduced from the formula (3.17) with the highest root $\theta = \alpha_4 + 2\alpha_3 + 2\alpha_{12}$ in B_3.

Finally, we construct the affine Weyl group for the parameter space in the case of the E_7 curve. As in the D_5 case, the correspondence between the curve and the brane configuration reduces the symmetry of the curve $W(E_7)$ to $W(F_4)$. And the Weyl group $W(F_4)$ acts to the four-dimensional vector

$$v = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ G_1 \end{pmatrix}, \tag{3.26}$$

as follows,

$$\begin{cases} v - s_4 v = G_1 \alpha_4, \\ v - s_3 v = -\frac{1}{2}(3F_1 - F_2 - F_3 + G_1)\alpha_3, \\ v - s_{25} v = 2(F_1 - F_2)\alpha_{25}, \\ v - s_{16} v = 2(F_2 - F_3)\alpha_{16}, \\ v - s_{07} v = 2F_3 \alpha_{07}. \end{cases} \tag{3.27}$$

where α_4, α_3, α_{25}, α_{16} and α_{07} are the simple root vectors corresponding to the reflections s_4, s_3, s_{25}, s_{16} and s_{07} respectively. With the metric

$$(g_{ij}) = \frac{1}{2}\begin{pmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.28}$$

these simple root vectors are given as

$$\alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \alpha_{25} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_{16} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix}, \quad \alpha_{07} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}, \tag{3.29}$$

and the Cartan matrix is obtained as

$$(A_{ij}^{(F_4)}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{3.30}$$

By considering the reflection about the boundary plane as the extra reflection in the affine Weyl group, the extra root and the Cartan matrix are given as

$$\tilde{\alpha}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad (A_{ij}^{(\widehat{F}_4)}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{3.31}$$

and see figure 7 about the Dynkin diagram. Also, the fundamental affine Weyl chamber

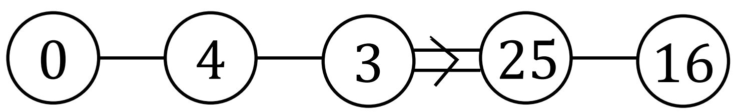

Figure 7: Affine F_4 Dynkin diagram.

shares the area surrounded by the four points:

$$(F_1, F_2, F_3, G_1) = \left(\frac{k}{2}, \frac{k}{2}, k, 0\right), \left(\frac{k}{2}, \frac{3k}{4}, \frac{3k}{4}, 0\right), \left(\frac{2k}{3}, \frac{2k}{3}, \frac{2k}{3}, 0\right), \left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, -\frac{k}{2}\right), \tag{3.32}$$

with the fundamental domain of the duality cascades, namely it is given as

$$-3F_1 + F_2 + F_3 - G_1 \le 0, \quad F_1 \le F_2 \le F_3, \quad G_1 \le 0, \quad F_1 + F_2 + F_3 - G_1 \le 2k. \tag{3.33}$$

This region fills the four-dimensional space $C_{\mathrm{B}}^{(2,4)} \cap C_{\mathrm{P}}^{E_7}$ with the actions of the affine Weyl group. And the reflection corresponding to the simple root vector $\tilde{\alpha}_0$ is concretely represented as

$$\begin{aligned} \tilde{s}_0 : (F_1, F_2, F_3, G_1) \mapsto &\Big(\frac{1}{2}(F_1 - F_2 - F_3 + G_1) - k, \frac{1}{2}(-F_1 + F_2 - F_3 + G_1) - k, \\ &\frac{1}{2}(-F_1 - F_2 + F_3 + G_1) - k, \frac{1}{2}(F_1 + F_2 + F_3 + G_1) + k\Big), \\ \tilde{s}_0 : N \mapsto &N + \frac{1}{2}(F_1 + F_2 + F_3 - G_1) + k. \end{aligned} \tag{3.34}$$

3.4 Translations and brane transitions

In the previous subsection, the affine Weyl groups are given by considering the fundamental domains in addition to the Weyl groups which are the symmetries of the curves. In this subsection, let us discuss that the affine Weyl group is not only a mathematically interesting structure but also a natural structure from the viewpoint of the brane transitions.

The extra reflection $\tilde{s}_0$ in the affine Weyl group is the reflection about the boundary plane of the fundamental domain, and this reflection changes the overall rank (3.17). If the overall rank decreases after the reflection, we expect it to match with the duality cascade. At the same time, since all translations of the affine Weyl group are always represented with at least one $\tilde{s}_0$, these translations also change the overall rank. In this subsection, conversely, from the transitions which change the overall rank, we reproduce the translations obtained from the affine Weyl group. The transitions changing the reference rank for sequence of the 5-branes are given as the rotation for the 5-brane at the left end to the right end. These transitions mean that we replace the reference after cyclically moving 5-branes to the S^1 direction, thus we will call these transitions the cyclic transitions. Since the cyclic transitions mean the redefinition of the reference rank, if the reference rank decreases after the cyclic transitions, the duality cascade occurs. Also, since the cyclic transitions are obtained by turning the sequence of the 5-branes to S^1 direction, they are natural transitions considered as extensions of the similarity transitions in the case without difference ranks.

First, in the case of the A_1 curve, the cyclic transitions for the NS5-brane $\overset{\mathrm{i}}{\bullet}\ c_{\mathrm{i}}$ and for the $(1,k)$5-brane $\overset{1}{\circ}\ c_1$ act to the brane configuration $\langle N_1 \overset{\mathrm{i}}{\bullet} N_2 \overset{1}{\circ}\rangle = \langle N \overset{\mathrm{i}}{\bullet} N + M + \frac{k}{2}\overset{1}{\circ}\rangle$ as

$$\begin{aligned} c_{\mathrm{i}} :& \langle N \overset{\mathrm{i}}{\bullet} N + M + \frac{k}{2}\overset{1}{\circ}\rangle \mapsto \langle N + M + \frac{k}{2} \overset{1}{\circ} N \overset{\mathrm{i}}{\bullet}\rangle \\ &\overset{\mathrm{HW}}{=} \langle N + M + \frac{k}{2} \overset{\mathrm{i}}{\bullet} N + 2M + 2k \overset{1}{\circ}\rangle = \langle N' \overset{\mathrm{i}}{\bullet} N' + M + \frac{3k}{2}\overset{1}{\circ}\rangle, \end{aligned} \tag{3.35}$$

$$\begin{aligned} c_1 :& \langle N \overset{\mathrm{i}}{\bullet} N + M + \frac{k}{2}\overset{1}{\circ}\rangle \overset{\mathrm{HW}}{=} \langle N \overset{1}{\circ} N - M + \frac{k}{2}\overset{\mathrm{i}}{\bullet}\rangle \\ &\mapsto \langle N - M + \frac{k}{2} \overset{\mathrm{i}}{\bullet} N \overset{1}{\circ}\rangle = \langle N' \overset{\mathrm{i}}{\bullet} N' + M - \frac{k}{2}\overset{1}{\circ}\rangle, \end{aligned} \tag{3.36}$$

where the equals labeled HW mean the dualities under the Hanany-Witten transitions and the unlabeled equals mean the redefinition of the reference rank. Thus, these transitions act to the one-dimensional space of the brane configurations $C_{\mathrm{B}}^{(1,1)} = \{M\}$ as

$$c_{\mathrm{i}} : M \mapsto M + k, \quad c_1 : M \mapsto M - k, \quad c_{\mathrm{i}}c_1 : M \mapsto M. \tag{3.37}$$

These transitions clearly mean the translations on $C_{\rm B}^{(1,1)} = \{M\}$ and are described in terms of the affine Weyl group $W(\widehat{A}_1) = \{s_1, s_0\}$ as

$$c_{\rm i} = \tilde{s}_0 s_1, \quad c_1 = s_1 \tilde{s}_0. \tag{3.38}$$

Also, these translations act to the overall rank N as

$$c_{\rm i} : N \mapsto N + M + \frac{k}{2}, \quad c_1 : N \mapsto N - M + \frac{k}{2}. \tag{3.39}$$

Next, in the case of the D_5 curve, the translations act to the brane configuration $\langle N_1 \overset{\rm ii}{\bullet} N_2 \overset{\rm i}{\bullet} N_3 \overset{1}{\circ} N_4 \overset{2}{\circ} \rangle = \langle N \overset{\rm ii}{\bullet} N + M_1 - M_2 + M_3 + k \overset{\rm i}{\bullet} N + 2M_1 + 2k \overset{1}{\circ} N + M_1 - M_2 - M_3 + k \overset{2}{\circ} \rangle$, for example, the translation for an NS5-brane $\overset{\rm ii}{\bullet}$ $c_{\rm ii}$ is given as

$$\begin{aligned}
c_{\rm ii} :& \langle N \overset{\rm ii}{\bullet} N + M_1 - M_2 + M_3 + k \overset{\rm i}{\bullet} N + 2M_1 + 2k \overset{1}{\circ} N + M_1 - M_2 - M_3 + k \overset{2}{\circ} \rangle \\
&\mapsto \langle N + M_1 - M_2 + M_3 + k \overset{\rm i}{\bullet} N + 2M_1 + 2k \overset{1}{\circ} N + M_1 - M_2 - M_3 + k \overset{2}{\circ} N \overset{\rm ii}{\bullet} \rangle \\
&= \langle N' \overset{\rm i}{\bullet} N' + M_1 + M_2 - M_3 + k \overset{1}{\circ} N' - 2M_3 \overset{2}{\circ} N' - M_1 + M_2 - M_3 - k \overset{\rm ii}{\bullet} \rangle \\
&\overset{\rm HW}{=} \langle N' \overset{\rm ii}{\bullet} N' + M_1 - M_2 + M_3 + 3k \overset{\rm i}{\bullet} N' + 2M_1 + 4k \overset{1}{\circ} N' + M_1 - M_2 - M_3 + 2k \overset{2}{\circ} \rangle.
\end{aligned} \tag{3.40}$$

When we regard this cyclic transition $c_{\rm ii}$ as the transition on the three-dimensional space of the brane configurations $C_{\rm B}^{(2,2)} = \{(M_1, M_2, M_3)\}$, this transition is represented as

$$c_{\rm ii} : (M_1, M_2, M_3) \mapsto (M_1 + k, M_2 - \frac{1}{2}k, M_3 + \frac{1}{2}k), \tag{3.41}$$

thus this cyclic transition for the NS5-brane $\overset{\rm ii}{\bullet}$ means the translation on the parameter space. Also the inverse of $c_{\rm ii}$ is expressed as the inverse cyclic transition for the NS5-brane $\overset{\rm ii}{\bullet}$ in the brane configurations,

$$\begin{aligned}
c_{\rm ii}^{-1} :& \langle N \overset{\rm ii}{\bullet} N + M_1 - M_2 + M_3 + k \overset{\rm i}{\bullet} N + 2M_1 + 2k \overset{1}{\circ} N + M_1 - M_2 - M_3 + k \overset{2}{\circ} \rangle \\
&\overset{\rm HW}{=} \langle N \overset{\rm i}{\bullet} N + M_1 + M_2 - M_3 + k \overset{1}{\circ} N - 2M_3 + k \overset{2}{\circ} N - M_1 + M_2 - M_3 + k \overset{\rm ii}{\bullet} \rangle \\
&\mapsto \langle N - M_1 + M_2 - M_3 + k \overset{\rm ii}{\bullet} N \overset{\rm i}{\bullet} N + M_1 + M_2 - M_3 + k \overset{1}{\circ} N - 2M_3 + k \overset{2}{\circ} \rangle \\
&= \langle N' \overset{\rm ii}{\bullet} N' + M_1 - M_2 + M_3 - k \overset{\rm i}{\bullet} N' + 2M_1 \overset{1}{\circ} N' + M_1 - M_2 - M_3 \overset{2}{\circ} \rangle,
\end{aligned} \tag{3.42}$$

and is represented on the space of the brane configurations as

$$c_{\rm ii} : (M_1, M_2, M_3) \mapsto (M_1 - k, M_2 + \frac{1}{2}k, M_3 - \frac{1}{2}k). \tag{3.43}$$

When $c_{\rm i}$, $c_{\rm ii}$, c_1 and c_2 denote the cyclic transitions for the 5-branes $\overset{\rm i}{\bullet}$, $\overset{\rm ii}{\bullet}$, $\overset{1}{\circ}$ and $\overset{2}{\circ}$ respectively, the transitions c_i ($i = {\rm i}, {\rm ii}, 1, 2$) are represented on the space of the brane configurations as

$$
\begin{aligned}
c_{\rm ii} &: (M_1, M_2, M_3) \mapsto (M_1 + k, M_2 - \frac{1}{2}k, M_3 + \frac{1}{2}k), \\
c_{\rm i} &: (M_1, M_2, M_3) \mapsto (M_1 + k, M_2 + \frac{1}{2}k, M_3 - \frac{1}{2}k), \\
c_1 &: (M_1, M_2, M_3) \mapsto (M_1 - k, M_2 - \frac{1}{2}k, M_3 - \frac{1}{2}k), \\
c_2 &: (M_1, M_2, M_3) \mapsto (M_1 - k, M_2 + \frac{1}{2}k, M_3 + \frac{1}{2}k), \\
c_{\rm i}c_{\rm ii}c_1c_2 &: (M_1, M_2, M_3) \mapsto (M_1, M_2, M_3),
\end{aligned}
\tag{3.44}
$$

and mean the translations on the space $C_{\rm B}^{(2,2)} = \{(M_1, M_2, M_3)\}$. These translations are represented in terms of generators of the affine Weyl group $W(\widehat{B}_3)$ as

$$
\begin{aligned}
c_{\rm ii} &= s_3 s_{12} s_3 s_4 s_3 s_{12} s_3 \tilde{s}_0, \\
c_{\rm i} &= s_3 s_{12} s_3 \tilde{s}_0 s_3 s_{12} s_3 s_4, \\
c_1 &= s_{12} s_4 s_3 \tilde{s}_0 s_3 s_{12} s_4 s_3, \\
c_2 &= s_4 s_3 \tilde{s}_0 s_3 s_{12} s_4 s_3 s_{12}.
\end{aligned}
\tag{3.45}
$$

Here, since the Weyl group $W(B_3)$ does not change the reference rank and these cyclic transitions change the reference rank, we need to use at least one extra reflection $\tilde{s}_0$ in the affine Weyl group to describe these transitions with the affine Weyl group.

These cyclic transitions are geometrically interpreted as the translations in the direction perpendicular to the boundary planes of the fundamental domain of the duality cascades. We give all combinations of these cyclic transitions in table 7. The transitions in table 7 mean the translations perpendicular to the planes described by the inequalities (3.6). The extra inequalities in (3.6) are $-k \le M_1 \le k$, the transitions associated with these are given by one for the two NS5-branes $c_{\rm ii}c_{\rm i}$ and one for the two $(1,k)$5-branes c_1c_2. Namely, the cyclic transitions for the 5-branes of the same type are not the translations perpendicular to the boundary planes in the fundamental domain (3.7).

Also, the cyclic transitions are redefinitions of the reference rank that chooses the number of the D3-branes connecting to the 5-brane on the left end from the right side as the new reference rank. For example, the cyclic transition $c_{\rm ii}c_2 = c_1^{-1}c_{\rm i}^{-1}$ corresponds to the redefinition of the reference to the number of the D3-branes at $\cdot$ on the brane configuration $\langle \overset{\rm ii}{\bullet} \overset{2}{\circ} \cdot \overset{\rm i}{\bullet} \overset{1}{\circ} \rangle$.

Cyclic transitions	(M_1, M_2, M_3)	N
c_{ii}	$(M_1+k, M_2-\frac{k}{2}, M_3+\frac{k}{2})$	$N+M_1-M_2+M_3+k$
c_{i}	$(M_1+k, M_2+\frac{k}{2}, M_3-\frac{k}{2})$	$N+M_1+M_2-M_3+k$
c_1	$(M_1-k, M_2-\frac{k}{2}, M_3-\frac{k}{2})$	$N-M_1-M_2-M_3+k$
c_2	$(M_1-k, M_2+\frac{k}{2}, M_3+\frac{k}{2})$	$N+M_1+M_2+M_3+k$
$c_{\mathrm{ii}}c_{\mathrm{i}}$	(M_1+2k, M_2, M_3)	$N+2M_1+2k$
$c_{\mathrm{ii}}c_1$	(M_1, M_2-k, M_3)	$N-2M_2+2k$
$c_{\mathrm{i}}c_1$	(M_1, M_2, M_3-k)	$N-2M_3+2k$
$c_{\mathrm{ii}}c_2$	(M_1, M_2, M_3+k)	$N+2M_3+2k$
$c_{\mathrm{i}}c_2$	(M_1, M_2+k, M_3)	$N+2M_2+2k$
c_1c_2	(M_1-2k, M_2, M_3)	$N-2M_1+2k$
$c_{\mathrm{ii}}c_{\mathrm{i}}c_1$	$(M_1+k, M_2-\frac{k}{2}, M_3-\frac{k}{2})$	$N+M_1-M_2-M_3+k$
$c_{\mathrm{ii}}c_{\mathrm{i}}c_2$	$(M_1+k, M_2+\frac{k}{2}, M_3+\frac{k}{2})$	$N+M_1+M_2+M_3+k$
$c_{\mathrm{ii}}c_1c_2$	$(M_1-k, M_2-\frac{k}{2}, M_3+\frac{k}{2})$	$N-M_1-M_2+M_3+k$
$c_{\mathrm{i}}c_1c_2$	$(M_1-k, M_2+\frac{k}{2}, M_3-\frac{k}{2})$	$N-M_1+M_2-M_3+k$

Table 7: List of the cyclic transitions which are interpreted as the translations perpendicular to the boundary planes of the fundamental domain.

Finally, in the case of the E_7 curve, we discuss the cyclic transitions acting to the brane configuration $\langle N_1 \overset{\mathrm{ii}}{\bullet} N_2 \overset{1}{\circ} N_3 \overset{2}{\circ} N_4 \overset{\mathrm{i}}{\bullet} N_5 \overset{3}{\circ} N_6 \overset{4}{\circ}\rangle = \langle N \overset{\mathrm{ii}}{\bullet} N+G_1+k \overset{1}{\circ} N+\frac{1}{2}(-3F_1+F_2+F_3+G_1)+k \overset{2}{\circ} N-F_1-F_2+F_3+k \overset{\mathrm{i}}{\bullet} N-F_1-F_2+F_3+G_1+2k \overset{3}{\circ} N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k\overset{4}{\circ}\rangle$. In this case, as we mentioned in the previous section, we place the NS5-brane $\overset{\mathrm{ii}}{\bullet}$ on the left and the NS5-brane $\overset{\mathrm{i}}{\bullet}$ on the right and that the Hanany-Witten transition between NS5-branes is restricted. However the labels for the 5-branes are introduced to consider the correspondence between the brane configurations and the quantum curves, thus they are not physical. Therefore if we rotate the NS5-brane cyclically, we relabel for the NS5-branes so that the NS5-brane $\overset{\mathrm{ii}}{\bullet}$ is placed to the left of the NS5-brane $\overset{\mathrm{i}}{\bullet}$. Also, when we consider the cyclic transition for the NS5-brane $\overset{\mathrm{i}}{\bullet}$, it acts to the brane configuration after relabeling to the NS5-branes. The cyclic transitions are represented in the four-dimensional space $C_{\mathrm{B}}^{(2,4)} \cap C_{\mathrm{P}}^{E_7} = \{(F_1, F_2, F_3, G_1)\}$ as

$$
\begin{aligned}
c_\bullet := c_{\mathrm{ii}} = c_{\mathrm{i}} &: (F_1, F_2, F_3, G_1) \mapsto (F_1, F_2, F_3, G_1+2k),\\
c_1 &: (F_1, F_2, F_3, G_1) \mapsto (F_1-k, F_2, F_3, G_1-k),\\
c_2 &: (F_1, F_2, F_3, G_1) \mapsto (F_1, F_2-k, F_3, G_1-k),\\
c_3 &: (F_1, F_2, F_3, G_1) \mapsto (F_1, F_2, F_3-k, G_1-k),
\end{aligned}
$$

$$
\begin{aligned}
c_4 : (F_1, F_2, F_3, G_1) &\mapsto (F_1 + k, F_2 + k, F_3 + k, G_1 - k), \\
c_{\mathrm{i}} c_{\mathrm{ii}} c_1 c_2 c_3 c_4 : (F_1, F_2, F_3, G_1) &\mapsto (F_1, F_2, F_3, G_1),
\end{aligned}
\tag{3.46}
$$

and mean the translations on the space. In table 8, we give the combinations of the cyclic transitions geometrically interpreted as the translations perpendicular to the boundary planes of the fundamental domain (3.9). The cyclic transitions not included in table 8, $c_\bullet c_\bullet$, $c_i c_j$, $c_i c_j c_k$ and $c_i c_j c_k c_\ell$ $(i, j, k, \ell = 1, \cdots, 4)$ are not the translations associated with the boundary planes of the fundamental domain (3.9). Especially, among them, $c_i c_j$ are the translations perpendicular to the extra separations described by the extra inequalities in (3.8).

4 Discussions

4.1 Space-filling with Hanany-Witten transitions

In this subsection, we show that the boundary planes facing each other in the fundamental domain are parallel in the general case corresponding to the super Chern-Simons matrix model. This fact is derived from the Hanany-Witten transitions independently of the affine Weyl groups.

In the case with p NS5-branes and q $(1, k)$5-branes, the Hanany-Witten transitions lead

$$
\begin{aligned}
&\langle 0 \overset{1}{\bullet} N_2 \overset{2}{\bullet} N_3 \overset{3}{\bullet} \cdots N_x \overset{x}{\bullet} N_{x+1} \cdots \overset{p-1}{\bullet} N_p \overset{p}{\bullet} N_{p+1} \\
&\overset{1}{\circ} N_{p+2} \overset{2}{\circ} \cdots N_{p+y} \overset{y}{\circ} N_{p+y+1} \cdots \overset{q-2}{\circ} N_{p+q-1} \overset{q-1}{\circ} N_{p+q} \overset{q}{\circ} \rangle \\
\overset{\mathrm{HW}}{=} &\langle 0 \overset{q}{\circ} pk - N_{p+q} \overset{q-1}{\circ} 2pk - N_{p+q-1} \overset{q-2}{\circ} \cdots \overset{2}{\circ} (q-1)pk - N_{p+2} \overset{1}{\circ} qpk - N_{p+1} \\
&\overset{p}{\bullet} q(p-1)k - N_p \overset{p+1}{\bullet} \cdots qxk - N_{x+1} \overset{x}{\bullet} q(x-1)k - N_x \cdots \overset{3}{\bullet} 2qk - N_3 \overset{2}{\bullet} qk - N_2 \overset{1}{\bullet} \rangle,
\end{aligned}
\tag{4.1}
$$

where the RR-charges are invariant on both sides. From this equation, we find that if there is $0 \le N_a$ as the condition to describe the fundamental domain, there is always the condition $0 \le f(k) - N_a$ corresponding to it independently of the affine Weyl groups where f is the linear function of k.This fact means that the boundary planes facing each other in the fundamental domain are paralell. And the fundamental domain is obtained by the cutting out the rectangle described by $0 \le N_a \le f(k)$.

As a simple example, let us consider the brane configuration with one $(1, k)$5-brane and two NS5-brane: $\langle N \overset{\mathrm{ii}}{\bullet} N + M_1 \overset{\mathrm{i}}{\bullet} N + M_2 \overset{1}{\circ} \rangle$. Then the space of the brane configurations

Cyclic transitions	(F_1, F_2, F_3, G_1)	N
$c_\bullet$	$(F_1, F_2, F_3, G_1 + 2k)$	$N + G_1 + k$
c_1	$(F_1 - k, F_2, F_3, G_1 - k)$	$N + \frac{1}{2}(-3F_1 + F_2 + F_3 - G_1) + k$
c_2	$(F_1, F_2 - k, F_3, G_1 - k)$	$N + \frac{1}{2}(F_1 - 3F_2 + F_3 - G_1) + k$
c_3	$(F_1, F_2, F_3 - k, G_1 - k)$	$N + \frac{1}{2}(F_1 + F_2 - 3F_3 - G_1) + k$
c_4	$(F_1 + k, F_2 + k, F_3 + k, G_1 - k)$	$N + \frac{1}{2}(F_1 + F_2 + F_3 - G_1) + k$
$c_\bullet c_1$	$(F_1 - k, F_2, F_3, G_1 + k)$	$N + \frac{1}{2}(-3F_1 + F_2 + F_3 + G_1) + k$
$c_\bullet c_2$	$(F_1, F_2 - k, F_3, G_1 + k)$	$N + \frac{1}{2}(F_1 - 3F_2 + F_3 + G_1) + k$
$c_\bullet c_3$	$(F_1, F_2, F_3 - k, G_1 + k)$	$N + \frac{1}{2}(F_1 + F_2 - 3F_3 + G_1) + k$
$c_\bullet c_4$	$(F_1 + k, F_2 + k, F_3 + k, G_1 + k)$	$N + \frac{1}{2}(F_1 + F_2 + F_3 + G_1) + k$
$c_\bullet c_1 c_2$	$(F_1 - k, F_2 - k, F_3, G_1)$	$N - F_1 - F_2 + F_3 + k$
$c_\bullet c_1 c_3$	$(F_1 - k, F_2, F_3 - k, G_1)$	$N - F_1 + F_2 - F_3 + k$
$c_\bullet c_1 c_4$	$(F_1, F_2 + k, F_3 + k, G_1)$	$N - F_1 + F_2 + F_3 + k$
$c_\bullet c_2 c_3$	$(F_1, F_2 - k, F_3 - k, G_1)$	$N + F_1 - F_2 - F_3 + k$
$c_\bullet c_2 c_4$	$(F_1 + k, F_2, F_3 + k, G_1)$	$N + F_1 - F_2 + F_3 + k$
$c_\bullet c_3 c_4$	$(F_1 + k, F_2 + k, F_3, G_1)$	$N + F_1 + F_2 - F_3 + k$
$c_\bullet c_1 c_2 c_3$	$(F_1 - k, F_2 - k, F_3 - k, G_1 - k)$	$\frac{1}{2}(-F_1 - F_2 - F_3 - G_1) + k$
$c_\bullet c_1 c_2 c_4$	$(F_1, F_2, F_3 + k, G_1 - k)$	$\frac{1}{2}(-F_1 - F_2 + 3F_3 - G_1) + k$
$c_\bullet c_1 c_3 c_4$	$(F_1, F_2 + k, F_3, G_1 - k)$	$\frac{1}{2}(-F_1 + 3F_2 - F_3 - G_1) + k$
$c_\bullet c_2 c_3 c_4$	$(F_1 + k, F_2, F_3, G_1 - k)$	$\frac{1}{2}(3F_1 - F_2 - F_3 - G_1) + k$
$c_\bullet c_\bullet c_1 c_2 c_3$	$(F_1 - k, F_2 - k, F_3 - k, G_1 + k)$	$N + \frac{1}{2}(-F_1 - F_2 - F_3 + G_1) + k$
$c_\bullet c_\bullet c_1 c_2 c_4$	$(F_1, F_2, F_3 + k, G_1 + k)$	$N + \frac{1}{2}(-F_1 - F_2 + 3F_3 + G_1) + k$
$c_\bullet c_\bullet c_1 c_3 c_4$	$(F_1, F_2 + k, F_3, G_1 + k)$	$N + \frac{1}{2}(-F_1 + 3F_2 - F_3 + G_1) + k$
$c_\bullet c_\bullet c_2 c_3 c_4$	$(F_1 + k, F_2, F_3, G_1 + k)$	$N + \frac{1}{2}(3F_1 - F_2 - F_3 + G_1) + k$
$c_\bullet c_1 c_2 c_3 c_4$	$(F_1, F_2, F_3, G_1 - 2k)$	$N - G_1 + k$

Table 8: List of the cyclic transitions interpreted as the translations perpendicular to the boundary planes of the fundamental domain.

is the two-dimensional space $C_{\mathrm{B}}^{(2,1)} = \{(M_1, M_2)\}$. In this case, firstly we find that the fundamental domain is at least inside the rectangle described by $0 \le M_1 \le k$ and $0 \le M_2 \le 2k$ from the equation $\langle N \overset{\mathrm{ii}}{\bullet} N + M_1 \overset{\mathrm{i}}{\bullet} N + M_2 \overset{1}{\circ} \rangle \overset{\mathrm{HW}}{=} \langle N \overset{1}{\circ} N + 2k - M_2 \overset{\mathrm{i}}{\bullet} N + k - M_1 \overset{\mathrm{ii}}{\bullet} \rangle$. And the fundamental domain is obtained as the parallelogram as in figure 8,

$$0 \le M_1 \le k, \quad 0 \le -M_1 + M_2 \le k, \tag{4.2}$$

by cutting out the rectangle with the condition $0 \le -M_1 + M_2 \le k$ from the Hanany-Witten

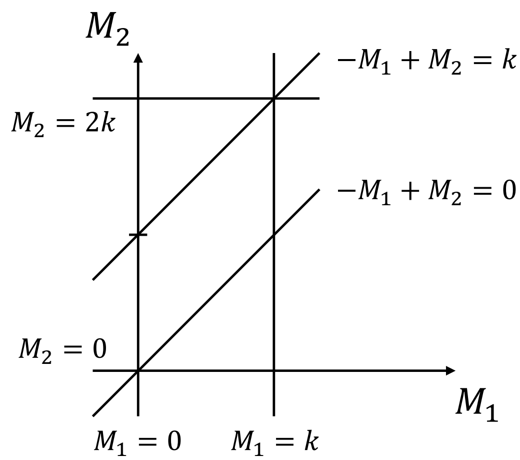

Figure 8: Fundamental domain of the $(2,1)$ model.

transitions for $\langle N \overset{\mathrm{ii}}{\bullet} N + M_1 \overset{\mathrm{i}}{\bullet} N + M_2 \overset{1}{\circ} \rangle$ (see table 9 for all the inequalities).

brane configurations	separations in $C_{\mathrm{B}}^{(2,1)} = \{(M_1, M_2)\}$
$\langle N \overset{\mathrm{ii}}{\bullet} N + M_1 \overset{\mathrm{i}}{\bullet} N + M_2 \overset{1}{\circ} \rangle$	$0 \le M_1, \quad 0 \le M_2$
$\langle N \overset{\mathrm{ii}}{\bullet} N + M_1 \overset{1}{\circ} N + k + M_1 - M_2 \overset{\mathrm{i}}{\bullet} \rangle$	$0 \le M_1, \quad -M_1 + M_2 \le k$
$\langle 0 \overset{1}{\circ} 2k - M_2 \overset{\mathrm{ii}}{\bullet} k + M_1 - M_2 \overset{\mathrm{i}}{\bullet} \rangle$	$M_2 \le 2k, \quad -M_1 + M_2 \le k$
$\langle N \overset{\mathrm{i}}{\bullet} N - M_1 + M_2 \overset{\mathrm{ii}}{\bullet} N + M_2 \overset{1}{\circ} \rangle$	$0 \le -M_1 + M_2, \quad 0 \le M_2$
$\langle N \overset{\mathrm{i}}{\bullet} N - M_1 + M_2 \overset{1}{\circ} N + k - M_1 \overset{\mathrm{ii}}{\bullet} \rangle$	$0 \le -M_1 + M_2, \quad M_1 \le k$
$\langle N \overset{1}{\circ} N + 2k - M_2 \overset{\mathrm{i}}{\bullet} N + k - M_1 \overset{\mathrm{ii}}{\bullet} \rangle$	$M_2 \le 2k, \quad M_1 \le k$

Table 9: Comparison between the brane configurations and the separations in $C_{\mathrm{B}}^{(2,1)}$ for the $(2,1)$ model.

Moreover, the fundamental domain in this case can fill $C_{\mathrm{B}}^{(2,1)} = \{(M_1, M_2)\}$ with the translations. Now we cannot use the affine Weyl group to obtain the translations, however by considering the cyclic transitions, these translations are obtained as

$$\begin{aligned} c_1 &: (M_1, M_2) \mapsto (M_1 - k, M_2 - 2k), \\ c_{\rm ii} &: (M_1, M_2) \mapsto (M_1 + k, M_2 + k), \\ c_{\rm i} &: (M_1, M_2) \mapsto (M_1, M_2 + k), \\ c_1 c_{\rm ii} c_{\rm i} &: (M_1, M_2) \mapsto (M_1, M_2). \end{aligned} \tag{4.3}$$

In general, when the fundamental domain is found, it is nontrivial whether it can fill the space of the brane configurations with the transitions. But if the fundamental domain cannot fill the space, it means that the space of the brane configurations has a special subspace whose configurations are not included in the fundamental domain of the duality cascades.

4.2 Deformations with FI parameters

In the previous section, we regard the Weyl group for the parameters of the curves as the brane transitions from the correspondences between the brane configurations and the quantum curves. Then, we fix the lowest rank of all ranks as the reference. And by considering the duality cascades that the reference rank decreases by the Hanany-Witten transitions, we can define the affine Weyl groups in the space of the brane configurations. In these discussions, the Weyl groups of the curves are reduced to the smaller Weyl groups from the correspondence between the brane configurations and the quantum curves. Concretely, in the case of the D_5 and E_7 curve, $W(B_3)$ and $W(F_4)$ act to the space of the brane configurations, respectively. In this subsection, by introducing the super determinant operator we regard full Weyl groups of the curves as the brane transitions. The insertion of this operator is interpreted as the introduce of Fayet-Iliopoulos parameters [19, 50–52].

Let us introduce the super determinant operator defined by

$$X^{(N_{a-1},N_a)} = e^{-\sum_{m=1}^{N_{a-1}} x_m^{(a-1)} + \sum_{n=1}^{N_a} x_n^{(a)}}. \tag{4.4}$$

The vacuum expectation value is denoted by

$$\langle X \rangle_{k,(Z_1,\cdots,Z_r)}^{(p_1,q_1,\cdots)}(N_1,\cdots,N_r) = \int \prod_{a=1}^{r} \frac{D^{N_a} x^{(a)}}{N_a!(2\pi)^{N_a}} \Delta^{(N_{a-1},N_a)} (X^{(N_{a-1},N_a)})^{Z_a}, \tag{4.5}$$

with

$$D^{N_a} x^{(a)} = \prod_{\ell=1}^{N_a} dx_\ell^{(a)} e^{\frac{ik_a}{4\pi}(x_\ell^{(a)})^2 - \zeta_a x_\ell^{(a)}}, \quad \zeta_a = Z_a - Z_{a-1}. \tag{4.6}$$

If $Z_1 = \cdots = Z_r = 0$, this value matches the partition function as

$$Z_k^{(p_1,q_1,\cdots)}(N_1,\cdots,N_r) = \langle X\rangle_{k,(0,\cdots,0)}^{(p_1,q_1,\cdots)}(N_1,\cdots,N_r). \tag{4.7}$$

In the case without difference ranks $N_1 = \cdots = N_r = N$, the vacuum expectation value in the grand canonical ensemble is denoted by

$$\langle X\rangle_{k,(Z_1,\cdots,Z_r)}^{(p_1,q_1,\cdots)}(z) = \sum_{N=0}^{\infty} z^N \langle X\rangle_{k,(Z_1,\cdots,Z_r)}^{(p_1,q_1,\cdots)}(N). \tag{4.8}$$

With the Fredholm determinant, this value is represented as

$$\langle X\rangle_{k,(Z_1,\cdots,Z_r)}^{(p_1,q_1,\cdots)}(z) = \det\Big(1 + z\widehat{H}^{-1}\Big), \tag{4.9}$$

and then the quantum curve is modified as

$$\widehat{H}^{-1} = e^{-\frac{i}{2\hbar}\widehat{p}^2}\prod_{a=1}^{r}\Big(e^{-\frac{i}{2\hbar}s_a\widehat{q}^2}e^{-\frac{2\pi}{\hbar}Z_a\widehat{q}}\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{2\pi}{\hbar}Z_a\widehat{q}}e^{\frac{i}{2\hbar}s_a\widehat{q}^2}\Big)e^{\frac{i}{2\hbar}\widehat{p}^2}, \tag{4.10}$$

with the appropriate similarity transition and goes as

$$\widehat{H}^{-1} = \prod_{a=1}^{r}\frac{1}{2\cosh\frac{\widehat{r}-2\pi iZ_a}{2}}, \tag{4.11}$$

where $\widehat{r}$ denote $\widehat{p}$ for an NS5-brane and $\widehat{q}$ for a $(1,k)$5-brane.

In the case of the D_5 curve, the quantum curve $\widehat{H}_{(2,2)}$ is modified as

$$\widehat{H}'_{(2,2)} = \Big(2\cosh\frac{\widehat{q}-2\pi iZ_4}{2}\Big)\Big(2\cosh\frac{\widehat{q}-2\pi iZ_3}{2}\Big)\Big(2\cosh\frac{\widehat{p}-2\pi iZ_2}{2}\Big)\Big(2\cosh\frac{\widehat{p}-2\pi iZ_1}{2}\Big). \tag{4.12}$$

By redefining $\widehat{p}$ and $\widehat{q}$, we can set Z_2 and Z_4 as

$$Z_2 = Z_4 = 0. \tag{4.13}$$

Then, the parameters of D_5 curve (h_1,h_2,e_1,e_3,e_5) and (M_1,M_2,M_3,Z_1,Z_3) correspond as

$$(h_1,h_2,e_1,e_3,e_5) = \Big(\frac{m_2m_3}{m_1},\frac{m_1m_3}{m_2},\frac{m_3}{m_2z_1},m_2m_3z_3,\frac{m_3z_1}{m_2}\Big), \tag{4.14}$$

with $m_i = e^{2\pi iM_i}$ $(i=1,2,3)$ and $z_j = e^{2\pi iZ_j}$ $(j=1,3)$. And the Weyl group $W(D_5)$ acts to $C_{\mathrm{B}}^{(2,2)}$ as

$$s_1 : (m_1,m_2,m_3,z_1,z_2) \mapsto \Big(m_1,\sqrt{\frac{m_2z_3}{m_3}},\sqrt{\frac{m_3z_3}{m_2}},z_1,m_2m_3\Big),$$

$$
\begin{aligned}
s_2 &: (m_1, m_2, m_3, z_1, z_2) \mapsto \Big(m_1, \sqrt{\frac{m_2}{m_3 z_3}}, \sqrt{\frac{m_3}{m_2 z_3}}, z_1, \frac{1}{m_2 m_3}\Big), \\
s_3 &: (m_1, m_2, m_3, z_1, z_2) \mapsto \Big(\frac{1}{m_2 m_3}, \sqrt{\frac{m_2}{m_1 m_3}}, \sqrt{\frac{m_3}{m_1 m_2}}, z_1, z_3\Big), \\
s_4 &: (m_1, m_2, m_3, z_1, z_2) \mapsto \Big(\frac{m_3}{m_2}, \sqrt{\frac{m_2 m_3}{m_1}}, \sqrt{m_1 m_2 m_3}, z_1, z_3\Big), \\
s_5 &: (m_1, m_2, m_3, z_1, z_2) \mapsto \Big(m_1, \sqrt{\frac{m_2 m_3}{z_1}}, \sqrt{m_2 m_3 z_1}, \frac{m_3}{m_2}, z_3\Big), \\
s_0 &: (m_1, m_2, m_3, z_1, z_2) \mapsto \Big(m_1, \sqrt{m_2 m_3 z_1}, \sqrt{\frac{m_2 m_3}{z_1}}, \frac{m_2}{m_3}, z_3\Big),
\end{aligned}
\tag{4.15}
$$

namely, for the brane configuration

$$
\begin{aligned}
&\langle N_1 \overset{\text{ii}}{\underset{0}{\bullet}} N_2 \overset{\text{i}}{\underset{Z_1}{\bullet}} N_3 \overset{1}{\underset{Z_1}{\circ}} N_4 \overset{2}{\underset{0}{\circ}} \rangle \\
&= \langle N \overset{\text{ii}}{\underset{0}{\bullet}} N + M_1 - M_2 + M_3 + k \overset{\text{i}}{\underset{Z_1}{\bullet}} N + 2M_1 + 2k \overset{1}{\underset{Z_2}{\circ}} N + M_1 - M_2 - M_3 + k \overset{2}{\underset{0}{\circ}} \rangle,
\end{aligned}
\tag{4.16}
$$

the Weyl group $W(D_5)$ acts as

$$
\begin{aligned}
s_1 &: \mapsto \langle N \overset{\text{ii}}{\underset{0}{\bullet}} N + M_1 - M_2 + M_3 + k \overset{\text{i}}{\underset{Z_1}{\bullet}} N + 2M_1 + 2k \overset{1}{\underset{M_2+M_3}{\circ}} N + M_1 - Z_3 + k \overset{2}{\underset{0}{\circ}} \rangle, \\
s_2 &: \mapsto \langle N \overset{\text{ii}}{\underset{0}{\bullet}} N + M_1 - M_2 + M_3 + k \overset{\text{i}}{\underset{Z_1}{\bullet}} N + 2M_1 + 2k \overset{1}{\underset{-M_2-M_3}{\circ}} N + M_1 + Z_3 + k \overset{2}{\underset{0}{\circ}} \rangle, \\
s_5 &: \mapsto \langle N \overset{\text{ii}}{\underset{0}{\bullet}} N + M_1 + Z_1 + k \overset{\text{i}}{\underset{-M_2+M_3}{\bullet}} N + 2M_1 + 2k \overset{1}{\underset{Z_3}{\circ}} N + M_1 - M_2 - M_3 + k \overset{2}{\underset{0}{\circ}} \rangle, \\
s_0 &: \mapsto \langle N \overset{\text{ii}}{\underset{0}{\bullet}} N + M_1 - Z_1 + k \overset{\text{i}}{\underset{M_2-M_3}{\bullet}} N + 2M_1 + 2k \overset{1}{\underset{Z_3}{\circ}} N + M_1 - M_2 - M_3 + k \overset{2}{\underset{0}{\circ}} \rangle,
\end{aligned}
\tag{4.17}
$$

where we omit s_3 and s_4 since they have nothing to do with Z_1 and Z_3. Here there are s_3 and s_4 interpreted as the local rule but not s_1, s_2, s_5 and s_0 before inserting the super determinant operator. However, before inserting the operator, the products $s_1 s_2$ and $s_5 s_0$ of them exist and are interpreted as the Hanany-Witten transition between $(1, k)$5-branes and that between NS5-branes respectively. Therefore if we regard s_1, s_2, s_5 and s_0 as the brane transitions, these should be called half Hanany-Witten transitions.

Also, we can confirm that there are similar half Hanany-Witten transitions for the E_7 case. In the E_7 case, to consider Weyl group $W(E_7)$ acting to the brane configuration with the FI parameters

$$
\begin{aligned}
&\langle N_1 \overset{\text{ii}}{\underset{0}{\bullet}} N_2 \overset{1}{\underset{Z_1}{\circ}} N_3 \overset{2}{\underset{Z_2}{\circ}} N_4 \overset{\text{i}}{\underset{0}{\bullet}} N_5 \overset{3}{\underset{Z_3}{\circ}} N_6 \overset{4}{\underset{0}{\circ}} \rangle \\
&= \langle N \overset{\text{ii}}{\underset{0}{\bullet}} N + G_1 + k \overset{1}{\underset{Z_1}{\circ}} N + \frac{1}{2}(-3F_1 + F_2 + F_3 + G_1) + k \overset{2}{\underset{Z_2}{\circ}} N - F_1 - F_2 + F_3 + k \overset{\text{i}}{\underset{0}{\bullet}}
\end{aligned}
$$

$$N-F_1-F_2+F_3+G_1+2k \underset{Z_3}{\overset{3}{\circ}} N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k\underset{0}{\overset{4}{\circ}}\rangle, \tag{4.18}$$

we represent the Weyl group $W(E_7)$ (2.25) as

$$s_0:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(\frac{f_1}{\sqrt{f_3z_3}},\frac{f_2}{\sqrt{f_3z_3}},\frac{1}{z_3},g_1,\frac{z_1}{\sqrt{f_3z_3}},\frac{z_2}{\sqrt{f_3z_3}},\frac{1}{f_3}),$$
$$s_1:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(f_1,\sqrt{\frac{f_2f_3z_3}{z_2}},\sqrt{\frac{f_2f_3z_2}{z_3}},g_1,z_1,z_2,z_3),$$
$$s_2:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(\sqrt{\frac{f_1f_2z_2}{z_1}},\sqrt{\frac{f_1f_2z_1}{z_2}},f_3,g_1,\sqrt{\frac{f_2z_1z_2}{f_1}},\sqrt{\frac{f_1z_1z_2}{f_2}},z_3),$$
$$s_3:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(\sqrt{\frac{f_2f_3}{f_1g_1}},f_2,f_3,\frac{1}{f_1}\sqrt{\frac{f_2f_3g_1}{f_1}},z_1,z_2,z_3),$$
$$s_4:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(f_1,f_2,f_3,\frac{1}{g_1},z_1,z_2,z_3),$$
$$s_5:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(\sqrt{\frac{f_1f_2z_1}{z_2}},\sqrt{\frac{f_1f_2z_2}{z_1}},f_3,g_1,\sqrt{\frac{f_1z_1z_2}{f_2}},\sqrt{\frac{f_2z_1z_2}{f_1}},z_3),$$
$$s_6:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(f_1,\sqrt{\frac{f_2f_3z_2}{z_3}},\sqrt{\frac{f_2f_3z_3}{z_2}},g_1,z_1,\sqrt{\frac{f_2z_2z_3}{f_3}},\sqrt{\frac{f_3z_2z_3}{f_2}}),$$
$$s_7:(f_1,f_2,f_3,g_1,z_1,z_2,z_3)\mapsto(f_1\sqrt{\frac{z_3}{f_3}},f_2\sqrt{\frac{z_3}{f_3}},z_3,g_1,z_1\sqrt{\frac{f_3}{z_3}},z_2\sqrt{\frac{f_3}{z_3}},f_3), \tag{4.19}$$

under the parameter transitions

$$(f_1,f_2,f_3,g_1,h_1,h_2,h_3)\to(z_1f_1,z_2f_3,z_3f_3,g_1,\frac{1}{z_1}\sqrt{\frac{f_1}{f_2f_3g_1}},\frac{1}{z_2}\sqrt{\frac{f_2}{f_1f_3g_1}},\frac{1}{z_3}\sqrt{\frac{f_3}{f_1f_2g_1}}). \tag{4.20}$$

Indeed, this $W(E_7)$ act to (4.18) with $F_j=e^{2\pi i f_j}$, $G_1=e^{2\pi i g_1}$ and $Z_j=e^{2\pi i z_j}$ $(j=1,2,3)$ as

$$s_0:\mapsto\langle N\underset{0}{\overset{\text{ii}}{\bullet}}N+G_1+k\underset{Z_1}{\overset{1}{\circ}}N+\frac{1}{2}(-3F_1+F_2+F_3+G_1)+k\underset{Z_2}{\overset{2}{\circ}}N-F_1-F_2+F_3+k\underset{0}{\overset{\text{i}}{\bullet}}$$
$$N-F_1-F_2+F_3+G_1+2k\underset{\frac{1}{2}(-F_3+Z_3)}{\overset{3}{\circ}}N+\frac{1}{2}(-F_1-F_2+F_3+G_1)+Z_3+k\underset{0}{\overset{4}{\circ}}\rangle,$$
$$s_1:\mapsto\langle N\underset{0}{\overset{\text{ii}}{\bullet}}N+G_1+k\underset{Z_1}{\overset{1}{\circ}}N+\frac{1}{2}(-3F_1+F_2+F_3+G_1)+k\underset{\frac{1}{2}(-F_2+F_3+Z_2+Z_3)}{\overset{2}{\circ}}N-F_1+k+Z_2-Z_3\underset{0}{\overset{\text{i}}{\bullet}}$$
$$N-F_1+G_1+2k+Z_2-Z_3\underset{\frac{1}{2}(F_2-F_3+Z_2+Z_3)}{\overset{3}{\circ}}N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k\underset{0}{\overset{4}{\circ}}\rangle,$$
$$s_2:\mapsto\langle N\underset{0}{\overset{\text{ii}}{\bullet}}N+G_1+k\underset{\frac{1}{2}(-F_1+F_2+Z_1+Z_2)}{\overset{1}{\circ}}N+\frac{1}{2}(-F_1-F_2+F_3+G_1)+k+Z_1-Z_2\underset{\frac{1}{2}(F_1-F_2+Z_1+Z_2)}{\overset{2}{\circ}}$$
$$N-F_1-F_2+F_3+k\underset{0}{\overset{\text{i}}{\bullet}}N-F_1-F_2+F_3+G_1+2k\underset{Z_3}{\overset{3}{\circ}}N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k\underset{0}{\overset{4}{\circ}}\rangle,$$

$$
\begin{aligned}
s_5 :\mapsto \langle N \overset{\mathrm{ii}}{\underset{0}{\bullet}} N+G_1+k \overset{1}{\underset{\frac{1}{2}(F_1-F_2+Z_1+Z_2)}{\circ}} N+\frac{1}{2}(-F_1-F_2+F_3+G_1)+k-Z_1+Z_2 \overset{2}{\underset{\frac{1}{2}(-F_1+F_2+Z_1+Z_2)}{\circ}} \\
N-F_1-F_2+F_3+k \overset{\mathrm{i}}{\underset{0}{\bullet}} N-F_1-F_2+F_3+G_1+2k \overset{3}{\underset{Z_3}{\circ}} N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k \overset{4}{\underset{0}{\circ}}\rangle, \\
s_6 :\mapsto \langle N \overset{\mathrm{ii}}{\underset{0}{\bullet}} N+G_1+k \overset{1}{\underset{Z_1}{\circ}} N+\frac{1}{2}(-3F_1+F_2+F_3+G_1)+k \overset{2}{\underset{\frac{1}{2}(F_2-F_3+Z_2+Z_3)}{\circ}} N-F_1+k-Z_2+Z_3 \overset{\mathrm{i}}{\underset{0}{\bullet}} \\
N-F_1+G_1+2k-Z_2+Z_3 \overset{3}{\underset{\frac{1}{2}(-F_2+F_3+Z_2+Z_3)}{\circ}} N+\frac{1}{2}(-F_1-F_2-F_3+G_1)+k \overset{4}{\underset{0}{\circ}}\rangle, \\
s_7 :\mapsto \langle N \overset{\mathrm{ii}}{\underset{0}{\bullet}} N+G_1+k \overset{1}{\underset{Z_1}{\circ}} N+\frac{1}{2}(-3F_1+F_2+F_3+G_1)+k \overset{2}{\underset{Z_2}{\circ}} N-F_1-F_2+F_3+k \overset{\mathrm{i}}{\underset{0}{\bullet}} \\
N-F_1-F_2+F_3+G_1+2k \overset{3}{\underset{\frac{1}{2}(F_3+Z_3)}{\circ}} N+\frac{1}{2}(-F_1-F_2+F_3+G_1)-Z_3+k \overset{4}{\underset{0}{\circ}}\rangle.
\end{aligned}
\tag{4.21}
$$

Here s_3 and s_4 are omitted because they do not affect the FI parameters and the transitions interpreted as the local rule. And other transitions s_0, s_1, s_2, s_5, s_6 and s_7 are interpreted as the half Hanany-Witten transitions that give the Hanany-Witten transitions under $\mathbb{Z}_2$ folding.

From above discussion for the D_5 and E_7 Weyl groups, we find that the half Hanany-Witten transitions are summarized as

$$
s_\pm : \langle \cdots K(\underset{Z_1}{\bullet/\circ})L(\underset{Z_2}{\bullet/\circ})M\cdots\rangle \mapsto \langle \cdots K(\underset{Z_\pm}{\bullet/\circ})\frac{1}{2}(K+M)\pm(Z_2-Z_1)(\underset{Z_\mp}{\bullet/\circ})M\cdots\rangle, \quad (s_\pm)^2=1,
\tag{4.22}
$$

with $Z_\pm = \frac{1}{2}(\pm\frac{1}{2}(K-2L+M)+Z_1+Z_2)$. The products $s_+s_- = s_-s_+$ denote the Hanany-Witten transitions.

4.3 Fundamental domains for D_5 and E_7 Weyl groups

Next, let us find the fundamental domain obtain by the Weyl group $W(D_5)$. Before inserting the super determinant operator, the fundamental domain is obtained as that for the Weyl group $W(B_3)$

$$
N_1 \le N_2, \quad N_1 \le N_3, \quad N_1 \le N_4,
\tag{4.23}
$$

namely

$$
0 \le M_1 - M_2 + M_3 + k, \quad 0 \le 2M_1 + 2k, \quad 0 \le M_1 - M_2 - M_3 + k.
\tag{4.24}
$$

Acting the Weyl group $W(D_5)$ (4.15) to these inequalities, the following inequalities appear in addition to the inequalities (3.7) describing the fundamental domain for the Weyl group $W(B_3)$,

$$\begin{aligned} -k \le M_2 \pm M_3 \le k, \quad -k \le Z_1 \pm Z_3 \le k, \quad -k \le Z_i \le k, \\ -k \le M_1 \pm Z_i \le k, \quad -k \le M_2 + M_3 \pm Z_i \le k, \quad -k \le M_2 - M_3 \pm Z_i \le k, \end{aligned} \tag{4.25}$$

with $i = 1, 3$ and if we remove the extra inequalities, they become

$$\begin{aligned} -\frac{k}{2} \le M_2 \le \frac{k}{2}, \quad -\frac{k}{2} \le M_3 \le \frac{k}{2}, \quad -k \le Z_1 \pm Z_3 \le k, \\ -k \le M_1 + M_2 \pm M_3 \le k, \quad -k \le M_1 - M_2 \pm M_3 \le k, \\ -k \le M_1 \pm Z_i \le k, \quad -k \le M_2 + M_3 \pm Z_i \le k, \quad -k \le M_2 - M_3 \pm Z_i \le k. \end{aligned} \tag{4.26}$$

It is found that by the action of the Weyl group $W(D_5)$ to the vector on $C_{\mathrm{B}}^{(2,2)}$

$$v = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ Z_1 \\ Z_3 \end{pmatrix}, \tag{4.27}$$

with the metric $(g_{ij}) = \mathrm{diag}(1, 2, 2, 1, 1)$, the simple root vectors are represented as

$$\alpha_1 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \\ -1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \\ 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} -1 \\ -1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 \\ 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_5 = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix}. \tag{4.28}$$

And D_5 Weyl chamber is given as

$$\begin{aligned} (\alpha_1, v) = M_2 + M_3 - Z_3 \ge 0, \quad (\alpha_2, v) = M_2 + M_3 + Z_3 \ge 0, \quad (\alpha_3, v) = -M_1 - M_2 - M_3 \ge 0, \\ (\alpha_4, v) = M_1 + M_2 - M_3 \ge 0, \quad (\alpha_5, v) = -M_2 + M_3 - Z_1 \ge 0. \end{aligned} \tag{4.29}$$

If we define the new root $\tilde{\alpha}_0$ as $\tilde{\alpha}_0 = -\theta$ with the highest root of the D_5 Lie algebra

$$\theta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \tag{4.30}$$

the affine D_5 Weyl chamber is obtained by adding

$$(\tilde{\alpha}_0, v) + k = -M_2 + M_3 + Z_1 + k \ge 0, \tag{4.31}$$

to the above inequalities describing the D_5 Weyl chamber.

In the E_7 case, with the same argument as above, the structure as in the D_5 case can be obtained. Then we consider the brane configuration with the FI parameters,

$$
\begin{aligned}
&\langle N_1 \overset{\mathrm{ii}}{\underset{0}{\bullet}} N_2 \overset{1}{\underset{Z_1}{\circ}} N_3 \overset{2}{\underset{Z_2}{\circ}} N_4 \overset{\mathrm{i}}{\underset{0}{\bullet}} N_5 \overset{3}{\underset{Z_3}{\circ}} N_6 \overset{4}{\underset{0}{\circ}} \rangle \\
=&\langle N \overset{\mathrm{ii}}{\underset{0}{\bullet}} N + G_1 + k \overset{1}{\underset{Z_1}{\circ}} N + \frac{1}{2}(-3F_1 + F_2 + F_3 + G_1) + k \overset{2}{\underset{Z_2}{\circ}} N - F_1 - F_2 + F_3 + k \overset{\mathrm{i}}{\underset{0}{\bullet}} \\
&N - F_1 - F_2 + F_3 + G_1 + 2k \overset{3}{\underset{Z_3}{\circ}} N + \frac{1}{2}(-F_1 - F_2 - F_3 + G_1) + k \overset{4}{\underset{0}{\circ}} \rangle. \qquad (4.32)
\end{aligned}
$$

The parameters of the E_7 curve $(f_1, f_2, f_3, g_1, h_1, h_2, h_3)$ are corresponded to the parameters describing the brane configurations $(F_1, F_2, F_3, G_1, Z_1, Z_2, Z_3)$ as

$$
(f_1, f_2, f_3, g_1, h_1, h_2, h_3) \to (z_1 f_1, z_2 f_3, z_3 f_3, g_1, \frac{1}{z_1}\sqrt{\frac{f_1}{f_2 f_3 g_1}}, \frac{1}{z_2}\sqrt{\frac{f_2}{f_1 f_3 g_1}}, \frac{1}{z_3}\sqrt{\frac{f_3}{f_1 f_2 g_1}}), \qquad (4.33)
$$

with $F_j = e^{2\pi i f_j}$, $G_1 = e^{2\pi i g_1}$ and $Z_j = e^{2\pi i z_j}$ $(j = 1, 2, 3)$.

As in the D_5 case, the fundamental domain is obtained as

$$
\begin{aligned}
&-k \le F_i \le k, \quad -k \le G_1 \le k, \quad -k \le H_{i'} \le k, \quad -k \le F_i - F_j \le k, \quad -k \le H_{i'} - H_{j'} \le k, \\
&-k \le F_i - H_{i'} \le k, \quad -k \le H_{i'} + G_1 \le k, \quad -k \le F_i + G_1 + H_{i'} + H_{j'} \le k, \\
&-k \le F_i + F_j + G_1 + H_{i'} + H_{j'} \le k, \quad -k \le F_i + F_j + G_1 + \sum_{i'=1}^{3} H_{i'} \le k, \\
&-k \le F_i + F_j + 2G_1 + \sum_{i'=1}^{3} H_{i'} \le k, \quad -k \le \sum_{i=1}^{3} F_i + G_1 + \sum_{i'=1}^{3} H_{i'} \le k, \\
&-k \le \sum_{i=1}^{3} F_i + 2G_1 + \sum_{i'=1}^{3} H_{i'} \le k, \quad -k \le \sum_{i=1}^{3} F_i + 2G_1 + 2H_1 + H_2 + H_3 \le k, \\
&-k \le \sum_{i=1}^{3} F_i + 2G_1 + H_1 + 2H_2 + H_3 \le k, \quad -k \le \sum_{i=1}^{3} F_i + 2G_1 + H_1 + H_2 + 2H_3 \le k, \qquad (4.34)
\end{aligned}
$$

with $i, j, i', j' = 1, 2, 3$ ($i \neq j$ and $i' \neq j'$) in terms of $(F_1, F_2, F_3, G_1, H_1, H_2, H_3)$. In this notation, it is difficult to assess whether this fundamental domain contains the inequalities in the F_4 case (3.9), however if we represent this fundamental domain in terms of $(F_1, F_2, F_3, G_1, Z_1, Z_2, Z_3)$ as

$$
\begin{aligned}
&-k \le F_1 + F_2 - F_3 \le k, \quad -k \le -F_1 + F_2 + F_3 \le k, \quad -k \le F_1 - F_2 + F_3 \le k, \\
&-2k \le F_1 + F_2 - 3F_3 \pm G_1 \le 2k, \quad -2k \le -3F_1 + F_2 + F_3 \pm G_1 \le 2k,
\end{aligned}
$$

$$
\begin{aligned}
-2k \le F_1 - 3F_2 + F_3 \pm G_1 \le 2k, \quad -k \le G_1 \le k, \quad -2k \le \sum_{i=1}^{3} F_i \pm G_1 \le 2k, \\
-k \le F_i \pm Z_i \le k, \quad -k \le F_i - F_j + Z_k \le k, \quad -2k \le F_i - F_j - F_k \pm G_1 - 2Z_i \le 2k, \\
-2k \le F_i - Z_j + Z_k \le 2k, \quad -k \le (F_i - F_j) \pm (Z_i - Z_j) \le k, \\
-2k \le F_i + F_j - F_k \pm G_1 + 2Z_i - 2Z_j \le 2k, \\
-k \le Z_1 + Z_2 - Z_3 \le k, \quad -k \le -Z_1 + Z_2 + Z_3 \le k, \quad -k \le Z_1 - Z_2 + Z_3 \le k,
\end{aligned}
\tag{4.35}
$$

with $i, j, k = 1, 2, 3$ (all subscripts i, j and k are different), it is clear that the inequalities up to the third line are those in the F_4 case (3.9). Also, the $W(E_7)$ acts to $C_{\mathrm{B}}^{(2,4)}$ as (2.25). By acting these transitions to the vector v with the metric g_{ij} as

$$
v = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ G_1 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 1 & 1 & 1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 1 & 1 & 1 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}, \tag{4.36}
$$

with $H_j = e^{2\pi i h_j}$ and the simple root vectors of the E_7 are obtained as

$$
\alpha_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix},
$$
$$
\alpha_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \alpha_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \alpha_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.37}
$$

And the E_7 Weyl chamber is described by the inequalities

$$
(\alpha_1, v) = F_2 - F_3 \ge 0, \quad (\alpha_2, v) = F_1 - F_2 \ge 0, \quad (\alpha_3, v) = -F_1 - G_1 - H_1 \ge 0,
$$

$$(\alpha_4, v) = G_1 \geq 0, \quad (\alpha_5, v) = H_1 - H_2 \geq 0, \quad (\alpha_6, v) = H_2 - H_3 \geq 0,$$
$$(\alpha_7, v) = F_1 + F_2 + F_3 + 2G_1 + H_1 + H_2 + H_3 \geq 0. \tag{4.38}$$

Moreover, when we consider the highest root vector of the E_7

$$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \tag{4.39}$$

we can construct the extra root vector as $\tilde{\alpha}_0 = -\theta$ and the affine E_7 Weyl chamber is obtained by adding

$$(\tilde{\alpha}_0, v) + k = F_3 + k \geq 0, \tag{4.40}$$

to the inequalities describing the E_7 Weyl chamber (4.38).

5 Conclusion

Let us summarize the contents of this thesis.

Firstly, we consider some super Chern-Simons matrix models with the equal ranks described with the A_1, D_5 and E_7 quantum curves. And in the case with the rank differences, by fixing the lowest rank of all ranks as the reference, we can consider the correspondence between the quantum curves and the brane configurations. Then, the Weyl groups, namely the symmetries of the quantum curves, contain the transitions interpreted as the Hanany-Witten transitions. After separating them, if we also regard the remaining transitions as the brane transitions, we find the new brane transitions that exchange the number of the D3-branes stretching both sides of the 5-brane surrounded by the different 5-branes from the original 5-brane. Furthermore, we propose the brane transitions without referring to whole configurations and call it local rule [1].

Secondly, we consider the duality cascades for the brane configurations in a circle. If the ranks lower than the reference is obtained by the Hanany-Witten transitions, we redefine it as the new reference rank, which is the duality cascade in field theories. We can understand the redefinitions of the reference rank as the cyclic transitions in the brane configurations and as the translations in the space of the brane configurations. We obtain the affine Weyl groups in the space of the brane configurations by considering these translations in addition to the Weyl groups of the curves. It is found that the fundamental domain of the duality cascades is characterized by the affine Weyl chamber and is the polytope that can fill the spaces of the brane configurations. Namely, we can systematically understand the duality cascades as the affine Weyl groups.

When the quantum curves are related to the brane configurations, the Weyl groups of the curves are diminished by $\mathbb{Z}_2$ folding. Concretely, $W(D_5)$ and $W(E_7)$ respectively become $W(B_3)$ and $W(F_4)$ by $\mathbb{Z}_2$ folding. Therefore, thirdly, we consider the insertion of the super determinant operator which is interpreted as the introduction of the FI parameters. By this, we consider the full Weyl group of the curves as the brane transitions. Also in this case, the fundamental domains in the Weyl chambers are naturally obtained as the affine Weyl chambers.

Fourthly, more generally, by the Hanany-Witten transitions, it is found that the facing boundary planes of the fundamental domains of the duality cascades are parallel. Moreover, for brane configurations with NS5-branes and $(1,k)$5-branes aligned in a circle, the fundamental domains are obtained by cutting out the rectangle described the condition that the numbers of the D3-branes at the intervals of the 5-branes in the brane configuration of the standard order are positive.

In the appendices of this thesis, we give the reviews of our research results in my doctoral course including the mathematical structure of the quantum A-period of the D_5 quantum curve [11], the correspondence between the ABJM matrix model and two-dimensional Toda lattice hierarchy [12, 13] and the static force potential of non-Abelian gauge theory at a finite box in Coulomb gauge [59].

A Fermi gas formalism

The Fermi gas formalism [8] is the method that we regard the partition functions of the matrix models as those in free fermion model. We recognize the fermionic feature from the property of the determinant. Moreover, the computed partition function leads the Fermi surface in the WKB expansion $\hbar \to 0$. It is also found that the partition function is represented with the Airy function.

Here, we show that the grand canonical partition function is written with the Fredholm determinant. Namely when the partition function labeled by $p_1 + q_1 + p_2 + q_2 + \cdots = r$ is denoted by

$$Z_k^{(p_1,q_1,\cdots)}(N) = \int \prod_{a=1}^{r} \frac{D^N x^{(a)}}{N!} \prod_{a=1}^{r} \Delta^{(N)}(x^{(a-1)}, x^{(a)}), \tag{A.1}$$

with

$$D^N x^{(a)} = \prod_{\ell=1}^{N} \frac{dx_\ell^{(a)}}{2\pi} e^{\frac{is_a}{4\pi k}(x_\ell^{(a)})^2}, \tag{A.2}$$

$$\Delta^{(N)}(x^{(a-1)},x^{(a)})=\frac{\prod_{m<m'}^N 2k\sinh\frac{x_m^{(a-1)}-x_{m'}^{(a-1)}}{2k}\prod_{n<n'}^N 2k\sinh\frac{x_n^{(a)}-x_{n'}^{(a)}}{2k}}{\prod_{m=1}^N\prod_{n=1}^N 2k\cosh\frac{x_m^{(a-1)}-x_n^{(a)}}{2k}},\tag{A.3}$$

we show that the grand canonical partition function is expressed with the Fredholm determinant as

$$\Xi_k^{(p_1,q_1,\cdots)}(z)=\det\Big(1+z\widehat{H}^{-1}_{(p_1,q_1,\cdots)}\Big).\tag{A.4}$$

After we showed it for the ABJM matrix model and we generalize the result to that for the $(p_1,q_1,\cdots)$ model.

The $(1,1)$ model is the ABJM matrix model without the rank difference and the partition function in the model is represented by

$$Z_k^{\mathrm{ABJM}}(N)=Z_k^{(1,1)}(N)=\int\frac{D^Nx^{(1)}}{N!}\frac{D^Nx^{(2)}}{N!}\Delta^{(N)}(x^{(2)},x^{(1)})\Delta^{(N)}(x^{(1)},x^{(2)}),\tag{A.5}$$

with

$$D^Nx^{(1)}=\prod_{\ell=1}^N\frac{dx_\ell^{(1)}}{2\pi}e^{\frac{i}{2\pi}(x_\ell^{(1)})^2},\quad D^Nx^{(2)}=\prod_{\ell=1}^N\frac{dx_\ell^{(2)}}{2\pi}e^{\frac{-i}{2\pi}(x_\ell^{(2)})^2}.\tag{A.6}$$

The measure is expressed with the Cauchy determinant,

$$\Delta^{(N)}(x,y)=\det\left(\frac{1}{2k\cosh\frac{x_m-y_n}{2k}}\right)_{N\times N}.\tag{A.7}$$

For convenience, we define the function $\mathcal{P}$ and $\mathcal{Q}$ as

$$\mathcal{P}(x^{(+)},x^{(-)})=\frac{1}{2k\cosh\frac{x^{(+)}-x^{(-)}}{2k}},\quad \mathcal{Q}(x^{(-)},x^{(+)})=\frac{1}{2k\cosh\frac{x^{(-)}-x^{(+)}}{2k}},\tag{A.8}$$

where $x^{(\pm)}$ denotes the variable associated with $s_a=\pm1$. These functions $\mathcal{P}$ and $\mathcal{Q}$ are the same functional type, however it is convenient to distinguish by a different notation when we consider the quantization. In this notation, the partition function is represented as

$$Z_k^{(1,1)}(N)=\int\frac{D^Nx^{(1)}}{N!}\det\Big(H^{-1}_{(1,1)}(x_m^{(1)},x_n^{(1)})\Big)_{N\times N},\tag{A.9}$$

with

$$H^{-1}_{(1,1)}(x_m^{(1)},x_n^{(1)})=\int Dx^{(-)}\mathcal{P}(x_m^{(1)},x^{(-)})\mathcal{Q}(x^{(-)},x_n^{(1)}),\tag{A.10}$$

where we used the Cauchy-Binet formula. And the determinant of $H^{-1}_{(1,1)}$ is expanded as

$$Z_k^{(1,1)}(N)=\int\frac{D^Nx^{(1)}}{N!}\sum_{\sigma\in S_N}(-1)^\sigma\prod_{m=1}^N H^{-1}_{(1,1)}(x_m^{(1)},x_{\sigma(m)}^{(1)}).\tag{A.11}$$

Due to the nature of the permutation group, with the trace

$$\mathrm{tr}(H_{(1,1)}^{-1}) = \int Dx^{(1)} H_{(1,1)}^{-1}(x^{(1)}, x^{(1)}), \tag{A.12}$$

the partition function is represented as

$$Z_k^{(1,1)}(N) = \frac{1}{N!} \sum_{\sum_{j=1}^n j\ell_j = N} (-1)^{\sum_{j=1}^n (j-1)\ell_j} \frac{N!}{\prod_{j=1}^n (j^{\ell_j} \cdot \ell_j!)} \prod_{j=1}^n \Big(\mathrm{tr}(H_{(1,1)}^{-j})\Big)^{\ell_j}, \tag{A.13}$$

and finally becomes

$$Z_k^{(1,1)}(N) = \sum_{\sum_{j=1}^n j\ell_j = N} \prod_{j=1}^n \frac{1}{\ell_j!} \left(\frac{(-1)^{j-1}}{j} \mathrm{tr}(H_{(1,1)}^{-j})\right)^{\ell_j}, \tag{A.14}$$

where we calculate the sum for all ℓ_j satisfying the condition $\sum_{j=1}^n j\ell_j = N$. From this partition function, the grand canonical partition function in the ABJM matrix model without the rank difference is represented with the determinant as

$$\begin{aligned}
\Xi_k^{(1,1)}(z) &= \sum_{N=0}^\infty z^N Z_k^{(1,1)}(N) \\
&= \sum_{N=0}^\infty z^N \sum_{\sum_{j=1}^n j\ell_j = N} \prod_{j=1}^n \frac{1}{\ell_j!} \left(\frac{(-1)^{j-1}}{j} \mathrm{tr}(H_{(1,1)}^{-j})\right)^{\ell_j} \\
&= \sum_{N=0}^\infty \sum_{\sum_{j=1}^n j\ell_j = N} \prod_{j=1}^n \frac{1}{\ell_j!} \left(\frac{(-1)^{j-1}}{j} \mathrm{tr}\Big((zH_{(1,1)}^{-1})^j\Big)\right)^{\ell_j} \\
&= \prod_{j=1}^\infty \sum_{\ell_j=0}^\infty \frac{1}{\ell_j!} \left(\frac{(-1)^{j-1}}{j} \mathrm{tr}\Big((zH_{(1,1)}^{-1})^j\Big)\right)^{\ell_j} \\
&= \prod_{j=1}^\infty \exp\left(\frac{(-1)^{j-1}}{j} \mathrm{tr}\Big((zH_{(1,1)}^{-1})^j\Big)\right) \\
&= \exp\Big(\mathrm{tr}\log\big(1 + zH_{(1,1)}^{-1}\big)\Big) \\
&= \det\Big(1 + zH_{(1,1)}^{-1}\Big)_{N\times N}.
\end{aligned} \tag{A.15}$$

Let us introduce the operators $\widehat{q}$ and $\widehat{p}$ satisfying the canonical quantization condition $[\widehat{q}, \widehat{p}] = i\hbar$ and the eigenstates corresponding to them

$$\langle x|\widehat{q} = x\langle x|, \quad \langle\!\langle p|\widehat{p} = p\langle\!\langle p|. \tag{A.16}$$

In the normalization of the eigenstates

$$\langle x|x'\rangle = 2\pi\delta(x - x'), \quad \langle x|p\rangle\!\rangle = \sqrt{\frac{2\pi}{\hbar}} e^{\frac{ixp}{\hbar}}, \quad \int \frac{dx}{2\pi} |x\rangle\langle x| = 1,$$

$$\langle\!\langle p|p'\rangle\!\rangle = 2\pi\delta(p-p'), \quad \langle\!\langle p|x\rangle = \sqrt{\frac{2\pi}{\hbar}}e^{\frac{-ixp}{\hbar}}, \quad \int\frac{dp}{2\pi}|p\rangle\!\rangle\langle\!\langle p| = 1, \tag{A.17}$$

through the Fourier transformation, the following equation holds,

$$\frac{1}{2k\cosh\frac{x-x'}{2}} = \int\frac{dp}{2\pi}\frac{e^{\frac{i(x-x')p}{\hbar}}}{\cosh\frac{p}{2}} = \int\frac{dp}{2\pi}\frac{1}{\cosh\frac{p}{2}}\langle x|p\rangle\!\rangle\langle\!\langle p|x'\rangle = \langle x|\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x'\rangle. \tag{A.18}$$

Therefore the functions $\mathcal{P}$ and $\mathcal{Q}$ are written as

$$\mathcal{P}(x^{(+)},x^{(-)}) = \langle x^{(+)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x^{(-)}\rangle, \quad \mathcal{Q}(x^{(-)},x^{(+)}) = \langle x^{(-)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x^{(+)}\rangle, \tag{A.19}$$

and $H^{-1}_{(1,1)}$ becomes

$$\begin{aligned} H^{-1}_{(1,1)}(x^{(1)}_m,x^{(1)}_n) &= \int\frac{dx^{(2)}}{2\pi}e^{\frac{-i}{2\hbar}(x^{(2)})^2}\langle x^{(1)}_m|\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x^{(2)}\rangle\langle x^{(2)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x^{(1)}_n\rangle \\ &= \langle x^{(1)}_m|\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{-i}{2\hbar}\widehat{q}^2}\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x^{(1)}_n\rangle, \end{aligned} \tag{A.20}$$

in terms of the operators. Since the trace of the $H^{-1}_{(1,1)}$ is

$$\begin{aligned} \mathrm{tr}(H^{-1}_{(1,1)}) &= \int Dx^{(1)}H^{-1}_{(1,1)}(x^{(1)},x^{(1)}) \\ &= \int\frac{dx^{(1)}}{2\pi}e^{\frac{i}{2\hbar}(x^{(1)})^2}\langle x^{(1)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{-i}{2\hbar}\widehat{q}^2}\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x^{(1)}\rangle \\ &= \int\frac{dx^{(1)}}{2\pi}\langle x^{(1)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{-i}{2\hbar}\widehat{q}^2}\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{i}{2\hbar}\widehat{q}^2}|x^{(1)}\rangle, \end{aligned} \tag{A.21}$$

the similarity transition leads

$$\begin{aligned} \mathrm{tr}(H^{-1}_{(1,1)}) &= \int\frac{dx^{(1)}}{2\pi}\langle x^{(1)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{-i}{2\hbar}\widehat{p}^2}e^{\frac{-i}{2\hbar}\widehat{q}^2}\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{i}{2\hbar}\widehat{q}^2}e^{\frac{i}{2\hbar}\widehat{p}^2}|x^{(1)}\rangle \\ &= \int\frac{dx^{(1)}}{2\pi}\langle x^{(1)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}\frac{1}{2\cosh\frac{\widehat{q}}{2}}|x^{(1)}\rangle, \end{aligned} \tag{A.22}$$

where we use the formula,

$$e^{-\frac{i}{2\hbar}\widehat{q}^2}f(\widehat{q},\widehat{p})e^{\frac{i}{2\hbar}\widehat{q}^2} = f(\widehat{q},\widehat{p}+\widehat{q}), \quad e^{-\frac{i}{2\hbar}\widehat{p}^2}f(\widehat{q},\widehat{p})e^{\frac{i}{2\hbar}\widehat{p}^2} = f(\widehat{q}-\widehat{p},\widehat{p}). \tag{A.23}$$

By defining the operators

$$\widehat{H}^{-1}_{(1,1)} = \widehat{\mathcal{P}}^{-1}\widehat{\mathcal{Q}}^{-1}, \quad \widehat{\mathcal{P}}^{-1} = \frac{1}{2\cosh\frac{\widehat{p}}{2}}, \quad \widehat{\mathcal{Q}}^{-1} = \frac{1}{2\cosh\frac{\widehat{q}}{2}}, \tag{A.24}$$

the grand canonical partition function is derived as

$$\Xi_k^{(1,1)}(z) = \det\Big(1 + z\widehat{H}_{(1,1)}^{-1}\Big), \tag{A.25}$$

where the determinant is defined through the trace,

$$\mathrm{tr}(\widehat{H}_{(1,1)}^{-1}) = \int \frac{dx^{(1)}}{2\pi} \langle x^{(1)}|\widehat{H}_{(1,1)}^{-1}|x^{(1)}\rangle. \tag{A.26}$$

The operator $\widehat{H}_{(1,1)}^{-1}$ is nothing but the quantum curve.

Since the generalization of the ABJM matrix model only changes the measure, it is sufficient to generalize the quantum curve $\widehat{H}^{-1}$ as

$$\widehat{H}^{-1} = \prod_{a=1}^{r} \frac{1}{2\cosh\frac{\widehat{r}}{2}}, \tag{A.27}$$

where $\widehat{r}$ denote $\widehat{p}$ for an NS5-brane and $\widehat{q}$ for a $(1,k)$5-brane. For example, in the case of the $(p_1, q_1, \cdots)$ model, the grand canonical partition function is represented with the quantum curve,

$$\widehat{H}_{(p_1,q_1,\cdots)}^{-1} = \widehat{\mathcal{P}}^{-p_1}\widehat{\mathcal{Q}}^{-q_1}\cdots. \tag{A.28}$$

B Similarity transition for E_7 curve

In this section, we show the similarity transitions to derive the E_7 quantum curve $\widehat{H}_{(E_7)}$ in (2.18) [48]. First, we define the operator

$$\widehat{Q}' := \widehat{U}(-1)^{-1}\widehat{Q}, \tag{B.1}$$

with

$$\widehat{U}(u)^{-1} := \widehat{P}\Big(\widehat{P} + q^{\frac{u}{2}} g_1\Big)^{-1} \quad \text{for} \quad u \in \mathbb{Z}. \tag{B.2}$$

Then, the relation between $\widehat{Q}^n$ and $\widehat{Q}'^n$ can be represented as

$$\widehat{Q}^n = \Big(\widehat{U}(-1)\widehat{Q}'\Big)^n = \widehat{Q}'^{-n}\prod_{i=1}^{n}\widehat{U}(2i-1), \quad \widehat{Q}^{-n} = \Big(\widehat{U}(-1)\widehat{Q}'\Big)^{-n} = \widehat{Q}'^{-n}\prod_{i=1}^{n}\widehat{U}(1-2i)^{-1}. \tag{B.3}$$

With these properties, we rewrite $\widehat{H}_{(E_7)}$ in terms of $\widehat{Q}'$ as

$\widehat{H}_{(E_7)}/\alpha$

$$
\begin{aligned}
&= q^{-1}\widehat{Q}'^{2}\widehat{U}(3)\widehat{U}(1)\widehat{P} \\
&+ q^{-1/2}\widehat{Q}'\widehat{U}(1)\Big(F_1^{(+)}\widehat{P} + q^{1/2}H_1^{(-)}\Big) \\
&+ F_2^{(+)}\widehat{P} + \frac{E}{\alpha} + H_2^{(-)}\widehat{P}^{-1} \\
&+ q^{1/2}F_4^{(+)}\widehat{Q}'^{-1}\widehat{U}(-1)^{-1}\Big(\widehat{P} + q^{-1/2}g_1\Big)\Big(\widehat{P} + q^{-1/2}g_2\Big)\Big(F_1^{(-)}\widehat{P} + q^{-1/2}g_1g_2H_1^{(+)}\Big)\widehat{P}^{-2} \\
&+ qF_4^{(+)}\widehat{Q}'^{-2}\widehat{U}(-3)^{-1}\widehat{U}(-1)^{-1}\Big(\widehat{P} + q^{-3/2}g_1\Big)\Big(\widehat{P} + q^{-1/2}g_1\Big)\Big(\widehat{P} + q^{-3/2}g_2\Big)\Big(\widehat{P} + q^{-1/2}g_2\Big)\widehat{P}^{-3},
\end{aligned}
\tag{B.4}
$$

by replacing $\widehat{Q}'$ to $\widehat{Q}$, (2.18) is obtained as

$$
\begin{aligned}
&\widehat{H}_{(E_7)}/\alpha \\
&= q^{-1}\widehat{Q}^{2}\widehat{U}(3)\widehat{U}(1)\widehat{P} \\
&+ q^{-1/2}\widehat{Q}\widehat{U}(1)\Big(F_1^{(+)}\widehat{P} + q^{1/2}H_1^{(-)}\Big) \\
&+ F_2^{(+)}\widehat{P} + \frac{E}{\alpha} + H_2^{(-)}\widehat{P}^{-1} \\
&+ q^{1/2}F_4^{(+)}\widehat{Q}^{-1}\widehat{U}(-1)^{-1}\Big(\widehat{P} + q^{-1/2}g_1\Big)\Big(\widehat{P} + q^{-1/2}g_2\Big)\Big(F_1^{(-)}\widehat{P} + q^{-1/2}g_1g_2H_1^{(+)}\Big)\widehat{P}^{-2} \\
&+ qF_4^{(+)}\widehat{Q}^{-2}\widehat{U}(-3)^{-1}\widehat{U}(-1)^{-1}\Big(\widehat{P} + q^{-3/2}g_1\Big)\Big(\widehat{P} + q^{-1/2}g_1\Big)\Big(\widehat{P} + q^{-3/2}g_2\Big)\Big(\widehat{P} + q^{-1/2}g_2\Big)\widehat{P}^{-3} \\
&= q^{-1}\widehat{Q}^{2}\Big(\widehat{P} + q^{3/2}g_1\Big)\Big(\widehat{P} + q^{1/2}g_1\Big)\widehat{P}^{-1} \\
&+ q^{-1/2}\widehat{Q}\Big(\widehat{P} + q^{1/2}g_1\Big)\Big(F_1^{(+)}\widehat{P} + q^{1/2}H_1^{(-)}\Big)\widehat{P}^{-1} \\
&+ F_2^{(+)}\widehat{P} + \frac{E}{\alpha} + H_2^{(-)}\widehat{P}^{-1} \\
&+ q^{1/2}F_4^{(+)}\widehat{Q}^{-1}\Big(\widehat{P} + q^{-1/2}g_2\Big)\Big(F_1^{(-)}\widehat{P} + q^{-1/2}g_1g_2H_1^{(+)}\Big)\widehat{P}^{-1} \\
&+ qF_4^{(+)}\widehat{Q}^{-2}\Big(\widehat{P} + q^{-3/2}g_2\Big)\Big(\widehat{P} + q^{-1/2}g_2\Big)\widehat{P}^{-1}.
\end{aligned}
\tag{B.5}
$$

C Qunatum mirror map for del Pezzo geometries

In this section, let us review the results of [11] briefly.

The perturbative part of the grand canonical partition function in the ABJM matrix model leads to the fact that the partition function is the Airy function and the non-perturbative part consists of the worldsheet instantons, the membrane instantons and the bound states of them. On the other hand, we also understand the ABJM matrix model from a geometric viewpoint. The ABJM matrix model is represented with the Fredholm determinant of the quantum curve associated with the curve of genus one known as the local $\mathbb{P}^1 \times \mathbb{P}^1$. Then we can consider two cycles for the curve of one genus, integrations along them give the periods (called A- and B-periods) respectively and they are modified for the quantum curve as explained below. The B-period by integrating along the B-cycle give the derivatives of the free energy and it is found that the woldsheet instantons are written in terms of the BPS indices and have the multi-covering structure that the instantons in the higher order contain instantons in the lower order [7]. The membrane instantons also have similar multi-covering structure with the coefficients of the positive integer. Here, there is a natural question what role the A-period plays. The answer is that the A-period allows us to better understand the cancelation mechanism among the divergences of the instantons [4] that exactly give the non-perturbative part. This cancelation mechanism occurs among instantons including the bound states, however by redefining the fugacity with the A-period Π_A as

$$\log z_{\text{eff}} = \log z_E + \Pi_A(z), \tag{C.1}$$

with $z_E = z + E$, the bound state is included in the worldsheet instantons, as the result, the cancelation mechanism of the divergences occurs between the worldsheet and membrane instantons. This redefined chemical potential (or fugacity) is often called effective chemical potential. From such an important role, we expect that the A-period has the interesting structure and the structure is important to understand the non-perturbative effects. Therefore, in [11] to systematically examine the A-period, we have considered the D_5 quantum curve that enjoys the larger symmetries than the A_1 quantum curve describing the ABJM matrix model.

We can obtain the quantum A-period as the natural modification of the (classical) A-period by introducing the wave function and considering the Schrödinger equation for the quantum curve $\widehat{H}_{(D_5)}$ following [7, 30],

$$\Big[\frac{\widehat{H}_{(D_5)}}{\alpha} + \frac{z}{\alpha}\Big]\Psi(x) = 0. \tag{C.2}$$

The operators $\widehat{Q}$ and $\widehat{P}$ act to the wave function as

$$\widehat{Q}\Psi(x) = X\Psi[X], \quad \widehat{P}\Psi(x) = \Psi[q^{-1}X], \tag{C.3}$$

thus the quantum A-period is obtaind as

$$\Pi_A(z) = \text{Res}_{X=0}\,\frac{1}{X}\log\frac{P[X]}{-zX/\big(\alpha(X-e_3)(X-1)\big)}, \tag{C.4}$$

with

$$P[X] = \frac{\Psi[q^{-1}X]}{\Psi[X]}. \tag{C.5}$$

It is found that this quantum A-period is naturally given by comparing the classical A-period

$$\Pi_A^{\mathrm{cl}} = \oint p dx = \oint \frac{dX}{X} \log P. \tag{C.6}$$

Under the parameter transitions,

$$h_1 = q_3 q_4, \quad h_2 = \frac{1}{q_2 q_3}, \quad e_1 = \frac{q_1}{q_2}, \quad e_3 = \frac{q_4}{q_5}, \quad e_5 = \frac{1}{q_1 q_2}, \tag{C.7}$$

the quantum A-period is expanded in the large z as

$$\Pi_A(z) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{z^\ell} A_\ell(h_1, h_2, e_1, e_3, e_5; \alpha), \tag{C.8}$$

with

$$\begin{aligned}
A_1 &= 0, \\
A_2 &= q_1 + q_1^{-1} + q_2 + q_2^{-1} + q_3 + q_3^{-1} + q_4 + q_4^{-1} + q_5 + q_5^{-1}, \\
\frac{A_3}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} &= \frac{1}{\sqrt{q_1 q_2 q_3 q_4 q_5}} (q_1 + q_2 + q_3 + q_4 + q_5 \\
&\quad + q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_2 q_5 + q_1 q_3 q_4 + q_1 q_3 q_5 + q_1 q_4 q_5 + q_2 q_3 q_4 \\
&\quad + q_2 q_3 q_5 + q_2 q_4 q_5 + q_3 q_4 q_5 + q_1 q_2 q_3 q_4 q_5), \cdots.
\end{aligned} \tag{C.9}$$

Here we choose the overall factor α to

$$\alpha = \frac{h_2^{\frac{1}{2}} e_1^{\frac{1}{4}}}{e_3^{\frac{1}{2}} e_5^{\frac{1}{4}}}, \tag{C.10}$$

because the Weyl group $W(D_5)$ acts to not only parameters of the curve but also the overall factor α. It is found that the quantum A-period has the group theoretical structure, which is written in terms of the D_5 characters as

$$\begin{aligned}
&A_1 = 0, \quad A_2 = \chi_{\mathbf{10}}, \quad A_3 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \chi_{\mathbf{16}}, \\
&A_4 = (q^2 + q^{-2}) \chi_{\mathbf{1}} + (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + \frac{3}{2}\chi_{\mathbf{54}} + \frac{5}{2}\chi_{\mathbf{45}} + \frac{11}{2}\chi_{\mathbf{1}}, \cdots.
\end{aligned} \tag{C.11}$$

Also, as in [6], since the inverse quantum A-period has a simple structure, we solve the quantum A-period inversely

$$\log z_E = \log z + \sum_{\ell=1}^{\infty}(-1)^{\ell} E_{\ell} z_{\text{eff}}^{-\ell}. \tag{C.12}$$

Then we find the multi-covering structure in the quantum A-period

$$E_{\ell} = \sum_{n|\ell} \frac{(-1)^{n+1}}{n} \epsilon_{\frac{\ell}{n}}, \tag{C.13}$$

with the multi-covering components $\epsilon_d(q; q_1, q_2, q_3, q_4, q_5)$

$$\begin{aligned}
\epsilon_1 &= 0, \\
\epsilon_2 &= \chi_{\mathbf{10}}, \\
\epsilon_3 &= (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\chi_{\mathbf{16}}, \\
\epsilon_4 &= (q^2 + q^{-2})\chi_{\mathbf{1}} + (q + q^{-1})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}}, \\
\epsilon_5 &= (q^{\frac{5}{2}} + q^{-\frac{5}{2}})\chi_{\overline{\mathbf{16}}} + (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(\chi_{\overline{\mathbf{144}}} + 3\chi_{\overline{\mathbf{16}}}) + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})3\chi_{\overline{\mathbf{16}}}, \\
\epsilon_6 &= (q^4 + q^{-4})\chi_{\mathbf{10}} + (q^3 + q^{-3})(\chi_{\mathbf{120}} + 4\chi_{\mathbf{10}}) + (q^2 + q^{-2})(\chi_{\mathbf{320}} + \chi_{\overline{\mathbf{126}}} + 3\chi_{\mathbf{120}} + 9\chi_{\mathbf{10}}) \\
&\quad + (q + q^{-1})(3\chi_{\mathbf{120}} + 8\chi_{\mathbf{10}}) + (\chi_{\mathbf{320}} + 2\chi_{\mathbf{120}} + 9\chi_{\mathbf{10}}), \\
\epsilon_7 &= (q^{\frac{11}{2}} + q^{-\frac{11}{2}})\chi_{\mathbf{16}} + (q^{\frac{9}{2}} + q^{-\frac{9}{2}})(\chi_{\mathbf{144}} + 4\chi_{\mathbf{16}}) + (q^{\frac{7}{2}} + q^{-\frac{7}{2}})(\chi_{\mathbf{560}} + 4\chi_{\mathbf{144}} + 13\chi_{\mathbf{16}}) \\
&\quad + (q^{\frac{5}{2}} + q^{-\frac{5}{2}})(\chi_{\mathbf{720}} + 4\chi_{\mathbf{560}} + 9\chi_{\mathbf{144}} + 25\chi_{\mathbf{16}}) + (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(3\chi_{\mathbf{560}} + 8\chi_{\mathbf{144}} + 27\chi_{\mathbf{16}}) \\
&\quad + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(\chi_{\mathbf{720}} + 3\chi_{\mathbf{560}} + 9\chi_{\mathbf{144}} + 27\chi_{\mathbf{16}}), \\
\epsilon_8 &= (q^8 + q^{-8})\chi_{\mathbf{1}} + (q^7 + q^{-7})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + (q^6 + q^{-6})(\chi_{\mathbf{210}} + \chi_{\mathbf{54}} + 4\chi_{\mathbf{45}} + 10\chi_{\mathbf{1}}) \\
&\quad + (q^5 + q^{-5})(\chi_{\mathbf{945}} + 4\chi_{\mathbf{210}} + 4\chi_{\mathbf{54}} + 14\chi_{\mathbf{45}} + 25\chi_{\mathbf{1}}) \\
&\quad + (q^4 + q^{-4})(\chi_{\overline{\mathbf{1050}}} + 4\chi_{\mathbf{945}} + \chi_{\mathbf{770}} + 13\chi_{\mathbf{210}} + 10\chi_{\mathbf{54}} + 35\chi_{\mathbf{45}} + 54\chi_{\mathbf{1}}) \\
&\quad + (q^3 + q^{-3})(\chi_{\mathbf{1386}} + 4\chi_{\overline{\mathbf{1050}}} + 10\chi_{\mathbf{945}} + 3\chi_{\mathbf{770}} + 25\chi_{\mathbf{210}} + 19\chi_{\mathbf{54}} + 62\chi_{\mathbf{45}} + 84\chi_{\mathbf{1}}) \\
&\quad + (q^2 + q^{-2})(3\chi_{\overline{\mathbf{1050}}} + 8\chi_{\mathbf{945}} + 3\chi_{\mathbf{770}} + 27\chi_{\mathbf{210}} + 19\chi_{\mathbf{54}} + 68\chi_{\mathbf{45}} + 102\chi_{\mathbf{1}}) \\
&\quad + (q + q^{-1})(\chi_{\mathbf{1386}} + 3\chi_{\overline{\mathbf{1050}}} + 10\chi_{\mathbf{945}} + 3\chi_{\mathbf{770}} + 27\chi_{\mathbf{210}} + 22\chi_{\mathbf{54}} + 73\chi_{\mathbf{45}} + 105\chi_{\mathbf{1}}) \\
&\quad + (4\chi_{\overline{\mathbf{1050}}} + 10\chi_{\mathbf{945}} + 2\chi_{\mathbf{770}} + 28\chi_{\mathbf{210}} + 22\chi_{\mathbf{54}} + 72\chi_{\mathbf{45}} + 108\chi_{\mathbf{1}}).
\end{aligned}$$

The coefficients of ϵ_d are positive integers and the representations appearing are same ones in the B-period [32, 33] except for the trivial case $d = 1$.

For the above results, some questions remain. First, we do not understand whether the structure that the A-period is written in terms of the D_5 characters of the same representations (except for $\epsilon_{d=1}$) with the B-period is natural. In addition, we do not know why the trivial case $\epsilon_{d=1}$ is the only exception. Second, the coefficients of the multi-covering components are positive integers, however it is not clear what they count.

D ABJM/2DTL correspondence

In this section, we summarize the results of [12, 13] briefly.

The localization technique is known as the method to obtain the matix models from the supersymmetric gauge theories [3, 17]. Therefore, only when the supersymmetries are preserved, the vacuum expectation values in the supersymmetric gauge theories reduce to those in the matrix model by the localization technique. In the ABJM theory. The vacuum expectation values of the half BPS Wilson loops in the ABJM theory are represented as the insertions of the super Schur functions [53] in the ABJM matrix model. For the one-point functions in the ABJM matrix model, the characteristic structure is found and it is called shifted Giambelli relation [22]. And this structure leads that the one-point functions satisfy some relations well-known for the Schur functions [54], the Giambelli relations [23] and the Jacobi-Trudi relations [12].

Later, in [21], the two-point function in the ABJM matrix model is proposed as the generalization of the one-point function. This generalization is constructed so that the two-point functions do not reduce to the one-point functions through the Littlewood-Richardson rule trivially and satisfy the modification of the shifted Giambelli relation of one-point functions. The two-point functions are not results by the localization technique, however we expect that the definition of the two-point functions is natural and physical from the characteristic properties of the two-point functions.

In [13], we have studied the correspondence between the ABJM matrix model and the soliton equations. After J. Scott-Russell found the solitary waves in 1834, the many non-linear differential equations describing the solitons are found and solved. Furthermore, it is found that there are infinite non-linear differential equations compatible with the original equation and the whole set of them is called integrable hierarchy. The Sato theory claims that when the soliton solution (tau-function) is expanded with the Schur functions, the coefficients satisfy the Plücker relations and the whole set of the relations equivalent to the integrable hierarchy [55–58]. Interestingly, the two-dimensional Toda lattice (2DTL) hierarchy gives modified Kadomtsev-Petviashvili (mKP) hierarchy by simple reduction.

The coefficients of the tau function expanded with the Schur functions are represented as the expectation values of the product of the fermions in the Fermionic construction known in the integrable systems. Then, the Wick theorem for the fermions leads not only the Plücker relations but also the Giambelli relations and Jacobi-Trudi relations (see, for example, [58]). In [13], by generalizing the theorem, we have shown that the coefficients of the 2DTL tau function satisfy the shifted Giambelli relation modified for the two-point functions. By this fact, if we correspond the two-point functions to the coefficients of the 2DTL tau function for the hook representations, the correspondence for any representation is automatically constructed due to the shifted Giambelli relation.

This correspondence between ABJM matrix model and 2DTL hierarchy (ABJM/2DTL correspondence) gives us some advantages. First, the proofs that the one-point functions satisfy the Giambelli relation and the Jacobi-Trudi relation were complicated, however the ABJM/2DTL correspondence allows us to rederive them with the Wick theorem systematically. Also, we can find the Giambelli relation for the two-point functions as the new relation with the generalized Wick theorem. In this way, we expect that the Fermionic construction will play an important role when discussing the mathematical structure of the ABJM matrix model in the future.

Second, the ABJM/2DTL correspondence allows us to reevaluate the generalization from the one-point functions to two-point functions in the ABJM matrix model in the viewpoint of the integrable systems. In the integrable systems, the 2DTL hierarchy reduces mKP hierarchy by simple reduction. On the other hand, in the ABJM matrix model, if one Schur function inserted in the two-point function is trivial one, the two-point function reduces to the one-point function. These reductions agree in the ABJM/2DTL correspondence. Actually, the two-point functions correspond to the 2DTL hierarchy and the one-point functions correspond to the mKP hierarchy. Therefore the generalization to the two-point functions is considered natural from the viewpoint of the integrable system.

E Static force potential of non-Abelian gauge theory at a finite box in Coulomb gauge

Although the contents in this appendix do not relate directly to the main topic of this thesis, I have also studied static force potential at a finite box in my doctoral course. In this section, let us review the results of [59] briefly. We have reconsidered the force potential existing between two classical static sources of pure non-Abelian gauge theory in the Caulomb gauge[9] at a twisted [71–77] (and periodic [78, 79]) finite box. There we have calculated the interaction energy up to the one-loop level in the old-fashioned perturbation theory and analyzed it in the derivative expansion and the short-distance expansion. And as the result, the former expansion leads the negative beta-function and

[9]We give some of the references about non-Abelian gauge theory with the Coulomb gauge [60–70].

the Uehling potential in the non-Abelian gauge theory. The latter expansion is given by the convolution of the two Green functions being non-singular at $\boldsymbol{x}_1 = \boldsymbol{x}_2$ (see (E.14)). Also, it was found that the effect of the twist comes from Green function of the Laplacian in the twisted sector.

First, in the Coulomb gauge, the Hamiltonian consists of the physical degrees of freedom and all of the gauge degree of freedom being eliminated as

$$\begin{aligned} H &= \frac{1}{2}\sum_a \int d^3x \left(\mathcal{J}^{-1}\Pi_i^{(\mathrm{tr})a}\mathcal{J}\Pi_i^{(\mathrm{tr})a} + B_i^{(\mathrm{tr})a}B_i^{(\mathrm{tr})a} \right) + H_{\mathrm{Coul}}, \\ H_{\mathrm{Coul}} &= \frac{g_0^2}{2}\sum_{a,b}\int d^3x d^3x' \mathcal{J}^{-1}\rho^a(\boldsymbol{x})\langle \boldsymbol{x}| \left((\partial_i D_i)^{-1}(-\partial^2)(\partial_j D_j)^{-1} \right)^{ab} |\boldsymbol{x}'\rangle \mathcal{J}\rho^b(\boldsymbol{x}'), \end{aligned} \tag{E.1}$$

where $\Pi_i^{(\mathrm{tr})a}$, $B_i^{(\mathrm{tr})a}$, D_i, ρ^a and $\mathcal{J}$ are respectively the conjugate momentum of the transverse gauge field $A_i^{(\mathrm{tr})a}$, the transverse magnetic field, covariant derivative, the current density and Faddeev-Popov determinant $\mathcal{J} = \det(\partial_i D_i)$. Here the current density includes two external source terms as

$$\begin{aligned} \rho^a(\boldsymbol{x}) &= g_0 \epsilon^{abc} A_i^b(\boldsymbol{x})\Pi_i^c(\boldsymbol{x}) + \rho_{1,\mathrm{ex}}^a(\boldsymbol{x}) + \rho_{2,\mathrm{ex}}^a(\boldsymbol{x}), \\ \rho_{1,2,\mathrm{ex}}^a(\boldsymbol{x}) &= q_{1,2}^a \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_{1,2}), \end{aligned} \tag{E.2}$$

where λ is twisted sector and $|\boldsymbol{w}\rangle_\lambda$ is the ket vector in the momentum representation in the λ twisted sector. The twisted sector is derived from the twisted boundary condition for the gauge field [80],

$$A_i(x,y,z) = PA_i(x+L,y,z)P^{-1} = QA_i(x,y+L,z)Q^{-1} = A_i(x,y,z+L), \tag{E.3}$$

where P and Q are the constant matrices which satisfy for $SU(N)$,

$$PQ = QPe^{\frac{2\pi i}{N}}. \tag{E.4}$$

The Green function of the Laplacian in the $\boldsymbol{\lambda}$ twisted sector is

$$G^{\boldsymbol{\lambda}}(\boldsymbol{x}|\boldsymbol{x}') = -\frac{1}{4\pi}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^3}\frac{e^{2\pi i \boldsymbol{\lambda}\cdot\boldsymbol{\ell}}}{|\boldsymbol{x} - \boldsymbol{x}' + L\boldsymbol{\ell}|}, \tag{E.5}$$

where we use the Poisson resummation formula. In the limit $L \to \infty$, the dependence on the twisted sector disappears

$$G(\boldsymbol{x}|\boldsymbol{x}') = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{E.6}$$

With the Green function in the $\boldsymbol{\lambda}$ twisted sector, the part of the vacuum energy which depends on linearly upon both $\rho^a_{1,\text{ex}}(\boldsymbol{x})$ and $\rho^a_{2,\text{ex}}(\boldsymbol{x})$ at the lowest classical level is obtained as

$$\begin{aligned} E_{\text{tw}} &= -g_0^2 \sum_a \int d^3x d^3x' \rho^a_{1,\text{ex}}(\boldsymbol{x}) G^{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{x}|\boldsymbol{x}') \delta^{ab} \rho^b_{2,\text{ex}}(\boldsymbol{x}') \\ &= -g_0^2 \sum_a q_1^a q_2^a G^{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{x}_1|\boldsymbol{x}_2). \end{aligned} \tag{E.7}$$

Therefore, the twisted sector makes the interaction energy smaller than that in the periodic boundary condition at the classical level.

And in the old-fashioned perturbation theory well-known in the quantum mechanics, the perturbative expansion of the interaction energy $E(r_{12} = |\boldsymbol{x}_1 - \boldsymbol{x}_2|)$ is given up to second order by

$$E(r_{12}) = \sum_a \frac{g_0^2 q_1^a q_2^a}{4\pi^2 L} \sum_{\substack{\boldsymbol{n} \in \mathbb{Z}^3 + \boldsymbol{\lambda}^{(a)} \\ \boldsymbol{n} \neq \boldsymbol{0}}} \langle \boldsymbol{x}_1 | \boldsymbol{n} \rangle_{\boldsymbol{\lambda}^{(a)}} \frac{1}{\boldsymbol{n} \cdot \boldsymbol{n}} \left[1 + g_0^2 \left(\delta'_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n}) + \delta''_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n}) \right) \right]_{\boldsymbol{\lambda}^{(a)}} \langle \boldsymbol{n} | \boldsymbol{x}_2 \rangle, \tag{E.8}$$

where $\delta'_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n})$ and $\delta''_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n})$ are contributions included in the leading order and the second order respectively and the sum of them gives the correction at the one-loop level

$$\begin{aligned} \delta'_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n}) &= \frac{3}{16\pi^3} \frac{1}{\boldsymbol{n} \cdot \boldsymbol{n}} \sum_{c \neq a} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^3 + \boldsymbol{\lambda}^{(c)} \\ \boldsymbol{m} \neq \boldsymbol{0}}} \frac{(n+m)_i P(\boldsymbol{m})_{ij} n_j}{|\boldsymbol{n} + \boldsymbol{m}|^2 |\boldsymbol{m}|}, \\ \delta''_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n}) &= -\frac{1}{32\pi^3} \frac{1}{\boldsymbol{n} \cdot \boldsymbol{n}} \sum_{c \neq a} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^3 + \boldsymbol{\lambda}^{(c)} \\ \boldsymbol{m} \neq \boldsymbol{0}}} \frac{(|\boldsymbol{m}| - |\boldsymbol{n} - \boldsymbol{m}|)^2}{|\boldsymbol{m}| + |\boldsymbol{n} - \boldsymbol{m}|} \frac{P(\boldsymbol{m})_{ij} P(\boldsymbol{n} - \boldsymbol{m})_{ij}}{|\boldsymbol{m}||\boldsymbol{n} - \boldsymbol{m}|}. \end{aligned} \tag{E.9}$$

The zero mode $\boldsymbol{m} = \boldsymbol{0}$ in the summation is removed to prevent the divergence in the periodic case $\boldsymbol{\lambda} = \boldsymbol{0}$.

In the following, we investigate the result (E.8) in the long-distance and the short-distance expansions. At long-distance $r_{12} \sim L$, the quantum correction is expanded in n^i as

$$\begin{aligned} &\delta'_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n}) + \delta''_{\boldsymbol{\lambda}^{(a)}}(\boldsymbol{n}) \\ &= \frac{1}{32\pi^3} \sum_{c \neq a} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^3 + \boldsymbol{\lambda}^{(c)} \\ \boldsymbol{m} \neq \boldsymbol{0}}} \left[\frac{1}{2!} \left(\frac{12}{|\boldsymbol{m}|^3} - \frac{14(\boldsymbol{m} \cdot \boldsymbol{n})^2}{|\boldsymbol{m}|^5 |\boldsymbol{n}|^2} \right) + \frac{1}{3!} \left(-\frac{66(\boldsymbol{m} \cdot \boldsymbol{n})}{|\boldsymbol{m}|^5} + \frac{57(\boldsymbol{m} \cdot \boldsymbol{n})^3}{|\boldsymbol{m}|^7 |\boldsymbol{n}|^2} \right) \right. \\ &\left. + \frac{1}{4!} \left(-\frac{150|\boldsymbol{n}|^2}{|\boldsymbol{m}|^5} + \frac{822(\boldsymbol{m} \cdot \boldsymbol{n})^2}{|\boldsymbol{m}|^7} - \frac{714(\boldsymbol{m} \cdot \boldsymbol{n})^4}{|\boldsymbol{m}|^9 |\boldsymbol{n}|^2} \right) + (\text{higher orders of } n^i) \right]. \end{aligned} \tag{E.10}$$

Here if we consider the case of the periodic boundary condition $\boldsymbol{\lambda} = \mathbf{0}$, due to the cubic symmetry, the quantum correction reduces to

$$\delta'_{\boldsymbol{\lambda}=\mathbf{0}}(\boldsymbol{n}) + \delta''_{\boldsymbol{\lambda}=\mathbf{0}}(\boldsymbol{n}) = \frac{1}{16\pi^3} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^3 \\ \boldsymbol{m}\neq\mathbf{0}}} \left[\frac{11}{3} \frac{1}{|\boldsymbol{m}|^3} - \frac{47}{60} \frac{|\boldsymbol{n}|^2}{|\boldsymbol{m}|^5} + \mathcal{O}(|\boldsymbol{n}|^4) \right]. \tag{E.11}$$

We obtain the interaction energy at the periodic box as

$$\begin{aligned} E(r_{12}) =& \frac{1}{4\pi} \sum_a \sum_{\boldsymbol{n}\in\mathbb{Z}^3} \frac{g_0^2 q_1^a q_2^a}{|\boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{n}L|} \left[1 + g_0^2 \frac{11}{12\pi^2} \frac{1}{4\pi} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^3 \\ \boldsymbol{m}\neq\mathbf{0}}} \frac{6}{|\boldsymbol{m}|^3} \right] \\ & - \sum_a \frac{g_0^4 q_1^a q_2^a}{(4\pi)^2} \frac{47}{30} \frac{1}{(2\pi)^3} \left(L^2 \sum_{\boldsymbol{n}\in\mathbb{Z}^3} \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{n}L) - \frac{1}{L} \right) \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^3 \\ \boldsymbol{m}\neq\mathbf{0}}} \frac{1}{|\boldsymbol{m}|^5} \\ & + (\text{higher orders in the derivative expansion}), \end{aligned} \tag{E.12}$$

and the parts of the first and second line give the negative beta-function and the Uehling potential respectively in the large L.

In the expansion at short-distance $r_{12} \ll L$, in the limit $L \to \infty$ we can expand the interaction energy in the inverse of the momentum of the external charges $1/p$ as

$$E(r_{12}) = \sum_a \frac{g_0^2 q_1^a q_2^a}{(2\pi)^3} \int d^3p e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)} \left[\frac{1}{p^2} + g_0^2 \left(\frac{1}{3\pi^2} \frac{\Lambda^2}{p^4} + \mathcal{O}\left(\frac{\Lambda^3}{p^5} \right) \right) \right], \tag{E.13}$$

where Λ is UV cutoff. We note that the $1/p^4$-term is written as

$$\int d^3z G(\boldsymbol{x}_1|\boldsymbol{z}) G(\boldsymbol{z}|\boldsymbol{x}_2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^4} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}, \tag{E.14}$$

which is non-singular at $\boldsymbol{x}_1 = \boldsymbol{x}_2$,

$$\int d^3z G(\boldsymbol{x}|\boldsymbol{z}) G(\boldsymbol{z}|\boldsymbol{x}') = \left(\frac{-1}{4\pi} \right)^2 \int d^3z \frac{1}{|\boldsymbol{x} - \boldsymbol{z}|} \frac{1}{|\boldsymbol{z} - \boldsymbol{x}'|}. \tag{E.15}$$

Then in terms of the Green functions, the interaction energy $E(r_{12})$ is written as

$$E(r_{12}) = -g_0^2 \sum_a q_1^a q_2^a \left[G(\boldsymbol{x}_1|\boldsymbol{x}_2) - \frac{1}{3\pi^2} g_0^2 \Lambda^2 \int d^3z G(\boldsymbol{x}_1|\boldsymbol{z}) G(\boldsymbol{z}|\boldsymbol{x}_2) + \mathcal{O}(\Lambda^3) \right]. \tag{E.16}$$

The short-distance expansion begins with the convolution of the two Coulomb Green functions and is non-singular at the short distance limit of the two external sources $\boldsymbol{x}_1 = \boldsymbol{x}_2$. This term does not appear in the QED.

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