Gamma-Polynomial and its Generalization to a 2-string Tangle Polynomial

（ガンマ多項式と
その２ストリング・タングル多項式への一般化）

平成２７年度

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Abstract

The skein (HOMFLYPT) polynomial is a polynomial invariant of a link. It is one of the most general polynomial invariants in the link polynomial invariants which denote the equation of the skein relation.

The zeroth coefficient polynomial of the skein (HOMFLYPT) knot polynomial called the Γ-polynomial is studied from a viewpoint of regular homotopy of knot diagrams. In particular, an elementary existence proof of the knot invariance of the Γ-polynomial is given. After observing that there are three types for 2-string tangle diagrams, the Γ-polynomial is generalized to a polynomial invariant of a 2-string tangle. As an application, we have a new proof of the assertion that Kinoshita's θ-curve is not equivalent to the trivial θ-curve.

Acknowledgments

The author is deeply grateful to Professor Akio Kawauchi for giving her an opportunity to write this thesis and giving her lots of valuable suggestions and helpful comments throughout this research activity. The author would like to thank Professor Yuki Seo who assisted her in helpful advice and encouragement.

The author also truly grateful to Hirokazu Okamori, late Professor Emeritus at Osaka Kyoiku University who guided her to study mathematics education and gave a chance to study teaching knot theory with Professor Akio Kawauchi.

Contents

1 Introduction

Let D be a diagram of an oriented link L with r components. The skein (HOMFLYPT) polynomial of L in [3] is represented by a Laurent polynomial in two variables y and z which is denoted by:

$$P(L;y,z) = P(D;y,z) = (yz)^{-r+1} \sum_{n=0}^{+\infty} C_n(D;-y^2) z^{2n},$$

where $C_n(D;-y^2)$ is a Laurent polynomial in y^2 called the n-th coefficient polynomial of the skein polynomial $P(L;y,z) = P(D;y,z)$ (see [1]). [1]

In this paper, the zeroth coefficient polynomial $C_0(D;x) = C_0(D;-y^2)$ written as $x = -y^2$ is studied. Following the paper [5], we call the zeroth coefficient polynomial $C_0(D;x)$ the Γ-*polynomial* and denote it by $\Gamma(D) = \Gamma(D;x)$. Let D be a knot diagram, and p a crossing point of D. Let $\epsilon(p) = \pm$ be the sign of p as in Fig. 1. Let $D_{-\epsilon(p)}$ be the knot diagram obtained from

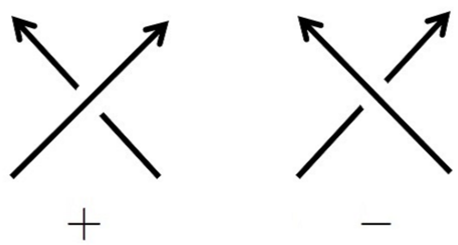


Figure 1: The signs of a crossing point.

D by the crossing change at p, and $D_{o(p)}$ the two-component link diagram obtained from D by the splice at p. Then, call $(D_{\epsilon(p)}, D_{-\epsilon(p)}, D_{o(p)})$ the *skein*

[1]See [2] for a general reference of terminologies in knot theory.

triple at p of D. When the crossing point p is not emphasized, the skein triple on D at p is denoted by (D_+, D_-, D_0) or (D_-, D_+, D_0) according to $\epsilon(p) = +$ or $-$. The crossing point p of the skein triple on D is as in Fig. 2. Let D_1 and D_2 be the knot component subdiagrams of D_0. The following

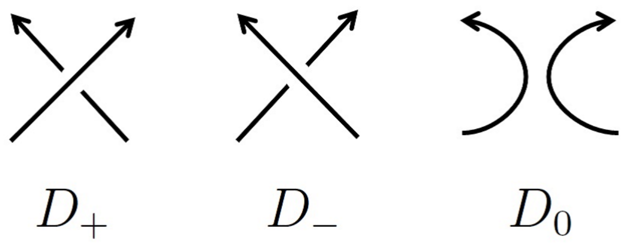


Figure 2: The skein triple on D.

skein relation is observed in Kawauchi [1].

$$-x\Gamma(D_+) + \Gamma(D_-) = (1-x)x^{-\mathrm{Link}(D_0)}\Gamma(D_1)\Gamma(D_2),$$

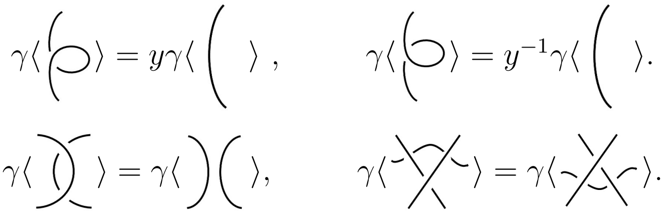
where $\mathrm{Link}(D_0)$ denotes the linking number $\mathrm{Link}(D_1, D_2)$. Let $w(D)$ be the writhe of an oriented knot diagram D. We define the γ-polynomial $\gamma(D; y)$ of D as follows:

$$\gamma(D; y) = y^{w(D)} C_0(D; -y^2),$$

which is a Laurent polynomial in y. Then the following result on $\gamma(D) = \gamma(D; y)$ is observed by Kawauchi [2].

Theorem 1.1. There is a Laurent polynomial $\gamma(D; y)$ defined on a knot diagram D which has the following 3 properties.

(1) The following equalities hold on Reidemeister Moves I, II, III:

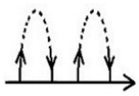


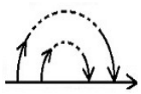
(2) If the crossing number $c(D) = 0$, then $\gamma(D) = 1$.

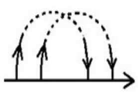
(3) For a skein triple (D_+, D_-, D_0) where D_+ and D_- are knot diagrams and $D_0 = D_1 \cup D_2$ is a two-component link diagram, we have

$$\gamma(D_+) + \gamma(D_-) = (y + y^{-1})(-1)^{-\mathrm{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

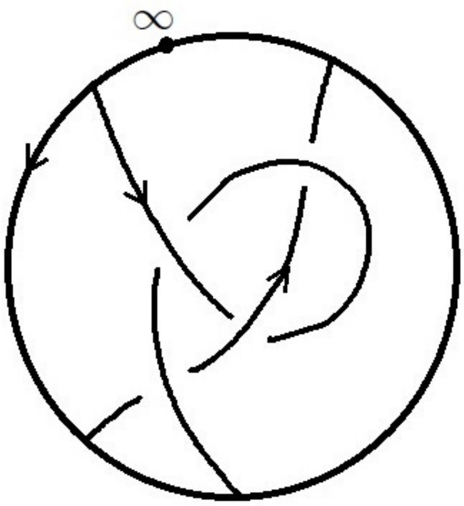
Although Theorem 1.1 is regarded as a known result as it is in principle included in the skein polynomial, we shall give an elementary direct proof of Theorem 1.1 in Section 3. In Section 4, a generalization of the Γ-polynomial to a 2-string tangle is shown. A tangle is a pair of a 3-ball and some strings in it whose ends are on the boundary sphere. Here, we consider a 2-string tangle which has two strings in a 3-ball. Let T and $D(T)$ be an oriented tangle and a tangle diagram. Let ∞ be a base point on the boundary circle of the disk underlying the tangle diagram. Although in general a tangle diagram $D(T)$ of a tangle T is denoted as in Fig. 3 (1), here it is denoted as a straight line obtained by cutting open the boundary circle of the disk at a base point ∞ as in Fig. 3 (2). From here, a tangle diagram $D(T)$ is denoted without the base point ∞. In the case that an oriented tangle diagram is denoted without an orientation of the boundary circle of the disk, we apply the same orientation as in Fig. 3 (2). For the string diagram $D(t_i)$ $(i = 1, 2)$ of an oriented tangle diagram $D(T)$, let s_i and a_i $(i = 1, 2)$ be the starting point and the arriving point of $D(t_i)$, respectively. The type A, B or C of an oriented tangle diagram $D(T)$, we mean the positional relations on s_i and a_i as in Fig. 4. We note that the positional relations of the end points of the tangle diagrams (1) − (4) in the same type B or C in Fig. 4 are deformed into a tangle diagram with the same positional relations of the end points by choosing other base points on the boundary circle of the disk underlying the tangle diagram $D(T)$. Similarly, the positional relations of the end points of the tangle diagram (1) and (2) of type A are deformed into the same ones. Further, as it will be shown in Lemma 4.1, $(1)'$ and $(2)'$ of type A are reduced to (1) and (2) of type A in the Γ-polynomial level although $(1)'$ and $(2)'$ are deformed into (1) and (2) with the reversed orientations, respectively. In these senses, we denote the tangle diagram $D(T)$ of type A, B or C as follows.

type A: $D(T) =$  ,

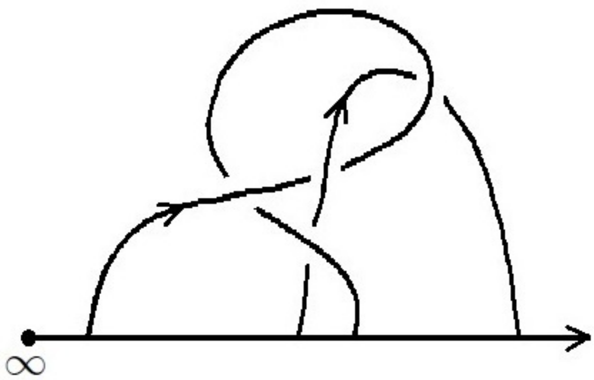
type B: $D(T) =$  ,

type C: $D(T) =$  .

For the generalization of the Γ-polynomial to a 2-string tangle, specific tangle



(1)



(2)

Figure 3: The denotations of a tangle diagram.

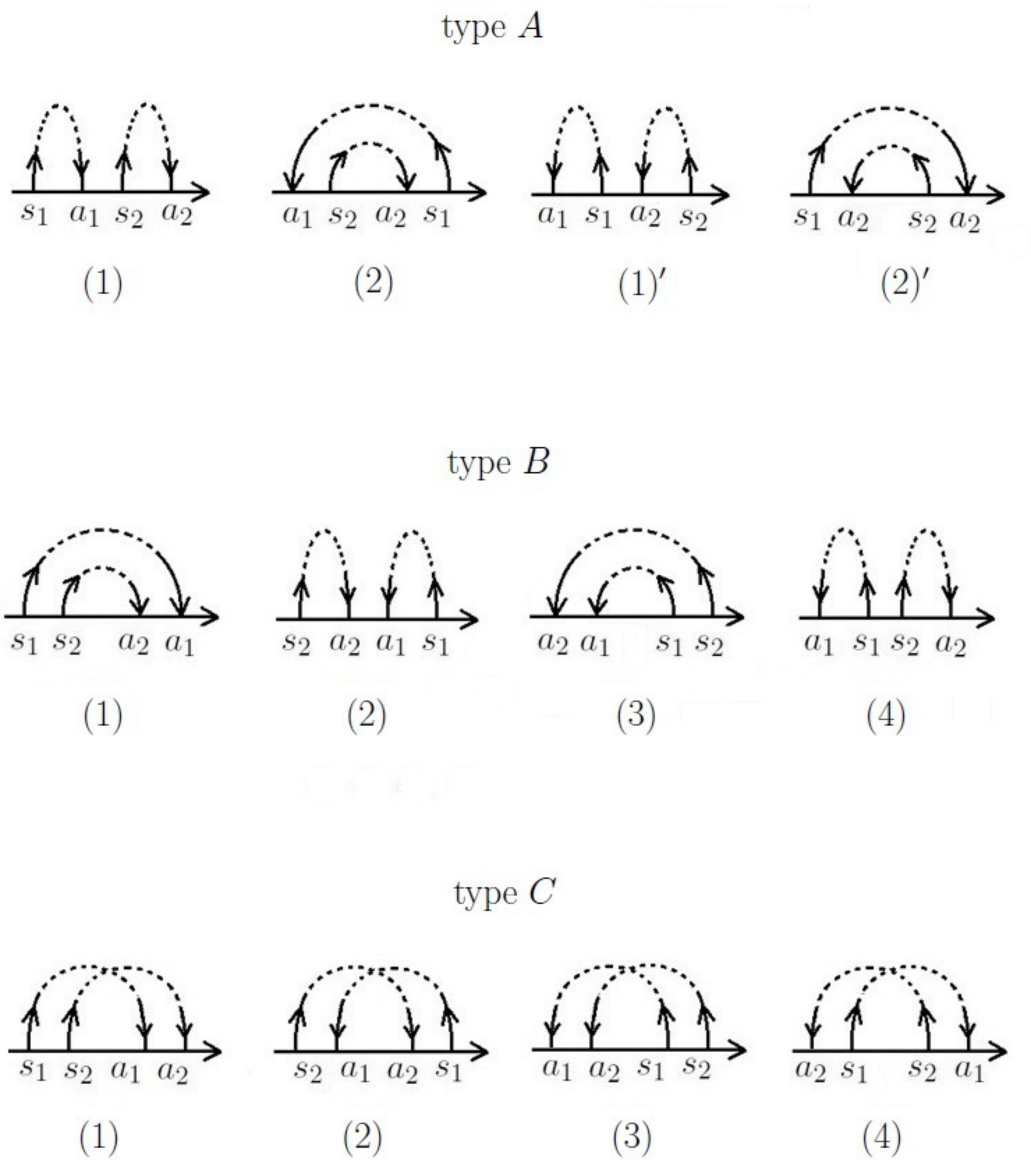
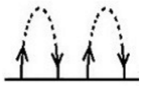
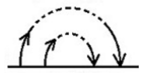
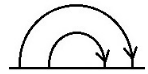
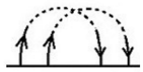
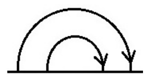


Figure 4: The types of an oriented tangle diagram.

diagrams $X_i\,(i = 0, 1)$ for the type A, B or C are fixed as follows:

type $A : D(T) =$  , $X_0 =$  , $X_1 =$  .

type $B : D(T) =$  , $X_0 =$  , $X_1 =$  .

type $C : D(T) =$  , $X_0 =$  , $X_1 =$  .

Let $D(T)$ be a diagram of a 2-string tangle T, and P a 2-string tangle diagram whose tangle sum $D(T) \cup P$ is a knot diagram. We call P a *complementary tangle diagram* for $D(T)$. Then the Γ-polynomial $\Gamma(D(T) \cup P)$ can be expressed as follows by applying the skein relations of the Γ-polynomial to $D(T)$ by induction on warping degree (see [7, 8]):

$$\Gamma(D(T) \cup P) = f_0(x)\Gamma(X_0 \cup P) + f_1(x)\Gamma(X_1 \cup P).$$

Then we define the Γ-polynomial $\Gamma(D(T))$ of $D(T)$ by

$$\Gamma(D(T)) = f_0(x)\Gamma(X_0) + f_1(x)\Gamma(X_1).$$

The well-definedness of this identity will be our main result. Two specific complementary tangle diagrams P_j $(j = 0, 1)$ of $D(T)$ depending on the type A, B or C of $D(T)$ are introduced in Fig. 5. We have the following theorem.

Theorem 1.2. For any 2-string tangle diagrams $D(T)$ and $D(T')$ and some complementary tangle diagrams P and P' for $D(T)$ and $D(T')$ respectively,

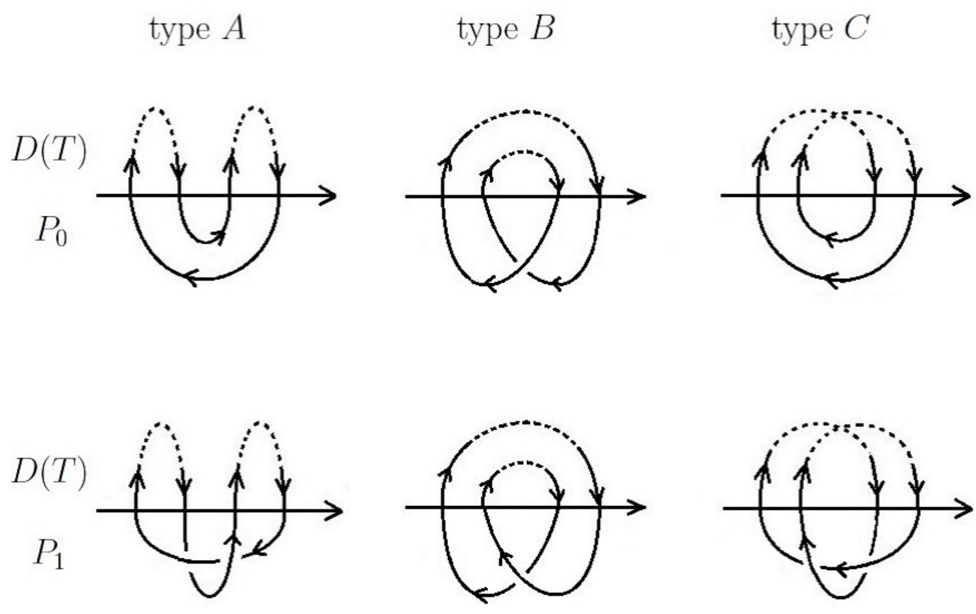


Figure 5: Two specific complementary tangle diagrams P_0 and P_1.

we take

$$\Gamma(D(T)) = f_0(x)\Gamma(X_0) + f_1(x)\Gamma(X_1),$$
$$\Gamma(D(T')) = f_0'(x)\Gamma(X_0) + f_1'(x)\Gamma(X_1).$$

If $D(T)$ and $D(T')$ are of the same type and

$$\Gamma(D(T) \cup P_j) = \Gamma(D(T') \cup P_j) \quad (j = 0, 1),$$

then we have $f_i(x) = f_i'(x)$ for any $i\,(i = 0, 1)$.

In particular, taking $D(T) = D(T')$, we have the following corollary:

Corollary 1.3. For every tangle diagram $D(T)$, the Laurent polynomials f_0 and f_1 in x are independent of choices of the skein relations on $D(T)$ and uniquely determined.

Remark: In these theorem and corollary, as it is shown in the proof of Lemma 4.1, the polynomials $f_i(x)\,(i = 0, 1)$ of the Γ-polynomial $\Gamma(D(T))$ are also independent of a choice of orientations of the boundary circle of the disk underlying an oriented tangle diagram $D(T)$. Corollary 1.3 is a version of a linear skein theory; cf. [6].

The proofs of Theorem 1.2 and Corollary 1.3 are given in Section 4. In Section 5, we apply the Γ-polynomial of a 2-string tangle to a θ-curve and give a new proof of the assertion that Kinoshita's θ-curve is not equivalent to the trivial θ-curve (see (5.1)) later.

2 Preliminary

In this section, we describe briefly some basic concepts related to this paper. The following concepts come from Kawauchi[2] and Shimizu[8]. An r-component link L is a smooth embedding image of the disjoint union of r simple closed curves into the 3-space R^3. In particular, a knot K is a link of one component. A *knot diagram* and a *link diagram* illustrated in Fig. 6, are projected images of a knot and a link, respectively to the plane R^2 with only double crossings, where the information of over-crossings and under crossings of the crossing points is given. We denote a knot diagram or a link diagram as D. The following local moves of a diagram in Fig. 7 are called *Reidemeister moves* of type I, II and III. Unless otherwise stated, we consider that a knot and a link have orientations.

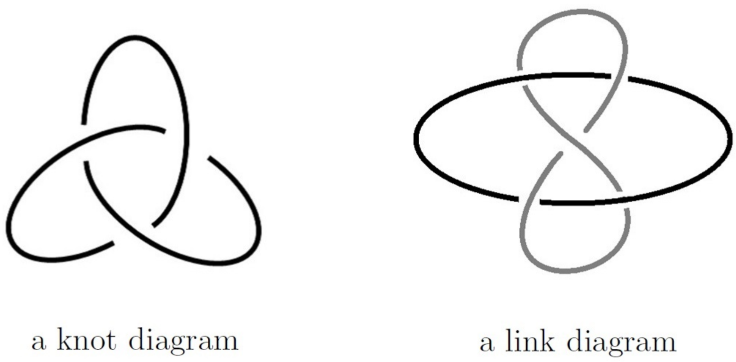


Figure 6: Examples of a knot diagram and a link diagram.

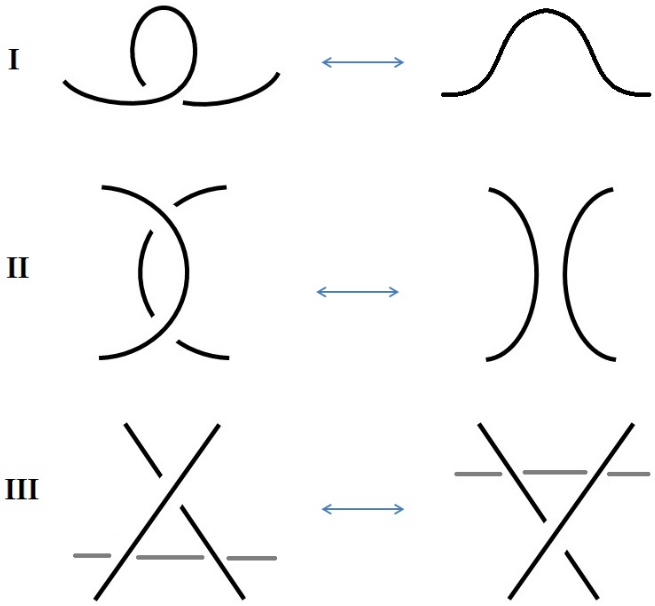


Figure 7: Reidemeister moves I, II and III.

The equivalence of several notions of "link equivalence"

We explain several conditions on the eqivalence of two links.

Definition 2.1. Two links L and L' in R^3 belong to *the same topological type* if there is an orientation-preserving self-homeomorphism h of R^3 such that $h(L) = L'$ and $h|_L : L \cong \ L'$ is orientation-preserving.

For a link L and a disk D in R^3 such that $L \cap D = L \cap \partial D$ which is an arc, the new link $L' = \mathrm{cl}(L - L \cap D) \cup (\partial D - L \cap D)$ is said to be obtained from L by a *disk move*. Here, the link L' is oriented so that the orientations on $L' - D$ and $L - D$ coincide (See Fig. 8).

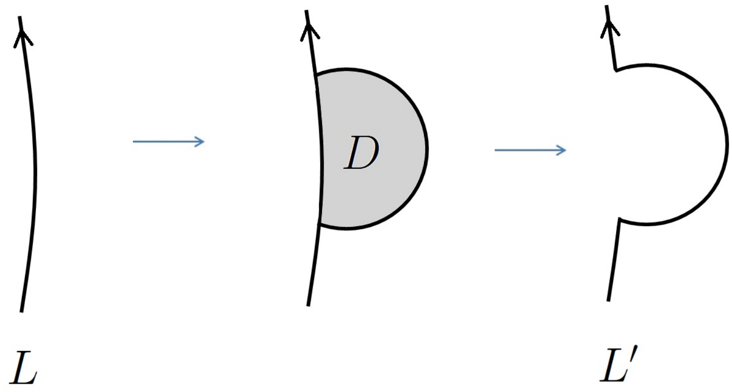


Figure 8: A disk move.

Definition 2.2. Two links L and L' in R^3 belong to *the same combinatorial type* if there is a sequence of links $L_i (i = 0, 1, ..., s+1)$ with $L_0 = L$ and $L_{s+1} = L'$ such that L_{i+1} is obtained from L_i by a disk move for each i.

Two links L and L' in R^3 are said to be *ambient isotopic* if there is an ambient isotopy h_t $(0 \leqq t \leqq 1)$ of R^3 such that h_0 =id and h_1 gives the same topological type $L \cong L'$. They are said to be *ambient isotopic with a compact support* if, in addition, there is a compact subset $X \subset R^3$ such that $h_t(x) = x$ for all $x \in R^3 - X$ and all $t \in [0, 1]$.

We are in a position to explain the equivalence of two links.

Theorem 2.3. For two links L and L' in R^3, the following (1)-(4) are mutually equivalent:

(1) L and L' belong to the same topological type.

(2) L and L' belong to the same combinatorial type.

(3) Any diagrams of L and L' are mutually related by a finite sequence of Redemeister moves I, II, II.

(4) L and L' are ambient isotopic with a compact support.

We say that two links L and L' are *equivalent* if one fo the conditions (1)-(4) on L and L' is stated.

Warping degree

Next, we explin the warping degree of a knot diagram. For an oriented diagram of a knot, the warping degree is defined by Kawauchi.[3] Let b be the point on a knot diagram D which is not any crossing point. We call it a *base point* of D. We denote the pair of D and b by (D, b). A crossing point of (D, b) is a *warping crossing point* if we meet the point first at the under-crossing when we go along the oriented diagram D by starting from b. For example, in Fig. 9, in the oriented knot diagram (D_1, b), q is a warping crossing point of (D_1, b), and p, r and s are non-warping crossing points of (D_1, b). On the other hand, in the knot diagram (D_2, b) with the same base point b and the opposite orientation of the knot diagram (D_1, b), p, r and s are warping crossing points, and q is a non-warping crossing points of (D_2, b). In relation to the warping crossing point, we define the warping degree.

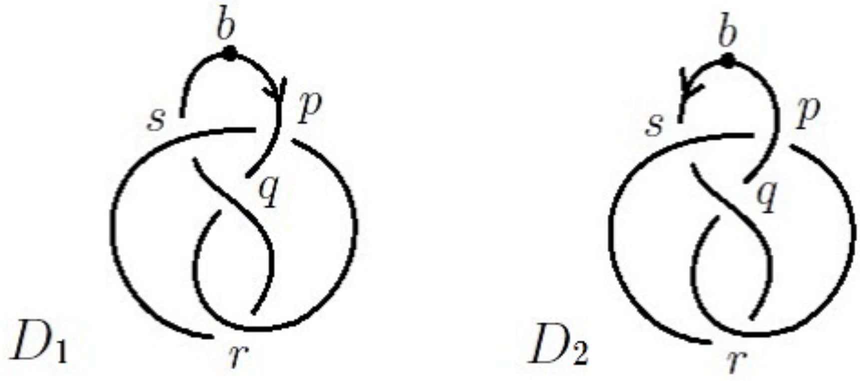


Figure 9:

Definition 2.3. The *warping degree* of (D, b), denoted by $d(D, b)$, is the number of warping crossing points of (D, b). The *warping degree* of D, denoted by $d(D)$, is the minimal warping degree for all base points of D.

For example, in Fig. 10, we have the warping degrees of the oriented knot

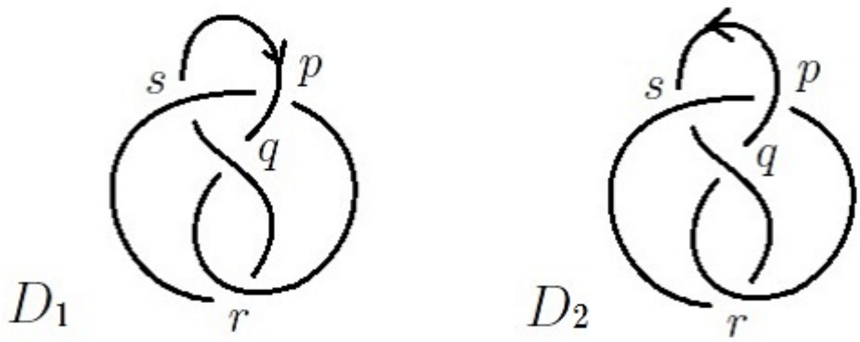


Figure 10:

diagrams $d(D_1) = 1$ and $d(D_2) = 2$.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1, elementary with a direct proof.

Let b be a base point in an edge of an oriented knot diagram D. We first define a Laurent polynomial $\gamma(D,b)$ in variable y, for an oriented based knot diagram (D,b). Then we prove that $\gamma(D,b)$ does not depend on any base point b. To do so, we use the mathematical induction on the crossing number $c(D)$ of a knot diagram D. Let $\gamma(D,b)=1$ when $c(D)=0$. Suppose that $\gamma(D',b')$ is defined for an oriented based knot diagram with $c(D')\leqq n-1$. We consider an oriented based knot diagram (D,b) with $c(D)\leqq n$. After we prove that $\gamma(D,b)$ does not depend on the base point b, the Laurent polynomial $\gamma(D,b)$ is simply denoted by $\gamma(D)$. For an oriented based knot diagram (D,b) with $c(D)\leqq n$, we define $\gamma(D,b)$ by mathematical induction on the warping degree $d(D,b)=m$ of the based knot diagram (D,b). We define the Laurent polynomial $\gamma(D,b)=y^{w(D)}$ when $d(D,b)=0$. Next, assume that $\gamma(D'',b'')$ is defined for an oriented based knot diagram $d(D'',b'')$ with $c(D'')\leqq n$ and the warping degree $d(D'',b'')=m-1$. We consider $\gamma(D,b)$ in the case that $d(D,b)=m\geqq 1$. For any warping crossing point p and the sign $\epsilon(p)=\pm$, the warping degree $d(D_{-\epsilon(p)},b)=m-1$. Here we note that for a link diagram $D=D_1\cup D_2\cup\cdots\cup D_r$ with $c(D)\leqq n-1$, we define

$$\gamma(D)=(y+y^{-1})^{r-1}(-1)^{-\mathrm{Link}(D)}\gamma(D_1)\gamma(D_2)\cdots\gamma(D_r)$$

where the total linking number of a link D, denoted by $\mathrm{Link}(D)$, is defined as follows:

$$\mathrm{Link}(D) = \sum_{1 \leqq i < j \leqq r} \mathrm{Link}(D_i, D_j).$$

Then, we can define the Laurent polynomial $\gamma(D, b)$ in y at a warping crossing point p as follows.

$$\gamma(D, b) = \gamma(D_{\epsilon(p)}, b) = -\gamma(D_{-\epsilon(p)}, b) + \gamma(D_{o(p)}).$$

The following lemmas are needed to reduce the proof of Theorem 1.1.

Lemma 3.1. The Laurent polynomial $\gamma(D, b)$ does not depend on any choices of the warping crossing points.

Lemma 3.2. The Laurent polynomial $\gamma(D, b)$ does not depend on the base point b.

Proof of Lemma 3.1. Let $d(D, b) \geqq 2$. Let p and q be warping crossing points of an oriented based knot diagram (D, b) with $c(D) \leqq n$. The Laurent polynomial $\gamma(D, b)$ at p is given as follows:

$$\gamma(D_{\epsilon(p)}, b) = -\gamma(D_{-\epsilon(p)}, b) + \gamma(D_{o(p)}).$$

In the same way, the Laurent polynomial $\gamma(D, b)$ is given at $q(\neq p)$ as follows.

$$\gamma(D_{\epsilon(q)}, b) = -\gamma(D_{-\epsilon(q)}, b) + \gamma(D_{o(q)}).$$

We show that the Laurent polynomial $\gamma(D_{\epsilon(p)}, b) = \gamma(D_{\epsilon(q)}, b)$. The proof is divided by the following two cases:

(i) The case when q is a self-crossing point after the splice at p.

(ii) The case when q is a non-self-crossing point after the splice at p.

Let $D_{o(q),o(p)}$ denotes the link diagram obtained from D by the splices at p and q.

Proof in the case of (i). The warping crossing point q is a self-crossing point after the splice at p (See Fig. 11).

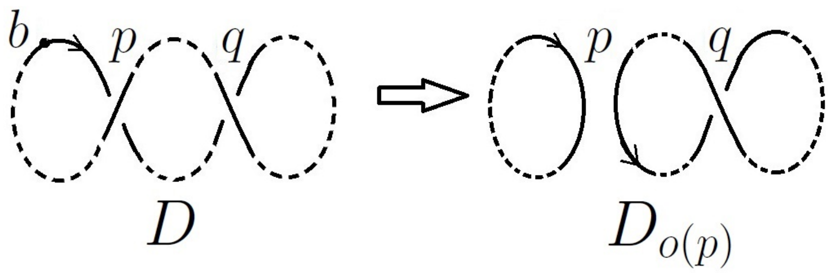


Figure 11:

We first consider the Laurent polynomial $\gamma(D_{o(q)})$. Let D_1 and D_2 be the

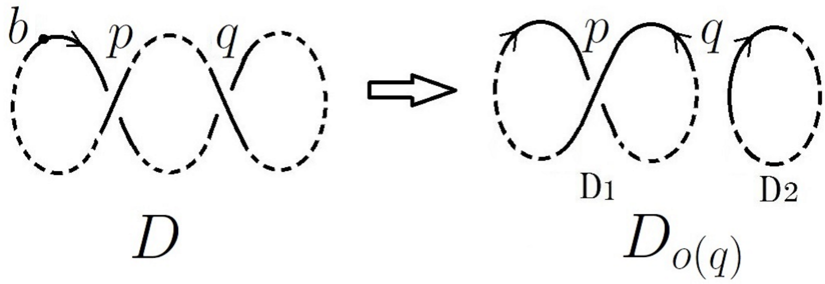


Figure 12:

knot components of the link diagram $D_{o(q)}$ (See Fig. 12). Let p be the warping crossing point of the knot diagram D_1. Since $c(D_i) \leqq n-1\,(i=1,2)$, we have

$$\begin{aligned}\gamma(D_{o(q)}) &= (y+y^{-1})(-1)^{-\mathrm{Link}(D_{o(q)})}\gamma(D_1)\gamma(D_2)\\ &= (y+y^{-1})(-1)^{-\mathrm{Link}(D_{o(q)})}(-\gamma((D_1)_{-\epsilon(p)})+\gamma((D_1)_{o(p)}))\gamma(D_2).\end{aligned}$$

Then we consider the Laurent polynomial $\gamma((D_1)_{o(p)})$. Let $(D_1)_1$ and $(D_1)_2$ be the knot components of the link diagram $(D_1)_{o(p)}$ (See Fig 13). Since

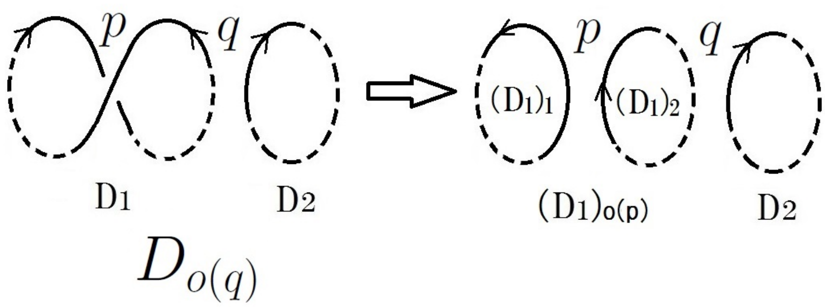


Figure 13:

$c((D_1)_i) \leqq c(D_1)$ $(i = 1, 2)$, we have

$$\gamma((D_1)_{o(p)}) = (y + y^{-1})(-1)^{-\mathrm{Link}((D_1)_{o(p)})}\gamma((D_1)_1)\gamma((D_1)_2).$$

Then, we have

$$\gamma(D_{o(q)}) = -(y + y^{-1})(-1)^{-\mathrm{Link}(D_{o(q)})}\gamma((D_1)_{-\epsilon(p)})\gamma(D_2)$$
$$+ (y + y^{-1})(-1)^{-\mathrm{Link}(D_{o(q)})}(y + y^{-1})(-1)^{-\mathrm{Link}((D_1)_{o(p)})}\gamma((D_1)_1)\gamma((D_1)_2)\gamma(D_2).$$

The right first term of the above identity is equal to the Laurent polynomial $\gamma((D_{o(q)})_{-\epsilon(p)})$ where $(D_{o(q)})_{-\epsilon(p)}$ denotes the link diagram obtained from $D_{o(q)}$ by the crossing change at p. The right second term of the above identity is equal to the Laurent polynomial $\gamma(D_{o(q),o(p)})$. Hence, we have

$$\gamma(D_{o(q)}) = -\gamma((D_{o(q)})_{-\epsilon(p)}) + \gamma(D_{o(q),o(p)}).$$

Since the link diagrams $(D_{o(q)})_{-\epsilon(p)}$ and $(D_{-\epsilon(p)})_{o(q)}$ are equal, we have

$$\gamma((D_{o(q)})_{-\epsilon(p)}) = \gamma((D_{-\epsilon(p)})_{o(q)})$$

and hence

$$\gamma((D_{-\epsilon(p)})_{o(q)}) = \gamma(D_{o(q),o(p)}) - \gamma(D_{o(q)}).$$

Because $d((D_{-\epsilon(p)})_{-\epsilon(q)}) = d(D_{-\epsilon(p)}) - 1$ and $c((D_{-\epsilon(p)})_{o(q)}) = c(D_{-\epsilon(p)}) - 1$, we have

$$
\begin{aligned}
\gamma(D_{\epsilon(p)}, b) &= -\gamma(D_{-}\epsilon(p), b) + \gamma(D_{o(p)}) \\
&= \gamma((D_{-\epsilon(p)})_{-\epsilon(q)}, b) - \gamma((D_{-\epsilon(p)})_{o(q)}) + \gamma(D_{o(p)}) \\
&= \gamma((D_{-\epsilon(p)})_{-\epsilon(q)}, b) - \gamma(D_{o(q),o(p)}) + \gamma(D_{o(q)}) + \gamma(D_{o(p)}).
\end{aligned}
$$

Then we see that $\gamma(D_{\epsilon(p)}, b) = \gamma(D_{\epsilon(q)}, b)$ if we interchange p and q. Hence, the identity $\gamma(D_{\epsilon(p)}, b) = \gamma(D_{\epsilon(q)}, b)$ is proved in the case of (i).

Proof in the case of (ii). The warping crossing point q is a non-self-crossing point by the splice at p (See Fig. 14). We first consider the relations between

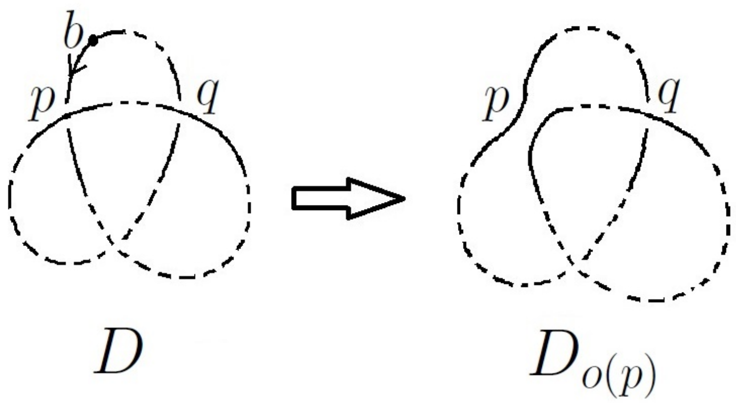


Figure 14:

the two Laurent polynomials $\gamma(D_{o(q)})$ and $\gamma(D_{o(q),-\epsilon(p)})$ (See Fig. 15). Let ℓ and $\ell_{-\epsilon(p)}$ be the linking numbers $\mathrm{Link}(D_{o(q)})$ and $\mathrm{Link}(D_{o(q),-\epsilon(p)})$ of the link diagrams $D_{o(q)}$ and $D_{o(q),-\epsilon(p)}$ respectively. Since $\ell - \ell_{-\epsilon(p)} = \pm 1$ and

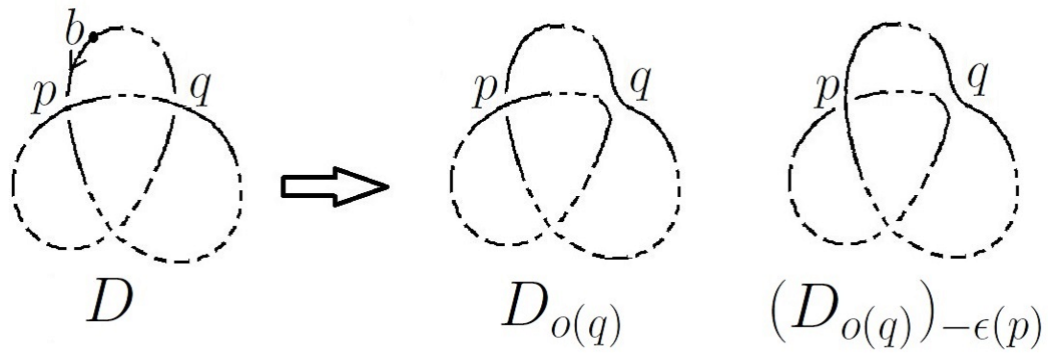


Figure 15:

$\gamma(D_{o(q),-\epsilon(p)}) = \gamma((D_{-\epsilon(p)})_{o(q)})$, we have

$$\gamma(D_{-\epsilon(p),o(q)}) = -\gamma(D_{o(q)}).$$

Since $d((D_{-\epsilon(p)})_{-\epsilon(q)}) = d(D_{-\epsilon(p)}) - 1$ and $c((D_{-\epsilon(p)})_{o(q)}) = c(D_{-\epsilon(p)}) - 1$, the Laurent polynomial $\gamma((D_{-\epsilon(p)})_{-\epsilon(q)}, b))$ is defined and we have

$$\begin{aligned}
\gamma(D_{\epsilon(p)}, b) &= -\gamma(D_{-\epsilon(p)}, b) + \gamma(D_{o(p)}) \\
&= -(-\gamma((D_{-\epsilon(p)})_{-\epsilon(q)}, b) + \gamma((D_{-\epsilon(p)})_{o(q)})) + \gamma(D_{o(p)}) \\
&= \gamma((D_{-\epsilon(p)})_{-\epsilon(q)}, b) + \gamma(D_{o(q)}) + \gamma(D_{o(p)}).
\end{aligned}$$

Then we see that $\gamma(D_{\epsilon(p)}, b) = \gamma(D_{\epsilon(q)}, b)$ if we interchange p and q. Thus, the identity $\gamma(D_{\epsilon(p)}, b) = \gamma(D_{\epsilon(q)}, b)$ is proved in the case of (ii). Hence, Lemma 3.1 is proved. □

Proof of Lemma 3.2. For the crossing point p of an oriented based knot diagram D, let b be a base point in front of the crossing point p, and b' a base point behind the crossing point p (See Fig. 16).

Then we consider the warping crossing points of the oriented based knot

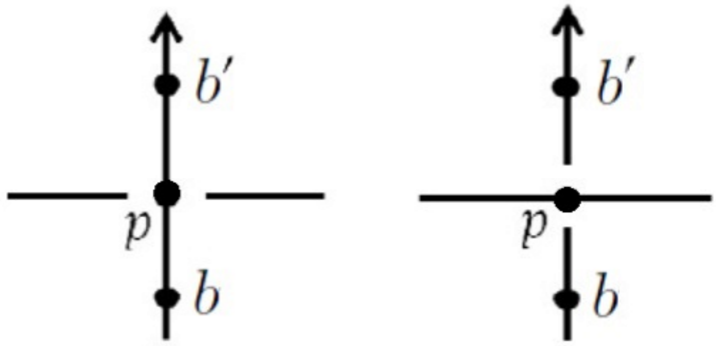


Figure 16:

diagrams (D, b) and (D, b'). The warping crossing points of (D, b) and (D, b') are consistent except the crossing point p. Let q be a warping crossing point of (D, b) except p. Since the warping degree $d(D_{-\epsilon(q)}, b) = d(D, b) - 1$ and the crossing number $c(D_{o(q)}) \leqq c(D) - 1$, we have

$$\gamma(D, b) + \gamma(D_{-\epsilon(q)}, b) = \gamma(D_{o(q)}),$$

$$\gamma(D, b') + \gamma(D_{-\epsilon(q)}, b') = \gamma(D_{o(q)}).$$

Then, we have

$$\gamma(D, b) - \gamma(D, b') = \gamma(D_{-\epsilon(q)}, b') - \gamma(D_{-\epsilon(q)}, b).$$

By continuing this process for the other warping crossing points, we have

$$\gamma(D,b) - \gamma(D,b') = \gamma(D',b') - \gamma(D',b)$$

for the oriented based knot diagram (D',b) without any warping crossing points except the crossing point p. Then we have the following two cases:

(i) The case when p is a warping crossing point of the based knot diagram (D',b).
(ii) The case when p is not a warping crossing point of the based knot diagram (D',b).

We show that the Laurent polynomial $\gamma(D,b) = \gamma(D,b')$ by showing that $\gamma(D',b) = \gamma(D',b')$ in each case. At the first, let D'_1 and D'_2 be the knot components of the link diagram $D'_{o(p)}$ obtained by the splice at p. Since the crossing number $c(D'_1) \leqq c(D') - 1$ and $c(D'_2) \leqq c(D') - 1$, we can define $\gamma(D'_{o(p)})$. Since the knot diagram (D',b) does not have any warping crossing points except the crossing point p, the linking number $\mathrm{Link}(D'_1, D'_2) = 0$.

Since $d(D'_1) = 0$ and $d(D'_2) = 0$, then we have

$$\begin{aligned}\gamma(D'_{o(p)}) &= (y + y^{-1})(-1)^{-\mathrm{Link}(D'_1, D'_2)}\gamma(D'_1)\gamma(D'_2) \\ &= (y + y^{-1})y^{w(D'_1)+w(D'_2)} \\ &= (y + y^{-1})y^{w(D')-\epsilon(p)1} \\ &= y^{w(D')-\epsilon(p)1+1} + y^{w(D')-\epsilon(p)1-1}.\end{aligned}$$

Proof in the case of (i). The positional relations of the base points b and b' between the crossing point p are given in Fig. 17. Then we have $d(D', b) = 1$ and $d(D', b') = 0$.

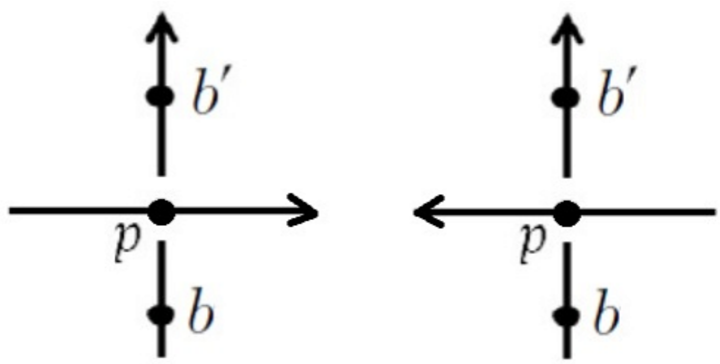


Figure 17:

Since $d(D'_{-\epsilon(p)}, b) = 0$, we have $\gamma(D'_{-\epsilon(p)}, b) = y^{w(D'_{-\epsilon(p)})}$ and

$$\begin{aligned}\gamma(D', b) &= \gamma(D'_{\epsilon(p)}, b)\\ &= -\gamma(D'_{-\epsilon(p)}, b) + \gamma(D'_{o(p)})\\ &= -y^{w(D'_{-\epsilon(p)})} + \gamma(D'_{o(p)})\\ &= -y^{w(D')-\epsilon(p)2} + y^{w(D')-\epsilon(p)1+1} + y^{w(D')-\epsilon(p)1-1}.\end{aligned}$$

Then, in spite of the sign $\epsilon(p) = \pm$ of the crossing point p, we have

$$\gamma(D', b) = y^{w(D')}.$$

On the other hand, since the warping degree $d(D', b') = 0$, we have $\gamma(D', b') = y^{w(D')}$. Then we have $\gamma(D', b) = \gamma(D', b')$, so that $\gamma(D, b) = \gamma(D, b')$.

Proof in the case of (ii). The positional relations between the crossing point p, the two base points b and b' is as follows (See Fig. 18). Then we have $d(D', b) = 0$ and $d(D', b') = 1$. Since $d(D'_{-\epsilon(p)}, b') = 0$, we have $\gamma(D'_{-\epsilon(p)}, b') =$

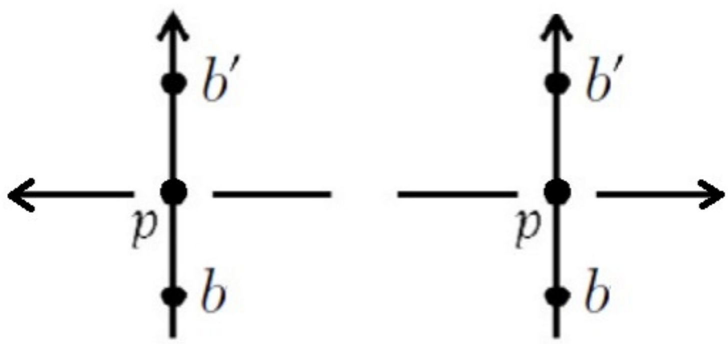


Figure 18:

$y^{w(D'_{-\epsilon(p)})}$ and

$$\begin{aligned}\gamma(D', b') &= \gamma(D'_{\epsilon(p)}, b') \\ &= -\gamma(D'_{-\epsilon(p)}, b') + \gamma(D'_{o(p)}) \\ &= -y^{w(D'_{-\epsilon(p)})} + \gamma(D'_{o(p)}) \\ &= -y^{w(D')-\epsilon(p)2} + y^{w(D')-\epsilon(p)1+1} + y^{w(D')-\epsilon(p)1-1}.\end{aligned}$$

Then, in spite of the sign $\epsilon(p) = \pm$ of the crossing point p, we have

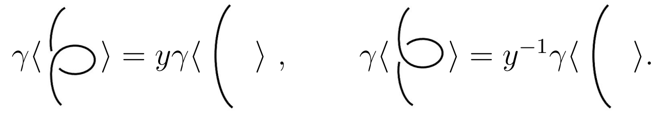
$$\gamma(D', b) = y^{w(D')}.$$

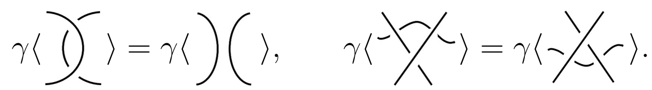
On the other hand, since the warping degree $d(D', b') = 0$, we have $\gamma(D', b') = y^{w(D')}$. Then we have $\gamma(D', b) = \gamma(D', b')$, so that $\gamma(D, b) = \gamma(D, b')$. $\square$

By Lemma 3.2, the Laurent polynomial $\gamma(D, b)$ of the oriented based knot diagram (D, b) is simply denoted by $\gamma(D)$. We return the proof of Theorem 1.1. We show the following Lemma 3.3:

Lemma 3.3. The Laurent polynomial $\gamma(D)$ defined for a knot diagram D with $c(D) \leqq n$ has the following three properties.

(i) The following equalities hold on Reidemeister Moves I, II, III.





(*ii*) If $c(D) = 0$, then $\gamma(D) = 1$.

(*iii*) For the skein triple (D_+, D_-, D_0) with a knot diagram $D_\pm$, we have

$$\gamma(D_+) + \gamma(D_-) = (y + y^{-1})(-1)^{-\mathrm{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2)$$

for D_1 and D_2 are the knot component diagrams of the link diagram D_0.

Proof of Lemma 3.3. The property (*ii*) is direct from the definition. Next, we show the property (*iii*). Let p be a warping crossing point of the oriented knot diagram D. Let D_1 and D_2 be the knot component diagrams of the link diagram$D_{o(p)}$. When the sign $\epsilon(p) = +$, we have the following Laurent polynomial by the definition.

$$\gamma(D_+) = -\gamma(D_-) + (y + y^{-1})(-1)^{-\mathrm{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

When the sign $\epsilon(p) = -$, we have the following Laurent polynomial by the definition.

$$\gamma(D_-) = -\gamma(D_+) + (y + y^{-1})(-1)^{-\mathrm{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

Thus, for the skein triple (D_+, D_-, D_0) of a knot diagram D, we have the following equality.

$$\gamma(D_+) + \gamma(D_-) = (y + y^{-1})(-1)^{-\mathrm{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

Let D' be a knot diagram which is applied Reidemeister move I, II or III to an oriented based knot diagram (D, b). We show the property (i) by using the mathematical induction on the warping degree $d(D', b) = m$ and the crossing point $c(D', b) = n$ of an oriented based knot diagram (D', b). Suppose that we have the equalities of Lemma 3.3.(i) on Reidemeister moves I, II, III when $c(D', b) \leqq n - 1$.

Proof on Reidemeister Move I. Let D' be a knot diagram which is applied Reidemeister move I to an oriented based knot diagram (D, b). Since Lemma 3.2, we take a base point b and a crossing point p as in Fig. 19. Then the warping crossing points of (D, b) and (D', b) are consistent. When the

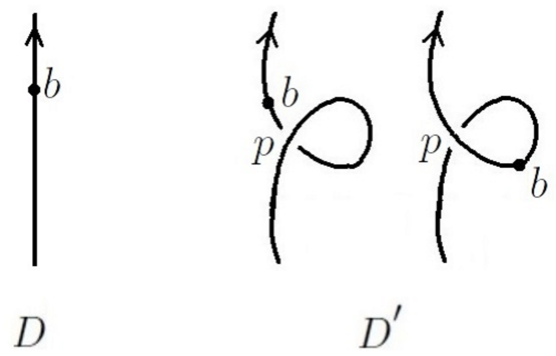
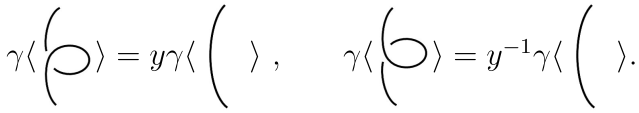
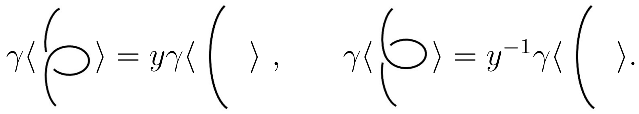


Figure 19:

crossing number $c(D', b) = 1$, we have $\gamma(D', b) = y^{\epsilon(p)}$ and $\gamma(D, b) = 1$. Then in the case that $\epsilon(p) = +$, we have $\gamma(D', b) = y\gamma(D, b)$. In the case that $\epsilon(p) = -$, we have $\gamma(D', b) = y^{-1}\gamma(D, b)$. Next, let the crossing number $c(D', b) = n \geqq 2$. Let the warping degree $d(D', b) = m = 0$. Then $\gamma(D', b) = y^{w(D')}$. Since $d(D, b) = 0$, $\gamma(D, b) = y^{w(D)}$. When the sign $\epsilon(p) = +$, we have $w(D') = w(D) + 1$ and $\gamma(D', b) = y\gamma(D, b)$, so that we have $\gamma(D') = y\gamma(D)$. When the sign $\epsilon(p) = -$, we have $w(D') = w(D) - 1$ and $\gamma(D', b) = y^{-1}\gamma(D, b)$, so that we have $\gamma(D') = y\gamma(D)$. Thus, we have



Let the warping degree $d(D', b) = m \geqq 1$. Suppose that we have the following equality on Reidemeister Move I for the oriented based knot diagram (D', b) with the warping degree $d(D', b) \leqq m-1$ or the crossing point $c(D', b) \leqq n-1$.



Let q be a warping crossing point of the oriented based knot diagram (D', b). Let D'_1 and D'_2 be the knot component diagrams of the link diagram $D'_{o(q)}$ and D'_1 has the crossing point by Reidemeister Move I. Then we have

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D'_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\mathrm{Link}(D'_1, D'_2)}\gamma(D'_1)\gamma(D'_2).$$

Then we show that the equality on Reidemeister Move I holds for the oriented based knot diagram (D', b) with the warping degree $d(D', b) = m \geqq 1$ in the case that $\epsilon(p) = \pm$, respectively. In the case that $\epsilon(p) = +$, the warping degree $d(D'_{-\epsilon(q)}, b) = m-1$, the crossing number $c(D'_i) \leqq n-1$ and $\text{Link}(D'_1, D_2) = \text{Link}(D_1, D_2)$, we have $\gamma(D'_{-\epsilon(q)}, b) = y\gamma(D_{-\epsilon(q)}, b)$ and

$$(y+y^{-1})(-1)^{\text{Link}(D'_1,D_2)}\gamma(D'_1)\gamma(D_2) = (y+y^{-1})(-1)^{\text{Link}(D_1,D_2)}y\gamma(D_1)\gamma(D_2).$$

Then we have

$$\gamma(D', b) = y\gamma(D_{-\epsilon(q)}, b) + y(y+y^{-1})(-1)^{\text{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

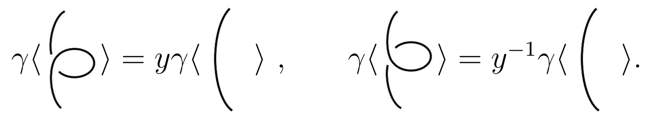
Thus, we have $\gamma(D', b) = y\gamma(D, b)$. In the case that $\epsilon(p) = -$, the warping degree $d(D'_{-\epsilon(q)}, b) = m-1$, the crossing number $c(D'_i) \leqq n-1$ and $\text{Link}(D'_1, D_2) = \text{Link}(D_1, D_2)$, we have $\gamma(D'_{-\epsilon(q)}, b) = y^{-1}\gamma(D_{-\epsilon(q)}, b)$ and

$$(y+y^{-1})(-1)^{\text{Link}(D'_1,D_2)}\gamma(D'_1)\gamma(D_2) = (y+y^{-1})(-1)^{\text{Link}(D_1,D_2)}y^{-1}\gamma(D'_1)\gamma(D_2).$$

Then we have

$$\gamma(D', b) = y^{-1}\gamma(D_{-\epsilon(q)}, b) + y^{-1}(y+y^{-1})(-1)^{\text{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2))$$

Thus, we have $\gamma(D', b) = y^{-1}\gamma(D, b)$ and



Proof on Reidemeister Move II. Let D' be a knot diagram which is applied Reidemeister move II to an oriented based knot diagram (D, b). We take a base point b and a crossing point p_1, p_2 as in Fig. 20. Then the warping crossing points of (D, b) and (D', b) are consistent. In spite of that the direction

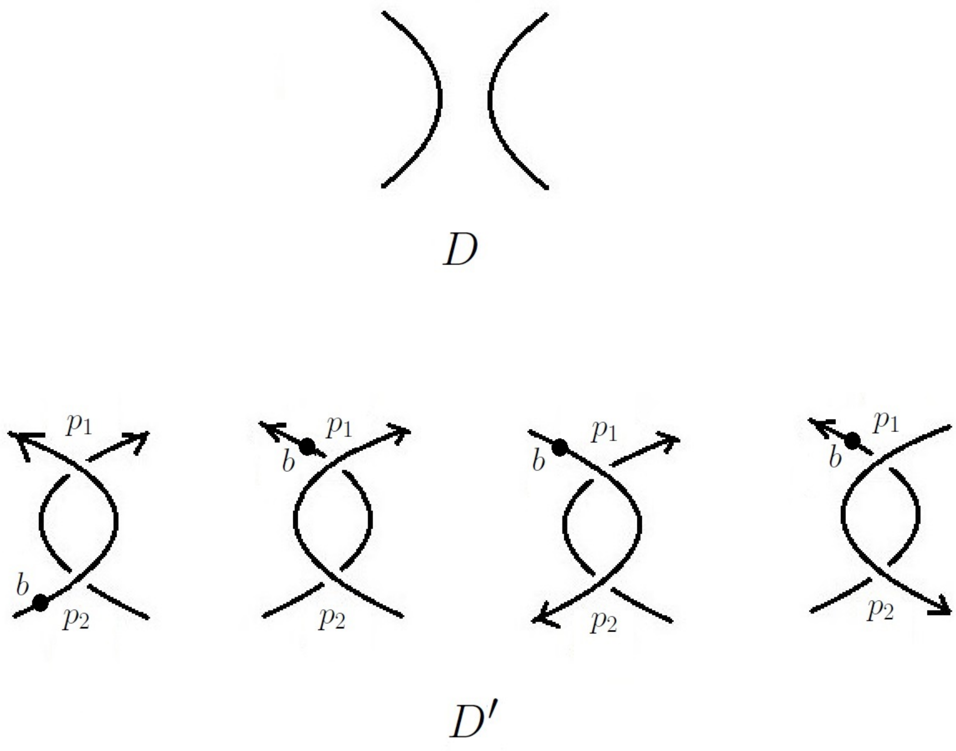


Figure 20:

of the two strings of the knot diagram D' has 4 patterns as in Fig. 20, we have $\epsilon(p_1)+\epsilon(p_2) = 0$ in every pattern. Then we have $w(D') = w(D)$. When the crossing number $c(D', b) = 2$, we have $\gamma(D', b) = 1$. Since $c(D, b) = 0$, we have $\gamma(D, b) = 1$. So that we have $\gamma(D', b) = \gamma(D, b)$. Next, let the crossing number $c(D', b) = n \geqq 3$. Let the warping degree $d(D', b) = m = 0$. Then $\gamma(D', b) = y^{w(D')} = y^{w(D)}$. Since $d(D, b) = 0$, $\gamma(D, b) = y^{w(D)}$. Thus, we have

$\gamma(D', b) = \gamma(D, b)$. Let the warping degree $d(D', b) = m \geqq 1$. Suppose that we have $\gamma(D', b) = \gamma(D, b)$ on Reidemeister Move II for an oriented based knot diagram (D', b) with the warping degree $d(D', b) \leqq m - 1$ or the crossing point $c(D', b) \leqq n - 1$. Let q be a warping crossing point of the oriented based knot diagram (D', b). Let D'_1 and D'_2 be the knot component diagrams of the link diagram $D'_{o(q)}$. Then we have

$$\gamma(D'_{-\epsilon(q)}, b) = -\gamma(D'_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\mathrm{Link}(D'_1, D'_2)}\gamma(D'_1)\gamma(D'_2).$$

Since $d(D'_{-\epsilon(q)}, b) = m - 1$, we have $\gamma(D'_{-\epsilon(q)}, b) = \gamma(D_{-\epsilon(q)}, b)$. In the case that p_1 and p_2 are non-self-crossing points, we have $\mathrm{Link}(D'_1, D'_2) = \mathrm{Link}(D_1, D_2)$. Since the crossing number $c(D'_i) \leqq n - 1$, we have $\gamma(D'_i) = \gamma(D_i)$ and

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\mathrm{Link}(D_1, D_2)}\gamma(D_1)\gamma(D_2).$$

In the case that p_1 and p_2 are self-crossing points of a knot component diagram D_i of the link diagram $D'_{o(q)}$, we have $\mathrm{Link}(D'_1, D'_2) = \mathrm{Link}(D_1, D_2)$. Since the crossing number $c(D'_i) \leqq n - 1$, we have $\gamma(D'_i) = \gamma(D_i)$ and

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\mathrm{Link}(D_1, D_2)}\gamma(D_1)\gamma(D_2).$$

Thus, we have $\gamma(D', b) = \gamma(D, b)$

Proof on Reidemeister Move III. Let D' be a knot diagram which is ap-

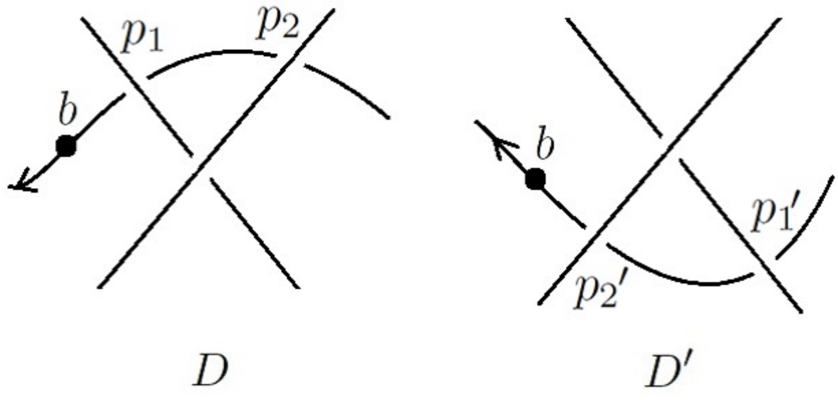


Figure 21:

plied Reidemeister move III to an oriented based knot diagram (D, b). Since Lemma 3.2, we take a base point b and crossing points $p_i, p'_i\,(i = 1, 2)$ as in Fig. 21. Then the warping crossing points of (D, b) and (D', b) are consistent and $\epsilon(p_1) = \epsilon(p'_1)$, $\epsilon(p_2) = \epsilon(p'_2)$.Thus we have $w(D') = w(D)$. When the crossing number $c(D', b) = 3$, we have $\gamma(D', b) = \gamma(D, b)$. Next, let the crossing number $c(D', b) = n \geqq 4$. If the warping degree $d(D', b) = m = 0$, then $\gamma(D', b) = y^{w(D')} = y^{w(D)}$. Since $d(D, b) = 0$, we have $\gamma(D, b) = y^{w(D)}$. Thus, we obtain $\gamma(D', b) = \gamma(D, b)$. Let the warping degree $d(D', b) = m \geqq 1$. Suppose that we have $\gamma(D', b) = \gamma(D, b)$ on Reidemeister Move III for an oriented based knot diagram (D', b) with the warping degree $d(D', b) \leqq m - 1$. Let q be a warping crossing point of the oriented based knot diagram (D', b). Let D'_1 and D'_2 be the knot component diagrams of the link diagram $D'_{o(q)}$. Then we have

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D'_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\mathrm{Link}(D'_1, D'_2)}\gamma(D'_1)\gamma(D'_2).$$

Since $d(D'_{-\epsilon(q)}, b) = m - 1$, we have $\gamma(D'_{-\epsilon(q)}, b) = \gamma(D_{-\epsilon(q)}, b)$. In the case that p_1 and p_2 are non-self-crossing points, we have $\text{Link}(D'_1, D'_2) = \text{Link}(D_1, D_2)$. Since the crossing number $c(D'_i) \leqq n - 1$, we have $\gamma(D'_i) = \gamma(D_i)$ $(i = 1, 2)$ and

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\text{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

In the case that p_1 and p_2 are self-crossing points of a knot component diagram D_i of the link diagram $D'_{o(q)}$, we have $\text{Link}(D'_1, D'_2) = \text{Link}(D_1, D_2)$. Because the crossing number $c(D'_i) \leqq n-1$, we have $\gamma(D'_i) = \gamma(D_i)$ $(i = 1, 2)$ and

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\text{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

In the case that one of p_i $(i = 1, 2)$ is a non-self-crossing point and the other is a self-crossing point of a knot component D_i of the link diagram $D'_{o(q)}$, we have $\text{Link}(D'_1, D'_2) = \text{Link}(D_1, D_2)$. Since the crossing number $c(D'_i) \leqq n-1$, we have $\gamma(D'_i) = \gamma(D_i)$ $(i = 1, 2)$ and

$$\gamma(D'_{\epsilon(q)}, b) = -\gamma(D_{-\epsilon(q)}, b) + (y + y^{-1})(-1)^{\text{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).$$

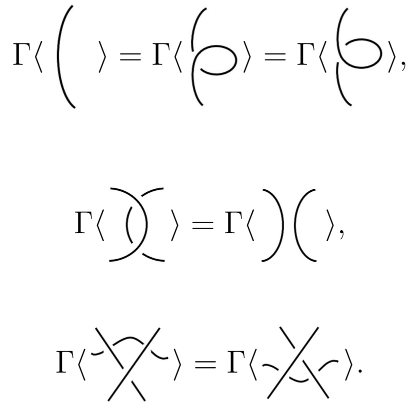
Thus, we have $\gamma(D', b) = \gamma(D, b)$. □

The proof of Theorem 1.1 is completed. □

We have the following corollary showing a reconstruction of the γ-polynomial stated in the introduction:

Corollary 3.4. For a knot diagram D, there is a Laurent polynomial $\Gamma(D;x)$ in x with the following three properties $(i), (ii)$ and (iii).

(i) The Laurent polynomial $\Gamma(D)$ is invariant on Reidemeister moves $\mathrm{I}, \mathrm{II}, \mathrm{III}$.



(ii) If $c(D) = 0$, then $\Gamma(D) = 1$.

(iii) For the skein triple (D_+, D_-, D_0) with a knot diagram $D_\pm$, we have

$$-x\Gamma(D_+) + \Gamma(D_-) = (1-x)x^{-\mathrm{Link}(D_1,D_2)}\Gamma(D_1)\Gamma(D_2)$$

for the knot component diagrams D_1 and D_2 in the link diagram D_0.

Proof of Corollary 3.4. Let $\Gamma'(D) = y^{-w(D)}\gamma(D)$ be the Laurent polynomial in y for a knot diagram D. We show the following lemma:

Lemma 3.5. For a knot diagram D, the Laurent polynomial $\Gamma'(D;y)$ has the following three properties $(i)', (ii)'$ and $(iii)'$.

$(i)'$ The Laurent polynomial $\Gamma'(D)$ is invariant on Reidemeister moves I, II, III.
$(ii)'$ If $c(D) = 0$, then $\Gamma'(D) = 1$.
$(iii)'$ For the skein triple (D_+, D_-, D_0) with a knot diagram $D_\pm$, we have

$$-y^2\Gamma'(D_+) - \Gamma'(D_-) = (-y^2 - 1)(-y^2)^{-\mathrm{Link}(D_1,D_2)}\Gamma'(D_1)\Gamma'(D_2)$$

for the knot component diagrams D_1 and D_2 in the link diagram D_0.

Proof of Lemma 3.5. We show $(i)'$. On Reidemeister moves II and III, $\Gamma'(D)$ is invariant since $\gamma(D)$ is invariant. Then we show the invariance on Reidemeister move I. Let D' and D'' be the knot diagrams which is applied Reidemeister move I to a knot diagram D as in Fig. 22. Then we have

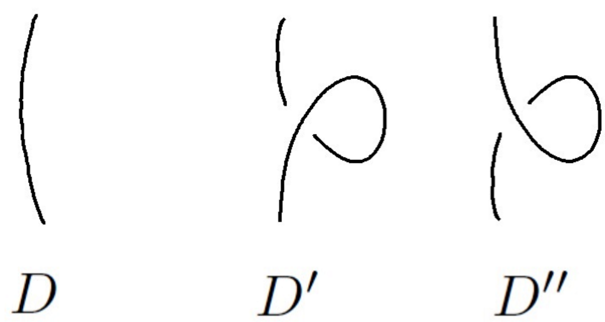


Figure 22:

$\Gamma'\langle$  $\rangle$ and $\Gamma'\langle$  $\rangle$ as follows.

$$\Gamma'\langle$$  $$\rangle = \Gamma'(D')$$

$$= y^{-w(D')}\gamma(D')$$

$$= y^{-(w(D)+1)}y\gamma(D)$$

$$= y^{-w(D)}y^{-1}y\gamma(D)$$

$$= y^{-w(D)}\gamma(D)$$

$$= \Gamma'(D).$$

$$\Gamma'\langle$$  $$\rangle = \Gamma'(D'')$$

$$= y^{-w(D'')}\gamma(D'')$$

$$= y^{-(w(D)-1)}y^{-1}\gamma(D)$$

$$= y^{-w(D)}yy^{-1}\gamma(D)$$

$$= y^{-w(D)}\gamma(D)$$

$$= \Gamma'(D).$$

Thus, the Laurent polynomial $\Gamma'(D)$ is invariable on Reidemeister Moves I, completing the proof of $(i)'$. The property $(ii)'$ follows from definition. We show $(iii)'$. Using the identity $\gamma(D_+) + \gamma(D_-) = \gamma(D_o)$, we have

$$y^{-w(D_-)}\gamma(D_+) + y^{-w(D_-)}\gamma(D_-) = y^{-w(D_-)}\gamma(D_o).$$

Since

$$w(D_-) = w(D_+) - 2,\ w(D_0) = w(D_1) + w(D_2) + 2\text{Link}(D_1, D_2)$$

and $w(D_-) = w(D_0) - 1$, we have

$$\begin{aligned}&y^{-w(D_+)+2}\gamma(D_+) + y^{-w(D_-)}\gamma(D_-)\\&= y^{-w(D_1)-w(D_2)-2\text{Link}(D_1,D_2)+1}(y + y^{-1})(-1)^{-\text{Link}(D_1,D_2)}\gamma(D_1)\gamma(D_2).\end{aligned}$$

Noting that $\Gamma'(D) = y^{-w(D)}\gamma(D)$, we have

$$\begin{aligned}y^2\Gamma'(D_+) + \Gamma'(D_-) &= y(y + y^{-1})y^{-2\text{Link}(D_1,D_2)}(-1)^{-\text{Link}(D_1,D_2)}\Gamma'(D_1)\Gamma'(D_2)\\&= (y^2 + 1)(-y^2)^{-\text{Link}(D_1,D_2)}\Gamma'(D_1)\Gamma'(D_2),\end{aligned}$$

completing the proof of $(iii)'$. Thus, the proof of Lemma 3.5 is completed.
□

Let $x = -y^2$ and $\Gamma(D; x) = \Gamma'(D; y)$. Thus, $\Gamma(D) = \Gamma(D; x)$ is a Laurent polynomial in x with the properties $(i), (ii)$ and (iii). Hence the proof of Corollary 3.4 is completed. □

If D is a diagram of a knot K, then we denote $\Gamma(D) = \Gamma(D; x)$ by $\Gamma(K) =$

$\Gamma(K;x)$ as an invariant of K, and call $\Gamma(D)=\Gamma(K)$ the Γ-polynomial of K. Let $D=D_1\cup D_2\cup\cdots\cup D_r$ be a diagram of a link $L=K_1\cup K_2\cup\cdots\cup K_r$ where D_i is a diagram of a knot component $K_i\,(i=1,2,\ldots,r)$. Then we define

$$\Gamma(D)=(1-x)^{r-1}x^{-\mathrm{Link}(D)}\Gamma(D_1)\Gamma(D_2)\cdots\Gamma(D_r)$$

which is an invariant of L and denoted by $\Gamma(L)$. We have the following corollary:

Corollary 3.6. For the skein triple (D_+,D_-,D_0) on a self-crossing point of a link diagram $D=D_1\cup D_2\cup\cdots\cup D_r\,(r\geqq 2)$, we have

$$-x\Gamma(D_+)+\Gamma(D_-)=\Gamma(D_0).$$

On the other hand, for the skein triple (D_+,D_-,D_0) on a non-self-crossing point of a link diagram $D=D_1\cup D_2\cup\cdots\cup D_r\,(r\geqq 2)$, we have

$$x\Gamma(D_+)=\Gamma(D_-).$$

Proof of Corollary 3.6. Let $D_\pm=D_1\cup\cdots\cup D_{i_\pm}\cup\cdots\cup D_r$ and $D_0=D_1\cup\cdots\cup D_{i_0}\cup\cdots\cup D_r$ be the link diagrams which is applied to a skein triple (D_+,D_-,D_0) on a self-crossing point of a knot component diagram D_i of a link diagram $D=D_1\cup D_2\cup\cdots\cup D_r$. Then $x\Gamma(D_+)$ and $\Gamma(D_-)$ are given as follows:

$$x\Gamma(D_+) = x(1-x)^{r-1}x^{-\mathrm{Link}(D)}\Gamma(D_1)\cdots\Gamma(D_{i_+})\cdots\Gamma(D_r),$$
$$\Gamma(D_-) = (1-x)^{r-1}x^{-\mathrm{Link}(D)}\Gamma(D_1)\cdots\Gamma(D_{i_-})\cdots\Gamma(D_r).$$

Thus we have

$$\begin{aligned}-x\Gamma(D_+)+\Gamma(D_-) &= (1-x)^{r-1}x^{-\mathrm{Link}(D)}\Gamma(D_1)\cdots(-x\Gamma(D_{i_+})+\Gamma(D_{i_-}))\cdots\Gamma(D_r)\\ &= (1-x)^{r-1}x^{-\mathrm{Link}(D)}\Gamma(D_1)\cdots\Gamma(D_{i_0})\cdots\Gamma(D_r).\end{aligned}$$

Let D_{i_1} and D_{i_2} be the knot component diagrams of the link diagram D_{i_0}. Since

$$\Gamma(D_{i_0}) = \Gamma(D_{i_1})\cup\Gamma(D_{i_2}) = (1-x)x^{-\mathrm{Link}(D_{i_1},D_{i_2})}\Gamma(D_{i_1})\Gamma(D_{i_2}),$$

we have

$$\begin{aligned}&-x\Gamma(D_+)+\Gamma(D_-)\\ &= (1-x)^{r-1}x^{-\mathrm{Link}(D)-\mathrm{Link}(D_{i_1},D_{i_2})}\Gamma(D_1)\cdots\Gamma(D_{i_1})\Gamma(D_{i_2})\cdots\Gamma(D_r)\\ &= (1-x)^{(r+1)-1}x^{-\mathrm{Link}(D_{i_0})}\Gamma(D_1)\cdots\Gamma(D_{i_1})\Gamma(D_{i_2})\cdots\Gamma(D_r)\\ &= \Gamma(D_0).\end{aligned}$$

For the skein triple (D_+, D_-, D_0) on a non-self-crossing point of a link diagram $D = D_1 \cup \cdots \cup D_r (r \geqq 2)$, we have

$$\Gamma(D_+) = (1-x)^{r-1} x^{-\mathrm{Link}(D_+)} \Gamma(D_1) \cdots \Gamma(D_{i_+}) \cdots \Gamma(D_r),$$
$$\Gamma(D_-) = (1-x)^{r-1} x^{-\mathrm{Link}(D_-)} \Gamma(D_1) \cdots \Gamma(D_{i_-}) \cdots \Gamma(D_r).$$

Since $\mathrm{Link}(D_+) = \mathrm{Link}(D_-) + 1$, we have

$$x\Gamma(D_+) = \Gamma(D_-).$$

Then, Corollary 3.6 is proved. □

4 A Generalization of the Γ-polynomial to a 2-string tangle

In this section, we generalize the Γ-polynomial $\Gamma(D)$ of a knot diagram D to a 2-string tangle diagram D.

For a 2-string tangle diagram $D(T)$ and a complementary tangle diagram P introduced in Section 1, we note that the patterns of the connections of the endpoints between $D(T)$ and P are uniquely determined, respectively as it is shown in Fig. 23. Let $D(T_0)$ be the oriented diagram which obtained from

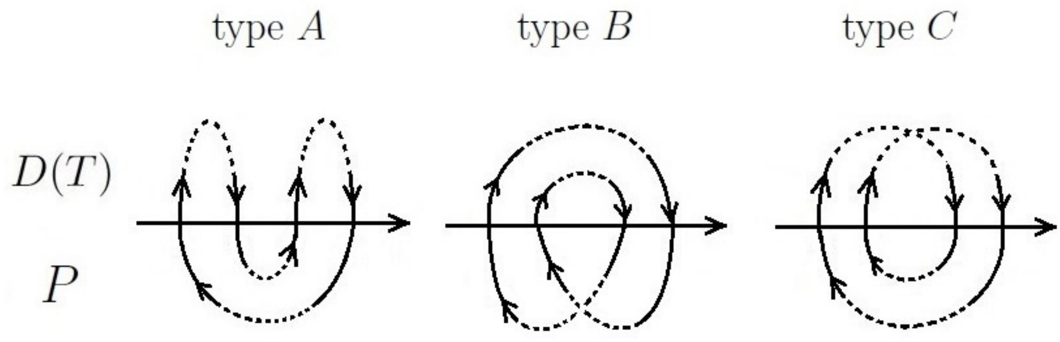


Figure 23: Knot diagrams $D(T) \cup P$.

$D(T)$ by the splice at a crossing point between the string diagrams $D(t_1)$ and $D(t_2)$ of $D(T)$. For the oriented knot diagram $D(T) \cup P$, we have an oriented link diagram $D(T_0) \cup P$ depending on the type A, B or C, respectively as in Fig 24. For an oriented tangle diagram $D(T)$ of type A, B or C, let

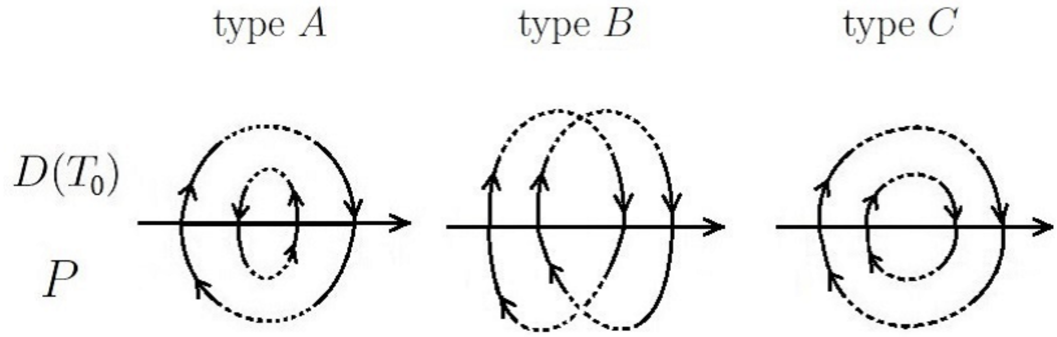
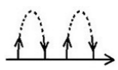
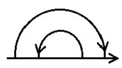
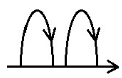
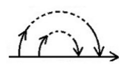
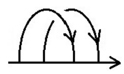
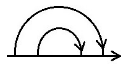
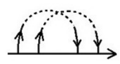
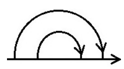
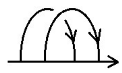


Figure 24: Link diagrams $D(T_0) \cup P$.

$X_i\,(i = 0, 1)$ be the following oriented tangle diagrams, respectively.

type A : $D(T) =$  , $X_0 =$  , $X_1 =$  .

type B : $D(T) =$  , $X_0 =$  , $X_1 =$  .

type C : $D(T) =$  , $X_0 =$  , $X_1 =$  .

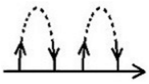
As it is shown in Section 1, for the type A, B or C of $D(T)$, the Γ-polynomial $\Gamma(D(T) \cup P)$ can be expressed as follows by applying the skein relation of the Γ-polynomial by induction on warping degree of $D(T)$:

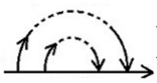
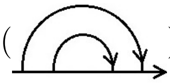
$$\Gamma(D(T) \cup P) = f_0(x)\Gamma(X_0 \cup P) + f_1(x)\Gamma(X_1 \cup P).$$

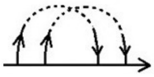
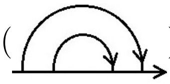
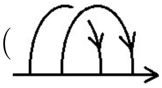
This well-definedness is shown soon later (Corollary 1.3). Then we denote it by

$$\Gamma(D(T)) = f_0(x)\Gamma(X_0) + f_1(x)\Gamma(X_1),$$

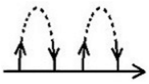
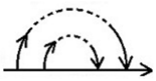
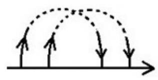
which means as follows:

type A : $\Gamma($  $) = f_0(x)\Gamma($  $) + f_1(x)\Gamma($  $)$.

type B : $\Gamma($  $) = f_0(x)\Gamma($  $) + f_1(x)\Gamma($  $)$.

type C : $\Gamma($  $) = f_0(x)\Gamma($  $) + f_1(x)\Gamma($  $)$.

For a tangle diagram $D(T)$ of type A, B or C and $i = 0, 1$, let P_i be the complementary tangle diagram described in Fig. 3. Here, we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. To prove Theorem 1.2, we first consider the three types, A : , B :  and C :  with the same orientation of the boundary circle of the disk of an oriented tangle diagram $D(T)$ as in Fig.23. On the two oriented tangle diagrams $D(T)$ and $D(T')$ of the type A, B, or C, we have $\Gamma(D(T) \cup P_0)$ and $\Gamma(D(T') \cup P_0)$ as follows:

$$\Gamma(D(T) \cup P_0) = f_0(x)\Gamma(X_0 \cup P_0) + f_1(x)\Gamma(X_1 \cup P_0),$$

$$\Gamma(D(T') \cup P_0) = f'_0(x)\Gamma(X_0 \cup P_0) + f'_1(x)\Gamma(X_1 \cup P_0).$$

On the other hand, we have $\Gamma(D(T) \cup P_1)$ and $\Gamma(D(T') \cup P_1)$ as follows:

$$\Gamma(D(T) \cup P_1) = f_0(x)\Gamma(X_0 \cup P_1) + f_1(x)\Gamma(X_1 \cup P_1),$$

$$\Gamma(D(T') \cup P_1) = f'_0(x)\Gamma(X_0 \cup P_1) + f'_1(x)\Gamma(X_1 \cup P_1).$$

Since $\Gamma(D(T) \cup P_0) = \Gamma(D(T') \cup P_0)$ and $\Gamma(D(T) \cup P_1) = \Gamma(D(T') \cup P_1)$, we have

$$(f_0(x) - f_0'(x))\Gamma(X_0 \cup P_0) + (f_1(x) - f_1'(x))\Gamma(X_1 \cup P_0) = 0,$$
$$(f_0(x) - f_0'(x))\Gamma(X_0 \cup P_1) + (f_1(x) - f_1'(x))\Gamma(X_1 \cup P_1) = 0.$$

Noting that

$$\Gamma(X_1 \cup P_0) = \Gamma(X_1 \cup P_1) = 1,$$

$$\Gamma(X_0 \cup P_0) = (1 - x) \quad \textit{and} \quad \Gamma(X_0 \cup P_1) = (1 - x)x^{-1},$$

we have

$$f_1(x) - f_1'(x) = (f_0(x) - f_0'(x))(1 - x),$$
$$f_1(x) - f_1'(x) = (f_0(x) - f_0'(x))(1 - x)x^{-1}.$$

Then we have

$$(f_0(x) - f_0'(x))(1 - x)(1 - x^{-1}) = 0$$

Since this equality holds for any x, we have $f_0(x) = f_0'(x)$ and then $f_1(x) = f_1'(x)$. □

Lemma 4.1. The Γ-polynomial $\Gamma(D(T))$ of a tangle diagram does not depend on a choice of orientations of the boundary circle of the disk underlying the tangle diagram.

Proof of Lemma 4.1. As in Fig. 25, let $D(T^{-})$ be the oriented tangle diagrams whose orientations of the boundary circles are the opposite orientations of the oriented tangle diagrams $D(T)$. For the type A, B or C of the

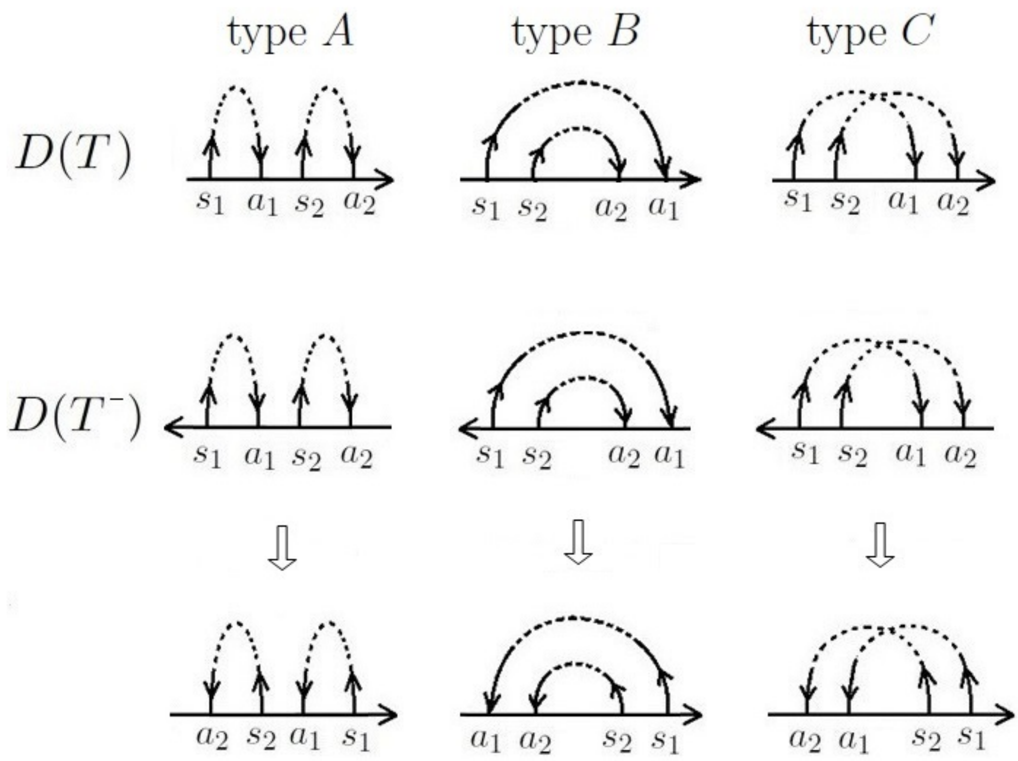
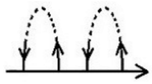
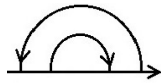
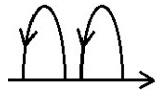
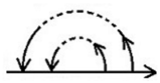
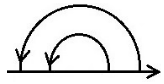


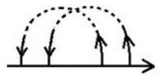
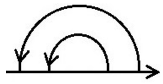
Figure 25: The tangle diagrams $D(T^{-})$ for the type A, B and C.

tangle diagram $D(T)$, the tangle diagram $D(T^{-})$ is consistent with the tangle diagram obtained by the 180^{o} rotation of the tangle T except the boundary circle. For the type A, B or C of the tangle diagram $D(T^{-})$, let X_i^{-} $(i = 1, 0)$

be the following tangle diagrams.

type $A : D(T^-) =$  , $X_0^- =$  , $X_1^- =$  .

type $B : D(T^-) =$  , $X_0^- =$  , $X_1^- =$  .

type $C : D(T^-) =$  , $X_0^- =$  , $X_1^- =$  .

For the type A, B or C of a tangle diagram $D(T^-)$, let P_0^- be the tangle diagrams as in Fig. 26.

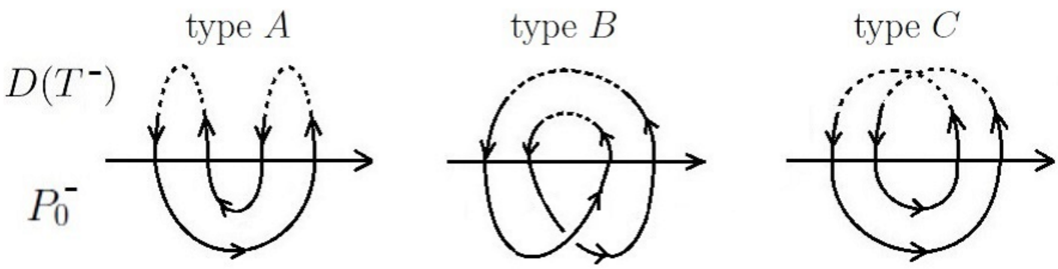


Figure 26: The complementary tangle diagrams P_0^- for $D(T^-)$

Since every crossing point of $D(T^-)$ and $D(T)$ is consistent and the signs of every consistent crossing points of $D(T^-)$ and $D(T)$ are equal, we have

$$\Gamma(D(T)) = f_0(x)\Gamma(X_0) + f_1(x)\Gamma(X_1),$$

$$\Gamma(D(T^-)) = f_0(x)\Gamma(X_0^-) + f_1(x)\Gamma(X_1^-).$$

Noting that $\Gamma(X_i \cup P_0) = \Gamma(X_i^- \cup P_0)$ and $\Gamma(X_i \cup P_1) = \Gamma(X_i^- \cup P_1)$ (i=0,1), we have $\Gamma(X_i) = \Gamma(X_i^-)\,(i = 0, 1)$ and then $\Gamma(D(T)) = \Gamma(D(T^-))$. □

Thus, the proof of Theorem 1.2 is completed. □

Proof of Corollary 1.3. Let $\Gamma(D(T)) = f_0(x)\Gamma(X_0)+f_1(x)\Gamma(X_1), \quad \Gamma(D(T')) = f_0'(x)\Gamma(X_0)+f_1'(x)\Gamma(X_1)$. For $j = 0, 1$, every crossing point of the tangle sum diagram of $D(T) \cup P_j$ is consistent with the crossing point of the tangle sum diagram $D(T') \cup P_j$. Then, we have $\Gamma(D(T) \cup P_j) = \Gamma(D(T') \cup P_j)\,(j = 0, 1)$. Hence, we have $f_i(x) = f_i'(x)\,(i = 0, 1)$ by Theorem 1.2. □

5 An application of the Γ-polynomial of a 2-string tangle to a theta-curve

In this section, we apply the Γ-polynomial of a 2-string tangle diagram to a theta-curve and show the fact that Kinoshita's θ-curve is not equivalent to the trivial θ-curve.

A θ-curve is one of spatial graphs, which has two vertices and three edges to connect the two vertices. The diagram (1) in Fig. 27 is the trivial θ-curve G^0 without crossing. The diagram (2) in Fig. 27 is a diagram of Kinoshita's θ-curve G with three edges (see [9, 10]). Though every constituent knot of Kinoshita's θ-curve G (i.e. every knot in G) is trivial, it is known that G is not equivalent to the trivial θ-curve G^0. As an application, we shall show

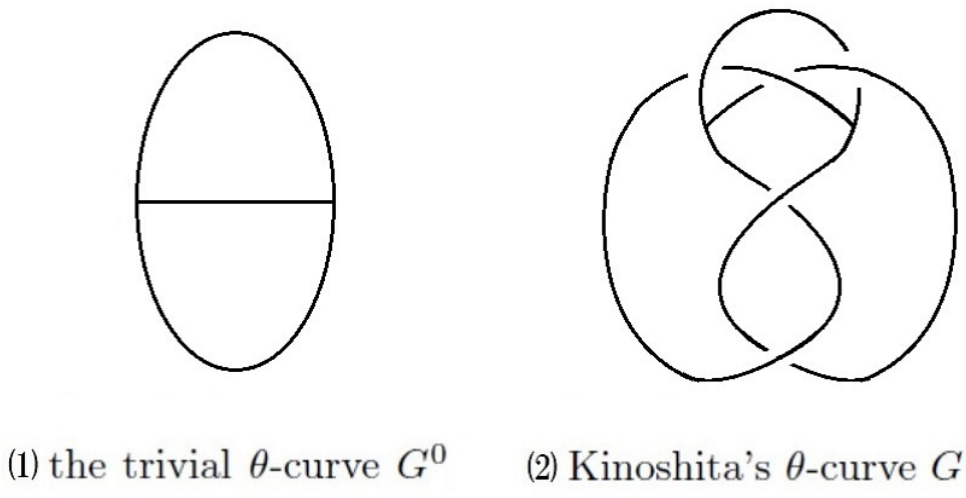


Figure 27: The trivial θ-curve and Kinoshita's θ-curve.

that the Γ-polynomial of a 2-string tangle is used to confirm this fact.

(5.1) Kinoshita's θ-curve is not equivalent to the trivial θ-curve.

Proof of (5.1). Let e_i and $e_i^0\,(i = 1, 2, 3)$ be the three edges of G and G^0, respectively. Suppose that there exists an orientation-preserving homeomorphism $h : R^3 \to R^3$ such that $h(G) = G^0$. We may consider that $h(e_i) = e_i^0\,(i = 1, 2, 3)$. For G^0, we take an oriented disk neighborhood R^0 of $D(e_3^0)$ as in Fig. 28 (1). Let $t_i^0\,(i = 1, 2)$ be the parts of the edges $e_i^0\,(i = 1, 2)$, obtained by removing $R^0 \cap e_i^0$ from the edges e_i^0, respectively. Let $a_i^0\,(i = 1, 2)$ be the oriented arcs in the boundary circle of the disk neighborhood R^0 which have the common end points of $t_i^0\,(i = 1, 2)$, respectively, as in Fig. 28 (1). Let $k_i^0 = t_i^0 \cup a_i^0$ be an oriented knot for $i = 1, 2$. Similarly, for G, we take an oriented disk neighborhood R of $D(e_3)$, t_i and $a_i\,(i = 1, 2)$ in $D(G)$ in Fig. 28 (2). Let $k_i = t_i \cup a_i$ be an oriented knot for $i = 1, 2$. Since we take the linking number $\mathrm{Link}(k_1, k_1) = 0$ as it is seen in Fig. 28 (2), the orientation-preserving homeomorphism $h : R^3 \to R^3$ can be assumed to satisfy that $h(R) = R^0$. Further, we may take $h(a_i) = a_i^0\,(i = 1, 2)$. As in Fig. 29, we give an orientation to the strings t_i and $t_i^0\,(i = 1, 2)$ of $D(T)$ and $D(T^0)$ so that $D(T)$ and $D(T^0)$ are the oriented tangle diagrams, which are seen to be of the same type B. Let $\Gamma(D(T))$ and $\Gamma(D(T^0))$ be the Γ-polynomials as follows.

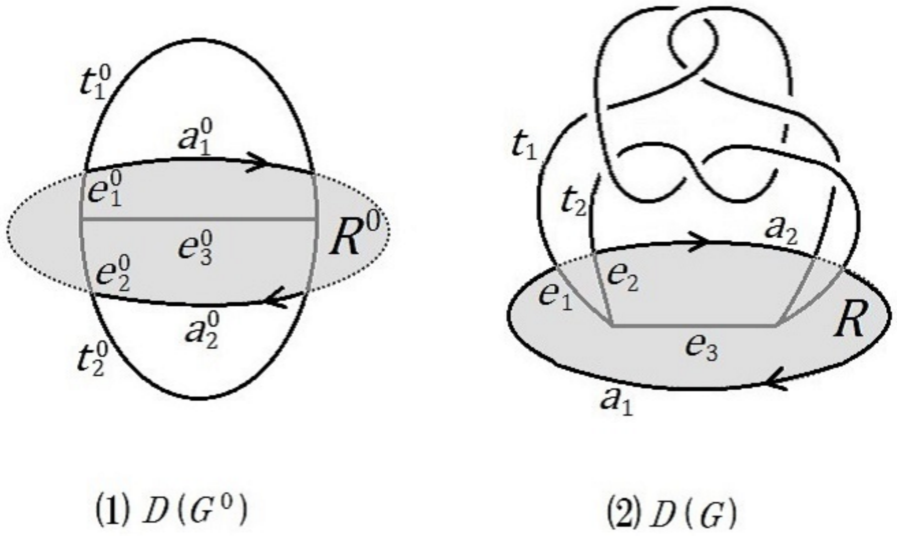


Figure 28: The oriented disk neighborhoods R^0 of $D(e_3^0)$ and R of $D(e_3)$.

$$\Gamma(D(T)) = f_0(x)\Gamma(X_0) + f_1(x)\Gamma(X_1),$$

$$\Gamma(D(T^0)) = f_0^0(x)\Gamma(X_0) + f_1^0(x)\Gamma(X_1).$$

For the oriented tangle diagrams $D(T)$ and any j $(j = 0, 1)$, since

$$\Gamma(D(T) \cup P_j) = (-1 + 2x^{-1} - x^{-2})\Gamma(X_0 \cup P_j) + \Gamma(X_1 \cup P_j),$$

we have

$$\Gamma(D(T)) = (-1 + 2x^{-1} - x^{-2})\Gamma(X_0) + \Gamma(X_1).$$

On the other hand, for the oriented tangle diagram $D(T^0)$ and any j $(j =$

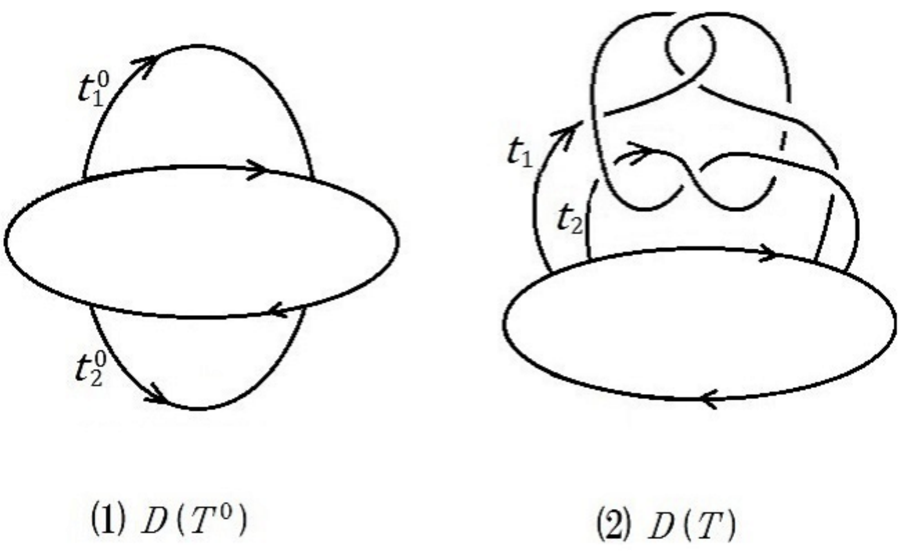


Figure 29: The tangle diagrams $D(T^0)$ and $D(T)$.

$0, 1)$, since

$$\Gamma(D(T^0) \cup P_j) = \Gamma(X_1 \cup P_j),$$

we have

$$\Gamma(D(T^0)) = \Gamma(X_1).$$

Because the knot diagrams $D(T) \cup P_j$ and $D(T^o) \cup P_j$ are equivalent and thus we have

$$\Gamma(D(T) \cup P_j) = \Gamma(D(T^o) \cup P_j) \quad (j = 0, 1),$$

we have $f_0(x) = f_0^0(x)$ by Theorem 1.2. Since actually we have

$$f_0(x) = -1 + 2x^{-1} - x^{-2} \neq 0 = f_0^0(x),$$

we conclude that such a homeomorphism h does not exist. Hence, the proof of (5.1) is completed. $\square$

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