One-loop superstring amplitudes and genus one super-Green's function

(1-ループ超弦振幅と種数1の超グリーン関数)

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Abstract

In this thesis, we reexamine genus one super-Green functions with general boundary conditions twisted by (α, β) for (σ, τ) directions in the eigenmode expansion and derive expressions as infinite series of hypergeometric functions. First of all we review the string theory in the operator and the path integral methods, and then, using the super-Green functions, we compute one-loop superstring amplitudes with non-maximal supersymmetry, taking an example of massless vector emissions of open string type I Z_2 orbifold.

Introduction

In the four fundamental forces of nature, the electromagnetic and the weak interactions have been unified by Glashow-Weinberg-Salam [1] as the electroweak force, and the Grand Unified Theory (GUT) [2] which unifies the strong interaction as well as the electromagnetic and the weak interactions is in progress. String theory have been developed as a candidate for the quantum gravity which unifies the all four forces [3, 4, 5, 6, 7, 8]. Although the discoveries such as the D-brane [9, 10], the T- and S-dualities [11, 12] and the AdS/CFT correspondence [13] have made some progresses, our understanding for the string theory is quite insufficient. A lot of effort has been devoted to computations of superstring amplitudes in the path integral formulation as well as in the operator formulation until now [14, 15, 4, 16, 17, 18, 19, 20, 21]. The string theory inhabits higher dimensional spacetimes than our familiar four-dimensional spacetime, whose extra dimensions are considered to be compactified [22, 23] and then not to be observed at the low-energy scale which we live in. Numerous models with orbifold compactification have been proposed [24, 25, 26, 27], where the partition functions have often been calculated. However, it appears that there are a few articles which calculate one-loop amplitudes for more than one point case on the orbifold. While Atick-Dixon-Sen have calculated the two and four point one-loop amplitudes on the orbifold in the heterotic string [28], there seems to be also a few such calculations in the type I superstring which is considered to be connected with the heterotic string by S-duality. In addition, the articles included the calculation of the amplitudes on the orbifold often use the operator formalism. In order to deal with the one-loop amplitudes on the orbifold in the path integral formalism, it is necessary to take the generalized Green function that does not satisfy ordinary periodicity or antiperiodicity on the genus one Riemann surfaces in σ or τ directions, but it seems that the number of articles dealt with such Green function is relatively small [29]. For these backgrounds, in this study, we discuss the Green function which satisfies the generalized boundary conditions, and then compute the one-loop superstring amplitudes on the orbifold as an example of using this Green function [30].

We apply the well-known eigenmode expansion in order to deal with the Green function. Additionally, it is important to exploit partial fractions in the bosonic part and to use Ramanujan's summation formula in the fermionic part. In general, our final expression is given by an infinite series consisting of a hypergeometric function (with its argument successively shifted), which is relevant to the genus zero Green function[1]. This is in accord with the picture that the genus one Green functions can be obtained from those of genus zero by putting an infinite number of image charges.

In this thesis, we consider orbifold compactification. Although it is Calabi-Yau or $K3$ that are important in elementary particle physics, in general we cannot calculate the amplitudes in such target space because still we have no exact representation for these on conformal field theory. Nevertheless, the amplitudes in the orbifold can be dealt exactly. In addition, $T^4/\mathbf{Z}_2$ orbifold we consider here gives us the toy model for those geometries, and we can calculate the perturbative amplitudes in string theory.

This paper consists of three parts, and we consider the Neveu-Schwarz-Ramond (NSR) superstring through all the three parts. First of all we review the construction of the string theory by the operator method in the part I and by the path integral method in the part II respectively. In the part III, we discuss the general Green function and then compute the one, two and three point one-loop superstring amplitudes on $T^4/\mathbf{Z}_2$ orbifold as an example of using the general Green function.

The part I consists of six sections. We review the bosonic string in the section 1, the relation between the Chan-Paton factor and the gauge group in the section 2 and the fermionic string in the section 3, respectively. In

[1]Such Green function in fact appears in string theory under constant B field [31].

addition, we introduce the $\hat{U}(1)$ character in the section 4. And then we calculate the one-loop partition functions on the flat spacetime in the section 5 and on the orbifold in the section 6. The $\hat{U}(1)$ character introduced in the section 4 help us to understand the partition functions in the section 5 and 6.

The part II consists of three sections. In turn by using the path integral method, we review the bosonic string in the section 7 and the fermionic string in the section 8 respectively, in which the partition functions are also calculated. In the section 9, we write down the N point one-loop superstring amplitudes in the path integral formalism. In the last part of the section 8, we recast the superstring one-loop partition functions in the worldsheet covariant path integrals with those of the light-cone operator formulation calculated in the section 5 in order to circumvent the nuisance of the overall normalization.

The part III consists of three sections. In the section 10, we construct the general Green function for the bosonic part and the fermionic part, and then, using these, the general super-Green function on the torus. After introducing our notations in the section 11, finally in the section 12 we compute the one, two, and three point amplitudes on the superannulus with maximal and non-maximal supersymmetries by using the super-Neumann function derived from the general super-Green function in the section 10 with the image method (or, involution) [32, 33, 34].

In appendix A-I, we summarize some notations, and give some details of computation and background materials quoted in the text.

Several parts of the review materials in this thesis are based on the unpublished lectures delivered [35] during the period 2013.11.11-2015.11.15.

Contents

Part I

Operator method: spectrum and partition function in the light-cone quantization

1 Bosonic string

We still don't know what string theory is, but we do know a lot about strings through first quantization, namely, quantization of string coordinates $X^M(\tau_{\rm M}, \sigma)$, $M = 0, 1, ..., D-1$, which is, at the same time, embedding of a surface swept by a string into D-dimensional Minkowski spacetime.

1.1 Action and mode expansion

We postulate the following action:

$$S^{(1)}\left[X^M, g_{mn}; \Sigma\right] \equiv \frac{1}{2\pi\alpha'}\frac{1}{2}\int_\Sigma {\rm d}\tau_{\rm M}{\rm d}\sigma\sqrt{-g}g^{mn}\partial_m X^M \partial_n X^N \eta_{MN}\ , \tag{1.1}$$

where

$$ {\rm d}^2\xi \equiv {\rm d}\tau_{\rm M}{\rm d}\sigma\ , \tag{1.2}$$

$$\eta_{MN} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \tag{1.3}$$

is a metric in the Minkowski spacetime. We denote by

$$X^M = X^M(\sigma, \tau_{\rm M}) \tag{1.4}$$

embedding of a string surface into two dimensional Minkowski spacetime.

$$g = \det g_{mn} \,, \tag{1.5}$$

we have denoted by

$$g_{mn}(\sigma, \tau_{\mathrm{M}}) : \text{two dimensional metric field. } m, n = 1, 2. \tag{1.6}$$

So action takes the form of two dimensional quantum field theory of D-massless scalars coupled to two dimensional metric fields and principle of general covariance is at work.

- both X^M, g_{mn} are dynamical variables. However, there is no kinetic term for g_{mn}, and the metric field is auxiliary field. This is the first remark.
- the second remark is that it is the first quantization, namely the quantization of coordinates as a string, but at the same it is the field (second) quantization as the two dimensional field theory.

First we regard two dimensional surface is also Minkowski like as well. As the two dimensional quantum field theory, we would like to add the local counterterm permitted by general covariance. So, such action will be

$$\begin{aligned} S_p \left[X^M, g_{mn}; \Sigma\right] = & (1 + A)\, S_p^{(1)} \left[X^M, g_{mn}; \Sigma\right] \\ & + \mu_0^2 \int_\Sigma \mathrm{d}\tau_{\mathrm{M}} \mathrm{d}\sigma \sqrt{-g} + \frac{\ln \lambda}{4\pi} \int_\Sigma \mathrm{d}\tau_M \mathrm{d}\sigma \sqrt{-g} R \,. \end{aligned} \tag{1.7}$$

One would like to quantize this theory by properly defining the path integral measure for both X^M and g_{mn}. However, in this first part we will proceed in a different way. We will do so by

1. first eliminating g_{mn} via equation of motion ("classically")
2. and then quantizing the theory in the light-cone gauge

Let us first derive equation of motion by the variation of the action. The variation of g^{mn} leads to

$$\delta g^{mn} : \partial_m X^M \partial_n X_M - \frac{1}{2} g_{mn} g^{pq} \partial_p X^M \partial_q X_M = 0 \,. \tag{1.8}$$

The variation of X^M leads to

$$\delta X^M : \frac{1}{\sqrt{-g}} \partial_m \left(\sqrt{-g} g^{mn} \partial_n X^N \right) \equiv \triangle_g X^N = 0 \,. \tag{1.9}$$

In order to get Nambu-Goto action, we eliminate g_{mn} from eq. (1.1). We have denoted by γ_{mn}, the induced metric, given by

$$\gamma_{mn} \equiv \partial_m X \cdot \partial_n X = \frac{1}{2} g_{mn} (g^{pq} \partial_p X \cdot \partial_q X) \,. \tag{1.10}$$

Then, after the elimination of g_{mn}, we obtain

$$\begin{aligned} S^{(1)} \left[X^M, g_{mn} \text{ eliminated}; \Sigma \right] &= \frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\xi \sqrt{-\det \gamma_{mn}} \\ &= \text{Nambu-Goto action for a string} \\ &= \frac{1}{2\pi\alpha'} \, (\text{area of the surface}) \,. \end{aligned} \tag{1.11}$$

This is the Nambu-Goto action which is nothing but the area of the surface divided by $2\pi\alpha'$.

From now on, we use the following notation:

$$\frac{\partial X^M}{\partial \tau_\mathrm{M}} \equiv \dot{X}^M \,, \qquad \frac{\partial X^M}{\partial \sigma} \equiv X'^M \,. \tag{1.12}$$

Then the Nambu-Goto action can be written as

$$\int \mathrm{d}\tau_\mathrm{M} \mathrm{d}\sigma \sqrt{-\det \gamma_{mn}} = \int \mathrm{d}\tau_\mathrm{M} \mathrm{d}\sigma \sqrt{-\left(\dot{X} \cdot \dot{X}\right) \cdot \left(X' \cdot X'\right) + \left(\dot{X} \cdot X'\right)^2} \,. \tag{1.13}$$

As is clear from the original form, this action is reparametrization invariant, namely, any change of coordinate

$$\begin{aligned} \tau_\mathrm{M} &= \tau_\mathrm{M}(\tau'_\mathrm{M}, \sigma') \\ \sigma &= \sigma(\tau'_\mathrm{M}, \sigma') \,, \end{aligned} \tag{1.14}$$

$$\int \mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma\sqrt{-\det\gamma_{mn}} = \int \mathrm{d}\tau'_{\mathrm{M}}\mathrm{d}\sigma'\sqrt{-\left(\dot{X}\cdot\dot{X}\right)\cdot\left(X'\cdot X'\right)+\left(\dot{X}\cdot X'\right)^2}\,. \tag{1.15}$$

But at the same time this action is non-polynomial. We would like to get rid of this square root by fixing this local symmetry. This is done by imposing the following "orthonormality conditions" as constraints:

$$\begin{cases} \dot{X}\cdot X' = 0 \\ \dot{X}^2 + X'^2 = 0 \end{cases} \tag{1.16}$$

or

$$\begin{cases} \left(\dot{X}+X'\right)^2 = 0 \\ \left(\dot{X}-X'\right)^2 = 0 \end{cases}\,. \tag{1.17}$$

Then Nambu-Goto action reduces to action for massless D-scalars in $1+1$ dimensions:

$$\begin{aligned} &S^{(1)}\left[X^M, g_{mn}\text{ eliminated}; \Sigma\right] \\ &\approx \frac{1}{2\pi\alpha'}\int \mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma\sqrt{-\left(\dot{X}\cdot\dot{X}\right)\cdot\left(X'\cdot X'\right)+\frac{1}{4}\left(\dot{X}^2+X'^2\right)^2} \\ &= \frac{1}{2\pi\alpha'}\frac{1}{2}\int \mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma\left(\dot{X}^2 - X'^2\right)\,. \end{aligned} \tag{1.18}$$

And the constraints takes the following form:

$$\begin{cases} \frac{\alpha'}{2}T(\tau_{\mathrm{M}},\sigma) \equiv \frac{1}{2}\left(\dot{X}+X'\right)^2 \approx 0 \\ \frac{\alpha'}{2}\bar{T}(\tau_{\mathrm{M}},\sigma) \equiv \frac{1}{2}\left(\dot{X}-X'\right)^2 \approx 0 \end{cases} \tag{1.19}$$

and it is identified as vanishing 2d energy-momentum tensor.

Conventions

Often used nowadays for closed string (Figure 1) itself is parametrized by

$$-\pi < \sigma < \pi\,, \qquad -\infty < \tau_{\mathrm{M}} < \infty\,, \tag{1.20}$$

while for open string (Figure 2) it is parametrized by

$$0 < \sigma < \pi\,, \qquad -\infty < \tau_{\mathrm{M}} < \infty\,. \tag{1.21}$$

And z and "$\bar{z}$" are introduced as follows:

$$z = e^{i(\tau_{\mathrm{M}}+\sigma)} = e^{\tau+i\sigma}$$
$$\text{"}\bar{z}\text{"} = e^{i(\tau_{\mathrm{M}}-\sigma)} = e^{\tau-i\sigma}\,. \tag{1.22}$$

In practice, most of the computation currently are carried out in the Euclidean signature by regarding

$$i\tau_{\mathrm{M}} \equiv \tau \tag{1.23}$$

to be real. In Euclidean signature, equation of motion is just a Laplace

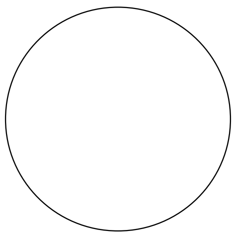

Figure 1: closed string

Figure 2: open string

equation and the mode expansion of the solution usually takes the following form in the 10d flat spacetime. For closed string,

$$X^M(\tau_{\mathrm{M}},\sigma) \equiv X_0^M - \frac{i\alpha'}{2}P_0^M \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\alpha_n^M\frac{z^{-n}}{n} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\tilde{\alpha}_n^M\frac{\bar{z}^{-n}}{n}\,, \tag{1.24}$$

where

$$P_0^M \equiv \sqrt{\frac{2}{\alpha'}}\alpha_0^M = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^M\,. \tag{1.25}$$

While for open string with free end $X' \mid_{\sigma=0,\pi} = 0$,

$$X^M(\tau_{\mathrm{M}},\sigma) \equiv X_0^M - i\alpha' P_0^M \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\alpha_n^M\frac{z^{-n}}{n} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\alpha_n^M\frac{\bar{z}^{-n}}{n}\,, \tag{1.26}$$

while an open string with fixed end $\delta X \mid_{\sigma=0,\pi} = 0$,

$$X^M \equiv c^M - \frac{i(c'^M - c^M)}{2\pi} \ln\left(\frac{z}{\bar{z}}\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^M \frac{z^{-n}}{n} - i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^M \frac{\bar{z}^{-n}}{n} . \tag{1.27}$$

1.2 Quantization in light-cone gauge

The choice of the gauge condition eq. (1.19) does not fix the invariance of the Nambu-Goto action completely: if eq. (1.19) is satisfied for τ_{M} and σ, then it is also satisfied for any change of variable to $\tilde{\tau}_{\mathrm{M}}$ and $\tilde{\sigma}$

$$\begin{aligned} \tilde{\tau}_{\mathrm{M}} &= \tilde{\tau}_{\mathrm{M}}(\tau_{\mathrm{M}}, \sigma) \\ \tilde{\sigma} &= \tilde{\sigma}(\tau_{\mathrm{M}}, \sigma) \end{aligned} \tag{1.28}$$

provided that they satisfy the following conditions:

$$\begin{cases} \tilde{\tau}_{\mathrm{M}} + \tilde{\sigma} = f(\tau_{\mathrm{M}} + \sigma) \\ \tilde{\tau}_{\mathrm{M}} - \tilde{\sigma} = \tilde{f}(\tau_{\mathrm{M}} - \sigma) \end{cases} . \tag{1.29}$$

So we can fix this local symmetry, namely, the function f and $\tilde{f}$ as well.

$$\tilde{\tau}_{\mathrm{M}} = \frac{1}{2}\Big(f(\tau_{\mathrm{M}} + \sigma) + \tilde{f}(\tau_{\mathrm{M}} - \sigma)\Big) , \qquad \tilde{\sigma} = \frac{1}{2}\Big(f(\tau_{\mathrm{M}} + \sigma) - \tilde{f}(\tau_{\mathrm{M}} - \sigma)\Big) . \tag{1.30}$$

We work out this procedure for the closed string case first.

Let us first introduce the light-cone coordinates which are denoted by

$$X^+ = \frac{X^0 + X^{D-1}}{\sqrt{2}} , \qquad X^- = \frac{X^0 - X^{D-1}}{\sqrt{2}} . \tag{1.31}$$

Then two dimensional inner product is written as

$$\begin{aligned} X \cdot Y = X^M Y_M &= -X^0 Y^0 + \sum_{i=1}^{D-1} X^i Y^i \\ &= \sum_{i=1}^{D-2} X^i Y^i - X^+ Y^- - X^- Y^+ \\ &= \eta_{M_{\mathrm{lc}} N_{\mathrm{lc}}} X^{M_{\mathrm{lc}}} Y^{N_{\mathrm{lc}}} \end{aligned} \tag{1.32}$$

$$M_{\rm lc}, N_{\rm lc} = +, -, 1, \cdots, D-2 \tag{1.33}$$

$$\eta_{M_{\rm lc}N_{\rm lc}} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{1.34}$$

$$X \cdot Y = X^i Y_i + X^+ Y_+ + X^- Y_- \tag{1.35}$$

Then the coordinate

$$\begin{aligned} X^+(\tau_{\rm M}, \sigma) = & X_0^+ + \frac{\alpha'}{2} P_0^+ \left((\tau_{\rm M} + \sigma) + (\tau_{\rm M} - \sigma) \right) \\ & + \frac{1}{2} \text{function}(\tau_{\rm M} + \sigma) + \frac{1}{2} \widetilde{\text{function}}(\tau_{\rm M} - \sigma) \end{aligned} \tag{1.36}$$

From this equation, we see that we choose $\tilde{\tau}_{\rm M}$ such that

$$X^+(\tau_{\rm M}, \sigma) = X_0^+ + \alpha' P_0^+ \tilde{\tau}_{\rm M} \tag{1.37}$$

and this fixes half of the residual local symmetry eq. (1.29). The remaining residual symmetry fixed by the following equation:

$$\alpha' P_0^+ \tilde{\sigma} = \int_0^\sigma {\rm d}\sigma' \frac{\partial X^+(\tau_{\rm M}, \sigma')}{\partial \tau_{\rm M}} \, . \tag{1.38}$$

So, by eqs. (1.37) and (1.38), the residual local symmetry eq. (1.29) is completely fixed.

To summarize the light-cone gauge condition corresponds to

$$X^+(\tau_{\rm M}, \sigma) = X_0^+ + \alpha' P_0^+ \tau_{\rm M} \, . \tag{1.39}$$

In this gauge,

$$\dot{X}^+ = \alpha' P_0^+ \tag{1.40}$$

$$X'^+ = \frac{\partial X^+}{\partial \sigma} = 0 \, . \tag{1.41}$$

Therefore X^+ is light-cone time synchronized at every point on a single string.

1.2.1 conjugate momenta, Lagrangian, Hamiltonian

Recall that

$$\begin{aligned}
S^{(1)} =& \frac{1}{2\pi\alpha'}\frac{1}{2}\int \mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma \left(-2\dot{X}^{+}\dot{X}^{-} + \dot{X}^{i}\dot{X}^{i} - X'^{i}X'^{i}\right) \\
=& -\frac{1}{2\pi}P_0^{+}\int \mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma\dot{X}^{-}(\tau_{\mathrm{M}},\sigma) + \frac{1}{4\pi\alpha'}\int_{-\pi}^{\pi}\mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma\left(\dot{X}^{i}\dot{X}^{i} - X'^{i}X'^{i}\right) \\
=& -\frac{P_0^{+}}{2\pi}\int \mathrm{d}X^{+}\int_{-\pi}^{\pi}\mathrm{d}\sigma\partial_{+}X^{-}(\tau_{\mathrm{M}},\sigma) \\
&\quad + \frac{1}{4\pi\alpha'}\frac{1}{\alpha' p_0^{+}}\int \mathrm{d}X^{+}\int_{-\pi}^{\pi}\mathrm{d}\sigma\left((\alpha' P^{+})^{2}\partial_{+}X^{i}\partial_{+}X^{i} - X'^{i}X'^{i}\right) .
\end{aligned} \tag{1.42}$$

Here we have denoted by

$$\partial_{+} = \frac{\partial}{\partial X^{+}} . \tag{1.43}$$

Therefore the Lagrangian becomes

$$\begin{aligned}
L(\tau_{\mathrm{M}}) =& -P_0^{+}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\partial_{+}X^{-}(\tau_{\mathrm{M}},\sigma) \\
&+\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left(P_0^{+}\partial_{+}X^{i}\partial_{+}X^{i} - \frac{1}{P_0^{+}\alpha'^{2}}X'^{2}X'^{2}\right) .
\end{aligned} \tag{1.44}$$

Let us introduce the momentum density conjugate to string written in the light-cone coordinate:

$$\mathcal{P}^{i}(\tau_{\mathrm{M}},\sigma) = \mathcal{P}_{i}(\tau_{\mathrm{M}},\sigma) \equiv \frac{\delta L}{\delta(\partial_{+}X^{i}(\tau_{\mathrm{M}},\sigma))} = \frac{1}{2\pi}P_0^{+}\partial_{+}X^{i} = \frac{1}{2\pi\alpha'}\dot{X}^{i} \tag{1.45}$$

$$-\mathcal{P}^{-}(\tau_{\mathrm{M}},\sigma) = \mathcal{P}_{+}(\tau_{\mathrm{M}},\sigma) = \frac{\delta L}{\delta\partial_{+}X^{+}(\tau_{\mathrm{M}},\sigma)} = 0 \tag{1.46}$$

$$-\mathcal{P}^{+}(\tau_{\mathrm{M}},\sigma) = \mathcal{P}_{-}(\tau_{\mathrm{M}},\sigma) = \frac{\delta L}{\delta\partial_{+}X^{-}(\tau_{\mathrm{M}},\sigma)} = -\frac{1}{2\pi}P_0^{+} . \tag{1.47}$$

The last equation tells us that P_0^{+} is the total momentum in the $+$ direction.

By the canonical procedure, the light-cone Hamiltonian is written as

$$\begin{aligned}
&\text{The light-cone Hamiltonian} \\
&= \int_{-\pi}^{\pi} \mathrm{d}\sigma \left(\mathcal{P}_i(\tau_\mathrm{M},\sigma)\partial_+ X^i(\tau_\mathrm{M},\sigma) + \mathcal{P}_-(\tau_\mathrm{M},\sigma)\partial_+ X^-(\tau_\mathrm{M},\sigma)\right) - L \\
&= \frac{1}{2P_0^+}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}4\pi^2(\mathcal{P}^i(\tau_\mathrm{M},\sigma))^2 + \frac{1}{2P_0^+}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\frac{1}{\alpha'^2}(X'^i X'^i) \\
&= \frac{1}{2P_0^+\alpha'^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left(\dot{X}^i\dot{X}^i + X'^i X'^i\right) . \qquad (1.48)
\end{aligned}$$

1.2.2 X^-

Let's see what happens to the coordinate X^-. Eq. (1.19) can be solved for X^- by the calculation below.

$$(\dot{X} \pm X')^i(\dot{X} \pm X')^i = 2(\dot{X} \pm X')^+(\dot{X} \pm X')^- = 2\alpha' P_0^+(\dot{X}^- \pm X^{-\prime}) \quad (1.49)$$

$$4\alpha' P_0^+ \dot{X}^- = (\dot{X} + X')^i(\dot{X} + X')^i + (\dot{X} - X')^i(\dot{X} - X')^i = 2\dot{X}^i\dot{X}^i + 2X'^i X'^i \quad (1.50)$$

$$4\alpha' P_0^+ X^{-\prime} = 4\dot{X}^i X'^i . \quad (1.51)$$

Therefore P_0^-, α_n^- and $\tilde{\alpha}_n^-$ are expressed quadratically in α_n^i and $\tilde{\alpha}_n^i$.

Recall

$$X^- = X_0^- + \alpha' P_0^- \tau_\mathrm{M} + (\text{oscillator}) , \quad (1.52)$$

and therefore

$$\dot{X}^- = \alpha' P_0^- + \frac{\partial}{2\partial\tau}(\text{oscillator}) . \quad (1.53)$$

Using eqs. (1.48) and (1.50), we see that the light-cone Hamiltonian is

$$\begin{aligned}
\text{The light-cone Hamiltonian} &= \frac{1}{2P_0^+\alpha'^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}2\alpha' P_0^+\dot{X}^- \\
&= \frac{1}{\alpha'}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\dot{X}^- \\
&\overset{\text{eq.(1.53)}}{=} P_0^- \qquad (1.54)
\end{aligned}$$

as expected. Namely, we have derived from eq. (1.48)

$$P_0^- = \frac{1}{2P_0^+\alpha'^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left(\dot{X}^i\dot{X}^i + X'^i X'^i\right) . \tag{1.55}$$

1.2.3 commutation relation

Now we are ready to carry out the canonical quantization. And equal time commutators read

$$\begin{aligned}[\mathcal{P}^i(\tau_{\mathrm{M}},\sigma), X^j(\tau_{\mathrm{M}},\sigma')] &= \left[\frac{1}{2\pi\alpha'}\dot{X}^i(\tau_{\mathrm{M}},\sigma), X^j(\tau_{\mathrm{M}},\sigma)\right] \\ &= -i\delta(\sigma-\sigma')\delta^{ij} . \end{aligned} \tag{1.56}$$

These give us a set of commutation relations for the center of mass transverse coordinates and the oscillation modes

$$\begin{aligned}[\alpha_n^i, \alpha_m^j] &= n\delta_{n+m,0}\delta^{ij} \\ \left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] &= n\delta_{n+m,0}\delta^{ij} \\ \left[P_0^i, X_0^i\right] &= -i\delta^{ij} , \end{aligned} \tag{1.57}$$

and all other commutators are vanishing. P_0^i are the total center of mass momenta for the ith transverse direction.

1.2.4 mass operator and number operator

The relativistic invariance tells us that the eigenvalues of the mass operator $\mathcal{M}$ play important roles in the analysis.

$$
\begin{aligned}
-\mathcal{M}^2 \equiv & P_0^{M_{\rm lc}} P_{0\,M_{\rm lc}} = -2P_0^+ P_0^- + P_0^i P_0^i \\
= & -\frac{1}{\alpha'^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}(\dot{X}^i\dot{X}^i + X'^i X'^i) + P_0^i P_0^i \\
= & -\frac{1}{2\alpha'^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left\{(\dot{X}+X')^2 + (\dot{X}-X')^2\right\} + P_0^i P_0^i \\
= & -\frac{1}{2\alpha'^2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left\{\left(i\sqrt{\frac{\alpha'}{2}}\right)^2\left(\sum_{n\neq 0}(-)\alpha_n 2iz^{-n}\right)^2\right. \\
& \qquad\qquad \left. + \left(i\sqrt{i\frac{\alpha'}{2}}\right)^2\left(\sum_{n\neq 0}(-)\tilde{\alpha}_n 2i\bar{z}^{-n}\right)^2\right\} \\
= & -\frac{1}{2\alpha'^2}\frac{\alpha'}{2}4\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left\{\left(\sum_{n\neq 0}\alpha_n z^{-n}\right)^2 + \left(\sum_{n\neq 0}\tilde{\alpha}_n \bar{z}^{-n}\right)^2\right\} . \qquad (1.58)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\alpha'\mathcal{M}^2 = & \sum_{i=1}^{D-2}\sum_{n\neq 0}\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right) \\
= & 2\left(\hat{N} + \hat{\tilde{N}} + \text{const.}\right) . \qquad (1.59)
\end{aligned}
$$

Here we have normal ordered the oscillator. The constant indicated is the constant which appear through the normal ordering process. The number operator $\hat{N}$ and $\hat{\tilde{N}}$ are defined by

$$
\begin{aligned}
\hat{N} = & \sum_{i=1}^{D-2}\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0} :\alpha_{-n}^i\alpha_n^i: \\
\hat{\tilde{N}} = & \sum_{i=1}^{D-2}\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0} :\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i: \ . \qquad (1.60)
\end{aligned}
$$

Therefore

$$\begin{aligned}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_{-n}^{i}\alpha_{n}^{i} &= 2\hat{N} + (D-2)\sum_{n=1}^{\infty} n \\ &= 2\left(\hat{N} + \frac{D-2}{2}\zeta(-1)\right) \\ &= 2\left(\hat{N} - \frac{D-2}{24}\right) . \end{aligned} \tag{1.61}$$

Here we have evaluated the normal ordering constant by zeta function regularization (appendix C).

$$\alpha' \mathcal{M}^2 = 2\left(\hat{N} + \hat{\tilde{N}} - \frac{D-2}{24}\cdot 2\right) . \tag{1.62}$$

By a separate argument, which we do not elaborate upon here, we find $D = 26$. Therefore

$$\alpha' \mathcal{M}^2 = 2\left(\hat{N} + \hat{\tilde{N}}\right) - 4 . \tag{1.63}$$

1.2.5 state space

In the state space of a closed string is given by

$$|P^i, \{\{n_{-n_k}^{i_k}\}\}\rangle = |P^i\rangle \otimes \alpha_{-n_1}^{i_1} \cdots \alpha_{-n_k}^{i_k}|\Omega\rangle \tag{1.64}$$

$$\alpha_n^{i_n}|\Omega\rangle = 0 , \qquad n > 0 . \tag{1.65}$$

Here the ground state level is denoted by $|P^i\rangle \otimes |\Omega\rangle$ and

$$\alpha' \mathcal{M}^2 |P^i\rangle \otimes |\Omega\rangle = -4|P^i\rangle \otimes |\Omega\rangle . \tag{1.66}$$

Another point which to note is the level matching condition. There is no boundary for a closed string, so there is no preferred point on the string. Quantum theory of a closed string must be invariant under the shift $\sigma \to$

$\sigma + \Delta$. From the expression of the mode expansion, this shift is the shift of the oscillators by

$$\begin{aligned} \alpha_n^j &\to \alpha_n^j \mathrm{e}^{-in\Delta} \\ \tilde{\alpha}_n^j &\to \tilde{\alpha}_n^j \mathrm{e}^{+in\Delta} \,. \end{aligned} \tag{1.67}$$

So with the unitary operator

$$U(\Delta) \equiv e^{i(\hat{N} - \hat{\tilde{N}})\Delta} \,, \tag{1.68}$$

this shift is generated. Namely,

$$U(\Delta) \begin{matrix} \alpha_n^j \\ \tilde{\alpha}_n^j \end{matrix} U^{-1}(\Delta) = \begin{matrix} e^{-in\Delta}\alpha_n^j \\ e^{in\Delta}\tilde{\alpha}_n^j \end{matrix} \,, \tag{1.69}$$

$$i.e. \ U(\Delta) X^i(\tau_{\mathrm{M}}, \sigma) U^{-1}(\Delta) = X^i(\tau_{\mathrm{M}}, \sigma + \Delta) \,. \tag{1.70}$$

So we want

$$U(\Delta)|\mathrm{phys}\rangle = |\mathrm{phys}\rangle \,, \tag{1.71}$$

which is satisfied by

$$(\hat{N} - \hat{\tilde{N}})|\mathrm{phys}\rangle = 0 \,. \tag{1.72}$$

The ground state level is the state called tachyon. The eigenvalue of the mass square operator in the unit of $\frac{1}{\alpha'}$ is -4. This is called state with "intercept" -4.

$$|\mathrm{tachyon}; P^i\rangle = |P^i\rangle \otimes |\Omega\rangle \tag{1.73}$$

$$|P^i\rangle = e^{iP^i X_0^i}|0\rangle \,. \tag{1.74}$$

The first excited state which satisfies the level matching condition of eq. (1.72) are

$$|P^i\rangle \otimes \ \alpha_{-1}^j \tilde{\alpha}_{-1}^k |\Omega\rangle \qquad j, k = 1, ..., 24 = D - 2 \tag{1.75}$$

and the eigenvalues of $\mathcal{M}^2$ is zero:

$$\alpha' \mathcal{M}^2 \left(|P^i\rangle \otimes \ \alpha_{-1}^j \tilde{\alpha}_{-1}^k |\Omega\rangle \right) = 0 \,. \tag{1.76}$$

These tensorial states are massless representing transverse degrees of symmetric traceless tensors and those of antisymmetric tensors, and a scalar from the trace part. Each of them act as an irreducible massless representation under $SO(D-1,1)$.

$$|P^i, \zeta^{jk}(p)\rangle = \zeta^{jk}(P)|P^i\rangle \otimes \ \alpha^j_{-1}\tilde{\alpha}^k_{-1}|\Omega\rangle \ . \tag{1.77}$$

Here $\zeta^{jk}(p)$ is a transverse polarization, namely, the wave function in the momentum space.

1.3 The case of an open string

The gauge condition expresses reflecting the factor 2 difference in the mode expansions eqs. (1.24) and (1.26),

$$X^+(\tau_{\rm M}, \sigma) = X_0^+ + 2\alpha' P_0^+ \tau_{\rm M} \ . \tag{1.78}$$

Therefore,

$$-\mathcal{P}^+(\tau_{\rm M}, \sigma) = -\frac{1}{2\pi} P_0^+ 2 = -\frac{1}{\pi} P_0^+ \ . \tag{1.79}$$

Therefore, in this case again, $\mathcal{P}^+$ has the meaning of the total momentum in the + direction.

The action is written as

$$\begin{aligned} S^{(1)} = &-\frac{1}{2\pi} 2P_0^+ \int \mathrm{d}\tau_{\rm M} \int_0^\pi \mathrm{d}\sigma \dot{X}^-(\tau_{\rm M}, \sigma) \\ &+ \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau_{\rm M} \int_0^\pi \mathrm{d}\sigma \left(\dot{X}^i \dot{X}^i - X'^i X'^i \right) \ . \end{aligned} \tag{1.80}$$

The canonical procedure is the same as that of the closed string. Let us just list a dictionary between the closed string case and the open string case.

$$\begin{array}{ccc} \text{closed} & & \text{open} \\ \alpha' P_0^+ & \to & 2\alpha' P_0^+ \\ \int_{-\pi}^{\pi} & \to & \int_0^{\pi} \end{array} \tag{1.81}$$

The light-cone Hamiltonian is therefore

$$\begin{aligned}\text{The light-cone Hamiltonian} &= \frac{1}{4P_0^+\alpha'^2}\int_0^\pi \frac{\mathrm{d}\sigma}{2\pi}\underbrace{\left(\dot{X}^i\dot{X}^i + X'^i X'^i\right)}_{4\alpha' P_0^+\dot{X}^-} \\ &= \frac{1}{\alpha'}\int_0^\pi \frac{\mathrm{d}\sigma}{2\pi}\dot{X}^- \\ &= P_0^- \end{aligned} \tag{1.82}$$

as expected. Here we have used the fact

$$\dot{X}^- = 2\alpha' P_0^- + \cdots \ . \tag{1.83}$$

Therefore,

$$P_0^- P_0^+ = \frac{1}{4\alpha'^2}\int_0^\pi \frac{\mathrm{d}\sigma}{2\pi}\left(\dot{X}^i\dot{X}^i + X'^i X'^i\right) \ . \tag{1.84}$$

The quantization carried out by those in the equal time commutators

$$\left[\mathcal{P}^i(\tau_{\mathrm{M}},\sigma), X^j(\tau_{\mathrm{M}},\sigma')\right] = -i\delta(\sigma-\sigma')\delta^{ij} \ , \qquad 0<\sigma,\sigma'<\pi \ . \tag{1.85}$$

Series to the commutation relations for the oscillators are

$$\left[\alpha_n^i, \alpha_m^j\right] = n\delta_{n+m,0}\delta^{ij} \ . \tag{1.86}$$

The mass operator square is computed as

$$\begin{aligned} -\mathcal{M}^2 &\equiv P_0^{M_{\mathrm{lc}}} P_{0M_{\mathrm{lc}}} = -2P_0^+P_0^- + P_0^iP_0^i \\ &= (-2)\frac{1}{4\alpha'^2}\int_0^\pi \frac{\mathrm{d}\sigma}{2\pi}\left(\dot{X}^i\dot{X}^i + X'^i X'^i\right) + P_0^iP_0^i \\ &= -\frac{1}{4\alpha'^2}\frac{\alpha'}{2}4\int_0^\pi \frac{\mathrm{d}\sigma}{2\pi}\left\{\left(\sum_{n\neq 0}\alpha_n z^{-n}\right)^2 + \left(\sum_{n\neq 0}\alpha_n \bar{z}^{-n}\right)^2\right\} \\ &= -\frac{1}{2\alpha'}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i \ . \end{aligned} \tag{1.87}$$

Therefore

$$\begin{aligned}
\alpha' m^2 &= \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n \\
&\overset{\text{normal order}}{=} \hat{N} + \frac{1}{2}(D-2)\sum_{n=1}^{\infty} n \\
&= \hat{N} - \frac{D-2}{24} \\
&\overset{D=26}{=} \hat{N} - 1\,.
\end{aligned} \tag{1.88}$$

The number operator has been normal ordered

$$\begin{aligned}
\hat{N} &= \sum_{i=1}^{D-2}\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n \\
&= \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0} : \alpha^i_{-n}\alpha^i_n : \,.
\end{aligned} \tag{1.89}$$

The normal ordering coefficient has been computed as before by the zeta function regularization.

m^2 of the ground state is -1 in the unit of $\frac{1}{\alpha'}$,

$$|\text{tachyon}; P^i\rangle = |P^i\rangle \otimes |\Omega\rangle\,. \tag{1.90}$$

These are tachyon states with "intercept" -1.

There is an eight component massless vector in the first excited level

$$|P^i, \zeta^i\rangle = \sum_j \zeta^j |P^i\rangle \otimes \alpha^j_{-1}|\Omega\rangle\,. \tag{1.91}$$

Here, we have denoted by ζ^i the transverse polarization of the massless vector.

1.4 Critical dmension

In the present formalism, the "critical dimension" is obtained from an anomaly in Lorentz generators

$$M^{M_{\rm lc}N_{\rm lc}} \equiv \int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi}\left(X^{M_{\rm lc}}\mathcal{P}^{N_{\rm lc}} - X^{N_{\rm lc}}\mathcal{P}^{M_{\rm lc}}\right)\,. \tag{1.92}$$

Those generator should satisfy the algebra

$$\left[M^{M_{\rm lc}N_{\rm lc}}, M^{K_{\rm lc}L_{\rm lc}}\right]$$
$$= i\eta^{N_{\rm lc}K_{\rm lc}}M^{M_{\rm lc}L_{\rm lc}} - i\eta^{M_{\rm lc}K_{\rm lc}}M^{N_{\rm lc}L_{\rm lc}} - i\eta^{N_{\rm lc}L_{\rm lc}}M^{M_{\rm lc}K_{\rm lc}} + i\eta^{M_{\rm lc}L_{\rm lc}}M^{N_{\rm lc}K_{\rm lc}} . \tag{1.93}$$

When all of the indices are transverse ($1 \leq i, j, k, \ell \leq 24$),

$$\left[M^{ij}, M^{k\ell}\right] , \tag{1.94}$$

it is easy to check.

The problem arises from the commutator where M^{-i} is involved.

- boost $X^+ \Rightarrow$ outside this gauge.
- M^{-i} actually consists of three oscillators, so the commutator associated with M^{-i} involves six oscillators, and naively $M^{-i}M^{-j}$ is zero but this can be anomalous due to an operator ordering problem.

[36] found

$$\left[M^{-i}, M^{-j}\right] = 0, \tag{1.95}$$

vanishes if $D = 26$ and intercept 4 for closed and 1 for open.

2 Orientation flip and Chan-Paton factors

2.1 Orientation flip (twist operator)

2.1.1 closed string

For any figure of an closed string, one can add arrows to indicate increasing order of σ-parametrization. So, these two figures, figure 3 and figure 4 are equivalent except for the direction of arrows, or the orientation.

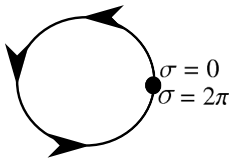

Figure 3: $X^M(\tau_{\rm M}, \sigma)$

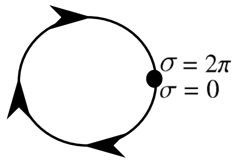

Figure 4: $X^M(\tau_{\rm M}, 2\pi - \sigma)$

Quantum mechanically, this can be stated as follows. Introduce

$$\Omega \qquad \text{such that} \qquad \Omega X^M(z,\bar{z})\Omega^{-1} = X^M(\bar{z}, z)\,. \tag{2.1}$$

This is equivalent to

$$\Omega \alpha_n^M \Omega^{-1} = \tilde{\alpha}_n^M\,. \tag{2.2}$$

Is our physical Hilbert space invariant under this operation, namely, can we have $\Omega|\text{phys}\rangle = |\text{phys}\rangle$?

I) if the answer is yes, the closed string we consider is said to be non-orientable. We called this string type I closed string. There is no way to assign an arrow for this string. And then only symmetric states under the interchange of α and $\tilde{\alpha}$ survive. Tachon survives. Symmetric traceless tensor and scalra survive, while antisymmetric tensor does not survive.

II) if $\Omega|\chi,\ \text{phys}\rangle = |\psi,\ \text{phys}\rangle$, $|\chi\rangle$ and $|\psi\rangle$ distinct, then

- the string theory is called orientable. And the context of super-strings is called type II closed string.
- orientable theory has more states than the unorientable states.
- type II theory must preserve orientability and the string surface swept by an orientable states own to be an orientable surface.

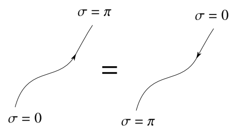

Figure 5:

2.1.2 open string

The same question can be raised for an open string as well. (See figure 5.) Namely, can theory be invariant by the change $\sigma \to \pi - \sigma$? And this can be stated as follows. Introduce

$$\Omega \qquad \text{such that} \qquad \Omega X^M(z,\bar{z})\Omega^{-1} = X^M(-\bar{z},-z)\,. \tag{2.3}$$

Namely,

$$\begin{aligned} \Omega\alpha_n^M\Omega^{-1} &= (-)^n\alpha_n^M && \text{for Neumann(N) boundary condition} \\ \Omega\alpha_n^M\Omega^{-1} &= (-)^{n+1}\alpha_n^M && \text{for Dirichlet(D) boundary condition} \end{aligned} \tag{2.4}$$

and

$$c'^M = c^M \qquad \text{for Dirichlet boundary condition}\,. \tag{2.5}$$

The question to be raised is $\Omega|\text{phys}\rangle = |\text{phys}\rangle$.

I) if the answer is yes, an open string is said to be non-orientable. Ω is realized as

$$\Omega = (-)^{\hat{N}}\,, \qquad \hat{N} = \sum_{n=1}^{\infty} \alpha_{-n}^M \alpha_{nM}\,. \tag{2.6}$$

The case where $\hat{N}$ acting on physical Hilbert state produces

$$\begin{aligned} \hat{N}|\text{phys}\rangle &= (\text{even})|\text{phys}\rangle && \text{are allowed for N}\,, \\ \hat{N}|\text{phys}\rangle &= (\text{odd})|\text{phys}\rangle && \text{are allowed for D}\,. \end{aligned} \tag{2.7}$$

Note that $\alpha_{-1}^M|\Omega\rangle$ is projected out in the Neumann boundary condition for bosonic string.

II) $\Omega|\chi, \text{phys}\rangle = |\psi, \text{phys}\rangle$ and χ and ψ are distinct then the string theory we consider is said to be orientable.

2.2 Chan-Paton factors

It is possible to add charges at two ends of an open string, and this is regarded as additional degrees of freedom for an open string. Graphically, it is represented as in figure 6.

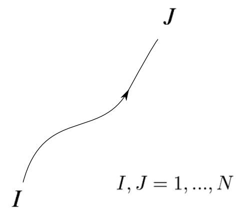

Figure 6:

In general, duality of an open string multi-particle diagrams requires

$$A(1,2,...,N) = A(\sigma(1),\sigma(2),...,\sigma(N)) \,. \tag{2.8}$$

Here we have denoted by $j \to \sigma(j)$, an element of cyclic permutation group C_n.

In the absence of this charge, the full S-matrix is written as

$$T\left(1,2,\cdots,n\right) = \sum_{\text{non-cyclic permutation } \sigma} A\left(\sigma(1),\sigma(2),\cdots,\sigma(n)\right) \ , \tag{2.9}$$

where the summation is over the non-cyclic permutation of n elements.

So the natural modification to include charges on the two ends of the string is

$$\begin{aligned} &T\left(1,2,\cdots,n\right) \\ &\quad = \sum_{\text{non-cyclic perm. } \sigma} \operatorname{tr}\left(\lambda^{\sigma(1)}\lambda^{\sigma(2)}\cdots\lambda^{\sigma(n)}\right) A\left(\sigma(1),\sigma(2),\cdots,\sigma(n)\right) \,. \end{aligned} \tag{2.10}$$

Here we have denoted by λ^a generators of some Lie algebra. The trace factor is called Chan-Paton factor and is cyclically symmetric by the trace property.

At the level of string states, this modification implies

$$\sum_{I,J} \lambda^a_{IJ} |k, \zeta^{i_1 \cdots i_n}(k); I, J\rangle \ , \tag{2.11}$$

where the k is the momentum of a string, ζ the polarization and I and J are additional labels of the string.

Now let's turn to the discussion for an unoriented string. Action of Ω is taken to be

$$\Omega \sum_{I,J} |k, \zeta^{i_1 \cdots i_n}(k); I, J\rangle \lambda^a_{IJ} = \sum_{I,J} \prod_i (-)^{m_i} |k, \zeta^{i_1 \cdots i_n}(k); I, J\rangle \left(M \lambda^{aT} M^{-1} \right)_{IJ} \ , \tag{2.12}$$

$$\Omega |k, \zeta^{i_1, \ldots, i_n}(k); I, J\rangle = |k, \zeta^{i_1, \ldots, i_n}(k); J, I\rangle \ , \tag{2.13}$$

where we permit M to be general $N \times N$ matrix.

e.g.1

$M = M^T = I_N$. In order for a vector to survive projection, λ must be $\lambda^T = -\lambda$. And then the gauge must be $SO(N)$.

e.g.2

If $M = -M^T = i \begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}$ then $\lambda^T = -M \lambda^T M$. And the gauge is $USp(N)$.

3 Fermionic string

3.1 Action

Let us start from a closed fermionic string. The action is

$$\begin{aligned}
&S[X^M, \psi^M] \\
&\quad = S_{\mathrm{B}}[X^M] + S_{\mathrm{F}}[\Psi^M] \\
&\quad = \frac{1}{2\pi\alpha'}\frac{1}{2}\int \mathrm{d}\tau_{\mathrm{M}}\mathrm{d}\sigma \left[(-)\eta^{mn}\partial_m X^M \partial_n X_M + i\bar{\Psi}^M \rho\cdot\partial\Psi_M\right] \;. \qquad (3.1)
\end{aligned}$$

Here Ψ^M's are D two dimensional Majorana fermions. Namely,

$$\Psi^M = \begin{pmatrix} \psi_1^M \\ \psi_2^M \end{pmatrix} \qquad \text{with } \psi_1^M, \psi_2^M \text{ real.} \qquad (3.2)$$

ρ^a's are two dimensional γ matrices satisfying

$$\left\{\rho^a, \rho^b\right\} = -2\eta^{ab} \,. \qquad (3.3)$$

One representation for these two dimensional γ matrices is

$$\rho^0 = \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \rho^2 = \rho^0\rho^1 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \,. \qquad (3.4)$$

$$\bar{\Psi}^M \equiv (\Psi^T)^M \rho^0 \,, \qquad (3.5)$$

$$\begin{aligned}
\bar{\Psi}\rho\cdot\partial\Psi &= (\Psi^T)^M(\partial_{\tau_{\mathrm{M}}} + \rho^0\rho^1\partial_\sigma)\Psi_M \\
&= {\psi_1}^M(\partial_{\tau_{\mathrm{M}}} - \partial_\sigma)\psi_{1M} + {\psi_2}^M(\partial_{\tau_{\mathrm{M}}} + \partial_\sigma)\psi_{2M} \,. \qquad (3.6)
\end{aligned}$$

${\psi_1}^M, {\psi_2}^M$ themselves can be either periodic or antiperiodic, they are called Neveu-Schwarz sector or Ramond sector:

$$\begin{aligned}
&{\psi_{1,2}}^M(\tau, \sigma = 0) = -{\psi_{1,2}}^M(\tau, \sigma = 2\pi) \qquad &&: \text{ Neveu-Schwarz sector} \\
&{\psi_{1,2}}^M(\tau, \sigma = 0) = +{\psi_{1,2}}^M(\tau, \sigma = 2\pi) \qquad &&: \text{ Ramond sector} \,. \qquad (3.7)
\end{aligned}$$

They are considered to be the two different sectors of the same Hilbert space. There are four sectors in total.

$$\begin{array}{cc} \psi_1 & \psi_2 \\ \text{NS} & \text{NS} \\ \text{NS} & \text{R} \\ \text{R} & \text{NS} \\ \text{R} & \text{R} \end{array} \tag{3.8}$$

3.2 Quantization in light-cone gauge

Now we fix this action to the light-cone gauge. The subsidiary condition is $\Psi^+ = 0$ in addition to $X^+ = X_0^+ + \alpha' P_0^+ \tau_{\mathrm{M}}$. And we repeat the same procedure as in the section 1.2.

3.2.1 momentum, Lagrangian, Hamiltonian

$$S = S_{\mathrm{B}}^{(\mathrm{L.C.})} + S_{\mathrm{F}}^{(\mathrm{L.C.})} , \tag{3.9}$$

where $S_{\mathrm{B}}^{(\mathrm{L.C.})} = \int \mathrm{d}X^+ L_{\mathrm{B}}^{(\mathrm{L.C.})}$ as before and

$$\begin{aligned} S_{\mathrm{F}}^{(LC)} &= \frac{1}{4\pi\alpha'} \frac{i}{\alpha' P_0^+} \int \mathrm{d}X^+ \int_{-\pi}^{\pi} \mathrm{d}\sigma (\Psi^T)^i (\alpha' P_0^+ \partial_+ + \rho^2 \partial_0) \Psi_i \\ &= \int \mathrm{d}X^+ L_{\mathrm{F}}^{(\mathrm{L.C.})} . \end{aligned} \tag{3.10}$$

$$\Pi_{\psi_{i\alpha}}(\tau_{\mathrm{M}}, \sigma) \equiv \frac{\delta L_{\mathrm{F}}^{(\mathrm{L.C.})}}{\delta(\partial_+ \psi_{i\alpha})} = -\frac{i}{4\pi\alpha'} \psi_{i\alpha} . \tag{3.11}$$

The light-cone Hamiltonian is introduced through by the canonical procedure.

$$\text{The light-cone Hamiltonian} = H_{\mathrm{B}}^{(\mathrm{L.C.})} + H_{\mathrm{F}}^{(\mathrm{L.C.})} = P_0^- . \tag{3.12}$$

The bosonic part is as before, while the fermionic part turns out to be

$$H_{\mathrm{F}}^{(\mathrm{L.C.})} = \frac{-i}{4\pi\alpha'^2 P_0^+} \int_{-\pi}^{\pi} \mathrm{d}\sigma \Psi^{t^i} \rho^2 \partial_\sigma \Psi_i . \tag{3.13}$$

3.2.2 commutation relation, mode expansion

Finally, quantization means

$$\{\Pi_{\psi_{i\alpha}}(\tau_{\rm M},\sigma),\psi_{j\beta}(\tau_{\rm M},\sigma')\}_+ = \frac{1}{2}(-i)\delta(\sigma-\sigma')\delta_{ij}\delta_{\alpha\beta}\ , \qquad (3.14)$$

$$i,j=1,...,D-2\ , \qquad \alpha,\beta=1,2\ .$$

$\frac{1}{2}$ of the light hand side of eq. (3.14) is due to the Majorana property. We obtain

$$\{\psi_{i\alpha}(\tau_{\rm M},\sigma),\psi_{j\beta}(\tau_{\rm M},\sigma')\}_+ = +2\pi\alpha'\delta(\sigma-\sigma')\delta_{ij}\delta_{\alpha\beta}\ . \qquad (3.15)$$

Let

$$\psi_1^i \equiv \sqrt{\alpha'}\psi^i \qquad \psi_2^i = \sqrt{\alpha'}\tilde{\psi}^i \qquad (3.16)$$

$$z = e^{i(\tau_{\rm M}+\sigma)} \qquad \bar{z} \equiv e^{i(\tau_{\rm M}-\sigma)}\ , \qquad (3.17)$$

$$\begin{aligned}
\psi^i &= \sum_{r\in\mathbf{Z}+1/2} b_r^i z^{-r} \quad :\mathrm{NS} \qquad & \tilde{\psi}^i &= \sum_{r\in\mathbf{Z}+1/2} \tilde{b}_r^i \bar{z}^{-r} \quad :\tilde{\mathrm{NS}} \\
\psi^i &= \sum_{n\in\mathbf{Z}} d_n^i z^{-n} \quad :\mathrm{R} \qquad & \tilde{\psi}^i &= \sum_{n\in\mathbf{Z}} \tilde{d}_n^i \bar{z}^{-n} \quad :\tilde{\mathrm{R}}\ ,
\end{aligned} \qquad (3.18)$$

$$\begin{aligned}
\{b_r, b_s\} &= \delta_{r+s,0} \qquad & \{\tilde{b}_r,\tilde{b}_s\} &= \delta_{r+s,0} \\
\{d_n, d_m\} &= \delta_{n+m,0} \qquad & \{\tilde{d}_n,\tilde{d}_m\} &= \delta_{n+m,0}\ .
\end{aligned} \qquad (3.19)$$

3.2.3 mass operator and number operator

The contribution of the fermionic oscillators to $\alpha'\widehat{m}^2$ is

$$\begin{aligned}
+2\alpha' P_0^+ H_{\rm F}^{\rm (L.C.)} &= -i\int_{-\pi}^{\pi}\frac{{\rm d}\sigma}{2\pi}\left(-\psi^i\partial_\sigma\psi^i + \tilde{\psi}^i\partial_\sigma\tilde{\psi}^i\right) \\
&= i\sum_{r\in\mathbf{Z}+\frac{1}{2}}\left(b_{-r}^i(-ir)b_r^i - \tilde{b}_{-r}^i(+ir)\tilde{b}_r^i\right) \\
&= \sum_{r\in\mathbf{Z}+\frac{1}{2}}\left(rb_{-r}^i b_r^i + r\tilde{b}_{-r}^i\tilde{b}_r^i\right) \qquad (\text{the case of NS, }\tilde{\mathrm{NS}})\ .
\end{aligned}$$

(3.20)

In general, eq. (3.20) is written as

$$+2\alpha' P_0^+ H_{\rm F}^{(\rm L.C.)} = \begin{pmatrix} \displaystyle\sum_{r\in\mathbf{Z}+\frac{1}{2}} r b^i_{-r} b^i_r \\ \displaystyle\sum_{n\in\mathbf{Z}} n d^i_{-n} d^i_n \end{pmatrix}^{\rm NS}_{\rm R} + \begin{pmatrix} \displaystyle\sum_{r\in\mathbf{Z}+\frac{1}{2}} r \tilde{b}^i_{-r} \tilde{b}^i_r \\ \displaystyle\sum_{n\in\mathbf{Z}} n \tilde{d}^i_{-n} \tilde{d}^i_n \end{pmatrix}^{\widetilde{\rm NS}}_{\tilde{\rm R}} . \tag{3.21}$$

Therefore mass operator is expressed as

$$\begin{aligned} \alpha' \widehat{m}^2 &= \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n + \begin{cases} \displaystyle\sum_{r\in\mathbf{Z}+\frac{1}{2}} r b^i_{-r} b^i_r & \text{NS} \\ \displaystyle\sum_{n\in\mathbf{Z}} n d^i_{-n} d^i_n & \text{R} \end{cases} \\ &\quad + \sum_{n\neq 0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \begin{cases} \displaystyle\sum_{r\in\mathbf{Z}+\frac{1}{2}} r \tilde{b}^i_{-r} \tilde{b}^i_r & \widetilde{\text{NS}} \\ \displaystyle\sum_{n\in\mathbf{Z}} n \tilde{d}^i_{-n} \tilde{d}^i_n & \tilde{\text{R}} \end{cases} \\ &= 2(\hat{N} + \hat{\tilde{N}} + \text{const.} + \widetilde{\text{const.}}) , \end{aligned} \tag{3.22}$$

$$\hat{N} = \sum_{n=1} \alpha^i_{-n}\alpha^i_n + \begin{cases} \displaystyle\sum_{r=\frac{1}{2},\dots} r b^i_{-r} b^i_r & \text{NS} \\ \displaystyle\sum_{m=1,\dots}^{\infty} m d^i_{-m} d^i_m & \text{R} \end{cases} . \tag{3.23}$$

We need to evaluate two constants, const. and $\widetilde{\text{const.}}$, associated with normal ordering. This can be done by the zeta function regularization and the computation is below.

<u>NS</u>

$$\begin{aligned} \text{const.} &= \frac{(D-2)}{2}\sum_{n=1}^{\infty} n + \frac{(D-2)}{2}\sum_{r=\frac{1}{2},\frac{3}{2},\dots} (-r) \\ &= \frac{(D-2)}{2}\left(\sum_{n=1}^{\infty} n - \frac{1}{2}\left(\sum_{n=1}^{\infty} n - 2\sum_{n=1}^{\infty} n\right)\right) \\ &= \frac{(D-2)}{2}\frac{3}{2}\zeta(-1) = -\frac{(D-2)}{16} . \end{aligned} \tag{3.24}$$

R

$$\text{const.} = \frac{(D-2)}{2}\sum_{n=1}^{\infty} n + \frac{(D-2)}{2}\sum_{m=1}^{\infty}(-m) = 0\,. \tag{3.25}$$

Similarly for $\widetilde{\text{const}}$.

Therefore

$$\alpha'\widehat{m}^2 = 2\left(\hat{N} + \hat{\tilde{N}} + \begin{array}{cccc} \frac{-(D-2)}{16} & \text{NS} & + & \frac{-(D-2)}{16}\ \tilde{\text{NS}} \\ 0 & \text{R} & + & 0 \quad \tilde{\text{R}} \end{array}\right)\,. \tag{3.26}$$

Invariance under the shift $\sigma \to \sigma + \Delta$ translates into

$$(\hat{N} - \hat{\tilde{N}})\,|\,\rangle = 0\,. \tag{3.27}$$

3.2.4 critical dimension

Again we check the Lorentz commutators. Recall that the Lorentz generators are

$$M^{MN} = \int \frac{\mathrm{d}\sigma}{2\pi}\left(X^M \mathcal{P}^N - X^N \mathcal{P}^M\right)\,. \tag{3.28}$$

And we examine

$$[M^{-i}, M^{-j}] = 0\,. \tag{3.29}$$

From that, we can derive

$$D = 10\,. \tag{3.30}$$

3.2.5 state space

Let's discuss the state space. Let ψ^i and $\tilde{\psi}^i$ be worldsheet fermions.

Neveu-Schwarz sector

NS sector is expanded

$$\psi^i = \sum_{r\in\mathbf{Z}+\frac{1}{2}} b^i_r z^{-r}\,. \tag{3.31}$$

The expansion was already indicated in eq. (3.18). The NS ground state is

$$|0\rangle_{\rm NS} \qquad \text{such that} \qquad b_r^i|0\rangle_{\rm NS} = 0\,, \qquad r = \frac{1}{2}, \frac{3}{2}, \ldots\,. \tag{3.32}$$

i is the vector index to label the $(D-2)$ transverse directions of our spacetime. $|0\rangle_{\rm NS}$ is not degenerate, namely, singlet. This state space

$$\mathcal{H}_{\rm NS} = \{\{b^{i_1}_{-r_1}\cdots b^{i_n}_{-r_n}\alpha^{j_1}_{-m_1}\cdots\alpha^{j_\ell}_{-m_\ell}|0\rangle_{\rm NS}\}\} \tag{3.33}$$

creates vector and in general rank n tensors from the singlet ground state $|0\rangle_{\rm NS}$. The spacetime consists of the spacetime bosons.

Ramond sector

As is indicated in eq. (3.18), the expansion is

$$\psi^i = \sum_{m\in\mathbf{Z}} d^i_m z^{-m}\,. \tag{3.34}$$

The ground state is

$$|0\rangle_{\rm R} \qquad \text{such that} \qquad d^i_m|0\rangle_{\rm R} = 0\,, \qquad m = 1, 2, 3, \ldots\,. \tag{3.35}$$

It is important to note

$$\{d^i_0, d^j_0\} = \delta^{ij}\,, \qquad {d^i_0}^\dagger = d^i_0\,. \tag{3.36}$$

If $|0\rangle_{\rm R}$ is a ground state, and then $d^i_0|0\rangle_{\rm R}$ is also a ground state. And therefore ground state is degenerate and must form a representation of the Clifford algebra. Therefore the Ramond ground state must carry a spinor index. And the Ramond ground state

$$|\,\rangle_{\rm R} = |\alpha\rangle_{\rm R} \tag{3.37}$$

must form a spinor representation of $SO(8)$. We can think of d^i_0 as Hermitian matrices obeying eq. (3.36). Therefore $2^{\frac{D-2}{2}} = 2^4 = 16$ dimension is required.

Let d^i_0 to be

$$d^i_0 = \frac{1}{\sqrt{2}}\gamma^i_{16}\,, \qquad i = 1, \ldots, 8\,. \tag{3.38}$$

γ^i_{16} is essentially 8 Euclidean dimensional gamma matrices. These matrices are known to be realized as a block diagonal form:

$$\gamma^i_{16} = \begin{pmatrix} 0 & \gamma^i_{8a\dot{a}} \\ (\gamma^T_8)^i{}_{\dot{b}b} & 0 \end{pmatrix} . \tag{3.39}$$

Therefore the 16 dimensional representation is actually reducible.

Let γ^9_{16} to be the product of d^i_0 then it anticommute with all γ^i_{16}:

$$\gamma^9_{16} \equiv \prod_{i=1}^{8} (\sqrt{2} d^i_0) \qquad \text{and} \qquad \left\{ \gamma^9_{16}, \gamma^i_{16} \right\}_+ = 0 . \tag{3.40}$$

Therefore one can assign + or − eigenvalue to the Ramond ground state:

$$\gamma^9_{16} |\,\rangle_{\mathrm{R}} = \begin{cases} +|\,\rangle_{\mathrm{R}} \\ -|\,\rangle_{\mathrm{R}} \end{cases} . \tag{3.41}$$

And this is considered to be the chirality of 8 dimensional spacetime in the transverse direction. So we can write this as

$$|\,\rangle_{\mathrm{R}} = |⑯\rangle = |⑧_{\mathrm{S}}\rangle \oplus |⑧_{\mathrm{C}}\rangle . \tag{3.42}$$

$|⑧_{\mathrm{S}}\rangle$ is a spinor representation with positive chirality while $|⑧_{\mathrm{C}}\rangle$ is another spinor representation of negative chirality.

To summarize, Ramond Hilbert space is

$$\mathcal{H}_{\mathrm{R}} = \{ \{ d^{i_1}_{-m_1} \cdots d^{i_r}_{-m_r} \alpha^{j_1}_{-n_1} \cdots \alpha^{j_s}_{-n_s} |⑯\alpha\rangle \} \} \tag{3.43}$$

created from the 16 dimensional spinorial states. It consists of spacetime fermions.

Let's list the low lying spectrum of the entire state space. From $(\mathrm{NS}, \tilde{\mathrm{NS}})$ sector, we obtain

- ground state scalar $|0\rangle_{\mathrm{NS}} \otimes |0\rangle_{\tilde{\mathrm{NS}}} \otimes |k\rangle$ with intercept $\alpha' m^2 = 2\left(-\frac{1}{2} - \frac{1}{2}\right) = -2$.

- massless rank 2 tensor $b^i_{-1/2}|0\rangle_{\rm NS}\otimes\tilde{b}^j_{-1/2}|0\rangle_{\tilde{\rm NS}}\otimes|k\rangle\zeta_{ij}(k)$ with polarization tensor $\zeta_{ij}(k)$, which is nothing but the wave function in k space.

From $(\mathrm{NS},\tilde{\mathrm{R}})$ sector, we get

- 16 dimensional fermion $|0\rangle_{\rm NS}\otimes|\textcircled{\scriptsize 16}\tilde{\alpha}\rangle_{\tilde{\rm R}}\otimes|k\rangle\psi_{\tilde{\alpha}}(k)$ with intercept $\alpha' m^2 = 2\left(-\frac{1}{2}\right) = -1$.
- massless vector spinor $b^i_{-1/2}|0\rangle_{\rm NS}\otimes|\textcircled{\scriptsize 16}\tilde{\alpha}\rangle_{\tilde{\rm R}}\otimes|k\rangle\chi^i_{\tilde{\alpha}}(k)$.

As for $(\mathrm{R},\tilde{\mathrm{NS}})$ sector, we replace tilde states by untilde states. For $(\mathrm{R},\tilde{\mathrm{R}})$ sector, we get massless bosonic bi-spinorial states $|\textcircled{\scriptsize 16}\alpha\rangle_{\rm R}\otimes|\textcircled{\scriptsize 16}\tilde{\alpha}\rangle_{\tilde{\rm R}}$, $\alpha' m^2 = 0$.

3.3 The case of open fermionic string

The action and the equation of motion is same as the case of closed string. We parametrize the length of the open string by σ. The range of σ is $0<\sigma<\pi$ this time. Fermionic part of the action is written as

$$\bar{\Psi}^M\rho\cdot\partial\Psi_M = \psi_1^M(\partial_{\tau_{\rm M}}-\partial_\sigma)\psi_{1M}+\psi_2^M(\partial_{\tau_{\rm M}}+\partial_\sigma)\psi_{2M} \tag{3.44}$$

as before. But there is a boundary in this case and variation of the action tells

$$\bar{\Psi}^M\rho^2\delta\Psi_M\,|_{\sigma=0}^{\sigma=\pi}=0\,. \tag{3.45}$$

This means

$$\bar{\Psi}^M\rho^2\delta\Psi_M\,|_{\sigma=0}^{\sigma=\pi}=-\psi_1\delta\psi_1\,|_{\sigma=0}^{\sigma=\pi}+\psi_2\delta\psi_2\,|_{\sigma=0}^{\sigma=\pi}=0 \tag{3.46}$$

and

$$\psi_1\delta\psi_1=\psi_2\delta\psi_2 \text{ at } \sigma=0,\pi \tag{3.47}$$

at the boundary. The equation of motion is the first order equation. So setting the boundary condition for ψ_1 and the boundary condition for ψ_2

independently is too strong. $\psi_1 = \pm\psi_2$ at $\sigma = 0$ implies $\delta\psi_1 = \pm\delta\psi_2$ at $\sigma = 0$ and the same is true for at $\sigma = \pi$. Without losing generality, we can state $\psi_1(\tau_{\rm M}, \sigma = 0) = \psi_2(\tau_{\rm M}, \sigma = 0)$. Then there is a choice for the sign at the other end $\sigma = \pi$. If

$$\psi_1(\tau_{\rm M}, \sigma = \pi) = -\psi_2(\tau_{\rm M}, \sigma = \pi) \, , \tag{3.48}$$

it is the Neveu-Schwarz sector. If

$$\psi_1(\tau_{\rm M}, \sigma = \pi) = \psi_2(\tau_{\rm M}, \sigma = \pi) \, , \tag{3.49}$$

it is the Ramond sector. Modes of ψ_1 and those of ψ_2 are no longer independent.

$$\psi_1^i = \sqrt{\alpha'}\psi^i \, , \qquad \psi_2^i = \sqrt{\alpha'}\tilde{\psi}^i \, . \tag{3.50}$$

The light-cone quantization is carried out as before but there is a slight change in the notation which we can tabulate in

$$\begin{array}{ccc} \text{closed} & & \text{open} \\ P_0^+ & \to & 2P_0^+ \\ \int_{-\pi}^{\pi} & \to & \int_0^{\pi} \to \int_{-\pi}^{\pi} \\ \begin{matrix} b \\ \tilde{b} \end{matrix} & \to & \begin{matrix} b \\ b \end{matrix} \to b \end{array} \tag{3.51}$$

and

$$\alpha' m^2 = \frac{1}{2}\left(\sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i + \begin{cases} \displaystyle\sum_{n,i\in \mathbf{Z}+1/2} r b_{-r}^i b_r^i & \text{NS} \\ \displaystyle\sum_{m,i\in\mathbf{Z}} m d_{-m}^i d_m^i & \text{R} \end{cases} \right) = \hat{N} + \text{const.} \, . \tag{3.52}$$

Ant the normal ordering coefficient is computed as before:

$$\text{const.} = \begin{cases} -\frac{(D-2)}{16} & \text{NS} \\ 0 & \text{R} \end{cases} . \tag{3.53}$$

Critical dimension is $D = 10$. The Neveu-Schwarz sector consists of spacetime bosons, and the Ramond sector consists of spacetime fermions.

The spectrum is as follows:

- NS sector: the ground state scalar $|0\rangle_{\rm NS}\otimes|k\rangle$ has an intercept $\alpha' m^2 = -\frac{1}{2}$. The next level is massless vector $b^i_{-1/2}|0\rangle_{\rm NS}\otimes|k\rangle\zeta_i(k)$ $(i=1,...,8)$.
- R sector: the ground state is massless 16 dimensional spinor state $|\alpha\rangle_{\rm R}\otimes|k\rangle$.

4 $\hat{U}(1)$ character

In general, state space is called module and the module of the Heisenberg algebra is essentially the Fock space of a free boson. The highest weight state of the Heisenberg algebra $|p;\{0\}\rangle$ is labelled by the momentum $(\alpha_0|p\rangle = \sqrt{\frac{\alpha'}{2}}p|p\rangle)$ with zero occupation number $\{0\}$. And the states built up from this heightest weight state are of the form

$$\alpha_{-1}^{n_1}\alpha_{-2}^{n_2}\cdots|p;\{0\}\rangle \propto |p:n_1,n_2,...\rangle\,. \tag{4.1}$$

Let me recall the definition of character in finite dimensional Lie algebra $\mathfrak{g}$. We denote by the pair (ρ,V) representation of this finite dimensional Lie algebra.

Let H_i be the elements of Cartan subalgebra and suppose that the representation matrices $H_i=\rho(h_i)$ are all diagonal. Then we define the following object

$${\rm ch}V = {\rm tr}_V(x_1^{H_1}\cdots x_\ell^{H_\ell})\,, \qquad (x_1,\cdots,x_\ell)\in V\,. \tag{4.2}$$

We decompose V into the direct sum of the submodule:

$$V=\bigoplus_\mu V(\mu)\,. \tag{4.3}$$

$V(\mu)$ is defined by the vector with eigenvalue of $h_iv=\mu_iv$:

$$V(\mu)\equiv\{v\in V|\, h_iv=\mu_iv\}\,. \tag{4.4}$$

Then the ch$V$ is written like

$$\mathrm{ch}V = \sum_{\mu_1,\ldots,\mu_\ell} \mathrm{dim}V(\mu) x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell} \,. \tag{4.5}$$

So let L_0 be

$$L_0 \equiv \frac{1}{2} : \sum_n \alpha_{-n}\alpha_n := \sum_{n=1} \alpha_{-n}\alpha_n + \frac{1}{2}\alpha_0^2 \,. \tag{4.6}$$

Namely, this is the zeroth element of the Virasoro algebra acting on the plane. Then the commutator of L_0 is

$$[L_0, \alpha_{-m}] = \left[\sum \alpha_{-n}\alpha_n, \alpha_{-m}\right] = m\alpha_{-m} \,. \tag{4.7}$$

Therefore we may include L_0 as the derivation in the Cartan subalgebra of $\hat{U}(1)$.

As the restricted character, we define $\mathrm{ch}_{\hat{U}(1),p}(q)$ to be the trace over the Fock space with momentum p. If expand in the power series of q, we get the following expression.

$$\mathrm{ch}_{\hat{U}(1),p}(q) \equiv \mathrm{Tr}_{p,\mathrm{Fock}} q^{L_0} = \sum_{m=0}^{\infty} p(m) q^m q^{\frac{1}{2}\left(\sqrt{\frac{\alpha'}{2}}p\right)^2} \,. \tag{4.8}$$

Here we have denoted by $p(m)$ the number of partition of m into positive integers:

$$\underbrace{1+1+\cdots+1}_{n_1} + \underbrace{2+2+\cdots+2}_{n_2} + \cdots = m \,. \tag{4.9}$$

So,

$$\sum_{m=0}^{\infty} p(m) q^m = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{\varphi(q)} \tag{4.10}$$

and $\varphi(q)$ is called Euler function.

For r folded Heisenberg algebra, the appropriate character is written as

$$\mathrm{ch}_{(x_1)^r,p_1\ldots p_r}(q) = \sum_{m=0}^{\infty} p_r(m) q^m q^{\frac{\alpha'}{4}\sum_{i=1}^r (p^i)^2} \,. \tag{4.11}$$

Here $p_r(m)$ is number of ways of separating m into positive integers of r "colors". The sum is given by $\frac{1}{\varphi(q)^r}$.

5 Partition function

5.1 Bosonic string partition function on torus

One-loop vacuum amplitudes in string perturbation theory carry an essential information on the spectrum of a free string. The worldsheet geometry of a torus corresponds to the one-loop vacuum amplitude of an oriented closed string.

To determine the normalization of the amplitude, it is better to start from quantum field theory and let us take a one component real scalar field in D dimensions. We set $c = \hbar$=1 and then the action is

$$S = \int \mathrm{d}^D x \left(\frac{1}{2} \partial^M \phi \partial_M \phi - \frac{1}{2} M^2 \phi^2 \right) . \tag{5.1}$$

In this section we use the following metric:

$$\eta^{MN} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} . \tag{5.2}$$

In order to go to the Euclidean field theory, we set $ix_0 = x_\mathrm{E}$.

$$S_\mathrm{E} = \int \mathrm{d}^D x_\mathrm{E} \phi (-\Delta_\mathrm{E} + M^2) \phi . \tag{5.3}$$

Then one-loop partition function in quantum field theory which is written as

$$\begin{aligned} \mathrm{e}^{-\Gamma_\mathrm{E}} = \mathrm{e}^{i\Gamma_{1-\mathrm{loop}}} = Z^{\mathrm{QFT}}_{\mathrm{Vac.1-loop}} &= \int [\mathcal{D}\phi] \mathrm{e}^{iS} \\ &= \int [\mathcal{D}\phi] \mathrm{e}^{-S_\mathrm{E}} = (\mathrm{const.})\ \mathrm{Det}^{-\frac{1}{2}} (-\Delta_\mathrm{E} + M^2) . \end{aligned} \tag{5.4}$$

These obtained by the functional integral of the real scalar field theory which is a determinant $-\frac{1}{2}$. Using the identity

$$\log \frac{a}{b} = - \int_0^\infty \frac{\mathrm{d}t}{t} \left(\mathrm{e}^{-at} - \mathrm{e}^{-bt} \right) , \tag{5.5}$$

we obtain

$$
\begin{aligned}
-\Gamma_{\mathrm{E}} &= -\frac{1}{2}\log \mathrm{Det}(-\Delta_{\mathrm{E}} + M^2) + (\mathrm{const.})' \\
&= -\frac{1}{2}\mathrm{Tr}\ln\left[\frac{(-\Delta_{\mathrm{E}} + M^2)}{N}\right] \\
&= \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{Tr}\left(\mathrm{e}^{-t(-\Delta_{\mathrm{E}}+M^2)} - \mathrm{e}^{-tN}\right) .
\end{aligned}
\tag{5.6}
$$

So we conclude that, up to the subtraction of ∞, Γ_{E} is expressed by

$$
\begin{aligned}
\Gamma_{\mathrm{E}} &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{Tr}\mathrm{e}^{-t(-\Delta_{\mathrm{E}}+M^2)} \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\mathrm{Tr}\mathrm{e}^{-t(-\Delta_{\mathrm{E}})} \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\sum_n \left\langle n\left|\mathrm{e}^{-t(-\Delta_{\mathrm{E}})}\right|n\right\rangle \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\int \mathrm{d}^D x_{\mathrm{E}}\int \mathrm{d}^D y_{\mathrm{E}}\sum_n \langle n|y\rangle \left\langle y\left|\mathrm{e}^{-t(-\Delta_{\mathrm{E}})}\right|x\right\rangle \langle x|n\rangle \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\int \mathrm{d}^D x_{\mathrm{E}}\int \mathrm{d}^D y_{\mathrm{E}} \left\langle y\left|\mathrm{e}^{-t(-\Delta_{\mathrm{E}})}\right|x\right\rangle \delta^{(D)}(x-y) \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\int \mathrm{d}^D x_{\mathrm{E}} \left\langle x\left|\mathrm{e}^{-t(-\Delta_{\mathrm{E}})}\right|x\right\rangle \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\int \mathrm{d}^D x_{\mathrm{E}}\ \mathrm{e}^{-t(-\Delta_{\mathrm{E}})}\delta^{(D)}(x-y)\big|_{y=x} \\
&= -\frac{1}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}V_{\mathrm{E}}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\mathrm{e}^{-tp^2} .
\end{aligned}
\tag{5.7}
$$

t is a parameter called proper time and V_{E} is the spacetime volume. And UV divergences contain in $t = 0$ end which we need to regulate. Carrying out the momentum integration, we obtain

$$
\begin{aligned}
\Gamma_{\mathrm{E}} &= -\frac{V_{\mathrm{E}}}{2}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tM^2}\int \frac{\mathrm{d}^D p_{\mathrm{E}}}{(2\pi)^D}\mathrm{e}^{-tp_{\mathrm{E}}^2} \\
&= -\frac{V_{\mathrm{E}}}{2(4\pi)^{\frac{D}{2}}}\int_\varepsilon^\infty \frac{\mathrm{d}t}{t^{\frac{D}{2}+1}}\mathrm{e}^{-tM^2} .
\end{aligned}
\tag{5.8}
$$

In the case of a Dirac fermion in D dimension, we instead obtain

$$\Gamma_{\mathrm{E}}^{\mathrm{Dirac}} = +\frac{V_{\mathrm{E}} 2^{\left[\frac{D}{2}\right]}}{2(4\pi)^{\frac{D}{2}}} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{t^{\frac{D}{2}+1}} \mathrm{e}^{-tM^2} \,. \tag{5.9}$$

Let us now apply this formula, namely, formula eq. (5.8), to $D = 26$ dimensional closed bosonic string. Recall that mass operator

$$\alpha' \widehat{m}^2 = 2(\hat{N} + \hat{\tilde{N}} - 2) \,. \tag{5.10}$$

Then

$$\begin{aligned}
\Gamma_{\text{"closed bosonic"}} &= -\frac{V_{\mathrm{E}}}{2(4\pi)^{13}} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{t^{14}} \mathrm{tr}_{\mathrm{phys}} \mathrm{e}^{-tm^2} \\
&= -\frac{V_{\mathrm{E}}}{2(4\pi)^{13}} \frac{1}{\alpha'^{13}} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{t^{14}} \frac{\alpha'^{14}}{\alpha'} \mathrm{tr}_{\mathrm{phys}} \mathrm{e}^{-\frac{t}{\alpha'}\alpha' m^2} \\
&\overset{\frac{t}{\alpha'} \equiv \pi\tau_2}{=} -\frac{V_{\mathrm{E}}}{2(4\pi^2\alpha')^{13}} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^{14}} \mathrm{tr}_{\mathrm{phys}} \mathrm{e}^{-2\pi\tau_2 \cdot (\hat{N} + \hat{\tilde{N}} - 2)} \\
&= -\frac{V_{\mathrm{E}}}{2(4\pi^2\alpha')^{13}} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^{14}} \sum_{\mathrm{phys}} \left\langle \mathrm{phys} \left| \mathrm{e}^{-2\pi\tau_2 \cdot (\hat{N} + \hat{\tilde{N}} - 2)} \right| \mathrm{phys} \right\rangle \,.
\end{aligned} \tag{5.11}$$

Due to the level matching condition $(\hat{N} - \hat{\tilde{N}})|\mathrm{phys}\rangle = 0$, inside the trace, we can insert

$$\mathbf{1} = \delta_{N,\tilde{N}} = \int_{-1/2}^{1/2} \mathrm{d}\tau_1 \mathrm{e}^{2\pi i \tau_1 (\hat{N} - \hat{\tilde{N}})} \,. \tag{5.12}$$

Note that the eigenvalue of N and $\tilde{N}$ are integers. Therefore

$$\begin{aligned}
&\Gamma_{\text{"closed bosonic"}} \\
&= -\frac{V_{\mathrm{E}}}{2(4\pi^2\alpha')^{13}} \int_{-1/2}^{1/2} \mathrm{d}\tau_1 \int_{\varepsilon}^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^2} \frac{1}{\tau_2^{12}} \mathrm{tr} \mathrm{e}^{+2\pi i(\tau_1 + i\tau_2)(\hat{N}-1)} \mathrm{e}^{-2\pi i(\tau_1 - i\tau_2)(\hat{\tilde{N}}-1)} \\
&= -\frac{V_{\mathrm{E}}}{2(4\pi^2\alpha')^{13}} \int \frac{\mathrm{d}^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \mathrm{tr} q^{(\hat{N}-1)} \bar{q}^{(\hat{\tilde{N}}-1)} \,,
\end{aligned} \tag{5.13}$$

$$q = \mathrm{e}^{2\pi i \tau} \,, \quad \bar{q} = \mathrm{e}^{-2\pi i \bar{\tau}} \,, \quad \tau = \tau_1 + i\tau_2$$

is true.

We can undo the integrations of the 24 light-cone momenta by

$$\int \frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}} \mathrm{e}^{-t(p^i)^2} = \left(\frac{1}{4\pi t}\right)^{12} = \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^{12} . \tag{5.14}$$

Then the expression becomes

$$\Gamma_{\text{"closed bosonic"}} = -\frac{V_{\mathrm{E}}}{2}\frac{1}{4\pi^2\alpha'}\int\frac{\mathrm{d}^2\tau}{\tau_2^2}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\mathrm{tr}\left(\mathrm{e}^{-\pi\alpha'\tau_2(p^i)^2}q^{(\hat{N}-1)}\bar{q}^{(\hat{\tilde{N}}-1)}\right) . \tag{5.15}$$

Recalling

$$\hat{N} = \frac{1}{2} : \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n : , \qquad \hat{\tilde{N}} = \frac{1}{2} : \sum_{n\neq 0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n : \tag{5.16}$$

and

$$\alpha^i_0 = \tilde{\alpha}^i_0 \equiv \sqrt{\frac{\alpha'}{2}}p^i , \tag{5.17}$$

the $\Gamma_{\text{"closed bosonic"}}$ can be written as

$$\Gamma_{\text{"closed bosonic"}} = -\frac{V_{\mathrm{E}}}{2}\frac{1}{4\pi^2\alpha'}\int\frac{\mathrm{d}^2\tau}{\tau_2^2}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\mathrm{tr}q^{(\hat{L}_0-1)}\bar{q}^{(\hat{\tilde{L}}_0-1)} . \tag{5.18}$$

Here

$$\hat{L}_0 = \frac{1}{2}\sum_n : \alpha^i_{-n}\alpha^i_n : , \qquad \hat{\tilde{L}}_0 = \frac{1}{2}\sum_n : \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n : \tag{5.19}$$

$$\hat{L}_0 - 1 \equiv \hat{L}_0^{(\text{cylinder})} , \qquad \hat{\tilde{L}}_0 - 1 \equiv \hat{\tilde{L}}_0^{(\text{cylinder})} . \tag{5.20}$$

Using the notation of section 4, we can write eq. (5.18) as

$$\begin{aligned}&\Gamma_{\text{"closed bosonic"}}\\ &= -\frac{V_{\mathrm{E}}}{2}\frac{1}{4\pi^2\alpha'}\int\frac{\mathrm{d}^2\tau}{\tau_2^2}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\left(\mathrm{ch}_{\hat{U}(1)^{24},p^i}(q)\,q^{-1}\right)\left(\mathrm{ch}_{\hat{U}(1)^{24},p^i}(\bar{q})\,\bar{q}^{-1}\right) .\end{aligned} \tag{5.21}$$

Finally the modular invariance restricts the integration of the complex proper time $\int\frac{\mathrm{d}^2\tau}{\tau_2^2}$ into the fundamental domain $\int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2^2}$.

In order to evaluate eq. (5.18), we get back to eq. (5.13) to carry out the trace. And we obtain

$$\Gamma_{\text{closed bosonic}} = -\frac{V_{\rm E}}{2(4\pi^2\alpha')^{13}}\int_{\mathcal{F}}\frac{{\rm d}^2\tau}{\tau_2^2}\frac{1}{\tau_2^{12}|\eta(\tau)|^{48}}\,, \tag{5.22}$$

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{5.23}$$

The Virasoro character is defined as

$$\chi^{\rm Vir}_{(h_i,c)}(q) \equiv {\rm tr}_{h^i} q^{L_0-\frac{c}{24}} = q^{h_i-\frac{c}{24}}\sum_{m=0}^{\infty} d_m q^m\,, \tag{5.24}$$

where d_m is the number of degeneracies at level m.

Therefore we can recast Γ into

$$\Gamma = -\frac{V_{\rm E}}{2}\frac{1}{4\pi^2\alpha'}\mathcal{T}_{\text{torus}}\,, \tag{5.25}$$

$$\mathcal{T}_{\text{torus}} = \int_{\mathcal{F}}\frac{{\rm d}^2\tau}{\tau_2^2}\int\frac{{\rm d}^{24}p^i}{(2\pi)^{24}}\left|\chi^{\rm Vir}_{(h^i=\frac{\alpha'}{4}p^i,c=24)}(q)\right|^2\,. \tag{5.26}$$

In the more general case, one-loop torus partition function takes the following form:

$$\mathcal{T}_{\text{torus}} = \int_{\mathcal{F}}\frac{{\rm d}^2\tau}{\tau_2^2}\sum_{i,j}\bar{\chi}^{\rm Vir}_{h_i}(\bar{q})X_{ij}\chi^{\rm Vir}_{h_j}(q)\,. \tag{5.27}$$

$\sum_{i,j}$ implies generalization of momentum integration and X_{ij} denotes some matrix.

Let us finally mention that for $c=1$ Gaussian model at generic $h=\frac{\alpha'}{4}p^2$, Virasoro character and $U(1)$ character are related by

$$\chi^{\rm Vir}_{(h,c=1)}(q) = \frac{q^h}{\eta(q)} = {\rm ch}_{U(1),p}(q)q^{-\frac{1}{24}}\,. \tag{5.28}$$

First equality is true at generic h, and second equality is always true.

5.2 Fermionic string partition function on torus

5.2.1 GSO projection and spacetime supersymmetry

Now we turn to the fermionic string.

Let us consider the following trace.

$$\mathrm{Tr}^{(\mathrm{NS})} q^{L_0^{(\mathrm{NS})}} , \qquad \mathrm{Tr}^{(\mathrm{R})} q^{L_0^{(\mathrm{R})}} \tag{5.29}$$

or, after the momentum integration, we are left with the trace over the oscillators

$$\mathrm{tr}^{(\mathrm{NS})} q^{\hat{N} - \frac{D-2}{16}} , \qquad \mathrm{tr}^{(\mathrm{R})} q^{\hat{N}} . \tag{5.30}$$

The goal is to find a sensible projection which make sure of the integer spacing of the spectrum and this turned out to get rid of the tachyon from the spectrum. This is called "$\mathbf{Z}_2$ grading".

Introduce $(-)^F$ such that

$$\{(-)^F, \psi^i(\tau_{\mathrm{M}}, \sigma)\}_+ = 0 \tag{5.31}$$

is true. Explicitly this is done by

$$(-)^F = \begin{cases} -(-)^{\sum_{i=1}^{D-2} \sum_{r=\frac{1}{2},\frac{3}{2},\dots} b^i_{-r} b^i_r} & (\mathrm{NS}) \\ (+ \text{ or } -)\gamma^9_{16} (-)^{\sum_{i=1}^{D-2} \sum_{m=1}^{\infty} d^i_{-m} d^i_m} & (\mathrm{R}) \end{cases} . \tag{5.32}$$

And then we project the state space into $\mathbf{Z}_2$ even states.

$$\mathcal{H}_{\mathrm{GSO}} = \left\{ \{ |\psi\rangle \, \big| |\psi\rangle \in \mathcal{H}_{\mathrm{NS}} \oplus \mathcal{H}_{\mathrm{R}} \, , \ (-)^F |\psi\rangle = |\psi\rangle \} \right\} . \tag{5.33}$$

In the NS sector, tachyon is projected out and the lowest is eight dimensional vector $\mathbf{8}_V$. In the R sector, spinor states $\mathbf{8}_S$ are kept while conjugate spinor $\mathbf{8}_C$ is removed, or vice versa.

Now for $D = 10$, after this projection, trace becomes

$$\begin{aligned}
&\mathrm{tr}^{(\mathrm{NS})}\left(\frac{1+(-)^F}{2}\right)q^{\hat{N}-\frac{1}{2}} \\
&= \frac{1}{2q^{\frac{1}{2}}}\left[\prod_{m=1}^{\infty}\left(\frac{(1+q^{m-\frac{1}{2}})}{(1-q^m)}\right)^8 - \left(\prod_{m=1}^{\infty}\frac{(1-q^{m-\frac{1}{2}})}{(1-q^m)}\right)^8\right] \\
&\equiv f_{\mathrm{NS}}(q)\,.
\end{aligned} \tag{5.34}$$

Note that $(-)^F$ reverses the sign of all contributions with odd number of fermion oscillators.

$$\mathrm{tr}^{(\mathrm{R})}\left(\frac{1+(-)^F}{2}\right)q^{\hat{N}} = 8\prod_{m=1}^{\infty}\left(\frac{1+q^m}{1-q^m}\right)^8 \equiv f_{\mathrm{R}}(q)\,. \tag{5.35}$$

Now $f_{\mathrm{NS}}(q) = f_{\mathrm{R}}(q)$ is true. This is the Jacobi elliptic function identity.

$$\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}})^8 - \prod_{m=1}^{\infty}(1-q^{m-\frac{1}{2}})^8 \equiv 16q^{\frac{1}{2}}\prod_{m=1}^{\infty}(1+q^m)^8\,. \tag{5.36}$$

Often quoted as

$$\vartheta_3^4(0) = \vartheta_2^4(0) + \vartheta_4^4(0)\,. \tag{5.37}$$

So the spectrum is supersymmetric.

The ground state is $\mathbf{8}_V \oplus \mathbf{8}_S$ vector multiplet of $D = 10$, $\mathcal{N} = 1$ spacetime supersymmetry.

Now it turns to the closed string. We can just repeat the computation to obtain

$$\begin{aligned}
&\mathrm{tr}^{(\cdot,\tilde{\cdot})}\left(\frac{1+(-)^F}{2}\right)\left(\frac{1+(-)^{\tilde{F}}}{2}\right)q^{L_0^{(\mathrm{cylinder})}}\bar{q}^{\bar{L}_0^{(\mathrm{cylinder})}} \\
&\qquad = \mathrm{tr}^{(\cdot)}\left(\frac{1+(-)^F}{2}\right)q^{L_0^{(\mathrm{cyl})}}\mathrm{tr}^{(\tilde{\cdot})}\left(\frac{1+(-)^{\tilde{F}}}{2}\right)\bar{q}^{\bar{L}_0^{(\mathrm{cyl})}}\,,
\end{aligned} \tag{5.38}$$

where $(\cdot,\tilde{\cdot}) = (\mathrm{NS},\tilde{NS})$, $(\mathrm{NS},\tilde{R})$, $(\mathrm{R},\tilde{NS})$, $(\mathrm{R},\tilde{R})$.

Now

$$
(-)^F = \begin{cases} -(-)^{\sum_{i=1}^{D-2}\sum_{r=\frac{1}{2},\ldots} b^i_{-r} b^i_r} & \text{(NS)} \\ +\gamma^9_{16}(-)^{\sum_{i=1}^{D-2}\sum_{m=1}^{\infty} d^i_{-m} d^i_m} & \text{(R)} \qquad (+ : \text{our convention}) \end{cases} , \quad (5.39)
$$

$$
(-)^{\tilde F} = \begin{cases} -(-)^{\sum_{i=1}^{D-2}\sum_{r=\frac{1}{2},\ldots} \tilde b^i_{-r} \tilde b^i_r} & (\tilde{\text{NS}}) \\ +\gamma^9_{16}(-)^{\sum_{i=1}^{D-2}\sum_{m=1}^{\infty} \tilde d^i_{-m} \tilde d^i_m} & (\tilde{\text{R}}) \end{cases} \qquad \text{type IIB} \quad (5.40)
$$

$$
(-)^{\tilde F} = \begin{cases} -(-)^{\sum_{i=1}^{D-2}\sum_{r=\frac{1}{2},\ldots} \tilde b^i_{-r} \tilde b^i_r} & (\tilde{\text{NS}}) \\ -\gamma^9_{16}(-)^{\sum_{i=1}^{D-2}\sum_{m=1}^{\infty} \tilde d^i_{-m} \tilde d^i_m} & (\tilde{\text{R}}) \end{cases} \qquad \text{type IIA .} \quad (5.41)
$$

- (NS,$\tilde{NS}$) sector: tachyon is projected out. So $\mathbf{8}_V \otimes \mathbf{8}_V$ is the ground state.

- (R,$\tilde{NS}$) sector: tachyon is projected out. Massless states are $\mathbf{8}_S \otimes \mathbf{8}_V = \mathbf{8}_C \oplus \mathbf{56}_S$.

- (NS,$\tilde{R}$) sector: tachyon is projected out. Massless spectrum is $\mathbf{8}_V \otimes \mathbf{8}_S = \mathbf{8}_C \oplus \mathbf{56}_S$. Therefore this is the same as the spectrum of (R,$\tilde{NS}$). Therefore the spectrum is chiral. So this is called type IIB string. On the other hand, in the case of type IIA, the massless spectrum is $\mathbf{8}_V \otimes \mathbf{8}_C = \mathbf{8}_S \oplus \mathbf{56}_C$. Therefore this case is non-chiral.

- (R,$\tilde{R}$) sector: massless type IIB $\mathbf{8}_S \otimes \mathbf{8}_S = \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1}$. Massless type IIA $\mathbf{8}_S \otimes \mathbf{8}_C = \mathbf{8}_V \oplus \mathbf{56}_T$.

So the ground state supermultiplet is

$$\text{Type IIB:} \quad \text{boson}\begin{cases} (\mathrm{NS},\tilde{\mathrm{NS}})\ \mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1} \\ (\mathrm{R},\tilde{\mathrm{R}}) \quad \mathbf{8}_S \otimes \mathbf{8}_S = \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1} \end{cases}$$
$$\text{fermion}\begin{cases} (\mathrm{R},\tilde{\mathrm{NS}})\ \mathbf{8}_S \otimes \mathbf{8}_V = \mathbf{8}_C \oplus \mathbf{56}_S \\ (\mathrm{NS},\tilde{\mathrm{R}})\ \mathbf{8}_V \otimes \mathbf{8}_S = \mathbf{8}_C \oplus \mathbf{56}_S \end{cases} \tag{5.42}$$

$$\text{Type IIA:} \quad \text{boson}\begin{cases} (\mathrm{NS},\tilde{\mathrm{NS}})\ \mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1} \\ (\mathrm{R},\tilde{\mathrm{R}})\ \mathbf{8}_S \otimes \mathbf{8}_C = \mathbf{8}_V \oplus \mathbf{56}_T \end{cases}$$
$$\text{fermion}\begin{cases} (\mathrm{R},\tilde{\mathrm{NS}})\ \mathbf{8}_S \otimes \mathbf{8}_V = \mathbf{8}_C \oplus \mathbf{56}_S \\ (\mathrm{NS},\tilde{\mathrm{R}})\ \mathbf{8}_V \otimes \mathbf{8}_C = \mathbf{8}_S \oplus \mathbf{56}_C \end{cases} . \tag{5.43}$$

In both case, these are the $\mathcal{N}=2$ supergravity multiplet.

Now type IIB is symmetric under the flip, given by

$$\Omega(-)^F\Omega^{-1} = (-)^{\tilde{F}} \tag{5.44}$$

and

$$\Omega\alpha\Omega^{-1} = \tilde{\alpha}\,, \qquad \Omega b\Omega^{-1} = \tilde{b}\,, \qquad \Omega d\Omega^{-1} = \tilde{d}\,. \tag{5.45}$$

So we can make the theory unorientable by projecting the Ω invariant state $\Omega|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$. This is type I theory.

$$\text{Type IIB}/\Omega \equiv \text{Type I}\,. \tag{5.46}$$

- $(\mathrm{NS},\tilde{\mathrm{NS}})$ sector: $\mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1} \overset{\mathbf{28}\ \text{removed}}{\rightarrow} \mathbf{35} \oplus \mathbf{1}$.
- $(\mathrm{R},\tilde{\mathrm{R}})$ sector: $\mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1} \overset{\mathbf{35},\mathbf{1}\ \text{removed}}{\rightarrow} \mathbf{28}$.
- $(\mathrm{R},\tilde{\mathrm{NS}})$, $(\mathrm{NS},\tilde{\mathrm{R}})$ sector: $\mathbf{8}_\mathrm{S} \otimes \mathbf{8}_\mathrm{V} = \mathbf{8}_\mathrm{C} \oplus \mathbf{56}_\mathrm{S}$. Only one set survives after Ω projection.

This is $\mathcal{N}=1$, $D=10$ supergravity multiplet.

Spacetime anomaly cancellation or infinity cancellation implies type I closed string must be accompanied with open strings. Consistency requires the gauge group to be $SO(32)$.

Upshot is

$$\text{Type I} = \text{Type IIB}/\Omega + \text{open string} \tag{5.47}$$

and is unorientable.

5.2.2 modular invariance and fermionic string partition function

Recall that

$$\Gamma_{\text{closed bosonic}} = -\frac{V_{\text{E}}}{2(4\pi^2\alpha')^{13}} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \frac{1}{|\eta(\tau)|^{48}} \,. \tag{5.48}$$

We have restricted the domain to be the fundamental region of a torus by hand. The reason is that the integrand is invariant under the modular group of a torus.

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in SL(2, \mathbf{Z}) \tag{5.49}$$

which is generated by

$$T:\ \tau \to \tau + 1 \qquad \text{and} \qquad S:\ \tau \to -\frac{1}{\tau} = \frac{-\bar{\tau}}{|\tau|^2} \,. \tag{5.50}$$

In fact,

$$\begin{aligned}
T:&\ \eta(\tau) \to \eta(\tau+1) = \mathrm{e}^{\frac{i\pi}{12}}\eta(\tau) \\
S:&\ \mathrm{d}\tau \to \mathrm{d}\left(-\frac{1}{\tau}\right) = \frac{1}{\tau^2}\mathrm{d}\tau \\
&\mathrm{Im}\tau \to \mathrm{Im}\left(-\frac{1}{\tau}\right) = \frac{\mathrm{Im}\tau}{|\tau|^2} \\
&\frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{(\mathrm{Im}\tau)^2} \to \frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}|\tau|^4}{|\tau|^4(\mathrm{Im}\tau)^2} = \frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{(\mathrm{Im}\tau)^2} \\
&\frac{\mathrm{d}^2\tau}{\tau_2} = \frac{2\mathrm{Im}\ \mathrm{d}\tau\mathrm{d}\bar{\tau}}{(\mathrm{Im}\tau)^2} \text{ is invariant} \\
&\eta(\tau) \to \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) \,.
\end{aligned} \tag{5.51}$$

Therefore $(\mathrm{Im}\tau)^{\frac{1}{2}}|\eta(\tau)|^2$ is invariant under the modular group.

More generically, Γ, it is a torus partition function and at the same time the free energy, is given by

$$\Gamma = -\frac{V_{\mathrm{E}}}{2}\frac{1}{4\pi^2\alpha'}\mathcal{T}_{\mathrm{torus}} \tag{5.52}$$

and

$$\mathcal{T}_{\mathrm{torus}} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\sum_{i,j}\bar{\chi}^{\mathrm{Vir}}_{h_i}(\bar{q})X_{ij}\chi^{\mathrm{Vir}}_{h_j}(q)\,. \tag{5.53}$$

In the general form eq. (5.53), the integrand is not guaranteed to be modular invariant. In the context of superstrings, the projection into modular invariant integrand is called GSO projection [37].

Let us give the definition of the Jacobi theta function by the Gaussian sums:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau) = \sum_n q^{\frac{1}{2}(n+\alpha)^2}\mathrm{e}^{2\pi i(n+\alpha)(z+\beta)}\,. \tag{5.54}$$

It can be written also in terms of the infinite products

$$\begin{aligned}&\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau)\\ &= \mathrm{e}^{2\pi i\alpha(z+\beta)}q^{\frac{\alpha^2}{2}}\prod_{n=1}^{\infty}(1-q^n)(1+q^{n+\alpha-\frac{1}{2}}\mathrm{e}^{2\pi i(z+\beta)})(1+q^{n-\alpha-\frac{1}{2}}\mathrm{e}^{-2\pi i(z+\beta)})\,.\end{aligned} \tag{5.55}$$

The behavior under T and S as follows:

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau+1) &= \mathrm{e}^{-i\pi\alpha(\alpha-1)}\vartheta\begin{bmatrix}\alpha\\ \beta+\alpha-\frac{1}{2}\end{bmatrix}(z|\tau)\\ \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) &= (-i\tau)^{\frac{1}{2}}\mathrm{e}^{2\pi i\alpha\beta+\frac{i\pi z^2}{\tau}}\vartheta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(z|\tau)\,.\end{aligned} \tag{5.56}$$

Now

$$\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau) = -q^{\frac{\alpha^2}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-q^{n+1})(1-q^{n-1}) = 0\,. \tag{5.57}$$

A few different notations have been used for instance

$$\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\equiv\vartheta_2 \qquad \vartheta\begin{bmatrix}0\\0\end{bmatrix}\equiv\vartheta_3 \qquad \vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\equiv\vartheta_4\,. \tag{5.58}$$

What is of direct relevance to superstring vacuum amplitudes are the following expressions.

$$\begin{aligned}
\frac{\vartheta^4\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(0|\tau)}{\eta^{12}(\tau)} = \frac{\vartheta_2^4(0|\tau)}{\eta^{12}(\tau)} &= \frac{q^{\frac{1}{2}}\left(\prod_{n=1}^{\infty}(1-q^n)\right)^4\left(\prod_{n=1}^{\infty}(1+q^n)\right)^{4\cdot 2}2^4}{q^{\frac{1}{2}}\left(\prod_{n=1}^{\infty}(1-q^n)\right)^{12}}\\
&= 16\frac{\left(\prod_{n=1}^{\infty}(1+q^n)\right)^8}{\left(\prod_{n=1}^{\infty}(1-q^n)\right)^8}\\
\frac{\vartheta^4\begin{bmatrix}0\\0\end{bmatrix}(0|\tau)}{\eta^{12}(\tau)} = \frac{\vartheta_3^4(0|\tau)}{\eta^{12}(\tau)} &= \frac{\left(\prod_{n=1}^{\infty}(1-q^n)\right)^4\left(\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})\right)^8}{q^{\frac{1}{2}}\left(\prod_{n=1}^{\infty}(1-q^n)\right)^{12}}\\
&= q^{-\frac{1}{2}}\frac{\left(\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})\right)^8}{\left(\prod_{n=1}^{\infty}(1-q^n)\right)^8}\\
\frac{\vartheta^4\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(0|\tau)}{\eta^{12}(\tau)} = \frac{\vartheta_4^4(0|\tau)}{\eta^{12}(\tau)} &= q^{-\frac{1}{2}}\frac{\left(\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})\right)^8}{\left(\prod_{n=1}^{\infty}(1-q^n)\right)^8}\,.
\end{aligned} \tag{5.59}$$

Let us define O, V, S, C:

$$O_8 \equiv \frac{\vartheta^4 \left[{0 \atop 0} \right] (0) + \vartheta^4 \left[{0 \atop \frac{1}{2}} \right] (0)}{2\eta^4}$$

$$V_8 \equiv \frac{\vartheta^4 \left[{0 \atop 0} \right] (0) - \vartheta^4 \left[{0 \atop \frac{1}{2}} \right] (0)}{2\eta^4}$$

$$S_8 \equiv \frac{\vartheta^4 \left[{\frac{1}{2} \atop 0} \right] (0) + \vartheta^4 \left[{\frac{1}{2} \atop \frac{1}{2}} \right] (0)}{2\eta^4}$$

$$C_8 \equiv \frac{\vartheta^4 \left[{\frac{1}{2} \atop 0} \right] (0) - \vartheta^4 \left[{\frac{1}{2} \atop \frac{1}{2}} \right] (0)}{2\eta^4} \,. \tag{5.60}$$

They are regarded as the $SO(8)$ level 1 current algebra characters $\widehat{SO(8)}_1$ associated with the four integral representations with eight independent Majorana-Weyl fermions. This is generalized to
$\widehat{SO(N)}_1$:

$$O_N = \frac{\vartheta \left[{0 \atop 0} \right]^{\frac{N}{2}} (0) + \vartheta \left[{0 \atop \frac{1}{2}} \right]^{\frac{N}{2}} (0)}{2\eta^{N/2}} = \mathrm{ch}_{\widehat{SO(N)}_1, \hat{w}_0}(q)$$

$$V_N = \frac{\vartheta \left[{0 \atop 0} \right]^{\frac{N}{2}} (0) - \vartheta \left[{0 \atop \frac{1}{2}} \right]^{\frac{N}{2}} (0)}{2\eta^{N/2}} = \mathrm{ch}_{\widehat{SO(N)}_1, \hat{w}_1} \,. \tag{5.61}$$

$\widehat{SO(2n)}_1$:

$$S_{2n} = \frac{\vartheta \left[{\frac{1}{2} \atop 0} \right]^{n} (0) + i^{-n} \vartheta \left[{\frac{1}{2} \atop \frac{1}{2}} \right]^{n} (0)}{2\eta^n} = \mathrm{ch}_{\widehat{SO(2n)}_1, \hat{w}_{n-1}}$$

$$C_{2n} = \frac{\vartheta \left[{\frac{1}{2} \atop 0} \right]^{n} (0) - i^{-n} \vartheta \left[{\frac{1}{2} \atop \frac{1}{2}} \right]^{n} (0)}{2\eta^n} = \mathrm{ch}_{\widehat{SO(2n)}_1, \hat{w}_{n}} \,. \tag{5.62}$$

$\widehat{SO(2n+1)}_1$:

$$S_{2n+1} = C_{2n+1} = \frac{\left[{\frac{1}{2} \atop 0} \right]^{n+\frac{1}{2}} (0)}{\sqrt{2}\eta^{n+\frac{1}{2}}} = \mathrm{ch}_{\widehat{SO(2n+1)}_1, \hat{w}_n} \,. \tag{5.63}$$

T and S acting on the four characters are represented as the matrices [2]

$$T = \mathrm{e}^{-\frac{in\pi}{12}} \mathrm{diag}\left(1, -1, \mathrm{e}^{\frac{in\pi}{4}}, \mathrm{e}^{\frac{in\pi}{4}}\right) , \qquad S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & -i^{-n} \end{pmatrix} . \quad (5.64)$$

In the $n = 4$ case,

$$T = \mathrm{e}^{-\frac{\pi i}{3}} \mathrm{diag}(1, -1, -1, -1) , \qquad S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix} . \quad (5.65)$$

The action of T on a Dedekind eta function is given by

$$T : \ \eta^8 \to \mathrm{e}^{\frac{2}{3}\pi i} \eta^8 . \quad (5.66)$$

Basic building blocks are given by

$$\left(\frac{O_8}{\tau_2^2 \eta^8}, \frac{V_8}{\tau_2^2 \eta^8}, \frac{S_8}{\tau_2^2 \eta^8}, \frac{C_8}{\tau_2^2 \eta^8} \right)^{\mathrm{T}} \equiv \chi_{\text{closed, 10d flat, NSR}} . \quad (5.67)$$

Using eqs. (5.65) and (5.66), action of T on the $\chi_{\text{closed, 10d flat, NSR}}$ is represented as

$$\mathrm{diag}(-1, 1, 1, 1) . \quad (5.68)$$

The corresponding torus amplitude for more general superstring models is given by

$$\begin{aligned} \Gamma_{\text{closed, 10d flat superstring}} &= -\frac{V_{\mathrm{E}}}{2(4\pi^2\alpha')^5} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \sum_{i,j} \bar{\chi}_i X_{ij} \chi_j \\ &\equiv -\frac{V_{\mathrm{E}}}{2(4\pi^2\alpha')^5} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \bar{\chi} X \chi , \end{aligned} \quad (5.69)$$

where X_{ij} is a generalized GSO projection.

[2]See also appendix D.

- constraints of modular invariance is written as

$$\begin{cases} S^\dagger X S = X \\ T^\dagger X T = X \end{cases} . \tag{5.70}$$

- other physical constraints are
  - the theory must contain massless graviton.
  - by spin-statistics, boson and fermions contribute to eq. (5.69) with the opposite signs.

We will just list obvious possibilities

$$\begin{aligned}
(\bar{\chi} X \chi)_{\mathrm{IIA}} &= (\bar{V}_8 - \bar{S}_8)(V_8 - C_8)\frac{1}{\tau_2^4 |\eta|^{16}} \\
(\bar{\chi} X \chi)_{\mathrm{IIB}} &= |V_8 - C_8|^2 \frac{1}{\tau_2^4 |\eta|^{16}} \\
(\bar{\chi} X \chi)_{\mathrm{0A}} &= (|O_8|^2 + |V_8|^2 + \bar{S}_8 C_8 + \bar{C}_8 S_8)\frac{1}{\tau_2^4 |\eta|^{16}} \\
(\bar{\chi} X \chi)_{\mathrm{0B}} &= (|O_8|^2 + |V_8|^2 + |S_8|^2 + |C_8|^2)\frac{1}{\tau_2^4 |\eta|^{16}} .
\end{aligned} \tag{5.71}$$

5.3 Klein bottle, annulus, möbius strip

5.3.1 bosonic string partition function

Let's get back to the flat 26 dimensional oriented closed bosonic string. Recall that the torus free energy reads

$$\Gamma_{\substack{\text{closed} \\ \text{bosonic} \\ \text{oriented} \\ \text{flat}}} = -\frac{V_E}{2}\frac{1}{(4\pi^2\alpha')}\mathcal{T} , \tag{5.72}$$

where

$$\mathcal{T} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \int \frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}} \left| \chi^{\mathrm{Vir}}_{\left(\frac{\alpha'}{4}p^i,\, c=24\right)}(q) \right|^2 = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \frac{1}{(4\pi^2\alpha')^{12}} \frac{1}{\tau_2^{12}} \frac{1}{|\eta(\tau)|^{48}} . \tag{5.73}$$

In the expansion of

$$
\begin{aligned}
|\chi|^2_{\mathcal{T}} &= \mathrm{Tr} q^{L_0^{(\mathrm{cyl})}} \bar{q}^{\bar{L}_0^{(\mathrm{cyl})}} \\
&= \frac{1}{|\eta(\tau)|^{48}} \mathrm{e}^{-t(p^i)^2} \\
&= |q|^{-2}(1 + 24q + \cdots)(1 + 24\bar{q} + \cdots) \\
&= |q|^{-2}(1 + 24q + 24\bar{q} + 24^2|q|^2 + \cdots)\,, \qquad t = \pi\alpha'\tau_2\,, \qquad (5.74)
\end{aligned}
$$

$24q$ and $24\bar{q}$ disappear after τ_1 integration due to the level matching condition. Therefore after ground state, first excited states are massless states and number of massless states are 24^2.

Now let's discuss how to build the state space of the unoriented closed string. Namely, the closed string counterpart of the computation of eq. (5.74). Definition of unoriented closed string is to project onto Ω invariant states $\Omega|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$ given by

$$
\Omega \alpha_n^{M_{\mathrm{lc}}} \Omega^{-1} = \tilde{\alpha}_n^{M_{\mathrm{lc}}}\,. \qquad (5.75)
$$

Therefore a free energy for closed bosonic unoriented flat 26 dimensional string $\Gamma_{\substack{\text{closed}\\ \text{bosonic}\\ \text{unoriented}\\ \text{flat}}}$ is obtained by replacing $|\chi|^2$ by $\mathrm{Tr}\frac{1+\Omega}{2} q^{L_0^{\mathrm{cyl}}} \bar{q}^{\bar{L}_0^{\mathrm{cyl}}}$, namely replace $\mathcal{T}$ by $\frac{\mathcal{T}+\mathcal{K}}{2}$, where $\mathcal{K}$ is

$$
\mathcal{K} = \int_0^\infty \frac{\mathrm{d}\tau_2}{{\tau_2}^2} \int \frac{\mathrm{d}^{24} p_i}{(2\pi)^2} |\chi|_{\mathrm{KB}}\,. \qquad (5.76)
$$

And $|\chi|_{\mathrm{KB}}$ is represented (projected trace is computed in the following way)

$$
\begin{aligned}
&|\chi|_{\mathrm{KB}}\\
&= \mathrm{Tr}\,\Omega q^{L_0^{\mathrm{cyl}}}\bar{q}^{\bar{L}_0^{\mathrm{cyl}}}\\
&= \sum_{\mathrm{all}}\langle \alpha_n \text{Fock space}| \otimes \langle \tilde{\alpha}_m \text{Fock space}|\Omega q^{L_0^{\mathrm{cyl}}}\bar{q}^{\bar{L}_0^{\mathrm{cyl}}}|\alpha_n \text{Fock space}\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes|\tilde{\alpha}_m \text{Fock space}\rangle\\
&= \sum_{\mathrm{all}}\langle \tilde{\alpha}_m \text{Fock space}| \otimes \langle \alpha_n \text{Fock space}|q^{L_0^{\mathrm{cyl}}}\bar{q}^{\bar{L}_0^{\mathrm{cyl}}}|\alpha_n \text{Fock space}\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes|\tilde{\alpha}_m \text{Fock space}\rangle \mathrm{e}^{-t(p^i)^2}\\
&= \sum_{\mathrm{all}}\langle \tilde{m}_1, \tilde{m}_2, \cdots | \otimes \langle n_1, n_2, \cdots |q^{L_0^{\mathrm{cyl}}}\bar{q}^{\bar{L}_0^{\mathrm{cyl}}}|n_1, n_2, \cdots\rangle \otimes |\tilde{m}_1, \tilde{m}_2, \cdots\rangle \mathrm{e}^{-t(p^i)^2}\\
&= \sum_{(n_1,n_2,\cdots)=(\tilde{m}_1,\tilde{m}_2,\cdots)}\langle \tilde{m}_1, \tilde{m}_2, \cdots | \otimes \langle n_1, n_2, \cdots |q^{L_0^{\mathrm{cyl}}}\bar{q}^{\bar{L}_0^{\mathrm{cyl}}}|n_1, n_2, \cdots\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes|\tilde{m}_1, \tilde{m}_2, \cdots\rangle \mathrm{e}^{-t(p^i)^2}\\
&= \sum_{(n_1,n_2,\cdots)}\langle n_1, n_2, \cdots |q^{L_0^{\mathrm{cyl}}}|n_1, n_2, \cdots\rangle\langle n_1, n_2, \cdots |\bar{q}^{\bar{L}_0^{\mathrm{cyl}}}|n_1, n_2, \cdots\rangle \mathrm{e}^{-t(p^i)^2}\\
&\overset{\text{effectively}}{=} \sum_{(n_1,n_2,\cdots)}\langle n_1, n_2, \cdots |(q\bar{q})^{L_0^{\mathrm{cyl}}}|n_1, n_2, \cdots\rangle \mathrm{e}^{-t(p^i)^2}\\
&= \sum_{(n_1,n_2,\cdots)}\langle n_1, n_2, \cdots |\mathrm{e}^{2\pi i(2i\tau_2)}|n_1, n_2, \cdots\rangle \mathrm{e}^{-t(p^i)^2}\,. \qquad (5.77)
\end{aligned}
$$

So we conclude that $|\chi|_{\mathrm{KB}}$ is obtained from $|\chi|_{\mathcal{T}}$ (not $|\chi|^2_{\mathcal{T}}$) by replacing τ by $2i\tau_2$ and $|q|$ by $|q|^2$. So the $\mathcal{K}$ is no longer modular invariant. Concrete expression $|\chi|_{\mathrm{KB}}$ is given by

$$
|\chi|_{\mathrm{KB}} = \frac{1}{|\eta(2i\tau)|^{24}}\mathrm{e}^{-t(p_i)^2} = \frac{1}{\eta(2i\tau)^{24}}\mathrm{e}^{-t(p_i)^2}\,. \qquad (5.78)
$$

Therefore

$$
\int_{-1/2}^{1/2}\mathrm{d}\tau_1 = 1
$$

$$
\mathcal{K} = \int_0^\infty \frac{\mathrm{d}\tau_2}{{\tau_2}^2}\frac{1}{(4\pi^2\alpha')^{12}}\frac{1}{{\tau_2}^{12}}\frac{1}{|\eta(2i\tau)|^{24}}\,. \qquad (5.79)
$$

And then to compute the multiplicitys of the states of unoriented closed string

$$\frac{1}{2}|\chi|^2_{\mathcal{T}}+\frac{1}{2}|\chi|_{\mathrm{KB}}=\frac{1}{2}|q|^{-2}(1+24q+24\bar{q}+24^2|q|^2+\cdots)+\frac{1}{2}|q|^{-2}(1+24|q|^2+\cdots) \tag{5.80}$$

is relevant. From that we conclude that $\#(\text{massless states}) = \frac{1}{2}24^2 + \frac{1}{2}24 = \frac{1}{2}24(24+1) = \frac{25\cdot 24}{2}$.

Now the open string will act as a sort of "twisted" sector to Ω projection. But no invariance principle such as modular invariance. We go back to the light-cone Hmiltonian of the open bosonic string and the vacuum energy of the corresponding quantum field theory.

Next, we consider the open string sector. First, we go back to the torus partition function and adapt it to the open string. Recall

$$\Gamma_{\mathrm{E}} = -\frac{V_{\mathrm{E}}}{2}\int_0^\infty \frac{\mathrm{d}t}{t}e^{-tM^2}\int \frac{\mathrm{d}^D p^{M_{\mathrm{lc}}}}{(2\pi)^D}\mathrm{e}^{-t(p^{M_{\mathrm{lc}}})^2} \tag{5.81}$$

for $M_{\mathrm{lc}} = 0, 25$ and do the Gaussian integration. Then we get

$$\left(\frac{1}{\sqrt{4\pi^2\alpha'\tau_2}}\right)^2 . \tag{5.82}$$

Now

$$\alpha' m^2 = \hat{N} - 1\,, \qquad \hat{N} = \frac{1}{2} : \sum_{\substack{n\in\mathbf{Z}\\ n\neq 0}} \alpha^i_{-n}\alpha^i_n : \tag{5.83}$$

and for open string

$$-i\alpha' p_0^{M_{\mathrm{lc}}} = i\sqrt{\frac{\alpha'}{2}}(-)\alpha_0^{M_{\mathrm{lc}}} \tag{5.84}$$

$$\alpha_0^{M_{\mathrm{lc}}} = 2\sqrt{\frac{\alpha'}{2}}p_0^{M_{\mathrm{lc}}}\,. \tag{5.85}$$

Therefore

$$\alpha'(p^i)^2 = \frac{{\alpha_0^i}^2}{2}\,. \tag{5.86}$$

Using these, we can calculate

$$
\begin{aligned}
&\Gamma_{\mathrm{E}} \\
&\overset{\frac{t}{\alpha'}\equiv\pi\tau_2}{=} -\frac{V_{\mathrm{E}}}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2}\frac{1}{4\pi^2\alpha'\tau_2}\mathrm{tr}\,\mathrm{e}^{-\pi\tau_2(\hat{N}-1)}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\mathrm{e}^{-\pi\tau_2\alpha'(p^i)^2} \\
&= -\frac{V_{\mathrm{E}}}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2}\frac{1}{4\pi^2\alpha'\tau_2}\mathrm{tr}\,\mathrm{e}^{-\pi\tau_2(\hat{N}-1)}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\mathrm{e}^{-\pi\tau_2\frac{\alpha_0^{i\,2}}{2}} \\
&\overset{q=\mathrm{e}^{2\pi i\tau_1}\mathrm{e}^{-2\pi\tau_2},\,\tau=\tau_1+i\tau_2}{=} -\frac{V_{\mathrm{E}}}{2}\frac{1}{4\pi^2\alpha'}\int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^{1+1}}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\mathrm{Tr}_{p_i}(\sqrt{|q|})^{L_0^{\mathrm{cyl}}}\,,
\end{aligned}
\tag{5.87}
$$

where

$$
L_0^{\mathrm{cyl}} = \frac{1}{2}\sum_n :\alpha_{-n}^i\alpha_n^i: -1\,. \tag{5.88}
$$

To summarize,

$$
\Gamma_{\substack{\text{open}\\\text{bosonic}\\\text{oriented}\\\text{flat}}} = -\frac{V_{\mathrm{E}}}{2}\frac{1}{4\pi^2\alpha'}\mathcal{A}\,, \tag{5.89}
$$

where

$$
\mathcal{A} = \int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^2}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\chi^{\mathrm{Vir}}_{(\alpha' p^{i\,2}\ c=24)\ \mathcal{A}}(\sqrt{|q|})\cdot N^2 = \int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^2}\frac{1}{(4\pi^2\alpha')^{12}\tau_2^{12}}\frac{1}{\left(\eta\left(\frac{i}{2}\tau_2\right)\right)^{24}} \tag{5.90}
$$

and

$$
\begin{aligned}
\chi^{\mathrm{Vir}}_{(\alpha' p^{i\,2}\ c=24)\ \mathcal{A}}(\sqrt{|q|})\cdot N^2 &= \mathrm{Tr}_{p^i}(\sqrt{|q|})^{L_0^{\mathrm{cyl}}}\cdot N^2 = \frac{N^2}{\left(\eta\left(\frac{i}{2}\tau_2\right)\right)^{24}}\mathrm{e}^{-tp^{i\,2}} \\
&= N^2(\sqrt{|q|})^{-1}(1+24\sqrt{|q|}+\cdots)\mathrm{e}^{-tp^{i\,2}}\,.
\end{aligned}
\tag{5.91}
$$

To make it unoriented, replace $\chi_{\mathcal{A}}(\sqrt{|q|})\cdot N^2$ by $\mathrm{Tr}_{\mathrm{p}^{\mathrm{i}}}\left(\frac{1+\epsilon\Omega}{2}\right)(\sqrt{|q|})^{L_0^{\mathrm{cyl}}}$ *i.e.* replace $\mathcal{A}$ by $\frac{\mathcal{A}+\mathcal{M}}{2}$:

$$
\mathcal{M} = \int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^2}\int\frac{\mathrm{d}^{24}p^i}{(2\pi)^{24}}\chi^{\mathrm{Vir}}_{(\alpha' p^{i\,2}\ c=24)\ \mathcal{M}}(\sqrt{|q|})\cdot N\epsilon\,, \tag{5.92}
$$

where

$$
\chi^{\mathrm{Vir}}_{\mathcal{M}}(\sqrt{|q|})\cdot N\epsilon = \mathrm{Tr}_{p^i}\ \epsilon\Omega(\sqrt{|q|})^{L_0^{\mathrm{cyl}}}\cdot N \tag{5.93}
$$

and $\epsilon = \pm 1$. Recall Ω such that $\Omega X^{M_{\rm lc}}(z,\bar{z})\Omega^{-1} = X^{M_{\rm lc}}(-\bar{z},-z)$:

$$\begin{aligned} \Omega\alpha_n^{M_{\rm lc}}\Omega^{-1} &= (-)^n\alpha_n^{M_{\rm lc}} \ \text{ for } N \\ \Omega\alpha_n^{M_{\rm lc}}\Omega^{-1} &= (-)^{n+1}\alpha_n^{M_{\rm lc}} \ \text{ for } D\,, \end{aligned} \tag{5.94}$$

where $\Omega = (-)^{\hat{N}}$. Here

$$\epsilon\Omega(\sqrt{|q|})^{L_0^{\rm cyl}} = \epsilon {\rm e}^{i\pi\hat{N}}{\rm e}^{-\pi\tau_2 L_0^{\rm cyl}} \approx \epsilon{\rm e}^{-i\pi\left(h_i-\frac{c}{24}\right)}{\rm e}^{2\pi i\left(\frac{i}{2}\tau_2+\frac{1}{2}\right)L_0}\,. \tag{5.95}$$

So in $\mathcal{M}$, the momentum integrations unchanged

$$\mathcal{M} = \epsilon N\int_0^\infty \frac{{\rm d}\tau_2}{\tau_2^2}\frac{1}{(4\pi^2\alpha')^{12}\tau_2^{12}}\frac{1}{\left(\tilde{\eta}\left(\frac{i}{2}\tau_2+\frac{1}{2}\right)\right)^{24}}\,, \tag{5.96}$$

where

$$\tilde{\eta}\left(\frac{i}{2}\tau_2+\frac{1}{2}\right) = (\sqrt{|q|})^{\frac{1}{24}}\prod_{n=1}^\infty(1-(-\sqrt{q})^n) \neq \eta\left(\frac{i}{2}\tau_2+\frac{1}{2}\right)\,. \tag{5.97}$$

Note that $(\sqrt{|q|})^{\frac{1}{24}}$ is not modified. Therefore

$$\tilde{\eta}\left(\frac{i}{2}\tau_2+\frac{1}{2}\right)^{24} = -\eta\left(\frac{i}{2}\tau_2+\frac{1}{2}\right)^{24}\,. \tag{5.98}$$

Using the expansion

$$\chi_{\mathcal{M}}^{\rm Vir}(\sqrt{|q|})\cdot N\epsilon = \frac{\epsilon N}{\left(\tilde{\eta}\left(\frac{i}{2}\tau_2+\frac{1}{2}\right)\right)^{24}}{\rm e}^{-tp_i^2} = \epsilon N(\sqrt{|q|})^{-1}(1-24\sqrt{|q|}+\cdots)\,, \tag{5.99}$$

we obtain

$$\begin{aligned} &\frac{1}{2}\left(\chi_{\mathcal{A}}^{\rm Vir}(\sqrt{|q|})\cdot N^2 + \chi_{\mathcal{M}}^{\rm Vir}(\sqrt{|q|})\cdot N\epsilon\right) \\ &\qquad = (\sqrt{|q|})^{-1}\left(1+24\frac{N^2}{2}\sqrt{|q|}-24\frac{\epsilon N}{2}\sqrt{|q|}+\cdots\right)\,. \end{aligned} \tag{5.100}$$

Therefore, in the open string sector we can see

$$\#(\text{massless states}) = 24\frac{N(N-\epsilon)}{2}\,. \tag{5.101}$$

ϵ takes $+1$ for $SO(N)$ and -1 for $Sp(N)$.

	Grauge group
$\epsilon = +1$	$SO(N)$
$\epsilon = -1$	$Sp(N)$

We summarize the modular parameters as the table below.

T	$\mathcal{K}$	$\mathcal{A}$	$\mathcal{M}$
τ	$2i\tau_2$	$\frac{i}{2}\tau_2$	$\frac{i}{2}\tau_2 + \frac{1}{2}$

5.3.2 fermionic string partition function

To construct an unoriented string, namely the type I superstring , one first makes the closed string sector by the Ω (twist) projection: $\mathrm{Tr} q^{L_0^{(\mathrm{cyl})}} \bar{q}^{\bar{L}_0^{(\mathrm{cyl})}} \to \mathrm{Tr}\left(\frac{1+\Omega}{2}\right) q^{L_0^{(\mathrm{cyl})}} \bar{q}^{\bar{L}_0^{(\mathrm{cyl})}}$. We obtain

$$\Gamma^{\mathrm{I}}_{\mathrm{closed,\,one-loop}} = -\frac{V_{\mathrm{E}}}{2(4\pi\alpha')^5} \frac{\mathcal{T} + \mathcal{K}}{2} \tag{5.102}$$

$$\mathcal{T} = \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^2} (\bar{\chi} X \chi)_{\mathrm{IIB}} \tag{5.103}$$

$$\mathcal{K} = \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^2} \sum_{i=\mathrm{NS,R}} \chi_i(2i\tau_2)\,. \tag{5.104}$$

By the similar procedure, the vacuum amplitude of the open string sector reads

$$\Gamma^{\mathrm{I}}_{\mathrm{open,\,one-loop}} = -\frac{V_{\mathrm{E}}}{2(4\pi\alpha')^5} \frac{\mathcal{A} + \mathcal{M}}{2} \tag{5.105}$$

$$\mathcal{A} = \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^2} \sum_{i=\mathrm{NS,R}} \chi_i \left(\frac{i}{2}\tau_2\right) (\mathrm{cpf})^2 \tag{5.106}$$

$$\mathcal{M} = \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^2} \sum_{i=\mathrm{NS,R}} \tilde{\chi}_i \left(\frac{i}{2}\tau_2 + \frac{1}{2}\right) (\mathrm{cpf})\epsilon\,. \tag{5.107}$$

Here, cpf denotes the Chan-Paton factor, $\epsilon = \pm 1$ and $\tilde{\chi}\left(\frac{i}{2}\tau_2 + \frac{1}{2}\right)$ indicates that the replacement by $\tau \to \frac{i}{2}\tau_2 + \frac{1}{2}$ in the argument is to be made only for the oscillator part. These replacements $\mathcal{K} : \tau \to 2i\tau_2$, $\mathcal{A} : \tau \to \frac{i}{2}\tau_2$, $\mathcal{M} : \tau \to \frac{i}{2}\tau_2 + \frac{1}{2}$ are understood both by the twist projection in the operator formalism and by the worldsheet involutions of the worldsheet path integrals with torus as the double of the respective open Riemann surfaces.

Finally, the infrared stability seen as the cancellation of the massless poles in $\Gamma^{\mathrm{I}}_{\text{closed, one-loop}} + \Gamma^{\mathrm{I}}_{\text{open, one-loop}}$ in the transverse channel (or equivalently the cancellation of dilaton tadpoles [32, 38, 39, 40, 41, 42] or infinity cancellation [43, 32]) selects cpf $= 2^5 = 32$, $\epsilon = -1$ and the gauge group $SO(32)$ [44].

6 T^4/Z_2 orbifold

6.1 Circle compactification

6.1.1 bosonic string

Take a bosonic closed string still light-cone. Suppose that we compactify 24th dimension on a circle of radius R.

Recall for a non-compact directions

$$X^i(\tau_{\mathrm{M}}, \sigma) = X^i_0 - \frac{i\alpha'}{2} p^i_0 \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \left(\sum_{n\neq 0} \alpha^i_n \frac{z^{-n}}{n} + \sum_{n\neq 0} \tilde{\alpha}^i_n \frac{\bar{z}^{-n}}{n} \right) \quad (i = 1, ..., 23) \tag{6.1}$$

while the compact direction is written as

$$\begin{aligned} X^{24} = X &= \hat{X}_0 + \alpha' \hat{p}_0 \tau_{\mathrm{M}} + \alpha' \sigma \hat{w} + i\sqrt{\frac{\alpha'}{2}} \left(\sum_{n\neq 0} \alpha^i_n \frac{z^{-n}}{n} + \sum_{n\neq 0} \tilde{\alpha}^i_n \frac{\bar{z}^{-n}}{n} \right) \\ &= X_0 + \sqrt{\frac{\alpha'}{2}} (\alpha_0 + \tilde{\alpha}_0)\tau_{\mathrm{M}} + \sqrt{\frac{\alpha'}{2}} (\alpha_0 - \tilde{\alpha}_0)\sigma \\ &\qquad + i\sqrt{\frac{\alpha'}{2}} \left(\sum_{n\neq 0} \alpha^i_n \frac{z^{-n}}{n} + \sum_{n\neq 0} \tilde{\alpha}^i_n \frac{\bar{z}^{-n}}{n} \right) , \end{aligned} \tag{6.2}$$

where $\hat{w}$ represents winding. Now

$$\begin{aligned}\alpha_0 + \tilde{\alpha}_0 &= 2\sqrt{\frac{\alpha'}{2}}\hat{p}_0 \\ \alpha_0 - \tilde{\alpha}_0 &= 2\sqrt{\frac{\alpha'}{2}}\hat{w}\,.\end{aligned} \tag{6.3}$$

The condition of the circle compactification

$$X^{24}(\tau_{\rm M}, \sigma) = X^{24}(\tau_{\rm M}, \sigma + 2\pi) + 2\pi\ell R \tag{6.4}$$

leads

$$\begin{aligned}(\text{eigenvalue of } \hat{p}_0) &= \frac{m}{R}\,, \qquad m \in \mathbf{Z} \\ (\text{eigenvalue of } \hat{w}) &= \frac{2\pi\ell R}{2\pi\alpha'} = \frac{\ell R}{\alpha'}\,, \qquad \ell \in \mathbf{Z}\,.\end{aligned} \tag{6.5}$$

Also we can write

$$X = X_{\rm R}(z) + \bar{X}_{\rm L}(\bar{z})\,, \tag{6.6}$$

where

$$\begin{aligned}X_{\rm R}(z) &= \left(\frac{X_0}{2} + \frac{Y_0}{2}\right) - i\sqrt{\frac{\alpha'}{2}}\alpha_0 \ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n z^{-n}}{n} \\ \bar{X}_{\rm L}(\bar{z}) &= \left(\frac{X_0}{2} - \frac{Y_0}{2}\right) - i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0 \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n \bar{z}^{-n}}{n}\,.\end{aligned} \tag{6.7}$$

Recall

$$\mathcal{T}_{\rm torus} = \int_{\mathcal{F}} \frac{{\rm d}^2\tau}{\tau_2^2} \sum_{i,j} \bar{\chi}_{h_i}^{\rm Vir}(\bar{q}) X_{ij} \chi_{h_j}^{\rm Vir}(q)\,. \tag{6.8}$$

For a while, we concentrate on this part only:

$$\begin{aligned}\bar{\chi}_{c=1}^{\rm Vir}(\bar{q})\chi_{c=1}^{\rm Vir}(q) &\equiv {\rm Tr}_{\Gamma(R)}(q)^{L_0^{c=1,({\rm cyl})}}(\bar{q})^{\bar{L}_0^{c=1,({\rm cyl})}} \\ &= \frac{1}{\sqrt{\tau_2}|\eta(q)|^2} a\sqrt{\tau_2} \sum_{(h,\bar{h})\in\Gamma^{(1,1)}} (q)^{\frac{1}{2}h^2}(\bar{q})^{\frac{1}{2}\bar{h}^2}\,,\end{aligned} \tag{6.9}$$

where $a \equiv \frac{\sqrt{\alpha'}}{R}$,

$$L_0^{c=1,(\text{cyl})} = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \frac{1}{24}$$
$$\bar{L}_0^{c=1,(\text{cyl})} = \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24}\,. \tag{6.10}$$

h and $\bar{h}$ are eigenvalue of α_0 and eigenvalue of $\tilde{\alpha}_0$ respectively. We know

$$\sqrt{\frac{\alpha'}{2}}(h+\bar{h}) = \frac{\alpha' m}{R}$$
$$\sqrt{\frac{\alpha'}{2}}(h-\bar{h}) = \frac{\ell R}{\alpha'}\,. \tag{6.11}$$

Therefore

$$h = \frac{1}{\sqrt{2}}\left(\frac{\ell}{a} + ma\right)$$
$$\bar{h} = \frac{1}{\sqrt{2}}\left(-\frac{\ell}{a} + ma\right)\,. \tag{6.12}$$

So the general lattice vector is

$$\begin{pmatrix} h \\ \bar{h} \end{pmatrix} = \frac{\ell}{a}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + ma\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,. \tag{6.13}$$

What is radius a independent is

$$h^2 - \bar{h}^2 = (h,\bar{h})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} h \\ \bar{h} \end{pmatrix}$$
$$= \frac{1}{2}\cdot 2\ell m \cdot 2 = 2\ell m\,. \tag{6.14}$$

This can be regarded as a Lorentz invariant of a two-dimensional Minkowski space with signature $(1,-1)$.

At $a = 1$ ($R = \sqrt{\alpha'}$), the lattice is generated by the two lattice vectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which are light like:

$$\left(\tfrac{1}{\sqrt{2}} \pm \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix} = 0\,. \tag{6.15}$$

For a general radius, boost with rapidity $y = \log a$ is

$$\begin{aligned}\Gamma_R =& a\mathbf{Z}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \oplus \frac{1}{a}\mathbf{Z}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\\ =& \text{Lorentz boost of } \Gamma_{1,1} ,\end{aligned} \tag{6.16}$$

where

$$\Gamma_{1,1} = \mathbf{Z}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \oplus \mathbf{Z}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} . \tag{6.17}$$

Using these, we can reduce eq. (6.9) to [3]

$$\bar{\chi}^{c=1}_{\text{Vir}}(\bar{q})\chi^{c=1}_{\text{Vir}}(q) = \frac{1}{\sqrt{\tau_2}|\eta(q)|^2}a\sqrt{\tau_2}\sum_{\ell,m\in\mathbf{Z}}(q)^{\frac{1}{4}(ma+\frac{\ell}{a})^2}(\bar{q})^{\frac{1}{4}(ma-\frac{\ell}{a})^2} . \tag{6.18}$$

6.1.2 fermionic string

Let's first go over the GSO projection once again and go to the S-S compactification.

Recall

$$\Gamma_{\text{IIB,flat}} = -\frac{V_{\text{E}}}{2}\frac{1}{(4\pi^2\alpha')^5}\mathcal{T} , \tag{6.19}$$

where

$$\mathcal{T} = \int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2^2}\sum_{i,j}\bar{\chi}^{\text{Vir}}_{h_i}(\bar{q})X_{ij}\chi^{\text{Vir}}_{h_j}(q) . \tag{6.20}$$

The momentum integration has been done 8 times.

Let us first perform the circle compactification to IIB superstring

$$(\bar{\chi}X\chi)_{\text{IIB}} = |V_8 - S_8|^2\frac{1}{\tau_2^4|\eta|^{16}} . \tag{6.21}$$

In our notation, Virasoro characters are as follows:

$$\left.\begin{aligned} O_8\tfrac{1}{\eta^8} &= \text{Tr}_{\text{NS}}\tfrac{1-(-)^{\hat{F}}}{2}q^{L_0^{(\text{cylinder})}}\\ V_8\tfrac{1}{\eta^8} &= \text{Tr}_{\text{NS}}\tfrac{1+(-)^{\hat{F}}}{2}q^{L_0^{(\text{cylinder})}}\end{aligned}\right\} = \frac{q^{-\frac{1}{2}}}{2}\frac{\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})^8 \pm \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})^8}{\prod_{n=1}^{\infty}(1-q^n)^8} \tag{6.22}$$

[3]See the next subsubsection for the coefficient $a\sqrt{\tau_2}$.

$$\left.\begin{aligned} S_8\tfrac{1}{\eta^8} &= \mathrm{Tr}_{\mathrm{R}}\tfrac{1+(-)^{\hat{F}}}{2} q^{L_0^{(\text{cylinder})}} \\ C_8\tfrac{1}{\eta^8} &= \mathrm{Tr}_{\mathrm{R}}\tfrac{1-(-)^{\hat{F}}}{2} q^{L_0^{(\text{cylinder})}} \end{aligned}\right\} = 8\frac{\prod_{n=1}^{\infty}(1+q^n)^8}{\prod_{n=1}^{\infty}(1-q^n)^8}\,. \tag{6.23}$$

Note that $\frac{1}{\eta^8}$ represents bosonic contribution with normal ordering coefficient included, and $\mathrm{Tr}_{\mathrm{R}}\frac{(-)^{\hat{F}}}{2} q^{L_0^{(\text{cylinder})}}$ makes no contribution in this case. Recall each integration contributes

$$\int \frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-tp^2} = \left(\frac{1}{4\pi t}\right)^{1/2} \overset{\frac{t}{\alpha'}=\pi\tau_2}{=} \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}}\,. \tag{6.24}$$

The circle compactification eliminates $\frac{1}{\sqrt{4\pi^2\alpha'\tau_2}}$ and replace it by the lattice sum of the momentum and the winding number. Therefore

$$\Gamma_{\mathrm{IIB},S_1} = -\frac{V_{\mathrm{E}}}{2}\frac{1}{(4\pi^2\alpha')^{\frac{10}{2}}}\mathcal{J}^{S_1,\mathrm{IIB}}\,, \tag{6.25}$$

where

$$\mathcal{J}^{S_1,\mathrm{IIB}} = \int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2^2}(\bar{\chi}X\chi)^{S_1}_{\mathrm{IIB}}\,, \tag{6.26}$$

$$\begin{aligned} (\bar{\chi}X\chi)^{S_1}_{\mathrm{IIB}} &= |V_8 - S_8|^2\sqrt{4\pi^2\alpha'\tau_2}\frac{1}{\tau_2^4|\eta|^{16}}\frac{1}{2\pi R}\sum_{\ell,m\in\mathbf{Z}} q^{\frac{1}{4}(ma+\frac{\ell}{a})^2}\bar{q}^{\frac{1}{4}(ma-\frac{\ell}{a})^2} \\ &\equiv |V_8 - S_8|^2\frac{1}{\tau_2^4|\eta|^{16}}F_2(a,\tau)\,, \end{aligned} \tag{6.27}$$

$$F_2(a,\tau) \equiv a\sqrt{\tau_2}\sum_{\ell,m\in\mathbf{Z}} q^{\frac{1}{4}(ma+\frac{\ell}{a})^2}\bar{q}^{\frac{1}{4}(ma-\frac{\ell}{a})^2} \tag{6.28}$$

and $a \equiv \frac{\sqrt{\alpha'}}{R}$ (see also appendix E). Note that $\frac{1}{2\pi R} = \frac{\Delta m}{2\pi R} \to \frac{\mathrm{d}p}{2\pi}$ in the continuous limit.

6.2 Z_2 orbifold of a circle compactification

Recall the circle compactification, $X^{24}(z,\bar{z}) \equiv X$, focusing on $\alpha^{24}(z) \equiv \alpha$ and $\tilde{\alpha}^{24}(\bar{z}) \equiv \tilde{\alpha}$:

$$\mathcal{J}^{\text{"}c=1\text{"},S_1} \equiv \int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2^2}(\bar{\chi}X\chi)^{c=1,S_1}(q,\bar{q})\,, \tag{6.29}$$

where

$$
\begin{aligned}
(\bar{\chi}X\chi)^{c=1,S_1}(q,\bar{q}) &= \mathrm{Tr}_{\Gamma(R)} q^{L_0^{(\mathrm{cyl})}} \bar{q}^{\bar{L}_0^{(\mathrm{cyl})}} \\
&= \frac{1}{\sqrt{\tau_2}|\eta(q)|^2} a\sqrt{\tau_2} \sum_{(\alpha,\tilde{\alpha})\in\Gamma^{(1,1)}(R)} q^{\frac{1}{2}\alpha^2} \bar{q}^{\frac{1}{2}\bar{\alpha}^2} \ , \qquad (6.30)
\end{aligned}
$$

$$
\begin{aligned}
\alpha &: \text{ eigenvalue of } \alpha_0 \\
\bar{\alpha} &: \text{ eigenvalue of } \tilde{\alpha}_0 \ , \qquad (6.31)
\end{aligned}
$$

$$
\Gamma^{(1,1)}(R) = \text{Lorentz boost of } \Gamma^{(1,1)} \qquad (6.32)
$$

and

$$
\begin{aligned}
L_0^{(\mathrm{cyl})} &= \frac{1}{2}\sum_{n\in\mathbf{Z}} :\alpha_{-n}\alpha_n: - \frac{1}{24} \\
\bar{L}_0^{(\mathrm{cyl})} &= \frac{1}{2}\sum_{n\in\mathbf{Z}} :\tilde{\alpha}_{-n}\tilde{\alpha}_n: - \frac{1}{24} \ . \qquad (6.33)
\end{aligned}
$$

Recall

$$
\begin{aligned}
\hat{X}(z,\bar{z}) &= X - i\sqrt{\frac{\alpha'}{2}}(\alpha+\tilde{\alpha})\tau + \sqrt{\frac{\alpha'}{2}}(\alpha-\tilde{\alpha})\sigma + i\sqrt{\frac{\alpha'}{2}}\left(\sum_{n\neq 0}\alpha_n^i \frac{z^{-n}}{n} + \sum_{n\neq 0}\tilde{\alpha}_n^i \frac{\bar{z}^{-n}}{n}\right) \\
&= \hat{X} - i\alpha'\hat{p}\tau + \alpha'\sigma\hat{w} + i\sqrt{\frac{\alpha'}{2}}\left(\sum_{n\neq 0}\alpha_n^i \frac{z^{-n}}{n} + \sum_{n\neq 0}\tilde{\alpha}_n^i \frac{\bar{z}^{-n}}{n}\right) \ , \qquad (6.34)
\end{aligned}
$$

where

$$
\begin{aligned}
\text{eigenvalue of } \hat{p} &: \ \frac{m}{R}, \ m\in\mathbf{Z} \\
\text{eigenvalue of } \hat{w} &: \ \frac{\ell R}{\alpha'}, \ \ell\in\mathbf{Z} \ . \qquad (6.35)
\end{aligned}
$$

Using these, we can obtain (see subsubsection 6.1.1)

$$
\begin{cases}
\alpha = \frac{1}{\sqrt{2}}\left(\frac{\ell}{a} + ma\right) \\
\bar{\alpha} = \frac{1}{\sqrt{2}}\left(-\frac{\ell}{a} + ma\right)
\end{cases} , \qquad a = \frac{\sqrt{\alpha'}}{R} \ . \qquad (6.36)
$$

Note $X(z, \bar{z})$ is an angle variable.

Now we would like to consider Z_2 projection

$$Z_2: \; X(z, \bar{z}) \to -X(z, \bar{z}) \,. \tag{6.37}$$

Namely, we will introduce the operator

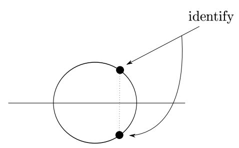

Figure 7:

$$\hat{\mathbf{Z}}_2 \quad \text{such that} \quad \hat{\mathbf{Z}}_2 \hat{X} \hat{\mathbf{Z}}_2 = -\hat{X}, \;\; \hat{\mathbf{Z}}_2^2 = \mathbf{1} \tag{6.38}$$

$$i.e. \quad \{\hat{\mathbf{Z}}_2, \hat{X}\}_+ = 0 \,. \tag{6.39}$$

Therefore

$$[\hat{\mathbf{Z}}_2, L_0^{(\mathrm{cyl})}] = [\hat{\mathbf{Z}}_2, \bar{L}_0^{(\mathrm{cyl})}] = 0 \,. \tag{6.40}$$

The diagonalization need not be changed.

$(\bar{\chi} X \chi)$ on the Z_2 orbifold of the circle compactification consists of the two sectors:

$$(\bar{\chi} X \chi)^{c=1, S_1, \mathrm{orb}}(q, \bar{q}) = (\bar{\chi} X \chi)^{\mathrm{orb}}_{\mathrm{untwisted}}(q, \bar{q}) + (\bar{\chi} X \chi)^{\mathrm{orb}}_{\mathrm{twisted}}(q, \bar{q}) \,. \tag{6.41}$$

6.2.1 untwisted sector

The mode expansion is unchanged in the untwisted sector. $(\bar{\chi} X \chi)$ in this sector can be written as

$$(\bar{\chi} X \chi)^{\mathrm{orb}}_{\mathrm{untwisted}} = \frac{1}{2} \mathrm{Tr}_{\Gamma(R)} (1 + \hat{\mathbf{Z}}_2) q^{L_0^{(\mathrm{cyl})}} \bar{q}^{\bar{L}_0^{(\mathrm{cyl})}} \,. \tag{6.42}$$

Supposed $\hat{\mathbf{Z}}_2$ action on the zero mode $|\Omega\rangle$ as

$$\hat{\mathbf{Z}}_2 |\Omega\rangle = |\Omega\rangle \,, \tag{6.43}$$

$$\hat{\mathbf{Z}}_2|m,\ell\rangle = |-m,-\ell\rangle \tag{6.44}$$

(therefore $\{\hat{\mathbf{Z}}_2,\alpha\}|m,\ell\rangle = \{\hat{\mathbf{Z}}_2,\tilde{\alpha}\}|m,\ell\rangle = 0$). The state space projected is

$$\mathcal{H}_{\text{untwisted}} = \mathcal{H}^{(+)}_{\text{untwisted}} \oplus \mathcal{H}^{(-)}_{\text{untwisted}}\,, \tag{6.45}$$

where

$$\begin{aligned}\mathcal{H}^{(+)}_{\text{untwisted}} = &\left\{\left\{\prod_{i,j}\alpha_{-m_i}\tilde{\alpha}_{-n_j}|0\rangle,\ \#(i+j)\in 2\mathbf{Z}\right\}\right\}\\ &\otimes\left\{\left\{\frac{|m,\ell\rangle+|-m,-\ell\rangle}{\sqrt{2}},(m,\ell)\neq\mathbf{0}_2,|0,0\rangle\right\}\right\}\end{aligned} \tag{6.46}$$

and

$$\begin{aligned}\mathcal{H}^{(-)}_{\text{untwisted}} = &\left\{\left\{\prod_{i,j}\alpha_{-m_i}\tilde{\alpha}_{-n_j}|0\rangle,\ \#(i+j)\in 2\mathbf{Z}+1\right\}\right\}\\ &\otimes\left\{\left\{\frac{|m,\ell\rangle-|-m,-\ell\rangle}{\sqrt{2}},(m,\ell)\neq\mathbf{0}_2\right\}\right\}\,.\end{aligned} \tag{6.47}$$

In calculation

$$\begin{aligned}&\hat{\mathbf{Z}}_2\alpha_{-n_1}\alpha_{-n_2}\cdots\alpha_{-n_{\dots}}\cdots\tilde{\alpha}_{-m_1}\tilde{\alpha}_{-m_2}\cdots\tilde{\alpha}_{-m_{\dots}}\cdots|\Omega\rangle\begin{pmatrix}\frac{|m,\ell\rangle+|-m,-\ell\rangle}{\sqrt{2}}\\ \frac{|m,\ell\rangle-|-m,-\ell\rangle}{\sqrt{2}}\end{pmatrix}\\ &=(-\alpha_{-n_1})(-\alpha_{-n_2})\cdots(-\alpha_{-n_{\dots}})\cdots(-\tilde{\alpha}_{-m_1})(-\tilde{\alpha}_{-m_2})\cdots(-\tilde{\alpha}_{-m_{\dots}})\cdots|\Omega\rangle\\ &\qquad\times\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\frac{|m,\ell\rangle+|-m,-\ell\rangle}{\sqrt{2}}\\ \frac{|m,\ell\rangle-|-m,-\ell\rangle}{\sqrt{2}}\end{pmatrix},\qquad (m,\ell)\neq\mathbf{0}_2\end{aligned} \tag{6.48}$$

but

$$\hat{\mathbf{Z}}_2|0,0\rangle = |0,0\rangle\,. \tag{6.49}$$

Taking trace, we obtain

$$\mathrm{Tr}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=0\,. \tag{6.50}$$

Then the lattice sum cancels exept the $|0,0\rangle$ contribution. Therefore

$$(\bar{\chi}X\chi)^{\text{orb}}_{\text{untwisted}} = \frac{1}{2}(\bar{\chi}X\chi) + \frac{1}{2}\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left| q^{-\frac{1}{24}}\frac{1}{\prod_{n=1}^{\infty}(1+q^n)}\right|^2$$
$$\equiv \frac{1}{2}(\bar{\chi}X\chi)_{(0,0)} + \frac{1}{2}(\bar{\chi}X\chi)_{(0,\frac{1}{2})}\,. \qquad (6.51)$$

Note

$$\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(0)}{\eta^3(\tau)} = 2\frac{\prod_n(1+q^n)^2}{\prod_n(1-q^n)^2} \qquad (6.52)$$

$$\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0)}{\eta^3(\tau)} = q^{-\frac{1}{8}}\frac{\prod_n(1+q^{n-\frac{1}{2}})^2}{\prod_n(1-q^n)^2} \qquad (6.53)$$

$$\frac{\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(0)}{\eta^3(\tau)} = q^{-\frac{1}{8}}\frac{\prod_n(1-q^{n-\frac{1}{2}})^2}{\prod_n(1-q^n)^2}\,. \qquad (6.54)$$

Now we can write $(\bar{\chi}X\chi)_{(0,\frac{1}{2})}$ as

$$(\bar{\chi}X\chi)_{(0,\frac{1}{2})} \overset{\text{eq.(6.52)}}{=} \frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left| q^{-\frac{1}{24}}\sqrt{2}\left(\frac{\eta^{\frac{3}{2}}}{\vartheta^{\frac{1}{2}}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}}\right)\frac{1}{\prod_n(1-q^n)}\right|^2 = \frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left|\sqrt{2}\sqrt{\frac{\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}}}\right|^2$$
$$= \frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left|\frac{2\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}}\right|\,. \qquad (6.55)$$

6.2.2 twisted sector

In the twisted sector, we impose

$$X(\tau,\sigma+2\pi) = \left(\hat{\mathbf{Z}}_2 X(\tau,\sigma)\hat{\mathbf{Z}}_2\right) + 2\pi\ell R = -X(\tau,\sigma) + 2\pi\ell R\,. \qquad (6.56)$$

The mode expansion is

$$X(z,\bar{z}) = X_0 - i\alpha'\hat{p}\tau + \alpha'\sigma\hat{w} + i\sqrt{\frac{\alpha'}{2}}\left(\sum_{r\neq 0}\alpha_r\frac{z^{-r}}{r} + \sum_{r\neq 0}\tilde{\alpha}_r\frac{\bar{z}^{-r}}{r}\right) . \quad (6.57)$$

Substituting into (6.56), we obtain

$$\begin{aligned} &2\hat{X}_0 - 2i\alpha'\hat{p}\tau + 2\alpha'\sigma\hat{w} + \alpha' 2\pi\hat{w} \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\left(\sum_{r\neq 0}\alpha_r\frac{(z\mathrm{e}^{2\pi i})^{-r}}{r} + \sum_{r\neq 0}\tilde{\alpha}_r\frac{(\bar{z}\mathrm{e}^{-2\pi i})^{-r}}{r} + \sum_{r\neq 0}\alpha_r\frac{z^{-r}}{r} + \sum_{r\neq 0}\tilde{\alpha}_r\frac{\bar{z}^{-r}}{r}\right) \\ &\quad = 2\pi R\,. \end{aligned} \quad (6.58)$$

We conclude

$$r \in \mathbf{Z} + 1/2\,, \qquad \text{eigenvalues of } \hat{p} \text{ and } \hat{w} \text{ are } 0 \quad (6.59)$$

($0 \leq$ eigenvalue of $\hat{X}_0 \leq \pi R$ is $\frac{2\pi R}{2} \Rightarrow 0, \pi R$). The string in the twisted sector must line in the fixed points. Hence there are two states in the coordinate representation: $\{\{|X^0 = 0\rangle,\ |X^0 = 2\pi R\rangle\}\}$. Calculation of the normal ordering coefficients can be dealt with

$$\begin{aligned} L_0^{\mathrm{cyl}} &= \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_{-r}\alpha_r = \frac{1}{2} : \sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_{-r}\alpha_r : + \frac{1}{2}\sum_{r=\frac{1}{2},\dots} r \\ &= \frac{1}{2} : \sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_{-r}\alpha_r : + \frac{1}{2} : \sum_{n=0}^{\infty}\left(n - \frac{1}{2}\right) \\ &= \frac{1}{2} : \sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_{-r}\alpha_r : + \frac{1}{2}\zeta_{\frac{1}{2}}(-1, x = 0) \\ &= \frac{1}{2} : \sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_{-r}\alpha_r : + \frac{1}{2}\frac{1}{24} = \frac{1}{2} : \sum_{r\in\mathbf{Z}+\frac{1}{2}}\alpha_{-r}\alpha_r : + \frac{1}{48}\,. \end{aligned} \quad (6.60)$$

Likewise

$$\bar{L}_0^{\mathrm{cyl}} = \frac{1}{2} : \sum_{r\in\mathbf{Z}+\frac{1}{2}}\tilde{\alpha}_{-r}\tilde{\alpha}_r : + \frac{1}{48}\,. \quad (6.61)$$

Therefore

$$
\begin{aligned}
(\bar{\chi}X\chi)^{\rm orb}_{\rm twisted} &= \frac{1}{2}{\rm Tr}_{\rm twisted}\left(1+\hat{\mathbf{Z}}_2\right) q^{L_0^{\rm (cyl)}}\bar{q}^{\bar{L}_0^{\rm (cyl)}} \\
&\equiv \frac{1}{2}(\bar{\chi}X\chi)_{(\frac{1}{2},0)} + \frac{1}{2}(\bar{\chi}X\chi)_{(\frac{1}{2},\frac{1}{2})}\,, \qquad (6.62)
\end{aligned}
$$

where, using eqs. (6.53) and (6.54),

$$
\begin{aligned}
(\bar{\chi}X\chi)_{(\frac{1}{2},0)} &= 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left| q^{\frac{1}{48}}\frac{1}{\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})}\right|^2 \\
&\overset{\rm eq.(6.54)}{=} 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left| q^{\frac{1}{48}-\frac{1}{16}}\frac{\eta^{\frac{3}{2}}}{\vartheta^{\frac{1}{2}}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}}\frac{1}{\prod_n(1-q^n)}\right|^2 \\
&= 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left|\frac{\eta}{\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}}\right| = 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\sqrt{\frac{\eta}{\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}}}\sqrt{\frac{\bar{\eta}}{\bar{\vartheta}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}}} \qquad (6.63)
\end{aligned}
$$

and

$$
\begin{aligned}
(\bar{\chi}X\chi)_{(\frac{1}{2},\frac{1}{2})} &= 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left| q^{\frac{1}{48}}\frac{1}{\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})}\right|^2 \\
&\overset{\rm eq.(6.53)}{=} 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left| q^{\frac{1}{48}-\frac{1}{16}}\frac{\eta^{\frac{3}{2}}}{\vartheta^{\frac{1}{2}}\begin{bmatrix}0\\ 0\end{bmatrix}}\frac{1}{\prod_n(1-q^n)}\right|^2 \\
&= 2\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}}\left|\frac{\eta}{\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}}\right|\,. \qquad (6.64)
\end{aligned}
$$

6.2.3 summary

The final answer is

$$\mathcal{J}^{c=1,\,S_1,\,\mathrm{orb}} \equiv \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} (\bar{\chi} X \chi)^{c=1,\,S_1,\,\mathrm{orb}}(q,\bar{q}) \,, \tag{6.65}$$

where

$$\begin{aligned}
&(\bar{\chi} X \chi)^{c=1,\,S_1\ \mathrm{orb}} \\
&\quad = \frac{1}{2}\frac{a\sqrt{\tau_2}}{\sqrt{\tau_2}} \left[\frac{1}{|\eta|^2} \sum_{(\alpha,\tilde{\alpha})\in\Gamma_{1,1}(R)} q^{\frac{1}{2}\alpha^2} \bar{q}^{\frac{1}{2}\tilde{\alpha}^2} + \left| \frac{2\eta}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right]} \right| + 2\left| \frac{\eta}{\vartheta\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right]} \right| + 2\left| \frac{\eta}{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \right| \right] \\
&\quad \equiv \frac{1}{2}\left[(\bar{\chi} X \chi)_{(0,0)} + (\bar{\chi} X \chi)_{(0,\frac{1}{2})} + (\bar{\chi} X \chi)_{(\frac{1}{2},0)} + (\bar{\chi} X \chi)_{(\frac{1}{2},\frac{1}{2})} \right] \,.
\end{aligned} \tag{6.66}$$

We can easily see that

$$\begin{aligned}
&\text{under } \hat{S}:\ (0,0),\ \left(\frac{1}{2},\frac{1}{2}\right) \text{ invariant and } \left(0,\frac{1}{2}\right) \leftrightarrow \left(\frac{1}{2},0\right) \\
&\text{under } \hat{T}:\ (0,0),\ \left(0,\frac{1}{2}\right) \text{ invariant and } \left(\frac{1}{2},0\right) \leftrightarrow \left(\frac{1}{2},\frac{1}{2}\right) \,,
\end{aligned} \tag{6.67}$$

and then eq. (6.66) is modular invariant.

6.3 IIB strings on T^4/Z_2 orbifold

Now we consider $T^4(=(S^1)^4)/\mathbf{Z}_2$ compactification.

6.3.1 bosonic case

The bosonic part is just a generalization of $S^1/\mathbf{Z}_2$ case. Namely,

$$\{\{X^i\}\} = \{\{X^{\tilde{i}}\}\} \oplus \{\{X^{\ell}\}\} \qquad \tilde{i} = 1,2,3,4,\ \ell = 5,6,7,8 \tag{6.68}$$

and $\mathbf{Z}_2$ action satisfies

$$\hat{\mathbf{Z}}_2 X^{\tilde{i}} \hat{\mathbf{Z}}_2 = X^{\tilde{i}},\ \ \hat{\mathbf{Z}}_2 X^{\ell} \hat{\mathbf{Z}}_2 = -X^{\ell},\ \ \left(\hat{\mathbf{Z}}_2\right)^2 = \mathbf{1} \,. \tag{6.69}$$

Then

$$\left[\hat{\mathbf{Z}}_2, L_0^{\rm cyl}\right] = \left[\hat{\mathbf{Z}}_2, \bar{L}_0^{\rm cyl}\right] = 0\,. \tag{6.70}$$

For the bosonic part,

$$\begin{aligned}
&(\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{\text{untwisted, bosonic}}\\
&= \frac{1}{2}\mathrm{Tr}_{\Gamma(R)}(1+\hat{\mathbf{Z}}_2)q^{L_0^{\rm cyl}}\bar{q}^{\bar{L}_0^{\rm cyl}}\\
&= \frac{1}{\tau_2^2|\eta|^{4\cdot 2}}\frac{1}{2}\\
&\times\left\{\frac{1}{\tau_2^2|\eta|^{4\cdot 2}}\left(\prod_I a_I\sqrt{\tau_2}\right)\sum_{(\alpha^I,\bar{\alpha}^J\in\Gamma_{4,4}(R^4))} q^{\frac{1}{2}(\alpha^I)^2}\bar{q}^{\frac{1}{2}(\bar{\alpha}^J)^2}\right.\\
&\qquad\qquad\left.+\frac{1}{\tau_2^2}\left(\prod_I a_I\sqrt{\tau_2}\right)\left|\frac{2\eta}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right]}\right|^4\right\}\\
&\equiv (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{(0,0)} + (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{(0,\frac{1}{2})}
\end{aligned} \tag{6.71}$$

and

$$\begin{aligned}
(\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{\text{twisted, bosonic}} &= \left(\frac{\prod_I a_I\sqrt{\tau_2}}{\tau_2^{2+2}|\eta|^8}\right)\frac{1}{2}\left\{2^4\left|\frac{\eta}{\vartheta\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right]}\right|^4 + 2^4\left|\frac{\eta}{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}\right|^4\right\}\\
&\equiv (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{(\frac{1}{2},0)} + (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{(\frac{1}{2},\frac{1}{2})}\,.
\end{aligned} \tag{6.72}$$

Note that $2^4 = 16$ in eq. (6.72) derives from the number of fixed points of T^4 under $\mathbf{Z_2}$ action. Since

$$\begin{aligned}
&\text{under } \hat{S}:\ (0,0),\ \left(\frac{1}{2},\frac{1}{2}\right)\ \text{invariant and}\ \left(0,\frac{1}{2}\right)\leftrightarrow\left(\frac{1}{2},0\right)\\
&\text{under } \hat{T}:\ (0,0),\ \left(0,\frac{1}{2}\right)\ \text{invariant and}\ \left(\frac{1}{2},0\right)\leftrightarrow\left(\frac{1}{2},\frac{1}{2}\right)\ ,
\end{aligned} \tag{6.73}$$

the modular invariance of

$$(\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{\text{bosonic}} = (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{\text{untwisted, bosonic}} + (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2}_{\text{twisted, bosonic}} \tag{6.74}$$

is preserved.

6.3.2 fermionic case

Now let's turn to the worldsheet fermions.

Recall

$$\begin{aligned} \begin{matrix} O_8 \\ V_8 \end{matrix} &= \mathrm{Tr}_{\mathrm{NS}} \frac{1 \mp (-)^{\hat{F}}}{2} q^{L_0^{\mathrm{fermion}}} , \qquad (-)^{\hat{F}}_{\mathrm{NS}} = -(-)^{\sum_i \sum_r b^i_{-r} b^i_r} \\ \begin{matrix} S_8 \\ C_8 \end{matrix} &= \mathrm{Tr}_{\mathrm{R}} \frac{1 \pm (-)^{\hat{F}}}{2} q^{L_0^{\mathrm{fermion}}} , \qquad (-)^{\hat{F}}_{\mathrm{R}} = \gamma^9_{16} (-)^{\sum_i \sum_m d^i_{-m} d^i_m} . \end{aligned} \tag{6.75}$$

Clearly

$$\begin{aligned} & \frac{1 \pm (-)^{\sum_i \sum_r b^i_{-r} b^i_r}}{2} \\ &= \left(\frac{1 + (-)^{\sum_{\tilde{i}} \sum_r b^{\tilde{i}}_{-r} b^{\tilde{i}}_r}}{2} \right) \left(\frac{1 \pm (-)^{\sum_{\ell} \sum_r b^{\ell}_{-r} b^{\ell}_r}}{2} \right) \\ &\quad + \left(\frac{1 - (-)^{\sum_{\tilde{i}} \sum_r b^{\tilde{i}}_{-r} b^{\tilde{i}}_r}}{2} \right) \left(\frac{1 \mp (-)^{\sum_{\ell} \sum_r b^{\ell}_{-r} b^{\ell}_r}}{2} \right) . \end{aligned} \tag{6.76}$$

This provides $SO(4)_{\tilde{i}\,\mathrm{part}} \otimes SO(4)_{\ell\,\mathrm{part}}$ decomposition of the $SO(8)$ current algebra character

$$\begin{aligned} O_8 &= O_{4\tilde{i}} O_{4\ell} + V_{4\tilde{i}} V_{4\ell} \\ V_8 &= O_{4\tilde{i}} V_{4\ell} + V_{4\tilde{i}} O_{4\ell} . \end{aligned} \tag{6.77}$$

Likewise, with $\gamma^9_{16} = \gamma^5_{16\tilde{i}} \gamma^5_{16\ell}$,

$$\begin{aligned} S_8 &= S_{4\tilde{i}} S_{4\ell} + C_{4\tilde{i}} C_{4\ell} \\ C_8 &= S_{4\tilde{i}} C_{4\ell} + C_{4\tilde{i}} S_{4\ell} . \end{aligned} \tag{6.78}$$

Now let's consider the $\mathbf{Z}_2$ action of the worldsheet fermions ψ^ℓ, $\ell = 5,6,7,8$. By worldsheet supersymmetry, we must have

$$\hat{\mathbf{Z}}_2 \psi^{\tilde{i}} \hat{\mathbf{Z}}_2 = \psi^{\tilde{i}}, \ \ \hat{\mathbf{Z}}_2 \psi^\ell \hat{\mathbf{Z}}_2 = -\psi^\ell \ \text{ for both NS \& R} . \tag{6.79}$$

Then

$$\begin{aligned} \hat{\mathbf{Z}}_2 \,|\text{holomorphic states in } \mathrm{O}_{4\ell}\rangle &= + \,|\text{holomorphic states in } \mathrm{O}_{4\ell}\rangle \\ \hat{\mathbf{Z}}_2 \,|\text{holomorphic states in } \mathrm{V}_{4\ell}\rangle &= - \,|\text{holomorphic states in } \mathrm{V}_{4\ell}\rangle \end{aligned} \tag{6.80}$$

in NS sector. As for R sector,

$$\hat{\mathbf{Z}}_2 d_0^{\tilde{i}} \hat{\mathbf{Z}}_2 = d_0^{\tilde{i}} , \qquad \hat{\mathbf{Z}}_2 d_0^\ell \hat{\mathbf{Z}}_2 = -d_0^\ell . \tag{6.81}$$

Therefore

$$\hat{\mathbf{Z}}_2 = \gamma^5_{16\,\ell} (-)^{\sum_{m=1}^{\infty} \sum_\ell d^\ell_{-m} d^\ell_m} . \tag{6.82}$$

So in this sector,

$$\begin{aligned} \hat{\mathbf{Z}}_2 \,|\text{states in } \mathrm{S}_{4\ell}\rangle &= - \,|\text{states in } \mathrm{S}_{4\ell}\rangle \\ \hat{\mathbf{Z}}_2 \,|\text{states in } \mathrm{C}_{4\ell}\rangle &= + \,|\text{states in } \mathrm{C}_{4\ell}\rangle \ . \end{aligned} \tag{6.83}$$

We can write the 1 inserted part in the untwisted sector as

$$V_8 - S_8 = (V_4 O_4 - C_4 C_4) + (O_4 V_4 - S_4 S_4) \equiv Q_O + Q_V . \tag{6.84}$$

On $\mathbf{Z}_2$ insertion in the untwisted sector, we get $Q_O - Q_V$. Therefore

$$\begin{aligned} &(\bar{\chi} X \chi)^{T^4/\mathbf{Z}_2 \text{ on IIB}}_{\text{untwisted}} \\ = &\frac{1}{\tau_2^2 |\eta|^8} \frac{1}{2} \\ &\times \left\{ \frac{1}{\tau_2^2 |\eta|^8} \frac{1}{2} F_{(4,4)}(a^I, \tau) \,|Q_O + Q_V|^2 + \frac{\left(\sum_I a_I \sqrt{\tau_2} \right)}{\tau_2^2} \left| \frac{2\eta}{\vartheta \left[\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix} \right]} \right|^4 |Q_O - Q_V|^2 \right\} , \end{aligned} \tag{6.85}$$

where

$$F_{(4,4)}(a_I,\tau) \equiv \left(\prod_I a_I\sqrt{\tau_2}\right) \sum_{(\alpha^I,\bar{\alpha}^J\in\Gamma_{4,4}(R^4))} q^{\frac{1}{2}(\alpha^I)^2}\bar{q}^{\frac{1}{2}(\bar{\alpha}^J)^2} . \tag{6.86}$$

In order to proceed to the twisted sector, we recall that

$$\begin{aligned}
\hat{S}O_{2n} &= \frac{1}{2}(O_{2n}+V_{2n}) + \frac{1}{2}(S_{2n}+C_{2n}) \\
\hat{S}V_{2n} &= \frac{1}{2}(O_{2n}+V_{2n}) - \frac{1}{2}(S_{2n}+C_{2n}) \\
\hat{S}S_{2n} &= \frac{1}{2}(O_{2n}-V_{2n}) + \frac{i^n}{2}(S_{2n}-C_{2n}) \\
\hat{S}C_{2n} &= \frac{1}{2}(O_{2n}-V_{2n}) - \frac{i^n}{2}(S_{2n}-C_{2n}) .
\end{aligned} \tag{6.87}$$

Since

$$\begin{aligned}
\hat{S}Q_O &= \frac{1}{2}\left\{(V-S)(O+S)+(O-C)(V+C)\right\} \\
\hat{S}Q_V &= \frac{1}{2}\left\{(V+S)(O-S)+(O+C)(V-C)\right\} ,
\end{aligned} \tag{6.88}$$

we can obtain

$$\hat{S}(Q_O-Q_V) = -SO+VS-CV+OC . \tag{6.89}$$

Therefore

$$\hat{S}(Q_O-Q_V) = (-S_{4\tilde{i}}O_{4\ell}+O_{4\tilde{i}}C_{4\ell})+(-C_{4\tilde{i}}V_{4\ell}+V_{4\tilde{i}}S_{4\ell}) \equiv Q_S+Q_C . \tag{6.90}$$

So the twisted sector with 1 insertion gets factor $|Q_S+Q_C|^2$. Under T,

$$\hat{T}(O_{2n},V_{2n},S_{2n},C_{2n})_j = (O_{2n},V_{2n},S_{2n},C_{2n})_i T_{ij} , \tag{6.91}$$

where

$$T^{(n)}_{ij} = \mathrm{e}^{\frac{-in\pi}{12}}\mathrm{diag}(1,-1,\mathrm{e}^{\frac{in\pi}{4}},\mathrm{e}^{\frac{in\pi}{4}}) \overset{n=2}{=} \mathrm{e}^{\frac{-i\pi}{6}}\mathrm{diag}(1,-1,i,i) . \tag{6.92}$$

Then

$$\hat{T}Q_S = \mathrm{e}^{-\frac{\pi i}{3}} iQ_S, \qquad \hat{T}Q_C = \mathrm{e}^{-\frac{\pi i}{3}}(-i)Q_C . \tag{6.93}$$

So the twisted sector with $\hat{\mathbf{Z}}_2$ insertion gets factor $|Q_S - Q_C|^2$. Therefore

$$(\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2 \text{ on IIB}} = (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2 \text{ on IIB}}_{\text{untwisted}} + (\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2 \text{ on IIB}}_{\text{twisted}} , \tag{6.94}$$

where $(\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2 \text{ on IIB}}_{\text{untwisted}}$ is the same as eq. (6.85) and

$$\begin{aligned}(\bar{\chi}X\chi)^{T^4/\mathbf{Z}_2 \text{ on IIB}}_{\text{twisted}} &= \frac{\left(\prod_I a_I\sqrt{\tau_2}\right)}{\tau_2^{2+2}|\eta|^8}\frac{1}{2} \\ &\times \left\{ 2^4 \left|\frac{\eta}{\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}}\right|^4 |Q_S + Q_C|^2 + 2^4 \left|\frac{\eta}{\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}}\right|^4 |Q_S - Q_C|^2 \right\} .\end{aligned} \tag{6.95}$$

Part II

Path integral method: bosonic and fermionic amplitudes

7 Bosonic string partition function

Here we set as follows:

- work in the Euclidean signature both for the worldsheet and spacetime.
- deal only with a closed bosonic string for a while.

String perturbation theory + path integrals read

$$Z = \sum_{h=0}^{\infty} \kappa^h \int \frac{[\mathcal{D}g_{mn}][\mathcal{D}X^M]}{[\text{vol.}(\text{Diff}(\mathrm{M}))]} \mathrm{e}^{-S_{\mathrm{E}}[X^M, g_{mn}]} \,, \tag{7.1}$$

where M is the worldsheet swept out by the string and [vol.(Diff(M))] is the volume of the group of the diffeomorphism. The Euclidean action is

$$S_{\mathrm{E}}[X^M, g_{mn}] = \frac{1}{2}\frac{1}{2\pi\alpha'} \int_{\mathrm{M}} \mathrm{d}^2\xi_{\mathrm{E}} \sqrt{g} g^{mn} \partial_m X^M \partial_n X_M + B \int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{g} \,, \tag{7.2}$$

where B is a counter term. B should be tuned so that the renormalized effective action has conformal invariance.

7.1 $\mathcal{D}X^M$

In the finite dimension case, N dimensional Riemannian space with metric is defined by

$$\mathrm{d}s^2 = G_{AB} \mathrm{d}Y^A \mathrm{d}Y^B \equiv \langle \mathrm{d}Y, \mathrm{d}Y \rangle \equiv ||\mathrm{d}Y||^2 \,. \tag{7.3}$$

The volume element is

$$\prod_{A=1}^{\infty} \mathrm{d}Y^A \sqrt{G} \equiv \mathcal{D}^N Y \; . \tag{7.4}$$

We want to construct an infinite dimensional analog of this.

We can write

$$\begin{aligned}
&\int [\mathcal{D}X^M] \mathrm{e}^{-S_{\mathrm{E}}[X,g]} \\
&= \mathrm{e}^{-B\int \mathrm{d}^2\xi_{\mathrm{E}}\sqrt{g}} \int [\mathcal{D}X_0^M] \int [\mathcal{D}X'^M] \mathrm{e}^{-\frac{1}{2}\langle X'|\Delta_g|X'\rangle} \\
&= \left(\int [\mathcal{D}X_0^M]\right) \frac{\int [\mathcal{D}X'^M] \mathrm{e}^{-\frac{1}{2}\langle X'|\Delta_g|X'\rangle}}{\int [\mathcal{D}X'^M] \mathrm{e}^{-\frac{1}{2}\langle X'|X'\rangle}} \mathrm{e}^{-B\int \mathrm{d}^2\xi_{\mathrm{E}}\sqrt{g}} \int [\mathcal{D}X'^M] \mathrm{e}^{-\frac{1}{2}\langle X'|X'\rangle} \\
&\equiv \left(\int [\mathcal{D}X_0^M]\right) \left(\frac{\det'\Delta_g}{\det'\mathbf{1}}\right)^{-\frac{D}{2}} \mathrm{e}^{-B\int \mathrm{d}^2\xi_{\mathrm{E}}\sqrt{g}} \int [\mathcal{D}X'^M] \mathrm{e}^{-\frac{1}{2}\langle X'|X'\rangle} \; .
\end{aligned} \tag{7.5}$$

Then we can tune B, albeit being divergent such that

$$\mathrm{e}^{-B\int \mathrm{d}^2\xi_{\mathrm{E}}\sqrt{g}} \int [\mathcal{D}X^M] \mathrm{e}^{-\frac{1}{2}||X||^2} \equiv 1 \; . \tag{7.6}$$

Using this,

$$\begin{aligned}
\int [\mathcal{D}X^M] \mathrm{e}^{-S[X,g]} &= (\text{spacetime volume}) \left(\frac{\det'\Delta_g}{\mathbf{1}}\right)^{-\frac{D}{2}} \frac{1}{\int [\mathcal{D}X_0^M] \mathrm{e}^{-\frac{1}{2}||X_0||^2}} \\
&= (\text{spacetime vol.}) \left(\frac{2\pi}{\int_{\mathrm{M}} \mathrm{d}^2\xi\sqrt{g}} \det{}'\Delta_g\right)^{-\frac{D}{2}} ,
\end{aligned} \tag{7.7}$$

where $||X_0||^2 = X_0^2 \int \mathrm{d}^2\xi_{\mathrm{E}}\sqrt{g}$. Therefore

$$\int [\mathcal{D}X^m] \mathrm{e}^{-\frac{1}{2}||X_0||^2} = \left(\frac{2\pi}{\int_{\mathrm{M}} \mathrm{d}^2\xi\sqrt{g}}\right)^{\frac{D}{2}} \; . \tag{7.8}$$

7.2 $\mathcal{D}g_{mn}$

Proceed to $\mathcal{D}g_{mn}$ in the same spirits. We consider

$$\begin{aligned}
\mathcal{M} &= \text{space of metrices on M} \\
&= \{\{g_{mn}(\xi) | g_{mn}(\xi) \text{ is a metric on M}\}\} \; .
\end{aligned} \tag{7.9}$$

A natural metric on $\mathcal{M}$ is

$$||\delta g||^2 \equiv \langle \delta g, \delta g \rangle = \int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{g}\, (G^{mnpq} + u g^{mn} g^{pq})\, \delta g_{mn} \delta g_{pq} \,, \tag{7.10}$$

where u is arbitrary and

$$G_{mn}{}^{pq} = \frac{1}{2}\, (\delta^p_m \delta^q_n + \delta^q_m \delta^p_n - g_{mn} g^{pq}) \tag{7.11}$$

is the projector onto the space of symmetric traceless tensors. We can write any metric variation as

$$\delta g_{mn} = \delta h_{mn} + \delta\rho g_{mn} \,. \tag{7.12}$$

Then

$$||\delta g||^2 \equiv \int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{g} G^{mnpq} \delta h_{mn} \delta h_{pq} + 4u \int \mathrm{d}^2\xi \sqrt{g} (\delta\rho)^2 \,. \tag{7.13}$$

Thus we conclude

$$[\mathcal{D} g_{mn}] = [\mathcal{D} h_{mn}][\mathcal{D}\rho] \,. \tag{7.14}$$

We must count each deformation once and for all.

The next strategy would be to trade δh_{mn} with two infinitesimal generator δv_m of diffeomorphism through

$$\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m \,. \tag{7.15}$$

Let δg_{mn} a diffeomorphism + Wely rescaling

$$\begin{aligned} \delta g_{mn} &= \nabla_m \delta v_n + \nabla_n \delta v_m + \delta\phi g_{mn} \\ &= \nabla_m \delta v_n + \nabla_n \delta v_m - g_{mn}(\nabla^p \delta v_p) + \delta\phi g_{mn} + g_{mn}(\nabla^p \delta v_p) \\ &\equiv (P_1 \delta v)_{mn} + (\delta\phi + \nabla^p \delta v_p) g_{mn} \,, \end{aligned} \tag{7.16}$$

where P_1 is the operator which maps a vector to a symmetric traceless rank

2 tensor. One would think

$$
\begin{aligned}
[\mathcal{D}g_{mn}] = [\mathcal{D}h_{mn}][\mathcal{D}\rho] &= [\mathcal{D}v_m][\mathcal{D}\phi] \left| \frac{\partial(h_{mn}\rho)}{\partial(v_m\phi)} \right| \\
&= [\mathcal{D}v_m][\mathcal{D}\phi] \left| \det \begin{pmatrix} P_1 & * \\ 0 & 1 \end{pmatrix} \right| \\
&= [\mathcal{D}v_m][\mathcal{D}\phi] \left(\det P_1^\dagger P_1 \right)^{\frac{1}{2}} \, .
\end{aligned} \tag{7.17}
$$

<u>Two catches</u>

1. Are all δh_{mn} obtained from δv_m?

2. $(P_1 \delta v) = 0$ possible?

Let $\delta k^\perp$ be in the orthogonal complement of $\mathrm{Im} P_1$ in $\{\{\delta h_{mn}\}\}$:

$$\langle \delta k^\perp, P_1 \delta v \rangle = \langle P_1^\dagger \delta k^\perp, \delta v \rangle = 0 \, , \tag{7.18}$$

where $P_1^\dagger$ is the operator which maps a symmetric traceless tensor to a vector. Therefore

$$(P_1^\dagger \delta k^\perp)_m = -2\nabla^n \delta k^\perp_{mn} = 0 \, . \tag{7.19}$$

Hence

$$\delta k^\perp \in \ker P_1^\dagger \, , \tag{7.20}$$

and then we can write

$$\{\{\delta g_{mn}\}\} = \{\{\delta\rho g_{mn}\}\} \oplus \mathrm{Im} P_1 \oplus \ker P_1^\dagger \, . \tag{7.21}$$

This is the answer to 1.

Next we consider 2.

$$
\begin{aligned}
(P_1 \delta v)_{mn} = 0 &\Leftrightarrow \delta v \in \ker P_1 \\
&\Leftrightarrow \nabla_m \delta v_n + \nabla_n \delta v_m - g_{mn} \nabla^p \delta v_p = 0 \, .
\end{aligned} \tag{7.22}
$$

This can be written as

$$\mathcal{L}_{(\delta v)} g_{mn} = (\nabla^p v_p) g_{mn} \; . \tag{7.23}$$

This is called conformal Killing equation, where δv is a conformal Killing vector (CKV). (7.23) means a diffeomorphism generated by CKV δv is equivalent to a Weyl rescaling. This implies doubly counting. We omit this in diffeomorphism. Then we conclude that $\left(\det P_1^\dagger P_1\right)^{\frac{1}{2}}$ should have been $\left(\det' P_1^\dagger P_1\right)^{\frac{1}{2}}$.

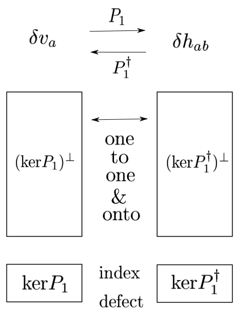

Figure 8:

This is the application of the Atiyah-Singer index theorem.

The Riemann-Roch

$$\dim \ker P_1 - \dim \ker P_1^\dagger = 3\chi(\mathrm{M}) \; . \tag{7.24}$$

The left hand side represents analytic side, while the right hand side does topological side.

Summary (figure 8):

1. extra integrations which belong to $\ker P_1^\dagger$; Theichmüller deformation.

2. $[\mathcal{D}v]$ is from $(\ker P_1)^\perp$ only.

So far, we have

$$Z = \sum_{h=0}^{\infty} \kappa^h \int [\mathcal{D}\phi] \prod_i \mathrm{d}t_i \frac{\mathrm{vol}(\mathrm{Diff}_0^{\perp})\mathrm{vol}(\mathrm{Diff}_0)}{\mathrm{vol}(\mathrm{Diff}_0)\mathrm{vol}(\mathrm{Diff})}$$
$$\times J[\det{}' P_1^{\dagger} P_1]^{\frac{1}{2}} \left(\frac{2\pi}{\int \mathrm{d}^2\xi \sqrt{g}} \det{}' \Delta_g \right)^{-\frac{D}{2}} (\text{spacetime vol.}) \,, \quad (7.25)$$

where $\prod_i \mathrm{d}t_i$ represents integration over Teichmüller deformation and J is Jacobian to correctly count the volume of Teichmüller deformation. Here

$$\frac{\mathrm{vol}(\mathrm{Diff}_0^{\perp})}{\mathrm{vol}(\mathrm{Diff}_0)} = \frac{1}{\mathrm{vol}(\mathrm{CKV})} \,. \quad (7.26)$$

Zero indicates connected component and

$$\frac{\mathrm{Diff(M)}}{\mathrm{Diff}_0\mathrm{(M)}} \equiv \Pi_0(\mathrm{Diff(M)}) = \text{mapping class group} = (\mathrm{MCG})_h \,. \quad (7.27)$$

Therefore

$$\frac{\mathrm{vol(Diff)}}{\mathrm{vol}(\mathrm{Diff}_0)} = \#\text{ of distinct path connected components of Diffeo. group} \,. \quad (7.28)$$

We will use following:

<u>Fact</u>

For any metric g_{mn} on M, $\exists$ a unique ϕ such that

$$g_{mn}(\xi) = \mathrm{e}^{\phi(\xi)} \hat{g}_{mn}(\xi)$$
$$R_{\hat{g}} = \text{const} = \begin{cases} 1 & h = 0 \\ 0 & h = 1 \\ -1 & h \geq 2 \end{cases} \,. \quad (7.29)$$

There exists a global "slice" for the action of the Weyl groups (Figure 9), which is

$$\hat{\mathcal{M}} \equiv \{\{\hat{g}_{mn}(\xi) | \hat{g}_{mn} \text{ on M}, R_{\hat{g}} = \text{const}\}\} \,. \quad (7.30)$$

Teichmüller space is defined by

$$\text{Teichmüller space with genus } h \equiv \hat{\mathcal{M}}/\mathrm{Diff}_0(\mathrm{M}) \equiv \mathcal{T}_h \tag{7.31}$$

and Moduli space is defined by

$$\text{Moduli space with } h \equiv \hat{\mathcal{M}}/\mathrm{Diff}(\mathrm{M}) \equiv (\text{moduli})_h = \mathcal{T}_h/(\mathrm{MCG})_h\ . \tag{7.32}$$

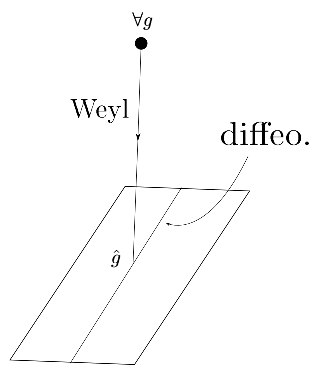

Figure 9:

Therefore

$$\{\{\text{tangent vectors at ĝ on } \hat{\mathcal{M}}\}\} = \{\{\nabla_m \delta v_n + \nabla_n \delta v_m\}\} \oplus \ker P_1^\dagger\ . \tag{7.33}$$

Hence

$$\{\{\text{tangent vectors at } \hat{g} \text{ on } (\text{moduli})_h\}\} = \ker P_1^\dagger\ . \tag{7.34}$$

Let's start over

$$\delta g_{mn}(\xi) = \delta\rho(\xi) g_{mn}(\xi) + \delta\ell_{mn}(\xi) + \sum_i \delta c_i \phi_{mn}^{(i)}\ , \tag{7.35}$$

where

$$\delta\ell_{mn}(\xi) \in \mathrm{Im} P_1\ . \tag{7.36}$$

$\{\{\phi^{(i)}_{mn}\}\}$ is a set of basis vector in $\ker P_1^\dagger$. $\langle\phi^{(i)}|\phi^{(j)}\rangle$ is defined through

$$\langle\phi^{(i)}|\phi^{(j)}\rangle_g \equiv \int \sqrt{g}G^{mnpq}\phi^{(i)}_{mn}\phi^{(j)}_{pq} \,. \tag{7.37}$$

Therefore

$$[\mathcal{D}g_{mn}] = [\mathcal{D}\rho][\mathcal{D}\ell_{mn}]\prod_i \mathrm{d}c_i\sqrt{\det\langle\phi^{(i)}|\phi^{(j)}\rangle} \,. \tag{7.38}$$

On $\hat{\mathcal{M}}$, we make a gauge slice $\hat{S}$ transverse to the orbit of diffeomorphism. Let

$$\hat{g} = (\text{action of Diff}_0)\hat{g}' \,, \qquad \hat{g}' \in \hat{S} \,, \tag{7.39}$$

$$g = \mathrm{e}^{\phi}(\text{action of Diff}_0)\hat{g}' \,. \tag{7.40}$$

Clearly

$$\hat{S} = \hat{\mathcal{M}}/\text{Diff}_0(\text{M}) = \mathcal{T}_n = \text{parametrized by } \tau_i \,. \tag{7.41}$$

Infinitesimal deformation of eq. (7.40) is

$$\begin{aligned}\delta g_{mn}(\xi) &= \delta\phi g_{mn} + (\nabla_m\delta v_n + \nabla_n\delta v_m) + \sum_i \frac{\partial g_{mn}}{\partial\tau_i}\delta\tau_i \\ &= \delta\phi(\xi)\mathrm{e}^{\phi(\xi)}\hat{g}_{mn}(\xi;\tau_i) + \nabla_m\delta v_n + \nabla_n\delta v_m + \sum_i \mathrm{e}^{\phi(\xi)}\frac{\partial\hat{g}_{mn}(\xi;\tau_i)}{\partial\tau_i}\delta\tau_i \,,\end{aligned} \tag{7.42}$$

where ∇_m indicates covariant derivative. Compare (7.35) = (7.42),

$$\begin{pmatrix}\delta\rho(\xi)\\ \delta\ell_{mn}(\xi)\\ \delta c_i\end{pmatrix} = \begin{pmatrix}1 & * & *\\ 0 & P_1 & *\\ 0 & 0 & M_i{}^j\end{pmatrix}\begin{pmatrix}\delta\phi(\xi)\\ \delta v_m(\xi)\\ \delta\tau_j\end{pmatrix} \,, \tag{7.43}$$

where $M_i{}^j$ is a matrix such that

$$\sum_i\langle\phi^{(\ell)}|\phi^{(i)}\rangle\delta c_i = \sum_j\langle\phi^{(\ell)}|\mathrm{e}^{\phi}\frac{\partial\hat{g}(\tau)}{\partial\tau_j}\rangle\delta\tau_j \tag{7.44}$$

$$\sum_i\langle\phi^{(\ell)}|\phi^{(i)}\rangle M_i{}^j = \sum_j\langle\phi^{(\ell)}|\mathrm{e}^{\phi}\frac{\partial\hat{g}(\tau)}{\partial\tau_j}\rangle \,. \tag{7.45}$$

Therefore

$$\prod_i \delta c_i = \det M_i{}^j \prod_i \delta\tau_i$$
$$= \det\left(\langle\phi^{(\ell)}|\mathrm{e}^{\phi}\frac{\partial\hat{g}(\tau)}{\partial\tau}\rangle\right)\left(\det\left(\langle\phi^{(\ell)}|\phi^{(i)}\rangle\right)\right)^{-1}\prod_i \mathrm{d}\tau_i\,. \qquad (7.46)$$

Finally

$$[\mathcal{D}g_{mn}]$$
$$= [\mathcal{D}\phi][\mathcal{D}v_m^{\perp}(\xi)]\det{}'(P_1^{\dagger}P_1)_g^{\frac{1}{2}}\prod_i \mathrm{d}\tau_i \det\langle\phi^{(\ell)}|\mathrm{e}^{\phi}\frac{\partial\hat{g}(\tau)}{\partial\tau_j}\rangle_g\left(\det\left(\langle\phi^{(\ell)}|\phi^{(i)}\rangle\right)\right)_g^{-1}\,. \qquad (7.47)$$

7.3 Summary

The previous formula for Z is refined to be

$$Z = \sum_{h=0}^{\infty}\kappa^h\int_{\mathcal{T}_h}\frac{1}{\mathrm{vol(CKV)}}\frac{\prod \mathrm{d}\tau_i}{||\mathrm{MCG}||}\int[\mathcal{D}\phi]\left(\det\langle\phi^{(\ell)}|\mathrm{e}^{\phi}\frac{\partial\hat{g}}{\partial\tau_j}\rangle_g\right)$$
$$\cdot\left(\frac{\det{}'(P_1^{\dagger}P_1)}{\det\langle\phi^{(\ell)}|\phi^{(\ell)}\rangle}\right)_g^{\frac{1}{2}}\left(\frac{2\pi}{\int_{\mathrm{M}}\mathrm{d}^2\xi\sqrt{g}}\det{}'\Delta_g\right)^{-\frac{D}{2}}(\text{spacetime volume})$$
$$= \sum_{h=0}^{\infty}\kappa^h\int_{(\mathrm{moduli})_h}\frac{\prod \mathrm{d}\tau_i}{\mathrm{vol(CKV)}}\int[\mathcal{D}\phi]\left(\det\langle\phi^{(\ell)}|\mathrm{e}^{\phi}\frac{\partial\hat{g}}{\partial\tau_j}\rangle_g\right)$$
$$\cdot\left(\frac{\det{}'(P_1^{\dagger}P_1)}{\det\langle\phi^{(\ell)}|\phi^{(i)}\rangle}\right)_g^{\frac{1}{2}}\left(\frac{2\pi}{\int_{\mathrm{M}}\mathrm{d}^2\xi\sqrt{g}}\det{}'\Delta_g\right)^{-\frac{D}{2}}(\text{spacetime volume})\,. \qquad (7.48)$$

7.4 Example: torus

We set $D = 26$ and look at $Z_{h=1}$.

Recall eq. (7.24), $\dim \ker P_1^\dagger = 2$, $\dim \ker P_1 = 2$. Using eq. (7.48),

$$Z_{h=1} = (\text{spacetime vol.}) \int_{(\text{moduli})_{h=1}} \mathrm{d}(\mathrm{WP})_{h=1} \frac{1}{\mathrm{vol}(\ker P_1)} \times (\det{}' P_1^\dagger P_1)^{\frac{1}{2}} \left(\frac{2\pi}{\int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{g}} \det{}' \Delta_{\hat{g}} \right)^{-\frac{D}{2}} , \tag{7.49}$$

where

$$\mathrm{d}(\mathrm{WP})_{h=1} = \frac{\det \langle \phi^{(\ell)} | \frac{\partial \hat{g}(\tau)}{\partial \tau_j} \rangle_{\hat{g}}}{\det \langle \phi^{(\ell)} | \phi^{(i)} \rangle_{\hat{g}}^{\frac{1}{2}}} \prod_{i=1,2} \mathrm{d}\tau_i \, . \tag{7.50}$$

Recall that for any metric g_{mn}, $\exists \phi$ such that

$$g_{mn} = \mathrm{e}^{\phi} \hat{g}_{mn} \tag{7.51}$$

$$R_{\hat{g}} = \text{const.} \tag{7.52}$$

Gauss-Bonnet theorem states $\chi(\mathrm{M}) = \frac{1}{4\pi} \int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{\hat{g}} R_{\hat{g}} = 2 - 2h - b$. When we set $h = 1$, $b = 0$,

$$R_{\hat{g}} \frac{1}{4\pi} \int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{\hat{g}} = 0 \tag{7.53}$$

Therefore

$$R_{\hat{g}} = 0 \tag{7.54}$$

and we can choose $\int_{\mathrm{M}} \mathrm{d}^2\xi \sqrt{\hat{g}} = \text{const} = 1$ for example.

We parametrize the torus by $0 \leq \xi^1 \leq 1$, $0 \leq \xi^2 \leq 1$ (Figure 10). The line element is written as

$$\begin{aligned} \mathrm{d}s^2 &= \hat{g}_{mn} \mathrm{d}\xi^m \mathrm{d}\xi^n \\ &= \mathrm{d}z \mathrm{d}\bar{z} \, , \end{aligned} \tag{7.55}$$

where $z = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$. Range of $\mathrm{Re}\, z$ or $\mathrm{Im}\, z$ is not $[0, 1]$. Let

$$\begin{aligned} \mathrm{Im}\, z &= \tau_2 \xi^2 \, , \qquad \mathrm{Im}\, z > 0 \\ \mathrm{Re}\, z &= \xi^1 + \tau_1 \xi^2 \, . \end{aligned} \tag{7.56}$$

Then

$$
\begin{aligned}
\mathrm{d}s^2 &= |\mathrm{d}\xi^1 + \tau_1 \mathrm{d}\xi^2 + i\tau_2 \mathrm{d}\xi^2|^2 \\
&= |\mathrm{d}\xi^1 + \tau \mathrm{d}\xi^2|^2 \,,
\end{aligned}
\tag{7.57}
$$

where $\tau = \tau_1 + i\tau_2$ (Figure 11). Now the metric can be written as

$$
\hat{g}_{mn} = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{bmatrix}
\tag{7.58}
$$

and $\int_{\mathrm{M}} \mathrm{d}\xi^2 \sqrt{\hat{g}} = \tau_2$. Note, in $D = 26$, $\int_{\mathrm{M}} \mathrm{d}\xi^2 \sqrt{\hat{g}} = \tau_2$ and $= 1$ will make no

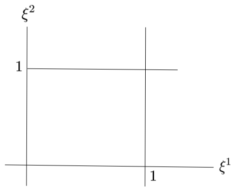

Figure 10:

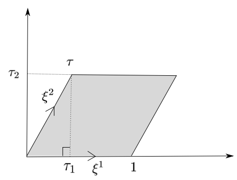

Figure 11:

difference because of Weyl invariance but $D \neq 26$ not clear.

Let's first get a concrete expression for $\mathrm{d(WP)}_{h=1}$, $\mathrm{vol}(\ker \hat{P}_1)$. Recall $\phi^{(\ell)}_{mn} \in \ker \hat{P}_1^\dagger$. We know $\ell = 1, 2$. In the complex basis

$$
P_1^\dagger = \begin{pmatrix} -\nabla^{z(+2)} & 0 \\ 0 & -\nabla_z^{(-2)} \end{pmatrix} \begin{pmatrix} \phi_{zz} \\ \phi^{zz} \end{pmatrix} = 0 \,,
\tag{7.59}
$$

where

$$
\nabla_z = \frac{\partial}{\partial z} \,, \qquad \nabla^z = g^{z\bar{z}} \nabla_{\bar{z}} = 2 \frac{\partial}{\partial \bar{z}} \,.
\tag{7.60}
$$

Therefore

$$
\begin{aligned}
&\phi_{zz} \text{ is a function of } z \text{ only} \\
&\phi^{zz} \text{ is a function of } \bar{z} \text{ only} \,.
\end{aligned}
$$

They must be periodic too. These mean ϕ_{zz}, $\phi_{\bar{z}\bar{z}}$ are both constant, and $\phi^{z\bar{z}} = 0$. $\hat{g}^{mn}$ is defined by the inverse of $\hat{g}_{mn}$:

$$\hat{g}^{mn} = \frac{1}{\tau_2^2} \begin{bmatrix} \tau_1^2 + \tau_2^2 & -\tau_1 \\ -\tau_1 & 1 \end{bmatrix} . \tag{7.61}$$

By invariance argument alone, we can conclude

$$\phi_{mn}^{(\ell)} = \frac{\partial}{\partial \tau^{(\ell)}} \hat{g}_{mn} - \frac{1}{2} \hat{g}_{mn} \hat{g}^{pq} \frac{\partial \hat{g}_{pq}}{\partial \tau^{(\ell)}} . \tag{7.62}$$

Therefore

$$\phi_{mn}^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 2\tau_1 \end{bmatrix} , \qquad \phi_{mn}^{(2)} = -\frac{1}{\tau_2} \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 - \tau_2^2 \end{bmatrix} . \tag{7.63}$$

From

$$\langle \phi^{(\ell)} | \phi^{(\ell')} \rangle_{\hat{g}} = \int \mathrm{d}^2\xi \sqrt{\hat{g}} G_{\hat{g}}^{abcd} \phi_{ab}^{(\ell)} \phi_{cd}^{(\ell')} = \int \mathrm{d}^2\xi \sqrt{\hat{g}} \hat{g}^{pm} \hat{g}^{nq} \phi_{mn}^{(\ell)} \phi_{qp}^{(\ell')} , \tag{7.64}$$

$$\langle \phi^{(1)} | \phi^{(1)} \rangle = \langle \phi^{(2)} | \phi^{(2)} \rangle = \frac{2}{\tau_2} , \qquad \langle \phi^{(1)} | \phi^{(2)} \rangle = 0 . \tag{7.65}$$

Hence

$$\det \langle \phi^{(\ell)} | \phi^{(j)} \rangle_{\hat{g}} = \frac{4}{\tau_2^2} , \tag{7.66}$$

while, from

$$\begin{aligned} \left\langle \phi^{(\ell)} | \frac{\partial}{\partial \tau^{(\ell')}} \hat{g} \right\rangle &= \int \mathrm{d}^2\xi \sqrt{\hat{g}} G^{mnpq} \phi_{mn}^{(\ell)} \frac{\partial}{\partial \tau^{(\ell')}} \hat{g}_{pq} \\ &= \langle \phi^{(\ell)} | \phi^{(\ell')} \rangle , \end{aligned} \tag{7.67}$$

$$\det \left\langle \phi^{(\ell)} | \frac{\partial}{\partial \tau^{(\ell')}} \hat{g} \right\rangle = \frac{4}{\tau_2^2} . \tag{7.68}$$

Next we consider vol(kerP_1). Again in the complex basis

$$P_1 = \begin{pmatrix} \nabla_z^{(+1)} & 0 \\ 0 & \nabla^{z(-1)} \end{pmatrix} . \tag{7.69}$$

We can take

$$\tilde{\phi}^{(1)m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \tilde{\phi}^{(2)m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.70}$$

as $\tilde{\phi}^{(\ell)m} \in \mathrm{ker}P_1$, $\ell = 1, 2$. Using these,

$$\begin{aligned} \det\langle\tilde{\phi}^{(\ell)}|\tilde{\phi}^{(\ell')}\rangle &= \det \int \mathrm{d}^2\xi \sqrt{\hat{g}}\hat{g}_{mn}\tilde{\phi}^{(\ell)m}\tilde{\phi}^{(\ell')n} \\ &= \det \int \mathrm{d}^2\xi \sqrt{\hat{g}}\hat{g}_{mn} \\ &= \tau_2^4 \end{aligned} \tag{7.71}$$

and then

$$\mathrm{vol}(\mathrm{ker}P_1) = \sqrt{\det\langle\tilde{\phi}^{(\ell)}|\tilde{\phi}^{(\ell')}\rangle} = \tau_2^2 \ . \tag{7.72}$$

Now we evaluate the determinant:

$$\begin{aligned} (P_1^\dagger P_1)\delta v_m &= (-2)\partial^n(P_1\delta v)_{nm} \\ &= -2\partial^n(\partial_m\delta v_n + \partial_n\delta v_m - \hat{g}_{mn}\partial^p\delta v_p) \\ &= -2\partial^n\partial_n\delta v_m = 2\Delta_{\hat{g}}\delta v_m \ . \end{aligned} \tag{7.73}$$

Therefore

$$\left(\det{}' P_1^\dagger P_1\right)^{\frac{1}{2}} = \det{}'(2\Delta_{\hat{g}}) \ . \tag{7.74}$$

We just quote the result for $\det{}'\Delta_{\hat{g}}$:

$$\det{}'\Delta_{\hat{g}} = \tau_2^2|\eta(\tau)|^4 \ . \tag{7.75}$$

Therefore

$$\begin{aligned} &Z_{h=1} \\ &= (\text{spacetime volume}) \int_{(\mathrm{moduli})_{\mathrm{h=1}}} \prod_{i=1,2} \mathrm{d}\tau_i \frac{\frac{4}{\tau_2^2}}{\left(\frac{4}{\tau_2^2}\right)^{\frac{1}{2}}} \frac{1}{\tau_2^2}(\det 2) \\ &\qquad\qquad \times(\tau_2^2|\eta(\tau)|^4)^{-\frac{D-2}{2}} \left(\frac{\tau_2}{2\pi}\right)^{\frac{D}{2}} \\ &= (\mathrm{vol.}) \int_{(\mathrm{moduli})_{\mathrm{h=1}}} \frac{2\prod_i \mathrm{d}\tau_i}{(2\pi)\tau_2^2}(\det 2)(2\pi\tau_2|\eta(\tau)|^4)^{-\left(\frac{D-2}{2}\right)} \ . \end{aligned} \tag{7.76}$$

Finally, we discuss the integration region of $\mathrm{d}^2\tau$. Again, Teichmüller space is written as

$$\begin{aligned}\mathcal{T}_{h=1} &= \text{genus 1 Teichmüllar space} \\ &= \{\tau = \tau_1 + i\tau_2 | \mathrm{Im}\tau \geq 0\} = \mathrm{UHP}\,. \end{aligned} \tag{7.77}$$

The consider

$$\begin{array}{ccc}\xi^1 \to \xi^{1\prime} = \alpha\xi^1 + \beta\xi^2 & \text{or} & \xi^1 = \alpha\xi^{1\prime} - \beta\xi^{2\prime} \\ \xi^2 \to \xi^{2\prime} = \gamma\xi^1 + \delta\xi^2 & & \xi^2 = -\gamma\xi^{1\prime} + \delta\xi^{2\prime}\end{array} \tag{7.78}$$

maintain periodicity. Then $\alpha, \beta, \gamma, \delta$ must be integer. Now that $\int_{\mathrm{M}} \mathrm{d}^2\xi\sqrt{g} =$ the same constant as before, $\alpha\delta - \beta\gamma = 1$. Then

$$|\mathrm{d}\xi^1 + \tau \mathrm{d}\xi^2|^2 = |\delta - \tau\gamma|^2 \left| \mathrm{d}\xi^{1\prime} + \left(\frac{\alpha\tau - \beta}{-\gamma\tau + \delta}\right) \mathrm{d}\xi^{2\prime} \right|^2 . \tag{7.79}$$

This motivates a transformation

$$\tau' = \frac{\alpha\tau - \beta}{-\gamma\tau + \delta}\,. \tag{7.80}$$

Let

$$SL(2, \mathbf{Z}) = \left\{ \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in \mathbf{Z}, \alpha\delta - \beta\gamma = 1 \right\} .$$

This is an genus one MCG

$$= (\mathrm{MCG})_{h=1}\,. \tag{7.81}$$

Note that the action of

$$SL(2, \mathbf{R}) = \left\{ \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in \mathbf{R}, \alpha\delta - \beta\gamma = 1 \right\} \tag{7.82}$$

on $\mathcal{T}_{h=1}$ preserves the boundary $\mathrm{Im}\,\tau = 0$, *i.e.* real line $\to$ real line. So the $\mathcal{T}_{h=1}$ is "stable" under $SL(2, \mathbf{R})$. But it is not "faithful" as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathcal{T}_{h=1}$:

$$\frac{(-1)\tau - 0}{-0 \cdot \tau + (-1)} = \tau\,. \tag{7.83}$$

The group $PSL(2,\mathbf{R}) \equiv SL(2,\mathbf{R})/\{\{\pm\mathbf{1}_2\}\}$ acts faithfully on $\mathcal{T}_{h=1}$. The elements of $SL(2,\mathbf{Z})$ which are in $PSL(2,\mathbf{R})$ form a group

$$\begin{aligned} PSL(2,\mathbf{Z}) &\equiv SL(2,\mathbf{Z})/\{\{\pm\mathbf{1}_2\}\} \\ &\equiv \text{the modular group}\,. \end{aligned} \tag{7.84}$$

Let

$$T = \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix} \qquad \text{or} \qquad \tau \to \tau' = \tau + 1\,, \tag{7.85}$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{or} \qquad \tau \to \tau' = -\frac{1}{\tau}\,, \tag{7.86}$$

and

$$F = \left\{ \tau \,\middle|\, \tau_2 \geq 0, |\tau| \geq 1, -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2} \right\} \tag{7.87}$$

as in figure 12.

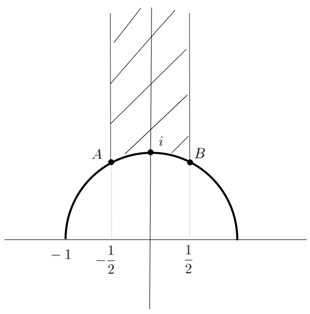

Figure 12:

<u>Facts</u>

i) $PSL(2,\mathbf{Z})$ is generated by S, T.

ii) for $\forall\tau \in \mathcal{T}_{h=1}$, $\exists g \in PSL(2,\mathbf{Z})$ such that $g\tau \in F$.

iii) $g\tau = \tau$, $\tau \in F$ has the solution $g = \mathbf{1}_2$ except fot the cases $\tau = A, i, B$.

ii), iii) $\equiv F$ is a fundamental region of the $PSL(2,\mathbf{Z})$. Therefore

$$\begin{aligned}(\text{moduli})_{h=1} &= \mathcal{T}_{h=1}/SL(2,\mathbf{Z}) = \mathcal{T}_{h=1}/(PSL(2,\mathbf{Z}))\cdot\{\{\pm\mathbf{1}_2\}\} \\ &= F/\{\{\pm\mathbf{1}_2\}\}\,. \end{aligned} \tag{7.88}$$

So this final formula is attained by

$$\int_{(\text{moduli})} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \cdots = \frac{1}{2}\int_{\mathcal{F}} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \cdots\,. \tag{7.89}$$

The final formula

$$Z_{h=1} = (\text{volume}) \int_{\mathcal{F}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{(2\pi)\tau_2^2} (\det 2) \left(2\pi\tau_2 |\eta(\tau)|^4\right)^{-12} \tag{7.90}$$

can be understood as coming from collection of free particle one loop diagrams. The τ_2 is understood as a "proper time". UV divergence correspond to ∞ at $\tau_2 \to 0$ but in eq. (7.90) it is cut off. The infinity at $\tau_2 \to \infty$ is IR and related to the presence of tachyon in the spectrum.

8 Fermionic string partition function

Variables are

$$\chi_\alpha{}^M,\ \psi_{\text{Maj}\,\alpha}{}^M\,, \tag{8.1}$$

where $\chi_a{}^m$ is a Rarita-Schwinger filed. The zwei bein $e_m{}^a$ and the metric g_{mn} are related by

$$g_{mn} = e_m{}^a e_n{}^b \delta_{ab}\,. \tag{8.2}$$

We use following indices:

$$\begin{aligned} M&: \text{ 10d vector} \\ \alpha&: \text{ 2d spinor} \\ m&: \text{ 2d Einstein} \\ a&: \text{ 2d local Lorentz}\,. \end{aligned} \tag{8.3}$$

Note that α, a are Euclidean indices. The action can be written as

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \left\{ \frac{1}{2} g^{mn} \partial_m X^M \partial_n X_M - \frac{i}{2} \psi_{\text{Maj}}{}^M \gamma^a \nabla_a \psi_{\text{Maj}\, M} \right.$$
$$\left. - \frac{1}{2} (\psi_{\text{Maj}}{}^M \gamma^a \gamma^b \chi_a)(\partial_b X^M - \frac{1}{4} \chi_b \psi_{\text{Maj}}{}^M) \right\} , \tag{8.4}$$

where

$$\chi_a = e_a{}^m \chi_m \tag{8.5}$$
$$\partial_b = e_b{}^m \partial_m \tag{8.6}$$
$$\nabla_a = e_a{}^m (\partial_m - \omega_m \frac{1}{2} \gamma^5) \tag{8.7}$$
$$\omega_m = e_m{}^a \varepsilon^{pq} \partial_p e_q{}^b \delta_{ab} \,. \tag{8.8}$$

The partition function we want to compute is

$$Z = \sum_{\text{topology}} \sum_{\text{spin structure}} \int \frac{\mathcal{D} e_m{}^a}{\Omega(\text{D})\Omega(\text{W})\Omega(\text{L})} \int \frac{\mathcal{D} \chi_m{}^a}{\Omega(\text{S})\Omega(\text{SW})} \int \mathcal{D} X^M \mathcal{D} \psi_{\text{Maj}}{}^M \mathrm{e}^{-S}, \tag{8.9}$$

where

$$\begin{aligned} \Omega(\text{D}) &: \text{ volume of 2d diffeo.} \\ \Omega(\text{W}) &: \text{ volume of 2d Weyl} \\ \Omega(\text{L}) &: \text{ volume of 2d local Lorentz} \\ \Omega(\text{S}) &: \text{ volume of 2d susy} \\ \Omega(\text{SW}) &: \text{ volume of 2d super Weyl .} \end{aligned} \tag{8.10}$$

There are following symmetries:
(1) 2d diffeomorphism

$$\delta e_m{}^a = \delta \eta^n \partial_n e_m{}^a + e_n{}^a \partial_m \eta^n \tag{8.11}$$
$$\delta \chi_m = \delta \eta^n \partial_n \chi_m + \chi_n \partial_m \eta^n \tag{8.12}$$
$$\delta X^M = \delta \eta^n \partial_n X^M \tag{8.13}$$
$$\delta \psi_{\text{Maj}}{}^M = \delta \eta^n \partial_n \psi_{\text{Maj}}{}^M \,. \tag{8.14}$$

Using this,

$$\delta(g_{mn}) = \delta({e_m}^a e_{na}) \overset{(8.11)}{=} \delta\eta_{m;n} + \delta\eta_{n;m} \,, \tag{8.15}$$

where ";" indicates the covariant derivative.

(2) Weyl transformation

$$\begin{aligned}
\delta {e_m}^a &= \Lambda {e_m}^a \; (\delta g_{mn} = 2\Lambda g_{mn}, \; \delta {e_a}^m = -\Lambda {e_a}^m) &(8.16)\\
\delta\chi_m &= \frac{1}{2}\Lambda\chi_m \; (\delta\chi_a = -\frac{1}{2}\Lambda\chi_a) &(8.17)\\
\delta X^M &= 0 &(8.18)\\
\delta {\psi_{\mathrm{Maj}}}^M &= -\frac{1}{2}\Lambda{\psi_{\mathrm{Maj}}}^M \,. &(8.19)
\end{aligned}$$

(3) local Lorentz transformation

$$\begin{aligned}
\delta {e_m}^a &= \ell\varepsilon^{ab}{e_m}^b &(8.20)\\
\delta\chi_m &= \frac{1}{2}\ell\gamma^5\chi_m &(8.21)\\
\delta X^M &= 0 &(8.22)\\
\delta {\psi_{\mathrm{Maj}}}^M &= \frac{1}{2}\ell\gamma^5{\psi_{\mathrm{Maj}}}^M \,. &(8.23)
\end{aligned}$$

(4) supersymmetry transformation

$$\begin{aligned}
\delta {e_m}^a &= i\zeta\gamma^a\chi_m &(8.24)\\
\delta\chi_m &= 2\nabla_m\zeta &(8.25)\\
\delta X^M &= \zeta{\psi_{\mathrm{Maj}}}^M &(8.26)\\
\delta {\psi_{\mathrm{Maj}}}^M &= -\frac{i}{2}\gamma^n\zeta(\chi_n{\psi_{\mathrm{Maj}}}^M) + i\gamma^m\zeta\partial_m X^M \\
&= i\gamma^n\zeta(\partial_n X^M - \frac{1}{2}(\chi_n{\psi_{\mathrm{Maj}}}^M)) \,. &(8.27)
\end{aligned}$$

(5) super Weyl transformation

$$\delta e_m{}^a = 0 \tag{8.28}$$

$$\delta \chi_m = \gamma_m \lambda \tag{8.29}$$

$$\delta X^M = 0 \tag{8.30}$$

$$\delta \psi_{\mathrm{Maj}}{}^M = 0\,. \tag{8.31}$$

Our procedure to compute eq. (8.9) is as follows:

1. define the inner product in the function space of $e_m{}^a$, χ_m.

2. find orthogonal bases.

3. change of integration.

8.1 $\mathcal{D}e_m^a$

The inner product can be defined by

$$\langle \delta e | \delta e \rangle = \int \mathrm{d}^2\sigma \sqrt{g} \left\{ e_a{}^n e_b{}^m \delta e_m{}^a \delta e_n{}^b + c e_a{}^n e_b{}^m \delta e_n{}^a \delta e_m{}^b + c' e_a{}^m e^{an} \delta e_m{}^b \delta e_{nb} \right\}. \tag{8.32}$$

The orthogonal decomposition is written as

$$\delta e_m{}^a = \delta\sigma e_m{}^a + (P_1 \delta\eta)_m{}^a + \delta\ell \varepsilon^{ab} e_{mb} + \sum_i \delta c_i \psi^i{}_m{}^a\,, \tag{8.33}$$

where

$$\begin{aligned} P_1 : (P_1\delta\eta)_m{}^a = & \left\{\delta\eta^n \partial_n e_m{}^a + e_n{}^a \partial_m \delta\eta^n\right\} - \frac{1}{2} e_m{}^a e_b{}^n \left\{\delta\eta^\ell \partial_\ell e_n{}^b + e_\ell{}^b \partial_n \delta\eta^\ell\right\} \\ & - \frac{1}{2} \varepsilon^{ab} e_m{}^b \varepsilon_{dc} e_c{}^n \left\{\delta\eta^\ell \partial_\ell e_n{}^d + e_\ell{}^d \partial_n \delta\eta^\ell\right\} \end{aligned} \tag{8.34}$$

$$\psi^i : \ker P_1^\dagger\,. \tag{8.35}$$

Now

$$
\begin{aligned}
e_a{}^m (P_1 \delta\eta)_m{}^a &= 0 \\
\varepsilon_{ab} e_b{}^m (P_1 \delta\eta)_m{}^a &= 0
\end{aligned} \tag{8.36}
$$

and

$$
\langle \delta\sigma e_m{}^a | (P_1 \delta\eta)_m{}^a \rangle = 0\,. \tag{8.37}
$$

Therefore

$$
\int \frac{\mathcal{D} e_m{}^a}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})} = \int \frac{\mathcal{D}\sigma \mathcal{D}(P_1\eta)\mathcal{D}\ell}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})} \prod_i \mathrm{d}c_i \det\langle \psi^i | \psi^j \rangle^{\frac{1}{2}}\,. \tag{8.38}
$$

Next we perform changing integration variables. We introduce following fiducial metric for gauge fixing:

$$
e_m{}^a = e^{\Lambda} \hat{e}_m{}^a(\tau_i)\,, \tag{8.39}
$$

where τ_i denotes bosonic moduli. Taking variation of eq. (8.39),

$$
\begin{aligned}
\delta e_m{}^a &= \delta\Lambda e_m{}^a + (\delta\eta^n \partial_n e_m{}^a + e_n{}^a \partial_m \delta\eta^n) + \delta L \varepsilon^{ab} e_m{}^b + \sum_i \delta\tau_i \left(\frac{\partial e_m{}^a}{\partial \tau} \right) \\
&= (\delta\Lambda + \frac{1}{2} e_b{}^n \left\{ \delta\eta^\ell \partial_\ell e_n{}^b + e_\ell{}^b \partial_n \delta\eta^\ell \right\}) e_m{}^a + (P_1 \delta\eta)_m{}^a \\
&\quad + \left[\delta L + \frac{1}{2} \varepsilon_{dc} e_c{}^n \left\{ \delta\eta^\ell \partial_\ell e_n{}^d + e_\ell{}^d \partial_n \delta\eta^\ell \right\} \right] \varepsilon^{ab} e_m{}^b + \sum_i \delta\tau_i \left(\frac{\partial e_m{}^a}{\partial \tau_i} \right)\,.
\end{aligned} \tag{8.40}
$$

By matrix form, this transformation can be describeed as

$$
\begin{bmatrix} \delta\sigma \\ \mathrm{d}\ell \\ P_1\delta\eta \\ \delta c_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & T_{ij} \end{bmatrix} \begin{bmatrix} \delta\Lambda \\ \mathrm{d}L \\ \delta\eta \\ \delta\tau_i \end{bmatrix}\,. \tag{8.41}
$$

To obtain T_{ij}, we consider the inner product of

$$
\sum_i \delta c_i \psi_{im}{}^a = \sum_i \delta\tau_i \frac{\partial e_m{}^a}{\partial \tau_i}\,. \tag{8.42}
$$

Then

$$\langle\psi_j|\psi_i\rangle\delta c_i = \langle\psi_j|\frac{\partial e_m{}^a}{\partial\tau_k}\rangle\delta\tau_k \,. \tag{8.43}$$

Therefore

$$\delta c_i = \langle\psi|\psi\rangle^{-1}_{ij}\langle\psi_j|\frac{\partial e_m{}^a}{\partial\tau_k}\rangle\delta\tau_k \equiv T_{ik}\delta\tau_k \,, \tag{8.44}$$

and hence

$$\mathrm{det}T = \mathrm{det}\langle\psi_i|\psi_j\rangle^{-1}\mathrm{det}\langle\psi_i|\frac{\partial e_m{}^a}{\partial\tau_j}\rangle \,. \tag{8.45}$$

Finally,

$$\begin{aligned}
&\int\frac{\mathcal{D}e_m{}^a}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})} \\
&= \int\frac{\mathcal{D}\sigma\mathcal{D}\ell\mathcal{D}(P_1\eta)}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})}\prod_i \mathrm{d}c_i\mathrm{det}\langle\psi_i|\psi_j\rangle^{\frac{1}{2}} \\
&= \int\frac{\mathcal{D}\Lambda\mathcal{D}L\mathcal{D}'\eta}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})}\prod_i \mathrm{d}\tau_i\mathrm{det}'(P_1^\dagger P_1)^{\frac{1}{2}}\mathrm{det}T\mathrm{det}\langle\psi_i|\psi_j\rangle^{\frac{1}{2}} \\
&= \int\frac{\mathcal{D}\Lambda\mathcal{D}L\mathcal{D}'\eta}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})}\prod_i \mathrm{d}\tau_i\mathrm{det}'(P_1^\dagger P_1)^{\frac{1}{2}}\mathrm{det}\langle\psi_i|\psi_j\rangle^{-\frac{1}{2}}\mathrm{det}\left\langle\psi_i\left|\frac{\partial e_m{}^a}{\partial\tau_j}\right.\right\rangle \\
&= \int\prod_i \mathrm{d}\tau_i\frac{1}{\Omega(\mathrm{CK})}\mathrm{det}'(P_1^\dagger P_1)^{\frac{1}{2}}\mathrm{det}\langle\psi_i|\psi_j\rangle^{-\frac{1}{2}}\mathrm{det}\left\langle\psi_i\left|\frac{\partial e_m{}^a}{\partial\tau_j}\right.\right\rangle \,,
\end{aligned} \tag{8.46}$$

where the conformal Killing vector is excluded in $\mathcal{D}'\eta$ and $\Omega(\mathrm{CK})$ denotes the volume of conformal Killing vector.

8.2 $\mathcal{D}\chi_m^a$

Similarly to the previous subsection, first we define the inner product by

$$\langle\delta\chi|\delta\chi\rangle = \int\mathrm{d}^2\sigma\sqrt{g}\left\{\delta\chi_m(g^{mn}-\frac{1}{2}\gamma^m\gamma^n)\delta\chi_n + c''\delta\chi_m\gamma^m\gamma^n\delta\chi_n\right\} \,. \tag{8.47}$$

The orthogonal decomposition is

$$\delta\chi_m = \gamma_m\delta\rho + (P_{1/2}\delta\zeta) + \sum_i\delta\epsilon_i\Psi^i_m \,, \tag{8.48}$$

where

$$
\begin{aligned}
\delta\zeta &: \text{spin } \frac{1}{2} \text{ (parameter)} \\
\delta\rho &: \text{spin } \frac{3}{2} \text{ (parameter)} \\
\delta\epsilon &: \text{Grassmann } \# \\
\Psi^i{}_m &\in \ker P_{1/2}^{\dagger}
\end{aligned}
\tag{8.49}
$$

and

$$
(P_{1/2}\delta\zeta)_m = 2\nabla_m\delta\zeta - \gamma_m\gamma^n\nabla_n\delta\zeta \ , \qquad \gamma^m(P_{1/2}\delta\zeta)_m = 0 \ . \tag{8.50}
$$

Then

$$
\langle\gamma_m\delta\rho|(P_{1/2}\delta\zeta)_m\rangle = 0 \qquad \text{etc.} \ . \tag{8.51}
$$

Therefore

$$
\int \frac{\mathcal{D}\chi_m{}^a}{\Omega(\mathrm{S})\Omega(\mathrm{SW})} = \int \frac{\mathcal{D}\rho\mathcal{D}(P_{1/2}\zeta)}{\Omega(\mathrm{S})\Omega(\mathrm{SW})} \prod_i \mathrm{d}\epsilon_i \det\langle\Psi_i|\Psi_j\rangle^{-\frac{1}{2}} \ . \tag{8.52}
$$

To consider the change of integration variables, we introduce the fiducial metric such as

$$
\chi_m = \gamma_m\lambda + \sum_i a_i\Phi_{i,m} \ , \tag{8.53}
$$

where a_i denotes supermoduli and $\Phi_{i,m} \in \ker P_{1/2}^{\dagger}$. The variation is

$$
\begin{aligned}
\delta\chi_m &= \gamma_m\delta\lambda + 2\nabla_m\delta\zeta + \sum_i \mathrm{d}a_i\Phi_i \\
&= \gamma_m\delta\lambda + \gamma_m\gamma^n\nabla_n\delta\zeta + (2\nabla_m\delta\zeta - \gamma_m\gamma^n\nabla_n\delta\zeta) + \sum_i \mathrm{d}a_i\Phi_i \ ,
\end{aligned}
\tag{8.54}
$$

where $\gamma_m\delta\lambda$, $2\nabla_m\delta\zeta$ and $\sum_i \mathrm{d}a_i\Phi_i$ denote super Weyl, local supersymmety and super moduli respectively. The transformation matrix can be written as

$$
\begin{bmatrix} \delta\rho \\ P_{1/2}\delta\zeta \\ \delta\epsilon_i \end{bmatrix} = \begin{bmatrix} 1 & * & 0 \\ 0 & P_{1/2} & 0 \\ 0 & 0 & S_{ij} \end{bmatrix} \begin{bmatrix} \delta\lambda \\ \delta\zeta \\ \mathrm{d}a_j \end{bmatrix} \ . \tag{8.55}
$$

From tihs

$$\delta\epsilon_i = S_{ij}\mathrm{d}a_j \ , \tag{8.56}$$

while from eqs. (8.48) and (8.54)

$$\delta\epsilon_i\Psi_i = \mathrm{d}a_i\Phi_i \ . \tag{8.57}$$

Multiplying Ψ on the left hand side,

$$\langle\Psi_j|\Psi_i\rangle\delta\epsilon_i = \langle\Psi_j|\Phi_i\rangle\mathrm{d}a_i \ , \tag{8.58}$$

and then

$$\mathrm{d}\epsilon_i = \langle\Psi|\Psi\rangle^{-1}_{ik}\langle\Psi_k|\Phi_j\rangle\mathrm{d}a_j \ . \tag{8.59}$$

Comparing with eq. (8.56),

$$S_{ij} = \langle\Psi|\Psi\rangle^{-1}_{ik}\langle\Psi_k|\Phi_j\rangle \ . \tag{8.60}$$

Therefore

$$\mathrm{det}S_{ij} = \mathrm{det}\langle\Psi|\Psi\rangle^{-1}_{ik}\mathrm{det}\langle\Psi_k|\Phi_j\rangle \ . \tag{8.61}$$

Finally,

$$\begin{aligned}
&\int \frac{\mathcal{D}\chi_m}{\Omega(\mathrm{S})\Omega(\mathrm{SW})} \\
&= \int \frac{\mathcal{D}\rho\mathcal{D}(P_{1/2}\zeta)}{\Omega(\mathrm{S})\Omega(\mathrm{SW})}\prod_i \mathrm{d}\epsilon_i\mathrm{det}\langle\Psi_i|\Psi_j\rangle^{-\frac{1}{2}} \\
&= \int \frac{\mathcal{D}\lambda\mathcal{D}'\zeta}{\Omega(\mathrm{S})\Omega(\mathrm{SW})}\mathrm{det}'(P^\dagger_{1/2}P_{1/2})^{-\frac{1}{2}}\int\prod_i \mathrm{d}a_i\mathrm{det}S^{-1}\mathrm{det}\langle\Psi|\Psi\rangle^{-\frac{1}{2}} \\
&= \int \frac{\mathcal{D}\lambda\mathcal{D}'\zeta}{\Omega(\mathrm{S})\Omega(\mathrm{SW})}\int\prod_i \mathrm{d}a_i\mathrm{det}'(P^\dagger_{1/2}P_{1/2})^{-\frac{1}{2}}\mathrm{det}\langle\Psi_i|\Psi_k\rangle^{+\frac{1}{2}}\mathrm{det}\langle\Psi_k|\Phi_j\rangle^{-1} \\
&= \int\prod_i \mathrm{d}a_i\frac{1}{\Omega(\mathrm{CKS})}\mathrm{det}'(P^\dagger_{1/2}P_{1/2})^{-\frac{1}{2}}\mathrm{det}\langle\Psi_i|\Psi_k\rangle^{+\frac{1}{2}}\mathrm{det}\langle\Psi_k|\Phi_j\rangle^{-1} \ ,
\end{aligned} \tag{8.62}$$

where the conformal killing spinor is exclued in $\mathcal{D}'\zeta$ and $\Omega(\mathrm{CKS})$ denotes the volume of conformal killing spinor.

8.3 Summary

To summarize subsections 8.1 and 8.2,

$$\int \frac{\mathcal{D}e_m{}^a}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})} \int \frac{\mathcal{D}\chi_m}{\Omega(\mathrm{S})\Omega(\mathrm{SW})}$$
$$= \int \prod_i \mathrm{d}\tau_i \frac{1}{\Omega(\mathrm{CK})} \mathrm{det}'(P_1^\dagger P_1)^{\frac{1}{2}} \mathrm{det}\langle\psi_i|\psi_j\rangle^{-\frac{1}{2}} \mathrm{det}\langle\psi_i|\frac{\partial e_m{}^a}{\partial\tau_j}\rangle$$
$$\times \int \prod_i \mathrm{d}a_i \frac{1}{\Omega(\mathrm{CKS})} \mathrm{det}'(P_{1/2}^\dagger P_{1/2})^{-\frac{1}{2}} \mathrm{det}\langle\Psi_i|\Psi_j\rangle^{+\frac{1}{2}} \mathrm{det}\langle\Psi_i|\Phi_j\rangle^{-1} \tag{8.63}$$

Therefore

$$Z = \sum_{\text{topology}} \sum_{\text{spin structure}} \int \prod_i \mathrm{d}\tau_i \frac{1}{\Omega(\mathrm{CK})} \mathrm{det}'(P_1^\dagger P_1)^{\frac{1}{2}} \mathrm{det}\langle\psi_i|\psi_j\rangle^{-\frac{1}{2}} \mathrm{det}\langle\psi_i|\frac{\partial e_m{}^a}{\partial\tau_j}\rangle$$
$$\times \int \prod_i \mathrm{d}a_i \frac{1}{\Omega(\mathrm{CKS})} \mathrm{det}'(P_{1/2}^\dagger P_{1/2})^{-\frac{1}{2}} \mathrm{det}\langle\Psi_i|\Psi_j\rangle^{+\frac{1}{2}} \mathrm{det}\langle\Psi_i|\Phi_j\rangle^{-1}$$
$$\times \int \mathcal{D}X^M \mathcal{D}\psi_{\mathrm{Maj}}{}^M \mathrm{e}^{-S} \,. \tag{8.64}$$

8.4 Example: torus

On torus, the metric is

$$g_{mn} = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{bmatrix} , \quad z = \sigma^1 + \tau\sigma^2 \,, \tag{8.65}$$

and $g_{mn} = e_m{}^a e_{na}$. The zwei bein can be written as

$$e_m{}^a = \begin{bmatrix} 1 & 0 \\ \tau_1 & \tau_2 \end{bmatrix} \tag{8.66}$$

because

$$\begin{bmatrix} 1 & 0 \\ \tau_1 & \tau_2 \end{bmatrix} \begin{bmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{bmatrix} = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{bmatrix} . \tag{8.67}$$

The inverse of ${e_m}^a$ is

$$ {e_a}^m = \begin{bmatrix} 1 & 0 \\ -\frac{\tau_1}{\tau_2} & \frac{1}{\tau_2} \end{bmatrix} . \tag{8.68} $$

Dirac matrices (worldsheet Euclidean) are

$$ \gamma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \ \gamma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \ \gamma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \ \gamma^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\gamma^2 . \tag{8.69} $$

Fermions on torus are classified into 4 sectors:

$$ \begin{aligned} \psi(\sigma^1+1, \sigma^2) &= r\psi(\sigma^1, \sigma^2) , \qquad & r = \pm 1 \\ \psi(\sigma^1, \sigma^2+1) &= s\psi(\sigma^1, \sigma^2) , \qquad & s = \pm 1 , \end{aligned} \tag{8.70} $$

where

$$ (r, s) = (+,+) \qquad \text{R R} \tag{8.71} $$
$$ (+,-) \qquad \text{R NS} \tag{8.72} $$
$$ (-,+) \qquad \text{NS R} \tag{8.73} $$
$$ (-,-) \qquad \text{NS NS} . \tag{8.74} $$
$$ \tag{8.75} $$

We consider how many supermoduli paraeters exist in (r, s) sector. To do this, we would like to solve

$$ P^\dagger_{1/2}\Psi_m = 0 , \quad \text{and} \quad \gamma^m\Psi_m \equiv {e_a}^m\gamma^a\Psi_m = 0 . \tag{8.76} $$

In general, this equation can be written as

$$ -2\nabla^m\Psi_m = 0 . \tag{8.77} $$

When the metric is conformal flat,

$$ 0 = \partial^m\Psi_m = g^{mn}\partial_n\Psi_m = 0 . \tag{8.78} $$

It is better to use complex notation:

$$\begin{aligned}\gamma^m\Psi_m &= \gamma^a e_a{}^m\Psi_m \\ &= \begin{bmatrix} 0 & e_1{}^m - ie_2{}^m \\ e_1{}^m + ie_2{}^m & 0 \end{bmatrix}\Psi_m \\ &= \gamma^z\Psi_z + \gamma^{\bar z}\Psi_{\bar z}\ ,\end{aligned} \tag{8.79}$$

where

$$\gamma^z = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix}\ ,\ \ \gamma^{\bar z} = \begin{bmatrix} 0\,0 \\ 1\,0 \end{bmatrix}\ . \tag{8.80}$$

Thus

$$\begin{aligned}\Psi_z &= (e_1{}^m - ie_2{}^m)\Psi_m \\ \Psi_{\bar z} &= (e_1{}^m + ie_2{}^m)\Psi_m\ .\end{aligned} \tag{8.81}$$

Since

$$\begin{aligned}\frac{\partial}{\partial\sigma^1} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial\bar z} \\ \frac{\partial}{\partial\sigma^2} &= \tau\frac{\partial}{\partial z} + \bar\tau\frac{\partial}{\partial\bar z}\ ,\end{aligned} \tag{8.82}$$

$$\partial^m\Psi_m = \frac{\partial}{\partial z}\Psi_{\bar z} + \frac{\partial}{\partial\bar z}\Psi_z\ . \tag{8.83}$$

So we conclude

$$\frac{\partial}{\partial z}\Psi_{\bar z} + \frac{\partial}{\partial\bar z}\Psi_z = 0\ , \qquad \gamma^z\Psi_z + \gamma^{\bar z}\Psi_{\bar z} = 0\ . \tag{8.84}$$

This solution is

$$\Psi_z = \begin{bmatrix} f(z) \\ 0 \end{bmatrix}\ , \qquad \Psi_{\bar z} = \begin{bmatrix} 0 \\ g(\bar z) \end{bmatrix}\ , \tag{8.85}$$

where $f(z)$ is an analytic function and $g(\bar z)$ antianalytic. On torus, there must be constant. Except for RR sector, *i.e.* $(++)$ sector, there is no supermoduli.

The conformal Killing spinor ζ satisfies

$$0 = P_{1/2}\zeta = 2\nabla_m\zeta - \gamma_m\gamma^n\nabla_n\zeta\ . \tag{8.86}$$

In the flat case, this reduces to

$$2\partial_m\zeta - \gamma_m\gamma^n\partial_n\zeta = 0 \tag{8.87}$$

and then

$$(2g_m{}^n - \gamma_m\gamma^n)\partial_n\zeta = 0 \; . \tag{8.88}$$

Therefore

$$\partial_n\zeta = 0 \; , \tag{8.89}$$

where ζ is constant spin $\frac{1}{2}$ spinor. In the $(+-), (-+), (--)$ sector there is no conformal Killing spinor, while in the $(++)$ there is one (two component) conformal Killing spinor.

The formula is eq. (8.64) as before:

$$\begin{aligned} &Z \\ &= \sum_{\text{topology}}\sum_{\text{spin structure}}\int\prod_i \mathrm{d}\tau_i \frac{1}{\Omega(\text{CK})}\det{}'(P_1^\dagger P_1)^{\frac{1}{2}}\det\langle\psi_i|\psi_j\rangle^{-\frac{1}{2}}\det\langle\psi_i|\frac{\partial e_m{}^a}{\partial\tau_j}\rangle \\ &\times\int\prod_i \mathrm{d}a_i\frac{1}{\Omega(\text{CKS})}\det{}'(P_{1/2}^\dagger P_{1/2})^{-\frac{1}{2}}\det\langle\Psi_i|\Psi_j\rangle^{+\frac{1}{2}}\det\langle\Psi_i|\Phi_j\rangle^{-1} \\ &\times\int\mathcal{D}X^M\mathcal{D}\psi_{\text{Maj}}{}^M\mathrm{e}^{-S} \; . \end{aligned} \tag{8.90}$$

But for RR it vanishes because of the matter fermion zero mode. Therefore, for $(+,-)$, $(-,+)$, $(-,-)$

- no supermoduli
- no CKS

and for $(+,+)$

- $(\mathrm{R},\mathrm{R})_{\text{vac. amp.}} = 0$.

The torus vacuum amplitude for IIB/IIA in flat ten dimensions is, therefore, simply written as

$$Z_{\rm flat}^{\rm IIB/IIA} = V_{\rm E} \sum_{(r,s)} \sum_{(r',s')} C_{rs} \bar{C}_{r's'} \frac{1}{2} \int_{\mathcal{F}} \prod_{i=1,2} \mathrm{d}\tau_i \det \left(P_1^\dagger P_1 \right)^{\frac{1}{2}} \left[\det \Delta_{\hat{g}} \right]^{-5}$$
$$\det \left\langle \psi_i \middle| \frac{\partial e_m{}^a}{\partial \tau_j} \right\rangle \det \langle \psi_i | \psi_j \rangle^{-\frac{1}{2}} \left(\int \mathrm{d}^2\sigma \sqrt{\hat{g}} \right)^5$$
$$\left[\det{}' \left(P_{1/2}^\dagger P_{1/2} \right)^{-\frac{1}{2}} \det (\gamma \cdot \partial)^5 \right]_{(r,s),(r',s')} , \qquad (8.91)$$

where we have chosen $C_{--} = -C_{-+} = -C_{+-} = \frac{1}{2}$, $\bar{C}_{--} = -\bar{C}_{-+} = -\bar{C}_{+-} = \frac{1}{2}$, $C_{++} = \bar{C}_{++} = 0$ in accordance with the GSO projection of the IIB superstring that implements the modular invariance. The Euclidean volume is denoted by $V_{\rm E}$. Omitting the calculations of the Weil Petersen measure factor and those of the functional determinants, we obtain

$$Z_{\rm flat}^{\rm IIB/IIA} = K V_{\rm E} \frac{1}{2} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\tau_2)^2} \frac{1}{\tau_2^4} \frac{1}{|\eta(\tau)|^{16}} |T(\tau)|^2, \qquad (8.92)$$

where

$$T(\tau) = \frac{1}{\eta(\tau)^4} \left(C_{--} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 + C_{-+} \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}^4 + C_{+-} \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}^4 \right) . \qquad (8.93)$$

We take a short cut to proceed further and to determine the normalization factor K by comparing the last expression eq. (8.92) with the vacuum amplitude evaluated in the light cone gauge operator formalism, written in terms of the $so(8)$ characters. (The overall normalization can also be seen by the one-loop free energy in local field theory):

$$\Gamma_{\rm flat}^{\rm IIB/IIA} = - \frac{V_{\rm E}}{2(4\pi^2 \alpha')^5} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} (\bar{\chi} X \chi)_{\rm IIB/IIA,\, flat} \, . \qquad (8.94)$$

Identifying eq. (8.92) with eq. (8.94), we obtain

$$K = - \frac{1}{(4\pi^2 \alpha')^5} \, . \qquad (8.95)$$

Note that, from the point of view of one-loop free energy in local field theory, $\frac{1}{(4\pi^2\alpha'\tau_2)^{\frac{1}{2}}}$ comes from a gaussian integration over one momentum, and $-\frac{1}{2}\int\frac{\mathrm{d}^2\tau}{\tau_2}\cdots$ comes from a proper time representation of $\log\mathrm{Det}$.

9 Fermionic string amplitudes at one-loop

The superstring scattering amplitudes are given in general by the functional integrals with the appropriately chosen vertex operators $\prod_I O_I$ over these worldsheet fields with respect to this action modulo the local symmetries

$$\begin{aligned}&\langle\prod_I O_I\rangle\\&=\sum_{\text{top.}}\sum_{\text{s.s.}}\int\frac{\mathcal{D}e_m{}^a}{\Omega(\mathrm{D})\Omega(\mathrm{W})\Omega(\mathrm{L})}\int\frac{\mathcal{D}\chi_m{}^a}{\Omega(\mathrm{S})\Omega(\mathrm{SW})}\int\mathcal{D}X^M\int\mathcal{D}\psi_{\mathrm{Maj}}{}^M e^{-S}\prod_I O_I\,.\end{aligned} \tag{9.1}$$

Therefore

$$\begin{aligned}\langle\prod_I O_I\rangle=&\sum_{\text{top.}}\sum_{\text{s.s.}}\int\prod_i\mathrm{d}\tau_i\frac{1}{\Omega(\mathrm{CKV})}\mathrm{det}'(P_1^\dagger P_1)^{\frac{1}{2}}\mathrm{det}\langle\psi_i|\psi_j\rangle^{-\frac{1}{2}}\mathrm{det}\left\langle\psi_i\left|\frac{\partial e_m{}^a}{\partial\tau_i}\right.\right\rangle\\&\times\int\prod_i\mathrm{d}a_i\frac{1}{\Omega(\mathrm{CKS})}\mathrm{det}'(P_{1/2}^\dagger P_{1/2})^{-\frac{1}{2}}\mathrm{det}\langle\Psi_i|\Psi_j\rangle^{\frac{1}{2}}\mathrm{det}\langle\Psi_i|\Phi_j\rangle^{-1}\\&\times\int\mathcal{D}X^M\int\mathcal{D}\psi_{\mathrm{Maj}}{}^M e^{-S}\prod_I O_I\,.\end{aligned} \tag{9.2}$$

Part III

Genus one Green's function with (α, β) boundary condition and superstring amplitudes

10 Genus one Green functions with (α, β) boundary condition

In this section, we compute the genus one Green functions with general boundary condition to be designated by (α, β), using the eigenmode expansion. We mainly consider the case of torus here. The other one-loop geometries, Klein bottle, annulus and Möbius band, can be constructed by the involution (or, the image method) as seen, for example, in [33, 34].

Let $z \equiv \sigma^1 + \tau\sigma^2$ and $\bar{z} = \sigma^1 + \bar{\tau}\sigma^2$ $(0 \leq \sigma_1, \sigma_2 \leq 1)$ be the complex coordinates on the worldsheet torus with modular parameter $\tau \equiv \tau_1 + i\tau_2$. The Laplacian is defined by $\Delta \equiv 4\partial_z\partial_{\bar{z}}$. We use the plane wave bases

$$\begin{aligned} \Phi_{n_1,n_2}\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\sigma^1,\sigma^2) &\equiv \frac{1}{\sqrt{\tau_2}}\mathrm{e}^{2\pi i(n_1+\alpha)\sigma^1}\mathrm{e}^{2\pi i(n_2+\beta)\sigma^2} \\ &= \frac{1}{\sqrt{\tau_2}}\mathrm{e}^{\frac{2\pi i}{\tau-\bar{\tau}}\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\}z}\mathrm{e}^{-\frac{2\pi i}{\tau-\bar{\tau}}\{(n_2+\beta)-(n_1+\alpha)\tau\}\bar{z}} \end{aligned} \tag{10.1}$$

as our eigenfunctions, where $n_1, n_2 \in \mathbf{Z}$, $0 \leq \alpha, \beta < 1$. We have imposed the orthonormality on $\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\sigma^1,\sigma^2)$ to determine the normalization factor $\frac{1}{\sqrt{\tau_2}}$. (See appendix A.6.) This function possesses the following quasiperiodicities:

$$\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1+1,\sigma^2)=\mathrm{e}^{2\pi i\alpha}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)$$
$$\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2+1)=\mathrm{e}^{2\pi i\beta}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)\,. \qquad (10.2)$$

In the subsequent subsections, we will first consider the bosonic and fermionic components and then use these components to provide the supertorus Green function. We will also consider the superannulus Neumann function as the involution of the supertorus Green function.

10.1 Bosonic part

Since the eigenequation is

$$\Delta\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)=\Lambda^{(\alpha,\beta)}_{n_1,n_2}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)\,, \qquad (10.3)$$

the eigenvalue reads

$$\begin{aligned}\Lambda^{(\alpha,\beta)}_{n_1,n_2}&\equiv\frac{4(2\pi)^2}{(\tau-\bar\tau)^2}\left|(n_2+\beta)-(n_1+\alpha)\tau\right|^2\\ &=-\frac{(2\pi)^2}{\tau_2^2}\left[\{(n_2+\beta)-(n_1+\alpha)\tau_1\}^2+\{(n_1+\alpha)\tau_2\}^2\right]\,. \qquad (10.4)\end{aligned}$$

Note that $\Lambda^{(0,0)}_{0,0}=0$. In the following, we consider the cases of $\alpha\neq 0$ and $(\alpha,\beta)=(0,0),(0,\frac{1}{2})$.

10.1.1 case of $\alpha\neq 0$

Now we would like to compute the Green function

$$\begin{aligned}G\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\bar z|0,0)&\equiv\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\Lambda^{(\alpha,\beta)}_{n_1,n_2}}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)\Phi^*_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0,0)\\ &=\frac{1}{\tau_2}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\Lambda^{(\alpha,\beta)}_{n_1,n_2}}\mathrm{e}^{2\pi i(n_1+\alpha)\sigma^1}\mathrm{e}^{2\pi i(n_2+\beta)\sigma^2}\,. \qquad (10.5)\end{aligned}$$

By translational invariance, we have chosen 0 in the second set of arguments.

Exploiting the partial fraction, we decompose $\frac{1}{\Lambda^{(\alpha,\beta)}_{n_1,n_2}}$ into

$$\frac{1}{\Lambda^{(\alpha,\beta)}_{n_1,n_2}} = \frac{\tau-\bar{\tau}}{4(2\pi)^2}\frac{1}{n_1+\alpha}\left\{\frac{1}{(n_2+\beta)-(n_1+\alpha)\tau} - \frac{1}{(n_2+\beta)-(n_1+\alpha)\bar{\tau}}\right\}, \tag{10.6}$$

which is permissible even for $n_1 = 0$, $\alpha \neq 0$. Using

$$\int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i\{(n_2+\beta)-(n_1+\alpha)\tau\}\sigma} = \frac{1-\mathrm{e}^{-2\pi i\beta}q^{n_1+\alpha}}{2\pi i\{(n_2+\beta)-(n_1+\alpha)\tau\}}, \tag{10.7}$$

we obtain

$$\frac{1}{\Lambda^{(\alpha,\beta)}_{n_1,n_2}} = \frac{i(\tau-\bar{\tau})}{4(2\pi)}\frac{1}{n_1+\alpha}\left[\frac{1}{1-\mathrm{e}^{-2\pi i\beta}q^{n_1+\alpha}}\int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i\{(n_2+\beta)-(n_1+\alpha)\tau\}\sigma} \right.$$
$$\left. - \frac{1}{1-\mathrm{e}^{-2\pi i\beta}\bar{q}^{-(n_1+\alpha)}}\int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\}\sigma}\right], \tag{10.8}$$

where $q \equiv \mathrm{e}^{2\pi i\tau}$. According to eq. (B.14), we have the following manipulation:

$$
\begin{aligned}
&\sum_{n_1,n_2=-\infty}^{\infty} \frac{1}{n_1+\alpha} \frac{1}{1-\mathrm{e}^{-2\pi i\beta} q^{n_1+\alpha}} \\
&\quad \times \int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i\{(n_2+\beta)-(n_1+\alpha)\tau\}\sigma} \mathrm{e}^{2\pi i(n_1+\alpha)\sigma^1} \mathrm{e}^{2\pi i(n_2+\beta)\sigma^2} \\
&\quad = \sum_{n_1=-\infty}^{\infty} \frac{1}{n_1+\alpha} \frac{1}{1-\mathrm{e}^{-2\pi i\beta} q^{n_1+\alpha}} \\
&\qquad \times \int_0^1 \mathrm{d}\sigma \delta(\sigma^2-\sigma) \mathrm{e}^{+2\pi i\beta(\sigma^2-\sigma)+2\pi i(n_1+\alpha)(\sigma^1+\tau\sigma)} \\
&\quad = \sum_{n_1=-\infty}^{\infty} \frac{1}{n_1+\alpha} \frac{\zeta^{n_1+\alpha}}{1-\mathrm{e}^{-2\pi i\beta} q^{n_1+\alpha}} \\
&\quad \overset{m\equiv n_1}{=} \sum_{m=0}^{\infty} \frac{1}{m+\alpha} \frac{\zeta^{m+\alpha}}{1-\mathrm{e}^{-2\pi i\beta} q^{m+\alpha}} + \sum_{m=1}^{\infty} \frac{1}{m-\alpha} \frac{\mathrm{e}^{2\pi i\beta}\left(\frac{q}{\zeta}\right)^{m-\alpha}}{1-\mathrm{e}^{2\pi i\beta} q^{m-\alpha}} \\
&\quad \overset{m'\equiv m-1,\, \alpha'\equiv 1-\alpha}{=} \sum_{m=0}^{\infty} \frac{1}{m+\alpha} \frac{\zeta^{m+\alpha}}{1-\mathrm{e}^{-2\pi i\beta} q^{m+\alpha}} \\
&\qquad\qquad + \mathrm{e}^{2\pi i\beta} \sum_{m'=0}^{\infty} \frac{1}{m'+\alpha'} \frac{\left(\frac{q}{\zeta}\right)^{m'+\alpha'}}{1-\mathrm{e}^{2\pi i\beta} q^{m'+\alpha'}} \\
&\quad \overset{\text{eq.}(B.14)}{=} \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left(\mathrm{e}^{-2\pi i\beta}\right)^n (\zeta q^n)^\alpha F(1,\alpha,1+\alpha;\zeta q^n) \\
&\qquad + \frac{\Gamma(\alpha')}{\Gamma(1+\alpha')} \sum_{n=0}^{\infty} (\mathrm{e}^{2\pi i\beta})^{n+1} \left(\frac{q^{n+1}}{\zeta}\right)^{\alpha'} F\left(1,\alpha',1+\alpha';\frac{q^{n+1}}{\zeta}\right) ,
\end{aligned}
\tag{10.9}
$$

where $\zeta \equiv \mathrm{e}^{2\pi iz} = \mathrm{e}^{2\pi i(\sigma^1+\tau\sigma^2)}$ and $\alpha' \equiv 1-\alpha$. Similarly,

$$\begin{aligned}
&\sum_{n_1,n_2=-\infty}^{\infty} \frac{1}{n_1+\alpha}\frac{1}{1-\mathrm{e}^{-2\pi i\beta}\bar{q}^{-(n_1+\alpha)}} \\
&\quad\times\int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\}\sigma}\mathrm{e}^{2\pi i(n_1+\alpha)\sigma^1}\mathrm{e}^{2\pi i(n_2+\beta)\sigma^2} \\
&\quad= -\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}(\mathrm{e}^{2\pi i\beta})^{n+1}\left(\frac{\bar{q}^{n+1}}{\bar{\zeta}}\right)^{\alpha} F\left(1,\alpha,1+\alpha;\frac{\bar{q}^{n+1}}{\bar{\zeta}}\right) \\
&\qquad -\frac{\Gamma(\alpha')}{\Gamma(1+\alpha')}\sum_{n=0}^{\infty}(\mathrm{e}^{-2\pi i\beta})^{n}\left(\bar{\zeta}\bar{q}^{n}\right)^{\alpha'} F\left(1,\alpha',1+\alpha';\bar{\zeta}\bar{q}^{n}\right) .
\end{aligned} \tag{10.10}$$

Substituting eqs. (10.9) and (10.10) into eq. (10.5), we obtain

$$\begin{aligned}
&G\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\bar{z}|0,0) \\
&= -\frac{1}{2(2\pi)}\left[\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\left(\mathrm{e}^{-2\pi i\beta}\right)^{n}(\zeta q^n)^{\alpha}F(1,\alpha,1+\alpha;\zeta q^n)\right. \\
&\qquad +\frac{\Gamma(\alpha')}{\Gamma(1+\alpha')}\sum_{n=0}^{\infty}(\mathrm{e}^{2\pi i\beta})^{n+1}\left(\frac{q^{n+1}}{\zeta}\right)^{\alpha'} F\left(1,\alpha',1+\alpha';\frac{q^{n+1}}{\zeta}\right) \\
&\qquad +\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}(\mathrm{e}^{2\pi i\beta})^{n+1}\left(\frac{\bar{q}^{n+1}}{\bar{\zeta}}\right)^{\alpha} F\left(1,\alpha,1+\alpha;\frac{\bar{q}^{n+1}}{\bar{\zeta}}\right) \\
&\qquad \left.+\frac{\Gamma(\alpha')}{\Gamma(1+\alpha')}\sum_{n=0}^{\infty}(\mathrm{e}^{-2\pi i\beta})^{n}\left(\bar{\zeta}\bar{q}^{n}\right)^{\alpha'} F(1,\alpha',1+\alpha';\bar{\zeta}\bar{q}^{n})\right] .
\end{aligned} \tag{10.11}$$

10.1.2 case of $(\alpha, \beta) = (0, 0)$

In this case, it is necessary to exclude $(n_1, n_2) = (0, 0)$ at the sum in eq. (10.5):

$$G_{++}(z, \bar{z}|0, 0) \equiv G \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] (z, \bar{z}|0, 0) = \frac{1}{\tau_2} \sum_{\substack{n_1, n_2 = -\infty \\ (n_1, n_2) \neq (0,0)}}^{\infty} \frac{1}{\frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} |n_2 - n_1 \tau|^2} \mathrm{e}^{2\pi i n_1 \sigma^1} \mathrm{e}^{2\pi i n_2 \sigma^2} . \tag{10.12}$$

As the result of the calculation in appendix G.1, we obtain

$$\begin{aligned} &G_{++}(z, \bar{z}|0, 0) \\ &\overset{\text{eq.}(G.12)}{=} \frac{1}{2\pi} \ln \left| \frac{\vartheta \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right] (z)}{\vartheta' \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right] (0)} \right| - \frac{1}{2} \frac{(\mathrm{Im}\, z)^2}{\tau_2} \\ &\qquad + \left[\frac{1}{2\pi} 2 \sum_{n=1}^{\infty} \ln |1 - q^n| - \frac{1}{2} (\mathrm{Im}\, z) + \frac{1}{2\pi} \ln(2\pi) + 2\tau_2 \cdot \frac{\pi^2}{6} \right] . \end{aligned} \tag{10.13}$$

The terms in the bracket [...] vanish when acting on $\Delta = 4\partial_z \partial_{\bar{z}}$.

10.1.3 case of $(\alpha, \beta) = (0, \frac{1}{2})$

Here we consider

$$G_{+-}(z, \bar{z}|0, 0) \equiv G \left[\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right] (z, \bar{z}|0, 0) = \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{\frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} \left| \left(n_2 + \frac{1}{2}\right) - n_1 \tau \right|^2} \mathrm{e}^{2\pi i n_1 \sigma^1} \mathrm{e}^{2\pi i \left(n_2 + \frac{1}{2}\right) \sigma^2} . \tag{10.14}$$

Now we divide this sum into $n_1 \neq 0$ part and $n_1 = 0$ part to use the partial fraction decomposition in eq. (10.6).

As the result of the calculation in appendix G.2, we obtain

$$G_{+-}(z,\bar{z}|0,0) \overset{\text{eq.}(G.24)}{=} \frac{1}{2\pi}\left[\ln|1-\zeta| + \sum_{m=1}^{\infty}(-1)^m \ln|1-\zeta q^m|\left|1-\frac{q^m}{\zeta}\right|\right] - \pi^2\tau_2 \,. \tag{10.15}$$

10.2 Fermionic part

The eigen-equations are

$$\begin{aligned}
(-i)\partial_{\bar{z}}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2) &= \kappa^{(\alpha,\beta)}_{n_1,n_2}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2) \\
(-i)\partial_{z}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2) &= \bar{\kappa}^{(\alpha,\beta)}_{n_1,n_2}\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2) \,.
\end{aligned} \tag{10.16}$$

The eigenvalues can be written as

$$\begin{aligned}
\kappa^{(\alpha,\beta)}_{n_1,n_2} &= -\frac{2\pi}{\tau-\bar{\tau}}\{(n_2+\beta)-(n_1+\alpha)\tau\} = +\frac{\pi i}{\tau_2}\{(n_2+\beta)-(n_1+\alpha)\tau\} \\
\bar{\kappa}^{(\alpha,\beta)}_{n_1,n_2} &= +\frac{2\pi}{\tau-\bar{\tau}}\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\} = -\frac{\pi i}{\tau_2}\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\} \,.
\end{aligned} \tag{10.17}$$

Note that $\kappa^{(0,0)}_{0,0} = \bar{\kappa}^{(0,0)}_{0,0} = 0$.

10.2.1 case of $(\alpha,\beta)\neq(0,0)$

Here we calculate the Green function

$$\begin{aligned}
\mathcal{S}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\bar{z}|0,0) &= \frac{1}{\tau_2}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\kappa^{(\alpha,\beta)}_{n_1,n_2}}\mathrm{e}^{2\pi i(n_1+\alpha)\sigma^1}\mathrm{e}^{2\pi i(n_2+\beta)\sigma^2} \\
\bar{\mathcal{S}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\bar{z}|0,0) &= \frac{1}{\tau_2}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\bar{\kappa}^{(\alpha,\beta)}_{n_1,n_2}}\mathrm{e}^{-2\pi i(n_1+\alpha)\sigma^1}\mathrm{e}^{-2\pi i(n_2+\beta)\sigma^2} \,.
\end{aligned} \tag{10.18}$$

We obtain

$$
\begin{aligned}
&\mathcal{S}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\bar{z}|0,0)\\
&\overset{\text{eqs. (10.7), (10.17), (10.18)}}{=} -\frac{1}{\tau_2}(\tau-\bar{\tau})i\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{1-\mathrm{e}^{-2\pi i\beta}q^{n_1+\alpha}}\\
&\qquad\times\int_0^1 \mathrm{d}\sigma \mathrm{e}^{2\pi i(\sigma^2-\sigma)n_2}\mathrm{e}^{2\pi i\beta(\sigma^2-\sigma)}\mathrm{e}^{2\pi i(n_1+\alpha)(\sigma^1+\tau\sigma^2)}\\
&= -\frac{(\tau-\bar{\tau})i}{\tau_2}\sum_{n_1=-\infty}^{\infty}\frac{\zeta^{n_1+\alpha}}{1-\mathrm{e}^{-2\pi i\beta}q^{n_1+\alpha}} \overset{\text{eq. }(B.47)}{=} \frac{i}{\pi}\frac{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}-\beta\end{bmatrix}(z|\tau)}{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}-\beta\end{bmatrix}(0|\tau)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}\,.
\end{aligned}
\tag{10.19}
$$

Similarly, using

$$
\overline{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)} = \vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|-\bar{\tau})\,, \tag{10.20}
$$

$$
\begin{aligned}
\bar{\mathcal{S}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\bar{z}|0,0) &= -\frac{i}{\pi}\frac{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}+\beta\end{bmatrix}(-\bar{z}|-\bar{\tau})}{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}+\beta\end{bmatrix}(0|-\bar{\tau})}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|-\bar{\tau})}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-\bar{z}|-\bar{\tau})}\\
&\overset{\text{eqs. }(B.38),\,(B.40)}{=} -\frac{i}{\pi}\frac{\overline{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}-\beta\end{bmatrix}(z|\tau)}}{\overline{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}-\beta\end{bmatrix}(0|\tau)}}\frac{\overline{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)}}{\overline{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}}\,.
\end{aligned}
\tag{10.21}
$$

This time, we have used

$$
\int_0^1 \mathrm{d}\sigma \mathrm{e}^{2\pi i\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\}\sigma} = -\frac{1-\mathrm{e}^{2\pi i\beta}\bar{q}^{n_1+\alpha}}{2\pi i\{(n_2+\beta)-(n_1+\alpha)\bar{\tau}\}} \tag{10.22}
$$

instead of eq. (10.7), avoiding getting $\delta(\sigma^2+\sigma)$ which vanishes in the original domain.

In particular, when $(\alpha,\beta)=(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0),(0,\frac{1}{2})$, we obtain

$$\mathcal{S}_{--}(z,\bar{z}|0,0)\equiv\mathcal{S}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z,\bar{z}|0,0)=\frac{i}{\pi}\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(z|\tau)}{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0|\tau)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}$$

$$\mathcal{S}_{-+}(z,\bar{z}|0,0)\equiv\mathcal{S}\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(z,\bar{z}|0,0)=\frac{i}{\pi}\frac{\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(z|\tau)}{\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0|\tau)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}$$

$$\mathcal{S}_{+-}(z,\bar{z}|0,0)\equiv\mathcal{S}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(z,\bar{z}|0,0)=\frac{i}{\pi}\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(z|\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0|\tau)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)} \tag{10.23}$$

and the complex conjugates of these.

10.2.2 case of $(\alpha,\beta)=(0,0)$

In this case, we need to exclude the zero mode $(n_1,n_2)=(0,0)$ in the sum:

$$\mathcal{S}_{++}(z,\bar{z}|0,0)\equiv\mathcal{S}\begin{bmatrix}0\\0\end{bmatrix}(z,\bar{z}|0,0)=\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\(n_1,n_2)\neq(0,0)}}^{\infty}\frac{1}{\left(-\frac{2\pi}{\tau-\bar{\tau}}\right)(n_2-n_1\tau)}\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i n_2\sigma^2}$$

$$\bar{\mathcal{S}}_{++}(z,\bar{z}|0,0)\equiv\bar{\mathcal{S}}\begin{bmatrix}0\\0\end{bmatrix}(z,\bar{z}|0,0)=\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\(n_1,n_2)\neq(0,0)}}^{\infty}\frac{1}{+\frac{2\pi}{\tau-\bar{\tau}}(n_2-n_1\bar{\tau})}\mathrm{e}^{-2\pi i n_1\sigma^1}\mathrm{e}^{-2\pi i n_2\sigma^2}\,. \tag{10.24}$$

Here we use the relation

$$\mathcal{S}_{++}(z,\bar{z}|0,0)\overset{\text{eqs.(10.12),(10.24)}}{=}4i\frac{\partial}{\partial z}G_{++}(z,\bar{z}|0,0) \tag{10.25}$$

to calculate eq. (10.24), because eq. (B.47) appears not to work well when $(\alpha,\beta)=(0,0)$. Eq. (10.25) can be easily understood by using the last line in eq. (10.1). From eqs. (10.25), (10.13), we obtain

$$\mathcal{S}_{++}(z,\bar{z}|0,0)=\frac{i}{\pi}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}-2\frac{(\mathrm{Im}\,z)}{\tau_2}-1\,. \tag{10.26}$$

In addition,

$$\bar{\mathcal{S}}_{++}(z,\bar{z}|0,0) = \overline{\mathcal{S}_{++}(z,\bar{z}|0,0)} \overset{(10.26)}{=} -\frac{i}{\pi}\frac{\overline{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z)}}{\overline{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z)}} - 2\frac{(\mathrm{Im}\, z)}{\tau_2} - 1\,. \quad (10.27)$$

The last term, namely, -1 in eqs. (10.26) and (10.27) vanishes when acting with $(-i)\partial_{\bar{z}}$ or $(-i)\partial_z$.

10.3 Supertorus Green function and superannulus Neumann function

10.3.1 supertorus Green function

We define the supertorus Green function ($\nu_{\mathrm{f}} = (-,-)$ or $(-,+)$ or $(+,-)$) by

$$\begin{aligned}
&\mathbf{G}^{\mathrm{supertorus}}_{\substack{+\pm\\ \nu_{\mathrm{f}}}}(z_I,\bar{z}_I|z_J,\bar{z}_J) \\
&\equiv G_{+\pm}(z_I,\bar{z}_I|z_J,\bar{z}_J) + \frac{\theta_I\theta_J}{4}\mathcal{S}_{\nu_{\mathrm{f}}}(z_I,\bar{z}_I|z_J,\bar{z}_J) - \frac{\bar{\theta}_I\bar{\theta}_J}{4}\overline{\mathcal{S}}_{\nu_{\mathrm{f}}}(z_I,\bar{z}_I|z_J,\bar{z}_J)\,,
\end{aligned} \quad (10.28)$$

where θ, $\bar{\theta}$ are Grassmann coordinates and G_{++}, G_{+-} and $\mathcal{S}_{\nu_{\mathrm{f}}}$ are given in eqs. (10.13), (10.15) and (10.23), respectively. According to appendix H, we can see that $\mathbf{G}^{\mathrm{supertorus}}_{\substack{+\pm\\ \nu_{\mathrm{f}}}} \sim \mathbf{G}^{\mathrm{supersphere}}$ when $z_I \sim z_J$, where $\mathbf{G}^{\mathrm{supersphere}}$ is the supersphere Green function. The worldsheet supersymmetry is broken in general by the boundary condition, but it is still useful to consider this object, which we demonstrate in section 12.

10.3.2 superannulus Neumann function

Using the image method as in [32] (appendix F), the superannulus Neumann function can be written as

$$
\begin{aligned}
&\mathbf{N}^{\text{superannulus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(z, \bar{z}'; z, \bar{z}' \middle| \frac{i\tau_2}{2}\right) \\
&= \frac{1}{2}\left\{ \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{z}{2}, \frac{z'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) + \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{\tilde{z}}{2}, \frac{z'}{2}; \frac{\tilde{\theta}}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) \right. \\
&\left. + \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{z}{2}, \frac{\tilde{z}'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\tilde{\theta}'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) + \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{\tilde{z}}{2}, \frac{\tilde{z}'}{2}; \frac{\tilde{\theta}}{\sqrt{2}}, \frac{\tilde{\theta}'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) \right\} \\
&= \frac{1}{2}\left\{ \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{z}{2}, \frac{z'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) + \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{-\bar{z}}{2}, \frac{z'}{2}; \frac{\pm i\bar{\theta}}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) \right. \\
&\left. + \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{z}{2}, \frac{-\bar{z}'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\pm i\bar{\theta}'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) + \mathbf{G}^{\text{supertorus}}_{\substack{+\pm \\ \nu_{\text{f}}}}\left(\frac{-\bar{z}}{2}, \frac{-\bar{z}'}{2}; \frac{\pm i\bar{\theta}}{\sqrt{2}}, \frac{\pm i\bar{\theta}'}{\sqrt{2}} \middle| \frac{i\tau_2}{2}\right) \right\} ,
\end{aligned}
\tag{10.29}
$$

where $\tilde{z}$, $\tilde{z}'$, $\tilde{\theta}$ and $\tilde{\theta}$ denote respectively the conjugate points of z, z', θ and θ'.

11 Box notation

In order to proceed even further and to prepare for calculation of string scattering amplitudes in section 12, we will introduce notation for the integrand of the string one-loop partition function.

11.1 IIB/IIA flat

Let us, in particular, write $(\bar{\chi}X\chi)_{\text{IIB/IIA flat}}$ as

$$
\begin{aligned}
&(\bar{\chi}X\chi)_{\text{IIB/IIA flat}} \\
&= \frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^{8} - \boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^{8} - \boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^{8} \mp \boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^{8}\right) \overline{\frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^{8} - \boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^{8} - \boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^{8} \mp \boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^{8}\right)} \\
&\equiv \sum_{\nu,\bar{\nu}} \mathcal{J}_{\nu,\bar{\nu},\text{IIB/IIA,flat}} \, .
\end{aligned}
\tag{11.1}
$$

Here we have introduced

$$
\boxed{\begin{smallmatrix}r_{\rm b} & s_{\rm b}\\ r_{\rm f} & s_{\rm f}\end{smallmatrix}} \left(\equiv \boxed{\boxed{\begin{smallmatrix}\alpha_{\rm b} & \beta_{\rm b}\\ \alpha_{\rm f} & \beta_{\rm f}\end{smallmatrix}}}\right)
\tag{11.2}
$$

in order to represent the contribution from a single chiral boson and fermion obeying the boundary conditions $(\alpha_{\rm b}, \beta_{\rm b})$ and $(\alpha_{\rm f}, \beta_{\rm f})$ respectively:

$$
\boxed{\begin{smallmatrix}+ & +\\ r_{\rm f} & s_{\rm f}\end{smallmatrix}} = \boxed{\boxed{\begin{smallmatrix}0 & 0\\ \alpha_{\rm f} & \beta_{\rm f}\end{smallmatrix}}} = \sqrt{\frac{\vartheta\begin{bmatrix}\frac{1}{2}+\alpha_{\rm f}\\ \frac{1}{2}+\beta_{\rm f}\end{bmatrix}(0)}{\eta^3}} \, .
\tag{11.3}
$$

r and s are the same as in eq. (8.70). In the notation of section 10, $r = \mathrm{e}^{2\pi i\alpha}$, $s = \mathrm{e}^{2\pi i\beta}$, so that

$$
\begin{aligned}
&r = +1 \Leftrightarrow \alpha = 0 \, , \qquad r = -1 \Leftrightarrow \alpha = \frac{1}{2} \\
&s = +1 \Leftrightarrow \beta = 0 \, , \qquad s = -1 \Leftrightarrow \beta = \frac{1}{2} \qquad \text{modulo } 1 \, .
\end{aligned}
\tag{11.4}
$$

The power $8 = 10 - 2$ seen in eq. (11.1) permits covariant interpretation as the 2d metric and 2d gravitino fields obey the same boundary condition as the worldsheet bosons and fermions do respectively.

11.2 IIB string on $T^4(=(S^1)^4)/\mathbf{Z}_2$

As a simple prototypical example, let us consider IIB string on $T^4(=(S^1)^4)/\mathbf{Z}_2$ with radii of S^1 being R_I, $I=5,6,7,8$.

$$
\begin{aligned}
&(\bar{\chi}X\chi)_{\mathrm{IIB},T^4/\mathbf{Z}_2} \\
&= \frac{1}{2}\left(\prod_{I=5,6,7,8} F_2(a_I,\tau)\right)(\bar{\chi}X\chi)_{\mathrm{IIB,\ flat}} \\
&+\frac{1}{2}\left(\prod_{I=5,6,7,8} a_I\sqrt{\tau_2}\right)\frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\-&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\+&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\+&-\end{smallmatrix}}^4\right) \\
&\qquad\times\frac{1}{2}\left(\overline{\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\-&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\+&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\+&-\end{smallmatrix}}^4}\right) \\
&+\frac{1}{2}\left(\prod_{I=5,6,7,8} a_I\sqrt{\tau_2}\right)\frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\+&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\+&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\-&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\-&+\end{smallmatrix}}^4\right) \\
&\qquad\times\frac{1}{2}\left(\overline{\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\+&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\+&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\-&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&+\\-&+\end{smallmatrix}}^4}\right) \\
&+\frac{1}{2}\left(\prod_{I=5,6,7,8} a_I\sqrt{\tau_2}\right)\frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\+&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\+&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\-&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\-&-\end{smallmatrix}}^4\right) \\
&\qquad\times\frac{1}{2}\left(\overline{\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\+&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\+&-\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\-&+\end{smallmatrix}}^4-\boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}-&-\\-&-\end{smallmatrix}}^4}\right) \\
&\equiv \sum_{\nu,\bar{\nu}} \mathcal{J}_{\nu,\bar{\nu},\mathrm{IIB},T^4/\mathbf{Z}_2}
\end{aligned}
\tag{11.5}
$$

where $F_2(a_I,\tau)\equiv a_I\sqrt{\tau_2}\sum_{\ell,m\in\mathbf{Z}} q^{\frac{1}{4}(ma_I+\frac{\ell}{a_I})^2}\bar{q}^{\frac{1}{4}(ma_I-\frac{\ell}{a_I})^2}$ and $a_I\equiv\frac{\sqrt{\alpha'}}{R_I}$. The first line represents the contribution from the T^4 compactification without $\mathbf{Z}_2$ insertion, the second, the third and the fourth lines represent the contributions from the untwisted sector with $\mathbf{Z}_2$ insertion, the $\mathbf{Z}_2$ twisted sector and the $\mathbf{Z}_2$ twisted sector with the $\mathbf{Z}_2$ insertion respectively. In each term inside the bracket, the first bin represents the spacetime part and the second bin the internal part. Referring to the character of $c=1$, $\mathbf{Z}_2$ orbifold, we are able to see

$$\boxed{\begin{smallmatrix} r_{\rm b} & s_{\rm b} \\ r_{\rm f} & s_{\rm f} \end{smallmatrix}} = \boxed{\boxed{\begin{smallmatrix} \alpha_{\rm b} & \beta_{\rm b} \\ \alpha_{\rm f} & \beta_{\rm f} \end{smallmatrix}}} = \sqrt{\frac{2\eta}{\vartheta\begin{bmatrix} \frac{1}{2}+\alpha_{\rm b} \\ \frac{1}{2}+\beta_{\rm b} \end{bmatrix}(0)}} \sqrt{\frac{\vartheta\begin{bmatrix} \frac{1}{2}+\alpha_{\rm f} \\ \frac{1}{2}+\beta_{\rm f} \end{bmatrix}(0)}{\eta}} = \sqrt{\frac{2\vartheta\begin{bmatrix} \frac{1}{2}+\alpha_{\rm f} \\ \frac{1}{2}+\beta_{\rm f} \end{bmatrix}(0)}{\vartheta\begin{bmatrix} \frac{1}{2}+\alpha_{\rm b} \\ \frac{1}{2}+\beta_{\rm b} \end{bmatrix}(0)}}$$

$$\text{for } (\alpha_{\rm b}, \beta_{\rm b}) = \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right) . \tag{11.6}$$

Here, the arguments of the theta constants are modulo 1 and the non-integer parts are understood to be taken. Note that, in this notation, we have included the contribution from the $2^4 = 16$ fixed points in the twisted sector in eq. (11.6).

11.3 open superstring on $T^4/\mathbf{Z}_2$

Another prototypical example which we will consider in the next section is the open string sector in the type I superstring on $T^4(= (S^1)^4)/\mathbf{Z}_2$. The partition function is

$$Z_{{\rm I},T^4/\mathbf{Z}_2} = -\frac{V_{\rm E}}{2} \frac{1}{(4\pi^2\alpha')^5} \frac{\mathcal{A}_{{\rm I},T^4/\mathbf{Z}_2} + \mathcal{M}_{{\rm I},T^4/\mathbf{Z}_2}}{2} \tag{11.7}$$

$$\mathcal{A}_{{\rm I},T^4/\mathbf{Z}_2} = \int_0^\infty \frac{{\rm d}\tau_2}{\tau_2} \frac{1}{\tau_2^5} \, \mathcal{J}_{{\rm I},T^4/\mathbf{Z}_2}\Big|_{\tau=\frac{i}{2}\tau_2} \tag{11.8}$$

$$\mathcal{M}_{{\rm I},T^4/\mathbf{Z}_2} = \int_0^\infty \frac{{\rm d}\tau_2}{\tau_2} \frac{1}{\tau_2^5} \, \tilde{\mathcal{J}}_{{\rm I},T^4/\mathbf{Z}_2}\Big|_{\tau=\frac{i}{2}\tau_2+\frac{1}{2}} . \tag{11.9}$$

Among the many possibilities discussed in [25, 42, 41], where the dilaton tadpoles cancel, we will consider the simplest case where the gauge group is $U(n = 16)_{(9)} \times U(d = 16)_{(5)}$ with all of the D5 branes at the same fixed point and the first and the second subscripts indicate D9 and D5 brane respectively[4].

$$
\begin{aligned}
&\mathcal{J}_{\mathrm{I},T^4/\mathbf{Z}_2}\\
&= \frac{1}{2}\mathcal{J}_{\mathrm{I},T^4} + \frac{1}{2}\mathcal{J}^{(\mathbf{Z}_2)}_{\mathrm{I},T^4}\\
&= (2n)^2\frac{1}{2}\left(\prod_{I=5,6,7,8} F_1(a_I,\tau_2)\right)\frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^8 - \boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^8 - \boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^8 - \boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^8\right)\\
&-(2d)^2\frac{1}{2}\left(\prod_{I=5,6,7,8}(a_I\sqrt{\tau_2})\right)\frac{1}{2}\left(\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}^4 - \boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\-&-\end{smallmatrix}}^4 - \boxed{\begin{smallmatrix}+&+\\+&-\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\+&+\end{smallmatrix}}^4 - \boxed{\begin{smallmatrix}+&+\\+&+\end{smallmatrix}}^4\boxed{\begin{smallmatrix}+&-\\+&-\end{smallmatrix}}^4\right)\\
&\equiv \frac{1}{2}\sum_{\nu}\mathcal{J}_\nu + \frac{1}{2}\sum_{\nu}\mathcal{J}^{(\mathbf{Z}_2)}_\nu\,, \qquad (11.10)
\end{aligned}
$$

where $F_1(a_I,\tau_2) = a_I\sqrt{\tau_2}\sum_{p_I}\mathrm{e}^{-tp_Ip^I}$, $\pi\tau_2 \equiv \frac{t}{\alpha'}$. See also [27, 26, 53, 54, 55].

12 One-loop superstring amplitudes with non-maximal supersymmetry

In this section, we apply the genus one super Green function constructed under the general twists in the (σ,τ) directions to superstring amplitudes. For simplicity, we illustrate this by the annulus contribution to the open superstring amplitudes of the compactification in section 11.3, but our procedure is applicable to a large class of toroidal models and their orbifolding of closed and open superstrings including heterotic string [56] compactifications.

12.1 Neumann functions with arguments on the boundary

In order to proceed to the computation, we need the Neumann function for the superannulus under a variety of boundary conditions for a worldsheet

[4] Other aspects of this series of model are discussed in [45]-[52].

boson and a worldsheet fermion specified by $\binom{\nu_{\rm b}}{\nu_{\rm f}}$ and with the arguments set on the same boundary. The Neumann function for the Möbius strip case can be read off from the annulus case by the change of the arguments in the theta functions and will not be discussed explicitly here. For the case of

$\binom{++}{\nu_{\mathrm{f}}}$, which is always needed,

$$
\begin{aligned}
&\mathbf{N}^{\text{superannulus}}_{\substack{++\\ \nu_{\mathrm{f}}}}\left(z_I,\bar z_I;z_J,\bar z_J\middle|\frac{i\tau_2}{2}\right)\\
&\overset{\text{on } z=\tilde z,\theta=\tilde\theta}{=}\frac{1}{2}\,4\mathbf{G}^{\text{supertorus}}_{\substack{++\\ \nu_{\mathrm{f}}}}\left(\frac{z_I}{2},\frac{z_J}{2};\frac{\theta_I}{\sqrt2},\frac{\theta_J}{\sqrt2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{(z,\theta)=(\tilde z,\tilde\theta)\equiv(-\bar z,\pm i\bar\theta)}\\
&=\frac{4}{2}\Bigg[G_{++}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)+\frac{\frac{\theta_I}{\sqrt2}\frac{\theta_J}{\sqrt2}}{4}\mathcal{S}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\\
&\qquad\qquad-\frac{\frac{\bar\theta_I}{\sqrt2}\frac{\bar\theta_J}{\sqrt2}}{4}\overline{\mathcal{S}}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg]\Bigg|_{(z,\theta)=(\tilde z,\tilde\theta)\equiv(-\bar z,\pm i\bar\theta)}\\
&=2\Bigg[G_{++}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde z\equiv-\bar z}+\frac{\frac{\theta_I}{\sqrt2}\frac{\theta_J}{\sqrt2}}{4}\,\mathcal{S}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde z\equiv-\bar z}\\
&\qquad\qquad-\frac{\frac{(\mp i\theta_I)}{\sqrt2}\frac{(\mp i\theta_J)}{\sqrt2}}{4}\,\overline{\mathcal{S}}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde z\equiv-\bar z}\Bigg]\\
&\overset{\text{eqs.}(B.38),(B.40),(10.20)}{=}2\Bigg[\frac{1}{2\pi}\ln\left|\frac{\vartheta\begin{bmatrix}\frac12\\ \frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)}{\vartheta'\begin{bmatrix}\frac12\\ \frac12\end{bmatrix}\left(0\middle|\frac{i\tau_2}{2}\right)}\right|+\frac{(z_I-z_J)^2}{4\tau_2}\\
&\qquad+\left\{\frac{1}{2\pi}2\sum_{n=1}^{\infty}\ln|1-q^n|-\frac{z_I-z_J}{2\cdot 2i}+\frac{1}{2\pi}\ln(2\pi)+2\frac{\tau_2}{2}\cdot\frac{\pi^2}{6}\right\}\\
&\qquad+\frac{\frac{\theta_I}{\sqrt2}\frac{\theta_J}{\sqrt2}}{4}\mathcal{S}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)+\frac{\frac{\theta_I}{\sqrt2}\frac{\theta_J}{\sqrt2}}{4}\mathcal{S}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg]\\
&=\frac{2}{2\pi}\ln\left|\frac{\vartheta\begin{bmatrix}\frac12\\ \frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)}{\vartheta'\begin{bmatrix}\frac12\\ \frac12\end{bmatrix}\left(0\middle|\frac{i\tau_2}{2}\right)}\right|+\frac{2(z_I-z_J)^2}{4\tau_2}+\frac{2\cdot 2}{4}\frac{\theta_I}{\sqrt2}\frac{\theta_J}{\sqrt2}\mathcal{S}_{\nu_{\mathrm{f}}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)\\
&\qquad+2\left\{\frac{1}{2\pi}2\sum_{n=1}^{\infty}\ln|1-q^n|-\frac{z_I-z_J}{2\cdot 2i}+\frac{1}{2\pi}\ln(2\pi)+2\frac{\tau_2}{2}\cdot\frac{\pi^2}{6}\right\}
\end{aligned}
$$

$$
= \frac{1}{\pi}\ln\left|\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(0\,\middle|\,\frac{i\tau_2}{2}\right)}\right| + \frac{(z_I-z_J)^2}{2\tau_2} + \frac{\theta_I}{\sqrt{2}}\frac{\theta_J}{\sqrt{2}}\mathcal{S}_{\nu_{\rm f}}\left(\frac{z_I}{2};\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)
$$

$$
+2\left\{\frac{1}{2\pi}2\sum_{n=1}^{\infty}\ln|1-q^n| - \frac{z_I-z_J}{2\cdot 2i} + \frac{1}{2\pi}\ln(2\pi) + 2\frac{\tau_2}{2}\cdot\frac{\pi^2}{6}\right\}. \tag{12.1}
$$

The last line of eq. (12.1) can be dropped in the calculation of amplitudes as the source $\mathbf{J}$ satisfies $\int \mathrm{d}^2z\mathrm{d}\theta\mathrm{d}\bar{\theta}\mathbf{J}=0$. We need the case $\binom{+-}{\nu_{\rm f}}$ as well:

$$
\mathbf{N}^{\text{superannulus}}_{\substack{+-\\ \nu_{\rm f}}}\left(z_J,\bar{z}_J;z_K,\bar{z}_K\,\middle|\,\frac{i\tau_2}{2}\right)
$$

$$
\overset{\text{on } z=\tilde{z},\theta=\tilde{\theta}}{=} 2\left[G_{+-}\left(\frac{z_J}{2};\frac{z_K}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde{z}\equiv-\bar{z}} + \frac{\frac{\theta_J}{\sqrt{2}}\frac{\theta_K}{\sqrt{2}}}{4}\mathcal{S}_{\nu_{\rm f}}\left(\frac{z_J}{2};\frac{z_K}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde{z}\equiv-\bar{z}}\right.
$$

$$
\left.-\frac{\frac{(\mp i\theta_J)}{\sqrt{2}}\frac{(\mp i\theta_K)}{\sqrt{2}}}{4}\overline{\mathcal{S}}_{\nu_{\rm f}}\left(\frac{z_J}{2};\frac{z_K}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde{z}\equiv-\bar{z}}\right]
$$

$$
= 2\,G_{+-}\left(\frac{z_J}{2};\frac{z_K}{2}\middle|\frac{i\tau_2}{2}\right)\Bigg|_{z=\tilde{z}\equiv-\bar{z}} + \frac{\theta_J}{\sqrt{2}}\frac{\theta_K}{\sqrt{2}}\mathcal{S}_{\nu_{\rm f}}\left(\frac{z_J}{2};\frac{z_K}{2}\middle|\frac{i\tau_2}{2}\right). \tag{12.2}
$$

Note that, for closed string models with $\mathbf{Z}_2$ insertion, $\mathbf{G}^{\text{supertorus}}_{\substack{-+\\ \nu_{\rm f}}}$ and $\mathbf{G}^{\text{supertorus}}_{\substack{--\\ \nu_{\rm f}}}$ are needed in order to evaluate the contributions from the twisted sectors. Likewise, $\mathbf{N}^{\text{superannulus}}_{\substack{-+\\ \nu_{\rm f}}}$, $\mathbf{N}^{\text{superannulus}}_{\substack{--\\ \nu_{\rm f}}}$ are needed in the case of a 5-9 string.

12.2 Koba-Nielsen type formula for genus one superstring amplitudes

Let $\prod_{I=1}^{N} O_I$ be the product of N vertex operators

$$
\zeta_I^{(\mathrm{P})}\cdot\int \mathrm{d}z_I\int \mathrm{d}\theta_I D_I\mathbf{X}(z_I,\theta_I)\mathrm{e}^{ik_I\cdot\mathbf{X}(z_I,\theta_I)}, \qquad I=1,...,N,
$$

for massless vector emission of an open superstring. It can be written as

$$\prod_{I=1}^{N} O_I = \prod_{J=1}^{N} \left(\zeta_J^{(\mathrm{P})} \cdot \int \mathrm{d}\eta_J \int \mathrm{d}z_J \int \mathrm{d}\theta_J \right) \times \exp\left[i \int \mathrm{d}^2 z \mathrm{d}\theta \mathrm{d}\bar{\theta} \mathbf{J}(z,\bar{z},\theta,\bar{\theta}) \cdot \mathbf{X}(z,\bar{z},\theta,\bar{\theta}) \right] , \quad (12.3)$$

$$\mathbf{J}^M(z,\bar{z},\theta,\bar{\theta}) = \sum_{I=1}^{N} (k_I^M - i\eta_I^M D_I) \delta^{(2)}(z - z_I)(\theta - \theta_I)(\bar{\theta} - \bar{\theta}_I) . \quad (12.4)$$

Here we have introduced the grassmann source η_J, $J = 1, 2, 3, ..., N$, for this representation. Following section 9, we carry out the gaussian integration[5] and the sum $\mathcal{S}$ over the boundary conditions.

Let $\mathcal{S} = \mathcal{S}' \oplus \mathcal{S}_{(++)}$, where $\mathcal{S}_{(++)}$ is the part of the sum which contains $(++)$ to some power in ν_{f}. For these parity-violating cases [62], it is well-known that the amplitudes for lower N vanish. Ignoring these cases in this thesis, let us denote the remaining part of the N point amplitude for the case labelled by $\bullet$ by

$$A'^{\bullet}_N = -\frac{1}{2} \frac{(V_{\mathrm{E}}\delta)\, g^N}{(4\pi^2\alpha')^5} \frac{\mathcal{A}'^{\bullet}_N + \mathcal{M}'^{\bullet}_N}{2} . \quad (12.5)$$

Here, we have denoted by $(V_{\mathrm{E}}\delta)$ a product of the momentum conserving delta functions $(2\pi)^{d_{\mathrm{dim}}} \delta^{(d_{\mathrm{dim}})} \left(\sum_I k_I \right)$ (that appear from the integrations of the zero modes of the bosonic coordinates) and the volume of the compactification V_{c}. The annulus and the Möbius strip contributions are denoted by $\mathcal{A}'^{\bullet}_N$ and $\mathcal{M}'^{\bullet}_N$ respectively, and

$$\mathcal{A}'^{\bullet}_N = \int_0^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^6} \sum_{\nu \in \mathcal{S}'} \mathcal{J}^{\bullet}_{\nu} \prod_{J=1}^{N} \left(\zeta_J^{(\mathrm{P})} \cdot \int \mathrm{d}\eta_J \int \mathrm{d}z_J \int \mathrm{d}\theta_J \right) \exp\left[\pi\alpha' \sum_{I,J=1}^{N} (k_I - i\eta_I D_I)^M (k_J - i\eta_J D_J)^L \mathbf{N}'_{ML,\nu} \right] \Bigg|_{\tau = \frac{i}{2}\tau_2} \quad (12.6)$$

[5] See [32]. See also [57, 58, 59, 60, 61] for different approaches of computation.

$$
\mathcal{M}'^{\bullet}_{N} = \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6} \sum_{\nu\in\mathcal{S}'} \tilde{\mathcal{J}}^{\bullet}_{\nu} \prod_{J=1}^{N} \left(\zeta_J^{(\mathrm{P})} \cdot \int \mathrm{d}\eta_J \int \mathrm{d}z_J \int \mathrm{d}\theta_J \right)
$$

$$
\exp\left[\pi\alpha' \sum_{I,J=1}^{N} (k_I - i\eta_I D_I)^M (k_J - i\eta_J D_J)^L \mathbf{N}'_{ML,\nu} \right] \Bigg|_{\tau=\frac{i}{2}\tau_2+\frac{1}{2}} \tag{12.7}
$$

in accordance with eqs. (5.107) and (11.9). We will restrict our attention to the annulus case from now on.

We have denoted by $\sum_{\nu\in\mathcal{S}'} \mathcal{J}^{\bullet}_{\nu}$ the part of the integrand which has appeared in the vacuum amplitude, (for instance, eq. (11.10)) and $\mathbf{N}'_{ML,\nu}$ indicates the superannulus Neumann function specified by the boundary condition $\nu = \binom{\nu_{\mathrm{b}}}{\nu_{\mathrm{f}}}$ which is determined by the spacetime indices M, L. The prime $'$ indicates the omission of the bosonic and fermionic zero modes.

Let us analyze the exponential part of the integrand in eq. (12.6)

$$
\begin{aligned}
\exp[\cdots] = \exp&\left[2\alpha' \sum_{1\le I<J\le N} k_I^M k_J^L \pi N'^{IJ}_{ML,\nu_{\mathrm{b}}} \right] \\
\exp&\left[2\alpha' \sum_{1\le I<J\le N} \left\{ i k_I^M \theta_I k_J^L \theta_J B^{IJ}_{ML,\nu_{\mathrm{f}}} \right.\right. \\
&+ (-\eta_I^M k_J^L \theta_J + \eta_J^L k_I^M \theta_I) B^{IJ}_{ML,\nu_{\mathrm{f}}} + (\eta_I^M \theta_I k_J^L - \eta_J^L \theta_J k_I^M) C^{IJ}_{ML,\nu_{\mathrm{b}}} \\
&- i\eta_I^M \eta_J^L B^{IJ}_{ML,\nu_{\mathrm{f}}} \\
&\left.\left. + \eta_I^M \eta_J^L \theta_I \theta_J E^{IJ}_{ML,\nu_{\mathrm{b}}} \right\} \right] ,
\end{aligned} \tag{12.8}
$$

where the $I = J$ part vanishes by the on-shell condition. Following [32], let us label the first, the second, the third and the fourth line of the exponent by the number of η's and by the number of θ's, namely, $[0,2]$, $[1,1]$, $[2,0]$, $[2,2]$ respectively. Also upon compactification, namely the division of the pair of indices (M, L) into the spacetime part (μ, λ) and the internal part (ℓ, ℓ'), we set the internal part of the momenta $k_I^{\ell} = 0$ for simplicity. Index structure of $\pi N^{IJ}_{ML,++}$, $\pi N^{IJ}_{ML,+-}$, $B^{IJ}_{ML,\nu_{\mathrm{f}}}$, $C^{IJ}_{ML,++}$, $C^{IJ}_{ML,+-}$, $E^{IJ}_{ML,++}$, $E^{IJ}_{ML,+-}$ are of the form $\sum_{\bullet\bullet} g_{\bullet\bullet} X^{IJ}_{\nu(\bullet\bullet)}$ and the expressions are read off from

$$\begin{aligned}
\pi N^{IJ}_{++} &\equiv \ln \left| \frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(0\,\middle|\,\frac{i\tau_2}{2}\right)} \right| + \frac{\pi}{2}\frac{(z_I-z_J)^2}{\tau_2} \\
B^{IJ}_{\nu_{\rm f}} &\equiv \frac{1}{2}\frac{\pi}{i}\mathcal{S}_{\nu_{\rm f}}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right) = \frac{1}{2}\frac{\vartheta_{\nu_{\rm f}}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta_{\nu_{\rm f}}\left(0\,\middle|\,\frac{i\tau}{2}\right)}\frac{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(0\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau}{2}\right)} \\
C^{IJ}_{++} &\equiv \frac{\partial}{\partial z_I}\left[\pi N^{IJ}_{++}\right]\Big|_{z=\tilde{z}\equiv-\bar{z}} = \frac{1}{2}\frac{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)} + \pi\frac{z_I-z_J}{\tau_2} \\
E^{IJ}_{++} &\equiv \frac{\partial}{\partial z_I}\frac{\partial}{\partial z_J}\left[\pi N^{IJ}_{++}\right]\Big|_{z=\tilde{z}\equiv-\bar{z}} \\
&= \frac{1}{4}\left\{\frac{\vartheta''\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)} - \left(\frac{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}\right)^2\right\} + \frac{\pi}{\tau_2}
\end{aligned} \tag{12.9}$$

$$
\begin{aligned}
\pi N_{+-}^{IJ} &\equiv 2\pi\, G_{+-}\left(\frac{z_I}{2};\frac{z_J}{2}\bigg|\frac{i\tau_2}{2}\right)\bigg|_{z=\tilde{z}\equiv-\bar{z}} - \left[2\pi \mathrm{Im}\frac{z_J}{2} - (2\pi)\pi^2\tau_2\right] \\
&= \ln\left|\sqrt{\zeta_I}-\sqrt{\zeta_J}\right| + \sum_{m=1}^{\infty}(-1)^m \ln\left|1-\frac{\sqrt{\zeta_I}(\sqrt{|q|})^m}{\sqrt{\zeta_J}}\right|\left|1-\frac{\sqrt{\zeta_J}(\sqrt{|q|})^m}{\sqrt{\zeta_I}}\right| \\
C_{+-}^{IJ} &\equiv \frac{1}{2}\left(\frac{\partial}{\partial z_I}-\frac{\partial}{\partial z_J}\right)\left[\pi N_{+-}^{IJ}\right]\big|_{z=\tilde{z}\equiv-\bar{z}} \\
&= \frac{\pi i}{2}\frac{\sqrt{\zeta_I}+\sqrt{\zeta_J}}{\sqrt{\zeta_I}-\sqrt{\zeta_J}} + \pi i \sum_{m=1}^{\infty}(-1)^m\left\{\frac{1}{1-\frac{\sqrt{\zeta_J}}{\sqrt{\zeta_I}(\sqrt{|q|})^m}} - \frac{1}{1-\frac{\sqrt{\zeta_I}}{\sqrt{\zeta_J}(\sqrt{|q|})^m}}\right\} \\
E_{+-}^{IJ} &\equiv (-1)\frac{1}{2}\left(\frac{\partial}{\partial z_I}-\frac{\partial}{\partial z_J}\right)\frac{1}{2}\left(\frac{\partial}{\partial z_J}-\frac{\partial}{\partial z_I}\right)\left[\pi N_{+-}^{IJ}\right]\big|_{z=\tilde{z}\equiv-\bar{z}} \\
&= -(\pi i)^2\frac{\sqrt{\zeta_I}\sqrt{\zeta_J}}{\left(\sqrt{\zeta_I}-\sqrt{\zeta_J}\right)^2} \\
&\quad -(\pi i)^2\sum_{m=1}^{\infty}(-1)^m\left\{\frac{\frac{\sqrt{\zeta_J}}{\sqrt{\zeta_I}(\sqrt{|q|})^m}}{\left(1-\frac{\sqrt{\zeta_J}}{\sqrt{\zeta_I}(\sqrt{|q|})^m}\right)^2} + \frac{\frac{\sqrt{\zeta_I}}{\sqrt{\zeta_J}(\sqrt{|q|})^m}}{\left(1-\frac{\sqrt{\zeta_I}}{\sqrt{\zeta_J}(\sqrt{|q|})^m}\right)^2}\right\}.
\end{aligned}
\tag{12.10}
$$

See appendix I for these properties.

12.3 Analysis and evaluation of $N = 1, 2, 3$ cases

We will now analyze a few simplest cases. Let us first obtain a few generic features of the amplitudes from the integral representation. First, in order to obtain a non-vanishing amplitude, all grassmann integrations must be saturated. Also, under the assumption made in the last subsection, $\sum_I k_I^M = 0$ for $M = 0, 1, ..., 9$. Note that, the zero mode is absent in the expansion of X^ℓ, $\ell = 5, 6, 7, 8$, and that momentum conservation is not ensured.

I) $N = 1$; the amplitude vanishes generally and trivially as such case is absent in the integrand.

II) $N=2$; there is no contribution from $[0,2][2,0]$ or from $[1,1]^2$ by $k_I \cdot k_J = k_I \cdot \zeta_J^{(\mathrm{P})} = 0$ for $I, J = 1, 2$. Neither is there any contribution from $[2,2]$ which does not involve ν_{f} by the same reason that the vacuum amplitude vanishes by the Jacobi identity or supersymmetry.

III) $N=3$; this case poses the general question of the presence or absence of the vertex correction. There is no contribution form the parts in which $[0,2]$ is involved as $k_I \cdot k_J = 0$ for $I, J = 1, 2, 3$. The remaining possibilities for a non-vanishing amplitude are $[1,1]^3$ and $[1,1][2,2]$. Among them, the parts which do not involve $B^{IJ}_{\bullet\bullet,\nu_{\mathrm{f}}}$ vanish after the summation over ν_{f} by the Jacobi identity or supersymmetry. We conclude that the possibilities are contained in $[1,1]$ of $B^{12}_{ML,\nu_{\mathrm{f}}} B^{23}_{ML,\nu_{\mathrm{f}}} B^{31}_{ML,\nu_{\mathrm{f}}}$ type and of $(B^{IJ}_{ML,\nu_{\mathrm{f}}})^2 C^{IJ}_{ML,\nu_{\mathrm{b}}}$ type.

12.3.1 case of maximal supersymmetry

In this case, namely, in the case of flat 10d and its toroidal compactifications, it is well-known that the vanishing of these two types after the summation over ν_{f} is established, (see, for example, [21]) by the Riemann identity eq. (B.44). In fact

$$\mathcal{J}_\nu = (2n)^2 \left(\prod_I F_1(a_I, \tau_2) \right) C_\nu \frac{\vartheta_\nu(0)^4}{\eta^{12}} \tag{12.11}$$

according to eqs. (11.3) and (11.10) and

$$
\begin{aligned}
&\frac{\eta^{12}}{(2n)^2\left(\prod_I F_I(a_I,\tau_2)\right)}\sum_{\nu_{\rm f}}\mathcal{J}_{\nu_{\rm f}}(B^{IJ}_{\nu_{\rm f}})^2\\
&=\sum_{\nu_{\rm f}} C_{\nu_{\rm f}}\vartheta_{\nu_{\rm f}}(0)^4(B^{IJ}_{\nu_{\rm f}})^2\\
&=\sum_{\nu_{\rm f}} C_{\nu_{\rm f}}\vartheta_{\nu_{\rm f}}(0)^4\left(\frac{1}{2}\frac{\vartheta_{\nu_{\rm f}}(\frac{z_I}{2}-\frac{z_J}{2})}{\vartheta_{\nu_{\rm f}}(0)}\frac{\vartheta'\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](0)}{\vartheta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](\frac{z_1}{2}-\frac{z_2}{2})}\right)^2\\
&\overset{\text{eq.}(B.44)}{=}\frac{1}{2}\left(\frac{1}{2}\frac{\vartheta'\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](0)}{\vartheta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](\frac{z_1}{2}-\frac{z_2}{2})}\right)^2\\
&\qquad\times 2\vartheta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right]\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\vartheta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right]\left(-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\right)\vartheta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](0)\vartheta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](0)\\
&=0\,,
\end{aligned}
\tag{12.12}
$$

where

$$
\begin{aligned}
x_1&=\frac{1}{2}\left\{0+0+\left(\frac{z_1}{2}-\frac{z_2}{2}\right)+\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\right\}=\frac{z_1}{2}-\frac{z_2}{2}\\
y_1&=\frac{1}{2}\left\{0+0-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\right\}=-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\\
u_1&=\frac{1}{2}\left\{0-0+\left(\frac{z_1}{2}-\frac{z_2}{2}\right)-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\right\}=0\\
v_1&=\frac{1}{2}\left\{0-0-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)+\left(\frac{z_1}{2}-\frac{z_2}{2}\right)\right\}=0\,.
\end{aligned}
\tag{12.13}
$$

and we have used $\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)=0$. Similarly,

$$
\begin{aligned}
&\frac{\eta^{12}}{(2n)^2\left(\prod_I F_I(a_I,\tau_2)\right)}\sum_{\nu_{\rm f}}\mathcal{J}_{\nu_{\rm f}}B^{12}_{\nu_{\rm f}}B^{23}_{\nu_{\rm f}}B^{13}_{\nu_{\rm f}} \\
&=\sum_{\nu_{\rm f}}C_{\nu_{\rm f}}\vartheta_{\nu_{\nu_{\rm f}}}(0)^4B^{12}_{\nu_{\rm f}}B^{23}_{\nu_{\rm f}}B^{13}_{\nu_{\rm f}} \\
&=\sum_{\nu_{\rm f}}C_{\nu_{\rm f}}\vartheta_{\nu_{\rm f}}(0)^4\left(\frac{1}{2}\right)^3\left(\frac{\vartheta_{\nu_{\rm f}}(\frac{z_1}{2}-\frac{z_2}{2})}{\vartheta_{\nu_{\rm f}}(0)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\frac{z_1}{2}-\frac{z_2}{2})}\right) \\
&\qquad\times\left(\frac{\vartheta_{\nu_{\rm f}}(\frac{z_2}{2}-\frac{z_3}{2})}{\vartheta_{\nu_{\rm f}}(0)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\frac{z_2}{2}-\frac{z_3}{2})}\right)\left(\frac{\vartheta_{\nu_{\rm f}}(\frac{z_1}{2}-\frac{z_3}{2})}{\vartheta_{\nu_{\rm f}}(0)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\frac{z_1}{2}-\frac{z_3}{2})}\right) \\
&\overset{\text{eq.}(B.44)}{=}\frac{1}{2}\left(\frac{1}{2}\right)^3\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)^3}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\frac{z_1}{2}-\frac{z_2}{2})\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\frac{z_2}{2}-\frac{z_3}{2})\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\frac{z_1}{2}-\frac{z_3}{2})} \\
&\qquad\times 2\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_1}{2}-\frac{z_3}{2}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_3}{2}-\frac{z_2}{2}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_2}{2}-\frac{z_1}{2}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0) \\
&=0\,, \qquad (12.14)
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= \frac{1}{2}\left\{0+\left(\frac{z_1}{2}-\frac{z_2}{2}\right)+\left(\frac{z_2}{2}-\frac{z_3}{2}\right)+\left(\frac{z_1}{2}-\frac{z_3}{2}\right)\right\}=\left(\frac{z_1}{2}-\frac{z_3}{2}\right) \\
y_1 &= \frac{1}{2}\left\{0+\left(\frac{z_1}{2}-\frac{z_2}{2}\right)-\left(\frac{z_2}{2}-\frac{z_3}{2}\right)-\left(\frac{z_1}{2}-\frac{z_3}{2}\right)\right\}=\left(\frac{z_3}{2}-\frac{z_2}{2}\right) \\
u_1 &= \frac{1}{2}\left\{0-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)+\left(\frac{z_2}{2}-\frac{z_3}{2}\right)-\left(\frac{z_1}{2}-\frac{z_3}{2}\right)\right\}=\left(\frac{z_2}{2}-\frac{z_1}{2}\right) \\
v_1 &= \frac{1}{2}\left\{0-\left(\frac{z_1}{2}-\frac{z_2}{2}\right)-\left(\frac{z_2}{2}-\frac{z_3}{2}\right)+\left(\frac{z_1}{2}-\frac{z_3}{2}\right)\right\}=0\,. \qquad (12.15)
\end{aligned}
$$

12.3.2 case of non-maximal supersymmetry

Finally, let us consider the case of type I superstring on $T^4/\mathbf{Z}_2$. Among the summation over $\nu_{\rm f}$, only the part belonging to $\frac{1}{2}\mathcal{J}^{(\mathbf{Z}_2)}_{\mathrm{I},T^4}$ in eq. (11.10) contributes to the $N = 3$ amplitude. So we will concentrate on this case. The integrations of the exponential factor eq. (12.8) over the grassmann coordinates $\prod_{J=1}^{3} \zeta_J^{(\mathrm{P})} \cdot \int \mathrm{d}\eta_J \int \mathrm{d}\theta_J \exp[\cdots]$ yield

$$(2\alpha')^3 k_3 \cdot \zeta_1^{(\mathrm{P})} k_1 \cdot \zeta_2^{(\mathrm{P})} k_2 \cdot \zeta_3^{(\mathrm{P})} \left\{ -2B^{12}_{\nu_\mathrm{f}^6} B^{23}_{\nu_\mathrm{f}^6} B^{31}_{\nu_\mathrm{f}^6} + \left(B^{12}_{\nu_\mathrm{f}^6}\right)^2 \left(C^{13}_{++} - C^{23}_{++}\right) \right. \\ \left. + \left(B^{23}_{\nu_\mathrm{f}^6}\right)^2 \left(C^{21}_{++} - C^{31}_{++}\right) + \left(B^{31}_{\nu_\mathrm{f}^6}\right)^2 \left(C^{32}_{++} - C^{12}_{++}\right) \right\} . \tag{12.16}$$

Coming back to eq. (12.6), we obtain the expression for $\mathcal{A}_3'^{\,T^4/\mathbf{Z}_2}$:

$$\begin{aligned} \mathcal{A}_3'^{\,T^4/\mathbf{Z}_2} = & (2n)\mathrm{Tr}\left(T^1 T^2 T^3\right) (2\alpha')^3 (k_3 \cdot \zeta_1^{(\mathrm{P})})(k_1 \cdot \zeta_2^{(\mathrm{P})})(k_2 \cdot \zeta_3^{(\mathrm{P})}) \\ & \times \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2} \frac{1}{\tau_2^5} \left(\prod_{I=1,2,3} \int \mathrm{d}z_I \right) \sum_{\nu \in \mathcal{S}'} \frac{1}{2} \mathcal{J}_\nu'^{(\mathbf{Z}_2)} \\ & \times \left\{ -2B^{12}_{\nu_\mathrm{f}^6} B^{23}_{\nu_\mathrm{f}^6} B^{31}_{\nu_\mathrm{f}^6} + \left(B^{12}_{\nu_\mathrm{f}^6}\right)^2 \left(C^{13}_{++} - C^{23}_{++}\right) \right. \\ & \left. \qquad + \left(B^{23}_{\nu_\mathrm{f}^6}\right)^2 \left(C^{21}_{++} - C^{31}_{++}\right) + \left(B^{31}_{\nu_\mathrm{f}^6}\right)^2 \left(C^{32}_{++} - C^{12}_{++}\right) \right\} , \end{aligned} \tag{12.17}$$

where $\mathcal{J}_\nu'^{(\mathbf{Z}_2)}$ has been introduced in eq. (11.10) and ν_f^6 refers to the spacetime part (bin) of ν_f. Let us recall from eqs. (11.3), (11.6)

$$\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^2 = \frac{\vartheta\left[{0 \atop 0}\right](0)}{\eta^3} , \qquad \boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}^2 = \frac{2\vartheta\left[{0 \atop \frac{1}{2}}\right](0)}{\vartheta\left[{\frac{1}{2} \atop 0}\right](0)} ,$$

$$\boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^2 = \frac{\vartheta\left[{0 \atop \frac{1}{2}}\right](0)}{\eta^3} , \qquad \boxed{\begin{smallmatrix}+&-\\-&-\end{smallmatrix}}^2 = \frac{2\vartheta\left[{0 \atop 0}\right](0)}{\vartheta\left[{\frac{1}{2} \atop 0}\right](0)} , \quad \text{and therefore} \tag{12.18}$$

$$\boxed{\begin{smallmatrix}+&+\\-&-\end{smallmatrix}}^4 \boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}^4 = \boxed{\begin{smallmatrix}+&+\\-&+\end{smallmatrix}}^4 \boxed{\begin{smallmatrix}+&-\\-&-\end{smallmatrix}}^4 = \frac{4}{\eta^6} \frac{\vartheta\left[{0 \atop 0}\right](0)^2 \vartheta\left[{0 \atop \frac{1}{2}}\right](0)^2}{\vartheta\left[{\frac{1}{2} \atop 0}\right](0)^2} \tag{12.19}$$

as well as from eq. (12.9)

$$
\begin{aligned}
B^{IJ}_{--} &= \frac{1}{2}\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(0\,\middle|\,\frac{i\tau}{2}\right)}\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}\left(0\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau}{2}\right)}\\
B^{IJ}_{-+} &= \frac{1}{2}\frac{\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}\left(0\,\middle|\,\frac{i\tau}{2}\right)}\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}\left(0\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau}{2}\right)}\\
C^{IJ}_{++} &= \frac{1}{2}\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)}+\pi\frac{z_I-z_J}{\tau_2}\,. \qquad (12.20)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\mathcal{A}'^{\,T^4/\mathbf{Z}_2}_3\\
&=(2n)\mathrm{Tr}\left(T^1T^2T^3\right)(2\alpha')^3(k_3\cdot\zeta_1^{(\mathrm{P})})(k_1\cdot\zeta_2^{(\mathrm{P})})(k_2\cdot\zeta_3^{(\mathrm{P})})\\
&\times\int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2}\frac{1}{\tau_2^5}\left(\prod_{I=1,2,3}\int\mathrm{d}z_I\right)\left(\frac12\right)^3\left(\prod_{I=5,6,7,8}(a_I\sqrt{\tau_2})\right)\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^2(0)\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}^2(0)}{\vartheta\begin{bmatrix}\frac12\\0\end{bmatrix}^2(0)\eta^6}\\
&\Bigg[\frac{(-2)\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}^3(0)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(1-2)\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(2-3)\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(3-1)}\\
&\cdot\left\{\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(1-2)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(2-3)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(3-1)}{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(0)}-\frac{\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}(1-2)\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}(2-3)\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}(3-1)}{\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}^3(0)}\right\}\\
&+\left\{\left(\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(1-2)}{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0)}\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(1-2)}\right)^2-\left(\frac{\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}(1-2)}{\vartheta\begin{bmatrix}0\\\frac12\end{bmatrix}(0)}\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(1-2)}\right)^2\right\}\\
&\qquad\times\left(\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(1-3)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(1-3)}-\frac{\vartheta'\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(2-3)}{\vartheta\begin{bmatrix}\frac12\\\frac12\end{bmatrix}(2-3)}+2\pi(z_1-z_2)\right)\\
&+\{1\to2,2\to3,3\to1\}\,\{1\to2,2\to3,3\to1\}\ \text{in the second term}\\
&+\{1\to2,2\to3,3\to1\}\,\{1\to2,2\to3,3\to1\}\ \text{in the second term}\Big]\Big|_{\tau=\frac{i\tau_2}{2}}\ , \qquad (12.21)
\end{aligned}
$$

where we have introduced shorthand notation $\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(I{-}J)\equiv\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\right)$. The second, third and fourth terms in eq. (12.21) can be further converted by using eq. (B.45):

$$
\begin{aligned}
&\sum_{\nu\in\mathcal{S}'}\frac{1}{2}\mathcal{J}'^{(\mathbf{Z}_2)}_{\nu,\mathcal{A}}(B^{IJ}_{\nu^6_{\rm f}})^2\\
&=\frac{1}{2}\left(\prod_{I=5,6,7,8}(a_I\sqrt{\tau_2})\right)\frac{1}{2}\frac{4}{\eta^6}\left(\frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^2(0)\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}^2(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}^2(0)}\right)\left\{(B^{IJ}_{--})^2-(B^{IJ}_{-+})^2\right\}\\
&=\frac{1}{2}\left(\prod_{I=5,6,7,8}(a_I\sqrt{\tau_2})\right)\frac{1}{2}\frac{4}{\eta^6}\left(\frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^2(0)\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}^2(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}^2(0)}\right)\\
&\times\left\{\frac{1}{2^2}\frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^2\left(I-J\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^2\left(0\,\middle|\,\frac{i\tau}{2}\right)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^2\left(0\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^2\left(I-J\,\middle|\,\frac{i\tau}{2}\right)}-\frac{1}{2^2}\frac{\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}^2\left(I-J\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}^2\left(0\,\middle|\,\frac{i\tau}{2}\right)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^2\left(0\,\middle|\,\frac{i\tau}{2}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^2\left(I-J\,\middle|\,\frac{i\tau}{2}\right)}\right\}\\
&\overset{\text{eq.}(B.45)}{=}-\left(\prod_{I=5,6,7,8}(a_I\sqrt{\tau_2})\right)\frac{1}{\eta^6}\frac{1}{2^2}\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^2\left(0\,\middle|\,\frac{i\tau}{2}\right)\,,
\end{aligned}
\tag{12.22}
$$

where in eq. (B.45),

$$
\begin{aligned}
x_1&=\frac{1}{2}\left\{\left(\frac{z_I}{2}-\frac{z_J}{2}\right)+\left(\frac{z_I}{2}-\frac{z_J}{2}\right)+0+0\right\}=\frac{z_I}{2}-\frac{z_J}{2}\\
y_1&=\frac{1}{2}\left\{\left(\frac{z_I}{2}-\frac{z_J}{2}\right)+\left(\frac{z_I}{2}-\frac{z_J}{2}\right)-0-0\right\}=\frac{z_I}{2}-\frac{z_J}{2}\\
u_1&=\frac{1}{2}\left\{\left(\frac{z_I}{2}-\frac{z_J}{2}\right)-\left(\frac{z_I}{2}-\frac{z_J}{2}\right)+0-0\right\}=0\\
v_1&=\frac{1}{2}\left\{\left(\frac{z_I}{2}-\frac{z_J}{2}\right)-\left(\frac{z_I}{2}-\frac{z_J}{2}\right)-0+0\right\}=0\,.
\end{aligned}
\tag{12.23}
$$

Finally, we obtain

$$
\begin{aligned}
&\mathcal{A}_3'^{\,T^4/\mathbf{Z}_2} \\
&= (2n)\mathrm{Tr}\left(T^1T^2T^3\right)(2\alpha')^3(k_3\cdot\zeta_1^{(\mathrm{P})})(k_1\cdot\zeta_2^{(\mathrm{P})})(k_2\cdot\zeta_3^{(\mathrm{P})}) \\
&\times\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2}\frac{1}{\tau_2^5}\left(\prod_{I=1,2,3}\int \mathrm{d}z_I\right)\left(\frac{1}{2}\right)^3\left(\prod_{I=5,6,7,8}(a_I\sqrt{\tau_2})\right)\frac{1}{\eta^6} \\
&\left[\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^2(0)\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}^2(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}^2(0)}\frac{(-2)\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^3(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(1-2)\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(2-3)\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(3-1)}\right. \\
&\cdot\left\{\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(1-2)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(2-3)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(3-1)}{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(0)}-\frac{\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(1-2)\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(2-3)\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(3-1)}{\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}^3(0)}\right\} \\
&\left.\left.+2\left(\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(1-2)}{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(1-2)}+\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(2-3)}{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(2-3)}+\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(3-1)}{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(3-1)}\right)\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}^2(0)\right]\right|_{\tau=\frac{i\tau_2}{2}}.
\end{aligned}
\tag{12.24}
$$

Unlike the case of maximal supersymmetry, after nationalizing, each term consists of the product of different ϑ functions and we do not find the use of the Riemann identity.

Conclusion

In this thesis, we have mainly provided following two things.

1. we found super-Green function with (α, β) twisted boundary condition.
2. we obtained finite value for the three point one-loop superstring amplitude with non-maximal supersymmetry.

Now we can calculate any amplitude with twisted boundary condition due to our first work. Here we have found the super-Green faction consists of the bosonic part which can be written as the infinite series of hypergeometric function and the fermionic part which is formed by the Jacobi theta functions by using Ramanujan's summation formula in analytic number theory. The second result insists that the amplitudes with non-maximal supersymmetry need not satisfy the non-renormalization theorem. Taking $T^4/\mathbf{Z}_2$ orbifold compactification, we have considered the toy model for Calabi-Yau or $K3$ compactifications which can be solved exactly.

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A Notations

A.1 Indices

$$
\begin{aligned}
M, N, ... = 0, 1, ..., D-1 &: \text{ spacetime vector} \\
M_{\rm lc}, N_{\rm lc}, ... = +, -, 1, ..., D-2 &: \text{ spacetime vector (light-cone)} \\
\alpha, \beta, ... = 1, 2 &: \text{ 2d worldsheet spinor} \\
m, n, ... = 1, 2 &: \text{ 2d worldsheet einstein} \\
a, b, ... = 1, 2 &: \text{ 2d worldsheet local Lorentz} \,. \qquad \text{(A.1)}
\end{aligned}
$$

A.2 Fields

$$
\begin{aligned}
X^M &: \text{ bosonic cordinate} \\
\psi_\alpha{}^M &: \text{ (worldsheet) fermionic cordinate} \\
e_a{}^m &: \text{ zwei bein} \\
\chi_\alpha{}^m &: \text{ Rarita} - \text{Schwinger field} \,. \qquad \text{(A.2)}
\end{aligned}
$$

A.3 Metric

$$
\eta_{MN} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} , \qquad \eta_{M_{\rm lc}N_{\rm lc}} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \mathbf{1} \end{pmatrix} . \qquad \text{(A.3)}
$$

A.4 Light-cone coordinates

$$
X^+ = \frac{X^0 + X^{D-1}}{\sqrt{2}} , \qquad X^- = \frac{X^0 - X^{D-1}}{\sqrt{2}} . \qquad \text{(A.4)}
$$

A.5 Superfield

We introduce a real superfield by

$$\mathbf{X}^M(z,\bar{z},\theta,\bar{\theta}) = X^M(z,\bar{z}) + \sqrt{\frac{1}{2}}\theta\psi^M(z,\bar{z}) + \sqrt{\frac{1}{2}}\bar{\psi}^M(z,\bar{z})\bar{\theta} + \frac{1}{2}\theta\bar{\theta}F^M(z,\bar{z}) \tag{A.5}$$

where θ and $\bar{\theta}$ are Grassmann numbers, $X^M(z,\bar{z})$ and $\psi^M(z,\bar{z})$ are bosonic and fermionic fields, and $F^M(z,\bar{z})$ is a auxiliary field. The super-derivatives are defined by

$$D = -\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial z}\,, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} - i\bar{\theta}\frac{\partial}{\partial\bar{z}}\,. \tag{A.6}$$

A.6 Normalization in eq. (10.1)

Let us determine the normalization N in

$$\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2) = N\mathrm{e}^{2\pi i(n_1+\alpha)\sigma^1}\mathrm{e}^{2\pi i(n_2+\beta)\sigma^2}\,. \tag{A.7}$$

The inner product with functions f, g is defined

$$(f,g) = \int_0^1 \mathrm{d}\sigma^1 \int_0^1 \mathrm{d}\sigma^2 \sqrt{\hat{g}} f^* g\,. \tag{A.8}$$

On a torus geometry

$$\hat{g}_{mn}(\tau) = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2+\tau_2^2 \end{bmatrix} \qquad \text{and} \qquad \sqrt{\hat{g}} = \sqrt{\det\,\hat{g}_{mn}} = \tau_2\,, \tag{A.9}$$

the orthonormality of $\Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)$ implies

$$\delta_{m_1,n_1}\delta_{m_2,n_2} = \left(\Phi_{m_1,m_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2), \Phi_{n_1,n_2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sigma^1,\sigma^2)\right) = |N|^2\tau_2\delta_{m_1,n_1}\delta_{m_2,n_2}. \tag{A.10}$$

Hence we take

$$N = \frac{1}{\sqrt{\tau_2}}\,. \tag{A.11}$$

B Some formulae

B.1 Gamma function

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} \mathrm{d}t \qquad [\mathrm{Re}x > 0] \ . \tag{B.1}$$

For example,

$$\Gamma(1) = 1 \ , \qquad \Gamma(2) = 1 \ , \qquad \Gamma(3) = 2 \ , \qquad \Gamma(4) = 3! = 6 \ . \tag{B.2}$$

B.2 Zeta function

B.2.1 definition

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} \ . \tag{B.3}$$

For example,

$$\zeta(0) = -\frac{1}{2} \ , \qquad \zeta(2) = \frac{\pi^2}{6} \ , \qquad \zeta(4) = \frac{\pi^4}{90} \ . \tag{B.4}$$

B.2.2 generalized zeta function

Generalized zeta function is defined by

$$\zeta(z, a) = \sum_{n=0}^\infty \frac{1}{(a+n)^z} \qquad [a: \ \text{const.}, \ \mathrm{Re}z > 1] \ . \tag{B.5}$$

This function satisfies

$$\zeta(z, 1) = \zeta(z) \ , \qquad \zeta\left(z, \frac{1}{2}\right) = (2^z - 1)\zeta(z) \ . \tag{B.6}$$

B.3 Gauss hypergeometric function

B.3.1 definition

$$
\begin{aligned}
F(\alpha,\beta,\gamma;z) = {}_2F_1(\alpha,\beta;\gamma;z) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\mathrm{d}t\,,
\end{aligned}
\tag{B.7}
$$

where

$$
(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,, \qquad (\alpha)_0 = 1\,. \tag{B.8}
$$

In order to obtain the second line in eq. (B.7), we must have

- $\mathrm{Re}\gamma > \mathrm{Re}\beta > 0$,
- z can not be the real number which is greater than 1,
- $(1-tz)^{-\alpha}$ takes the branch which goes to 1 as $t \to 0$.

B.3.2 specific cases

When we set $\alpha = 1$, $\beta = a$ (> 0), $\gamma = 1 + a$ $(\because\ \mathrm{Re}\gamma > \mathrm{Re}\beta > 0)$ and $z = x$, we obtain

$$
F(1,a,1+a;x) \overset{(B.2)}{=} \frac{\Gamma(1+a)}{\Gamma(a)}\int_0^1 \frac{t^{a-1}}{1-tx}\mathrm{d}t\,. \tag{B.9}
$$

Other cases are, for example,

$$
\log(1-z) = -zF(1,1,2;z)\,, \tag{B.10}
$$

$$
F(1,2,3;z) = -\frac{2}{z^2}\left\{z + \ln(1-z)\right\}\,. \tag{B.11}
$$

B.4 Computation of the infinite sum I

When $b = 1$, $k = 0$, the formula

$$\sum_{n=0}^{\infty} \frac{x^n}{(a+nb)(a+nb+1)\cdots(a+nb+k)} = \frac{1}{k!}\int_0^1 \frac{t^{a-1}(1-t)^k}{1-xt^b}\mathrm{d}t \ , \quad \text{(B.12)}$$

$$[a, b > 0,\ |x| < 1] \ ,$$

is converted into

$$\sum_{n=0}^{\infty} \frac{x^n}{n+a} = \int_0^1 \frac{t^{a-1}}{1-tx}\mathrm{d}t \overset{(B.9)}{=} \frac{\Gamma(a)}{\Gamma(1+a)} F(1, a, 1+a; x) \ . \quad \text{(B.13)}$$

Using this, with $a > 0$, $m \in \{0, \mathbf{N}\}$ and $|xy^m| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n+a}\frac{x^{n+a}}{1-Cy^{n+a}} &= \sum_{n=0}^{\infty} \frac{1}{n+a} x^{n+a} \sum_{m=0}^{\infty} \left(Cy^{n+a}\right)^m \\
&= \sum_{m=0}^{\infty} x^a (Cy^a)^m \sum_{n=0}^{\infty} \frac{(xy^m)^n}{n+a} \\
&\overset{(B.13)}{=} \sum_{m=0}^{\infty} x^a (Cy^a)^m \frac{\Gamma(a)}{\Gamma(1+a)} F(1, a, 1+a; xy^m) \\
&= \frac{x^a \Gamma(a)}{\Gamma(1+a)} \sum_{m=0}^{\infty} (Cy^a)^m F(1, a, 1+a; xy^m) \ .
\end{aligned}$$

(B.14)

Similarly, with $\tilde{a} \equiv 1 + a$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n+a} \frac{x^{n+a}}{1 - Cy^{n+a}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n+a} \frac{x^{n+a}}{1 - Cy^{n+a}} + \frac{1}{a} \frac{x^a}{1 - Cy^a} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)+(1+a)} \frac{x^{(n-1)+(1+a)}}{1 - Cy^{(n-1)+(1+a)}} + \frac{1}{a} \frac{x^a}{1 - Cy^a} \\
&\overset{m \equiv n-1,\, \tilde{a} \equiv 1+a}{=} \sum_{m=0}^{\infty} \frac{1}{m+\tilde{a}} \frac{x^{m+\tilde{a}}}{1 - Cy^{m+\tilde{a}}} + \frac{1}{a} \frac{x^a}{1 - Cy^a} \\
&\overset{(B.14)}{=} \frac{x^{\tilde{a}} \Gamma(\tilde{a})}{\Gamma(1+\tilde{a})} \sum_{m=0}^{\infty} (Cy^{\tilde{a}})^m F(1, \tilde{a}, 1+\tilde{a}; xy^m) + \frac{1}{a} \frac{x^a}{1 - Cy^a}
\end{aligned} \tag{B.15}$$

and, with $\tilde{\tilde{a}} \equiv 2 - a$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n-a} \frac{x^{n-a}}{1 - Cy^{n-a}} \\
&= \sum_{n=2}^{\infty} \frac{1}{n-a} \frac{x^{n-a}}{1 - Cy^{n-a}} + \frac{1}{1-a} \frac{x^{1-a}}{1 - Cy^{1-a}} \\
&= \sum_{n=2}^{\infty} \frac{1}{(n-2)+(2-a)} \frac{x^{(n-2)+(2-a)}}{1 - Cy^{(n-2)+(2-a)}} + \frac{1}{1-a} \frac{x^{1-a}}{1 - Cy^{1-a}} \\
&\overset{m \equiv n-2,\, \tilde{\tilde{a}} \equiv 2-a}{=} \sum_{n=2}^{\infty} \frac{1}{m+\tilde{\tilde{a}}} \frac{x^{m+\tilde{\tilde{a}}}}{1 - Cy^{m+\tilde{\tilde{a}}}} + \frac{1}{1-a} \frac{x^{1-a}}{1 - Cy^{1-a}} \\
&\overset{(B.14)}{=} \frac{x^{\tilde{\tilde{a}}} \Gamma(\tilde{\tilde{a}})}{\Gamma(1+\tilde{\tilde{a}})} \sum_{m=0}^{\infty} (Cy^{\tilde{\tilde{a}}})^m F(1, \tilde{\tilde{a}}, 1+\tilde{\tilde{a}}; xy^m) + \frac{1}{1-a} \frac{x^{1-a}}{1 - Cy^{1-a}} \; .
\end{aligned} \tag{B.16}$$

B.5 q-Pochhammer symbol

q-Pochhammer symbol is defined by

$$(a;q)_\infty \equiv \prod_{k=0}^{\infty}(1-aq^k)\,, \qquad (a;q)_n \equiv \frac{(a;q)_\infty}{(aq^n;q)_\infty}\,, \tag{B.17}$$

where $a, q \in \mathbf{C}$, $|q| < 1$ and $n \in \mathbf{Z}$. When $a = 0$, the former is

$$(0;q)_\infty = \prod_{k=0}^{\infty} 1 = 1\,. \tag{B.18}$$

The latter can be explicitly written as

$$(a;q)_n = \begin{cases} \prod_{k=0}^{n-1}(1-aq^k) & n>0 \\ 1 & n=0 \\ \prod_{k=n}^{-1}\frac{1}{(1-aq^k)} & n<0 \end{cases}\,. \tag{B.19}$$

This satisfies

$$(aq^{-n+1};q)_n = (-a)^n q^{-\frac{n(n-1)}{2}}\left(\frac{1}{a};q\right)_n\,. \tag{B.20}$$

B.6 Ramanujan's ${}_1\psi_1$ summation formula

B.6.1 definition

The Ramanujan's summation formula [63, 64] is

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(az;q)_\infty(q;q)_\infty(\frac{q}{az};q)_\infty(\frac{b}{a};q)_\infty}{(z;q)_\infty(b;q)_\infty(\frac{b}{az};q)_\infty(\frac{q}{a};q)_\infty}\,, \tag{B.21}$$

with $|\frac{b}{a}| < |z| < 1$, $|q| < 1$.

B.6.2 specific case

From eq. (B.21), one can derive

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{(az;q)_\infty (\frac{q}{az};q)_\infty (q;q)_\infty^2}{(a;q)_\infty (z;q)_\infty (\frac{q}{z};q)_\infty (\frac{q}{a};q)_\infty} \tag{B.22}$$

with $|q| < |z| < 1$.

Proof: substituting $b = aq$ into eq. (B.21), the left hand side is

$$\begin{aligned} \frac{(a;q)_n}{(b;q)_n} \overset{b=aq}{=} \frac{(a;q)_n}{(aq;q)_n} &= \frac{\prod_{k=0}^{n-1}(1-aq^k)}{\prod_{k=0}^{n-1}(1-aq^{k+1})} \\ &= \frac{(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-2})(1-aq^{n-1})}{(1-aq)(1-aq^2)\cdots(1-aq^{n-2})(1-aq^{n-1})(1-aq^n)} \\ &= \frac{1-a}{1-aq^n} \,. \end{aligned} \tag{B.23}$$

Therefore

$$\frac{1}{1-a}\frac{(a;q)_n}{(aq;q)_n} = \frac{1}{1-aq^n} \,. \tag{B.24}$$

The right hand side is [6]

$$\frac{1}{1-a}\frac{(az;q)_\infty (q;q)_\infty (\frac{q}{az};q)_\infty (\frac{b}{a};q)_\infty}{(z;q)_\infty (b;q)_\infty (\frac{b}{az};q)_\infty (\frac{q}{a};q)_\infty} \overset{b=aq}{=} \frac{(az;q)_\infty (\frac{q}{az};q)_\infty (q;q)_\infty^2}{(z;q)_\infty (a;q)_\infty (\frac{q}{z};q)_\infty (\frac{q}{a};q)_\infty} \,. \tag{B.25}$$

Therefore, substrituting $b = aq$ and dividing eq. (B.21) by $1-a$ on the both sides, we obtain eq. (B.22). □

[6] Note that $(1-a)(aq;q)_\infty = (1-a)\prod_{k=0}^{\infty}(1-aq^{k+1}) = (1-a)\times(1-aq)(1-aq^2)\cdots = \prod_{k=0}^{\infty}(1-aq^k) = (a;q)_\infty$.

B.6.3 Jacobi triple product

The Jacobi triple product is written as

$$\sum_{n=-\infty}^{\infty}(\mathrm{e}^{\pi i\tau})^{n^2}(\mathrm{e}^{2\pi iz})^n$$
$$=\prod_{m=1}^{\infty}(1-(\mathrm{e}^{\pi i\tau})^{2m})\prod_{m=0}^{\infty}(1+(\mathrm{e}^{\pi i\tau})^{(2m+1)}(\mathrm{e}^{2\pi iz}))(1+(\mathrm{e}^{\pi i\tau})^{(2m+1)}(\mathrm{e}^{2\pi iz})^{-1})\,, \tag{B.26}$$

where $z,\tau\in\mathbf{C}$ and $\mathrm{Im}\tau>0$.

Proof: When $b=0$, $q=q'^2$ and $z=-\frac{q'z'}{a}$, the left hand side of eq. (B.21) can be written as

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(a;q'^2)_n\left(-\frac{q'z'}{a}\right)^n\\
&=\sum_{n=-\infty}^{\infty}\left(\left(aq'^{2n-2}\right)(q'^2)^{-n+1};q'^2\right)_n\left(-\frac{q'z'}{a}\right)^n\\
&\overset{(B.20)}{=}\sum_{n=-\infty}^{\infty}\left(-aq'^{2n-2}\right)^n(q'^2)^{-\frac{n(n-1)}{2}}\left(\frac{1}{aq'^{2n-2}};q'^2\right)_n\left(-\frac{q'z'}{a}\right)^n\\
&=\sum_{n=-\infty}^{\infty}q'^{n^2}z'^n\left(\frac{1}{aq'^{2n-2}};q'^2\right)_n
\end{aligned}\tag{B.27}$$

and the right hand side

$$\frac{\left(a\frac{(-q'z')}{a};q'^2\right)_\infty(q'^2;q'^2)_\infty\left(\frac{q'^2}{a\frac{(-q'z')}{a}};q'^2\right)_\infty}{\left(-\frac{q'z'}{a};q'^2\right)_\infty\left(\frac{q'^2}{a};q'^2\right)_\infty}=\frac{(-q'z';q'^2)_\infty(q'^2;q'^2)_\infty\left(-\frac{q'}{z'};q'^2\right)_\infty}{\left(-\frac{q'z'}{a};q'^2\right)_\infty\left(\frac{q'^2}{a};q'^2\right)_\infty}. \tag{B.28}$$

Then, taking $a \to \infty$ on the both sides, [7] we obtain

$$\sum_{n=-\infty}^{\infty} q'^{n^2} z'^n = \prod_{m=1}^{\infty}(1-q'^{2m}) \prod_{m=0}^{\infty}(1+q'^{2m+1}z')(1+q'^{2m+1}z'^{-1}) \,. \tag{B.29}$$

Setting $q' = \mathrm{e}^{\pi i \tau}$ and $z' = \mathrm{e}^{2\pi i z}$, we can see

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi i n^2 \tau} \mathrm{e}^{2\pi i n z} \\
&= \prod_{m=1}^{\infty}(1-\mathrm{e}^{2m\pi i\tau}) \prod_{m=0}^{\infty}(1+\mathrm{e}^{(2m+1)\pi i\tau}\mathrm{e}^{2\pi i z})(1+\mathrm{e}^{(2m+1)\pi i\tau}\mathrm{e}^{-2\pi i z}) \,. \quad \square
\end{aligned} \tag{B.30}$$

B.7 Dedekind eta function

The Dedekind eta function is defined by

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \,. \tag{B.31}$$

B.8 Jacobi theta function

B.8.1 definition

We define the Jacobi theta function as

$$\begin{aligned}
\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) &= \sum_{n \in \mathbf{Z}} \mathrm{e}^{\pi i (n+\alpha)^2 \tau} \mathrm{e}^{2\pi i (n+\alpha)(z+\beta)} \\
&= \mathrm{e}^{2\pi i \alpha (z+\beta)} \mathrm{e}^{\pi i \alpha^2 \tau} \prod_{n=1}^{\infty} (1-\mathrm{e}^{2\pi i n \tau})(1+\mathrm{e}^{2\pi i (n+\alpha-\frac{1}{2})\tau}\mathrm{e}^{2\pi i (z+\beta)}) \\
&\qquad \times (1+\mathrm{e}^{2\pi i (n-\alpha-\frac{1}{2})\tau}\mathrm{e}^{-2\pi i (z+\beta)}) \,,
\end{aligned} \tag{B.32}$$

[7] $\frac{(-q'z';q'^2)_\infty (q'^2;q'^2)_\infty \left(-\frac{q'}{z'};q'^2\right)_\infty}{\left(-\frac{q'z'}{a};q'^2\right)_\infty (\frac{q'^2}{a};q'^2)_\infty} \overset{a\to\infty}{\to} (q'^2;q'^2)_\infty (-q'z';q'^2)_\infty \left(-\frac{q'}{z'};q'^2\right)_\infty = \prod_{m=0}^{\infty}(1-q'^{2m+2}) \prod_{m=0}^{\infty}(1+q'^{2m+1}z')(1+q'^{2m+1}z'^{-1}) = \prod_{m=1}^{\infty}(1-q'^{2m}) \prod_{m=0}^{\infty}(1+q'^{2m+1}z')(1+q'^{2m+1}z'^{-1})$.

where we have used eq. (B.26) in the second line. We also use the following notation:

$$\vartheta_{++} \equiv \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} , \qquad \vartheta_{+-} \equiv \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} , \qquad \vartheta_{--} \equiv \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \qquad \vartheta_{-+} \equiv \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} , \tag{B.33}$$

where

$$\begin{aligned} \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z|\tau) &= i e^{\pi i z} e^{\frac{\pi i \tau}{4}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 - e^{2\pi i n \tau} e^{2\pi i z})(1 - e^{2\pi i (n-1)\tau} e^{-2\pi i z}) \\ &= -2 e^{\frac{\pi i \tau}{4}} \sin \pi z \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 - e^{2\pi i n \tau} e^{2\pi i z})(1 - e^{2\pi i n \tau} e^{-2\pi i z}) , \end{aligned} \tag{B.34}$$

$$\begin{aligned} \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (z|\tau) &= e^{\pi i z} e^{\frac{\pi i \tau}{4}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 + e^{2\pi i n \tau} e^{2\pi i z})(1 + e^{2\pi i (n-1)\tau} e^{-2\pi i z}) \\ &= 2 e^{\frac{\pi i \tau}{4}} \cos \pi z \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 + e^{2\pi i n \tau} e^{2\pi i z})(1 + e^{2\pi i n \tau} e^{-2\pi i z}) , \end{aligned} \tag{B.35}$$

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z|\tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 + e^{2\pi i (n-\frac{1}{2})\tau} e^{2\pi i z})(1 + e^{2\pi i (n-\frac{1}{2})\tau} e^{-2\pi i z}), \tag{B.36}$$

$$\vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (z|\tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 - e^{2\pi i (n-\frac{1}{2})\tau} e^{2\pi i z})(1 - e^{2\pi i (n-\frac{1}{2})\tau} e^{-2\pi i z}). \tag{B.37}$$

B.8.2 properties

This function has following properties:

$$\vartheta \begin{bmatrix} \alpha+1 \\ \beta \end{bmatrix} (z|\tau) = \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) , \qquad \vartheta \begin{bmatrix} \alpha \\ \beta+1 \end{bmatrix} (z|\tau) = e^{2\pi i \alpha} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) , \tag{B.38}$$

$$\begin{aligned} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z+1|\tau) &= e^{2\pi i \alpha} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) , \\ \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z+\tau|\tau) &= e^{-2\pi i \beta} (q^{-\frac{1}{2}} e^{-2\pi i z}) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) , \end{aligned} \tag{B.39}$$

$$\overline{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau)} = \vartheta\begin{bmatrix}\alpha\\ -\beta\end{bmatrix}(-\bar{z}|-\bar{\tau})\,. \tag{B.40}$$

The theta function satisfies the heat equation:

$$\frac{\partial^2}{\partial z^2}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau) = 4\pi i\frac{\partial}{\partial\tau}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau)\,. \tag{B.41}$$

The derivative $\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)$ is expressed by the Dedekind eta function:

$$\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau) = -2\pi\{\eta(\tau)\}^3\,. \tag{B.42}$$

At $z \sim 0$,

$$\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)} \overset{z\sim 0}{\sim} z\,. \tag{B.43}$$

B.8.3 the Riemann identity

For $\alpha, \beta = 0, \frac{1}{2}$, this function satisfies the Riemann identity [65]

$$\begin{aligned}\sum_{\alpha,\beta=0,\frac{1}{2}} &(-1)^{2\alpha+2\beta}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(x)\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(y)\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(u)\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(v)\\ &= 2\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(x_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(y_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(v_1)\end{aligned} \tag{B.44}$$

and also

$$\begin{aligned}&\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(x\,|\tau)\,\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(y\,|\tau)\,\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(u\,|\tau)\,\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(v\,|\tau)\\ &\quad-\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(x\,|\tau)\,\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(y\,|\tau)\,\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(u\,|\tau)\,\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(v\,|\tau)\\ &\quad\quad-\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(x\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(y\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(v\,|\tau)\\ &\quad\quad\quad+\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(x\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(y\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(u\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(v\,|\tau)\\ &= -2\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(x_1\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(y_1\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u_1\,|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(v_1\,|\tau)\,,\end{aligned} \tag{B.45}$$

where

$$\begin{aligned}x_1 &= \frac{1}{2}(x+y+u+v)\,,\ \ y_1 = \frac{1}{2}(x+y-u-v)\,,\\ u_1 &= \frac{1}{2}(x-y+u-v)\,,\ \ v_1 = \frac{1}{2}(x-y-u+v)\,.\end{aligned} \tag{B.46}$$

B.9 Computation of the infinite sum II

Using eq. (B.22), we find

$$\sum_{n=-\infty}^{\infty} \frac{(\mathrm{e}^{2\pi i z})^n}{1-(\mathrm{e}^{-2\pi i\beta} q^{\alpha})q^n} = \frac{i\mathrm{e}^{-2\pi i\alpha z}}{2\pi} \frac{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}-\beta\end{bmatrix}(z|\tau)\,\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)}{\vartheta\begin{bmatrix}\alpha-\frac{1}{2}\\ \frac{1}{2}-\beta\end{bmatrix}(0|\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}\,. \qquad \text{(B.47)}$$

Proof: With $\zeta = \mathrm{e}^{2\pi i z}$ and $a = \mathrm{e}^{-2\pi i\beta} q^{\alpha}$,

$$\sum_{n=-\infty}^{\infty} \frac{\zeta^n}{1 - aq^n}$$

$$\overset{\text{eq.}(B.22)}{=} \frac{(a\zeta;q)_\infty (\frac{q}{a\zeta};q)_\infty (q;q)_\infty^2}{(a;q)_\infty (\zeta;q)_\infty (\frac{q}{\zeta};q)_\infty (\frac{q}{a};q)_\infty}$$

$$= \frac{\prod_{m=0}^{\infty}(1+\mathrm{e}^{2\pi i\{z+(\frac{1}{2}-\beta)\}} q^{m+\alpha})(1+\mathrm{e}^{-2\pi i\{z+(\frac{1}{2}-\beta)\}} q^{m-\alpha+1})}{\prod_{m=0}^{\infty}(1+\mathrm{e}^{2\pi i\{0+(\frac{1}{2}-\beta)\}} q^{m+\alpha})(1+\mathrm{e}^{2\pi i(z+\frac{1}{2})} q^{m})(1+\mathrm{e}^{-2\pi i(z+\frac{1}{2})} q^{m+1})}$$

$$\times \frac{\prod_{m=1}^{\infty}(1-q^m)^2}{\prod_{m=0}^{\infty}(1+\mathrm{e}^{2\pi i\{0+(\frac{1}{2}-\beta)\}} q^{m-\alpha+1})}$$

$$= \frac{\mathrm{e}^{2\pi i(\alpha-\frac{1}{2})(0+(\frac{1}{2}-\beta))} q^{\frac{(\alpha-\frac{1}{2})^2}{2}} \prod_{m=1}^{\infty}(1-q^m)}{\mathrm{e}^{2\pi i(\alpha-\frac{1}{2})(z+(\frac{1}{2}-\beta))} q^{\frac{(\alpha-\frac{1}{2})^2}{2}} \prod_{m=1}^{\infty}(1-q^m)}$$

$$\times \frac{\mathrm{e}^{2\pi i(\alpha-\frac{1}{2})(z+(\frac{1}{2}-\beta))} q^{\frac{(\alpha-\frac{1}{2})^2}{2}}}{\mathrm{e}^{2\pi i(\alpha-\frac{1}{2})(0+(\frac{1}{2}-\beta))} q^{\frac{(\alpha-\frac{1}{2})^2}{2}}}$$

$$\times \frac{\prod_{m=1}^{\infty}(1-q^m)(1+\mathrm{e}^{2\pi i\{z+(\frac{1}{2}-\beta)\}} q^{m+(\alpha-\frac{1}{2})-\frac{1}{2}})}{\prod_{m=1}^{\infty}(1-q^m)(1+\mathrm{e}^{2\pi i\{0+(\frac{1}{2}-\beta)\}} q^{m+(\alpha-\frac{1}{2})-\frac{1}{2}})}$$

$$\times \frac{\prod_{m=1}^{\infty}(1+\mathrm{e}^{-2\pi i\{z+(\frac{1}{2}-\beta)\}} q^{m-(\alpha-\frac{1}{2})-\frac{1}{2}})}{\prod_{m=1}^{\infty}(1+\mathrm{e}^{2\pi i\{0+(\frac{1}{2}-\beta)\}} q^{m-(\alpha-\frac{1}{2})-\frac{1}{2}})}$$

$$\times \frac{\mathrm{e}^{2\pi i(-\frac{1}{2})(z+\frac{1}{2})} q^{\frac{(-\frac{1}{2})^2}{2}} \prod_{m=1}^{\infty}(1-q^m)}{(-2\pi) q^{\frac{3}{24}} \prod_{m=1}^{\infty}(1-q^m)}$$

$$\times \frac{(-2\pi)\left\{q^{\frac{1}{24}} \prod_{m=1}^{\infty}(1-q^m)\right\}^3}{\mathrm{e}^{2\pi i(-\frac{1}{2})(z+\frac{1}{2})} q^{\frac{(-\frac{1}{2})^2}{2}} \prod_{m=1}^{\infty}(1-q^m)(1+\mathrm{e}^{2\pi i(z+\frac{1}{2})} q^{m+(-\frac{1}{2})-\frac{1}{2}})}$$

$$\times \frac{1}{\prod_{m=1}^{\infty}(1+\mathrm{e}^{-2\pi i(z+\frac{1}{2})} q^{m-(-\frac{1}{2})-\frac{1}{2}})}$$

$$\overset{\text{eqs.}(B.32),(B.31)}{=} \frac{\mathrm{e}^{-2\pi i(\alpha-\frac{1}{2})z} \mathrm{e}^{-\pi i(z+\frac{1}{2})}}{(-2\pi)} \frac{\vartheta \begin{bmatrix} \alpha-\frac{1}{2} \\ \frac{1}{2}-\beta \end{bmatrix} (z|\tau)}{\vartheta \begin{bmatrix} \alpha-\frac{1}{2} \\ \frac{1}{2}-\beta \end{bmatrix} (0|\tau)} \frac{(-2\pi)\{\eta(\tau)\}^3}{\vartheta \begin{bmatrix} -\frac{1}{2} \\ +\frac{1}{2} \end{bmatrix} (z|\tau)}$$

$$\overset{\text{eqs.}(B.38),(B.42)}{=} \frac{i\mathrm{e}^{-2\pi i\alpha z}}{2\pi} \frac{\vartheta \begin{bmatrix} \alpha-\frac{1}{2} \\ \frac{1}{2}-\beta \end{bmatrix} (z|\tau)}{\vartheta \begin{bmatrix} \alpha-\frac{1}{2} \\ \frac{1}{2}-\beta \end{bmatrix} (0|\tau)} \frac{\vartheta' \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (0|\tau)}{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z|\tau)} . \qquad \square \tag{B.48}$$

Note that, for $\tau_2 > 0$ and $0 \le \sigma^1, \sigma^2 \le 1$,

$$|q| < |\mathrm{e}^{2\pi i z}| < 1 \ . \tag{B.49}$$

C Zeta function regularization

In general,

$$\begin{aligned}\zeta_\alpha(-1,x) &\equiv \sum_{n=1}^{\infty}(n-\alpha)\mathrm{e}^{-(n-\alpha)x} \\ &= \frac{(1-\alpha)\mathrm{e}^{(n+\alpha)x}+\alpha\mathrm{e}^{\alpha x}}{(\mathrm{e}^x-1)^2} = \frac{2}{x^2}+\frac{1}{x}+\frac{\alpha(1-\alpha)}{2}-\frac{1}{12}+\mathcal{O}(x) \ .\end{aligned} \tag{C.1}$$

D Supplement to S, T

$$\begin{aligned} T: \eta(\tau) &\to \eta(\tau+1) = \mathrm{e}^{\frac{i\pi}{12}}\eta(\tau) \ , \qquad \tau_2 \to \text{ invariant} \ , \\ \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau) &\to \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau+1) = \mathrm{e}^{-i\pi\alpha(\alpha-1)}\vartheta\begin{bmatrix}\alpha\\ \alpha+\beta-\frac{1}{2}\end{bmatrix}(0|\tau) \ . \\ S: \eta(\tau) &\to \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\eta(\tau) \ , \qquad \tau_2 \to \frac{\tau_2}{|\tau|^2} \ , \\ \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau) &\to \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(0\middle|-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\mathrm{e}^{2\pi i\alpha\beta}\vartheta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(0|\tau) \ . \end{aligned} \tag{D.1}$$

Under T

$$\begin{aligned} &\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(0) \to \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0) \qquad \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0) \to \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(0) \\ &\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0) \to \mathrm{e}^{\frac{i}{4}\pi}\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0) \qquad \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0) \to \mathrm{e}^{\frac{i}{4}\pi}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0) \ , \end{aligned} \tag{D.2}$$

so

$$\hat{T}(O_{2n}, V_{2n}, S_{2n}, C_{2n})_j = (O_{2n}, V_{2n}, S_{2n}, C_{2n})_i T_{ij} \tag{D.3}$$

$$T_{ij}^{(n)} = \mathrm{e}^{\frac{-in\pi}{12}}\mathrm{diag}(1, -1, \mathrm{e}^{\frac{in\pi}{4}}, \mathrm{e}^{\frac{in\pi}{4}}) \ . \tag{D.4}$$

Under S

$$\frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0)}{\eta}\to\frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0)}{\eta}\qquad\frac{\vartheta\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right](0)}{\eta}\to\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right](0)}{\eta}$$
$$\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right](0)}{\eta}\to\frac{\vartheta\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right](0)}{\eta}\qquad\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\\frac{1}{2}\end{smallmatrix}\right](0)}{\eta}\to e^{\frac{\pi i}{2}}\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\\frac{1}{2}\end{smallmatrix}\right](0)}{\eta}\,, \tag{D.5}$$

so

$$\begin{aligned}
\hat{S}O_{2n} &= \frac{1}{2}(O_{2n}+V_{2n})+\frac{1}{2}(S_{2n}+C_{2n})\\
\hat{S}V_{2n} &= \frac{1}{2}(O_{2n}+V_{2n})-\frac{1}{2}(S_{2n}+C_{2n})\\
\hat{S}S_{2n} &= \frac{1}{2}\hat{T}\left(\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right]^n}{\eta^n}\right)+\frac{i^{-n}}{2}\hat{T}\left(\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\\frac{1}{2}\end{smallmatrix}\right]^n}{\eta^n}\right)\\
&= \frac{1}{2}\frac{\vartheta\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right]^n}{\eta^n}+\frac{i^{-n}}{2}i^n\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\\frac{1}{2}\end{smallmatrix}\right]^n}{\eta^n}\\
&= \frac{1}{2}(O_{2n}-V_{2n})+\frac{i^n}{2}(S_{2n}-C_{2n})\\
\hat{S}C_{2n} &= \frac{1}{2}(O_{2n}-V_{2n})-\frac{i^n}{2}(S_{2n}-C_{2n})\,.
\end{aligned} \tag{D.6}$$

Therefore

$$S_{ij}=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&i^n&-i^n\\1&-1&-i^n&i^n\end{bmatrix}\to\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{bmatrix}\,. \tag{D.7}$$

Set $n=4$,

$$\chi^T=(O_8,V_8,S_8,C_8)/\tau_2^2\eta^8 \tag{D.8}$$

is representation as

$$T=\mathrm{diag}(-1,1,1,1)\,. \tag{D.9}$$

Let's now check IIB case

$$X_{\rm IIB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{D.10}$$

Then

$$TX_{\rm IIB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = X_{\rm IIB}T$$

$$SX_{\rm IIB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = X_{\rm IIB}S\,. \tag{D.11}$$

E Modular invariance of the lattice sum

Consider the ratio

$$\begin{aligned} F_2(a,\tau) &\equiv \frac{\frac{1}{2\pi R}\sum_{\ell,m\in\mathbf{Z}} q^{\frac14(ma+\frac{\ell}{a})^2}\bar{q}^{\frac14(ma-\frac{\ell}{a})^2}}{\int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-tp^2}} \\ &= \frac{1}{2\pi R}2\pi\sqrt{\alpha'\tau_2}\sum_{\ell,m\in\mathbf{Z}} q^{\frac14(ma+\frac{\ell}{a})^2}\bar{q}^{\frac14(ma-\frac{\ell}{a})^2} \\ &= a\sqrt{\tau_2}\sum_{\ell,m}\mathrm{e}^{-\pi\tau_2(m^2a^2+\frac{\ell^2}{a^2})+2\pi i\tau_1 m\ell}\,. \end{aligned} \tag{E.1}$$

To prove its modular invariance, the basic identity is

$$\begin{aligned} \mathcal{F}(\mathbf{x}) &= \sum_{\mathbf{n}\in\mathbf{Z}^p}\exp(-\pi(\mathbf{n}+\mathbf{x})A(\mathbf{n}+\mathbf{x})) \\ &= (\det A)^{-1/2}\sum_{\mathbf{m}\in\mathbf{Z}^p}\exp(-\pi\mathbf{m}A^{-1}\mathbf{m}+2\pi i\mathbf{m}\cdot\mathbf{x})\,. \end{aligned} \tag{E.2}$$

Set $p = 2, \mathbf{x} = \mathbf{0}$, $A(\tau) = \begin{pmatrix} a^2\tau_2 & -i\tau_1 \\ -i\tau_1 & a^{-2}\tau_2 \end{pmatrix}$. Therefore

$$\begin{aligned} F_2(a,\tau) &= a\frac{\sqrt{\tau_2}}{|\tau|} \sum_{\mathbf{m}\in\mathbf{Z}^2} \exp(-\pi \mathbf{m} A^{-1} \mathbf{m}) \\ &= a\left(\frac{\tau_2}{|\tau|^2}\right)^{\frac{1}{2}} \sum_{\mathbf{m}\in\mathbf{Z}^2} \exp\left(-\pi \mathbf{m} A^{-1}\left(-\frac{1}{\tau}, \frac{1}{a}\right)\mathbf{m}\right) \\ &= F_2(-1/\tau, a) \; . \end{aligned} \tag{E.3}$$

F Image method in superspace

In this appendix, we apply the method of images in superspace to superannulus.

Let the conjugate point of (z,θ) be $(\tilde{z},\tilde{\theta})$. The involution acting on $f(z,\theta)$ associated with $(z,\theta) \to (\tilde{z},\tilde{\theta})$ is denoted by

$$\hat{i}f(z,\theta) = f(\tilde{z},\tilde{\theta}) = \text{fn}(\bar{z}, \text{ and } \bar{\theta} \text{ only}) \; . \tag{F.1}$$

Let the supersymmetry transformation of $f(z,\theta)$ be

$$\delta f = (\varepsilon Q - \bar{\varepsilon}\bar{Q}) f(z,\theta) \; . \tag{F.2}$$

We require

$$\begin{aligned} \hat{i}\delta f = \delta \hat{i} f(z,\theta) &= \delta f(\tilde{z},\tilde{\theta}) = -\bar{\varepsilon}\bar{Q} f(\tilde{z},\tilde{\theta}) \\ = \hat{i}\varepsilon Q f = \hat{i}\varepsilon &\left(i\theta\frac{\partial}{\partial z} + \frac{\partial}{\partial\theta}\right) f(z,\theta) = \tilde{\varepsilon}\tilde{Q} f(\tilde{z},\tilde{\theta}) \; . \end{aligned} \tag{F.3}$$

So we conclude

$$\tilde{\varepsilon}\tilde{Q} = -\bar{\varepsilon}\bar{Q} \tag{F.4}$$

$$\delta\tilde{\theta}\left(i\tilde{\theta}\frac{\partial}{\partial\tilde{z}} + \frac{\partial}{\partial\tilde{\theta}}\right) = -\delta\bar{\theta}\left(-i\bar{\theta}\frac{\partial}{\partial\bar{z}} - \frac{\partial}{\partial\bar{\theta}}\right) \; . \tag{F.5}$$

Therefore,

$$\begin{aligned}
\text{if} \quad \tilde{z} = \bar{z}, &\quad \text{then } \tilde{\theta} = \pm\bar{\theta} \quad \text{(UHP)}\,, \\
\tilde{z} = -\bar{z}, &\quad \text{then } \tilde{\theta} = \pm i\bar{\theta} \quad \text{(annulus)}\,, \\
\text{and} \quad \tilde{z} = \frac{1}{\bar{z}}, &\quad \text{then } \tilde{\theta} = \pm\frac{i\bar{\theta}}{\bar{z}} \quad \text{(disk)}\,.
\end{aligned} \tag{F.6}$$

G G_{++} and G_{+-}

G.1 $G_{++}(z, \bar{z}|0,0) = G\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z, \bar{z}|0,0)$

$$\begin{aligned}
&G_{++}(z, \bar{z}|0,0) \equiv G\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z, \bar{z}|0,0) \\
&= \frac{1}{\tau_2} \sum_{\substack{n_1,n_2=-\infty \\ (n_1,n_2)\neq(0,0)}}^{\infty} \frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}|n_2 - n_1\tau|^2} \mathrm{e}^{2\pi i n_1 \sigma^1} \mathrm{e}^{2\pi i n_2 \sigma^2} \\
&= \frac{1}{\tau_2} \sum_{\substack{n_1,n_2=-\infty \\ n_1\neq 0}}^{\infty} \frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}|n_2 - n_1\tau|^2} \mathrm{e}^{2\pi i n_1 \sigma^1} \mathrm{e}^{2\pi i n_2 \sigma^2} + \frac{1}{\tau_2} \sum_{\substack{n_2=-\infty \\ n_2\neq 0}}^{\infty} \frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}|n_2|^2} \mathrm{e}^{2\pi i n_2 \sigma^2}\,.
\end{aligned} \tag{G.1}$$

Using

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1-q^n} = -\sum_{m=0}^{\infty} \ln(1 - xq^m)\,, \tag{G.2}$$

the first term in the last line of eq. (G.1) can be computed as

$$\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar\tau)^2}|n_2-n_1\tau|^2}\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i n_2\sigma^2}$$

$$=\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{\tau-\bar\tau}{4(2\pi)^2}\frac{1}{n_1}\left\{\frac{1}{n_2-n_1\tau}-\frac{1}{n_2-n_1\bar\tau}\right\}\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i n_2\sigma^2}$$

$$=\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{i(\tau-\bar\tau)}{4(2\pi)}\frac{1}{n_1}\left[\frac{\int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i(n_2-n_1\tau)\sigma}}{1-q^{n_1}}-\frac{\int_0^1 \mathrm{d}\sigma \mathrm{e}^{-2\pi i(n_2-n_1\bar\tau)\sigma}}{1-\bar q^{-n_1}}\right]$$

$$\times\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i n_2\sigma^2}$$

$$=\frac{-2}{4(2\pi)}$$

$$\times\sum_{\substack{n_1=-\infty\\ n_1\neq 0}}^{\infty}\frac{1}{n_1}\left[\frac{\int_0^1 \mathrm{d}\sigma\delta(\sigma^2-\sigma)\mathrm{e}^{2\pi i n_1(\sigma^1+\tau\sigma)}}{1-q^{n_1}}-\frac{\int_0^1 \mathrm{d}\sigma\delta(\sigma^2-\sigma)\mathrm{e}^{2\pi i n_1(\sigma^1+\bar\tau\sigma)}}{1-\bar q^{-n_1}}\right]$$

$$=\frac{-2}{4(2\pi)}\sum_{\substack{n_1=-\infty\\ n_1\neq 0}}^{\infty}\left[\frac{1}{n_1}\frac{\zeta^{n_1}}{1-q^{n_1}}-\frac{1}{n_1}\frac{\bar\zeta^{-n_1}}{1-\bar q^{-n_1}}\right]$$

$$=\frac{-2}{4(2\pi)}\sum_{n_1=1}^{\infty}\left[\frac{1}{n_1}\frac{\zeta^{n_1}}{1-q^{n_1}}+\frac{1}{n_1}\frac{\left(\frac{\bar q}{\bar\zeta}\right)^{n_1}}{1-\bar q^{n_1}}+\frac{1}{n_1}\frac{\left(\frac{q}{\zeta}\right)^{n_1}}{1-q^{n_1}}+\frac{1}{n_1}\frac{\bar\zeta^{n_1}}{1-\bar q^{n_1}}\right]$$

$$\overset{\text{eq.}(G.2)}{=}\frac{-2}{4(2\pi)}$$

$$\times\sum_{m=0}^{\infty}\left[-\ln\left(1-\zeta q^m\right)-\ln\left(1-\frac{\bar q}{\bar\zeta}\bar q^m\right)-\ln\left(1-\frac{q}{\zeta}\bar q^m\right)-\ln\left(1-\bar\zeta\bar q^m\right)\right]$$

$$=\frac{2}{4(2\pi)}\left[\ln|\zeta-1|^2+\sum_{m=1}^{\infty}\ln|1-\zeta q^m|^2\left|1-\frac{q^m}{\zeta}\right|^2\right]. \tag{G.3}$$

Next, turning to the second term,

$$\frac{1}{\tau_2}\sum_{\substack{n_2=-\infty\\ n_2\neq 0}}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}|n_2|^2}\mathrm{e}^{2\pi i n_2\sigma^2}=\frac{1}{\tau_2}\frac{(\tau-\bar{\tau})^2}{4(2\pi)^2}\left(\sum_{n_2=1}^{\infty}\frac{1}{n_2^2}\mathrm{e}^{2\pi i n_2\sigma^2}+\sum_{n_2=1}^{\infty}\frac{1}{n_2^2}\mathrm{e}^{-2\pi i n_2\sigma^2}\right)$$
$$\equiv\frac{1}{\tau_2}\frac{(\tau-\bar{\tau})^2}{4(2\pi)^2}F(\sigma^2)\,. \tag{G.4}$$

Now

$$\frac{1}{(2\pi i)^2}\frac{\mathrm{d}^2F(\sigma^2)}{\mathrm{d}(\sigma^2)^2}=\sum_{n_2=1}^{\infty}\mathrm{e}^{2\pi i n_2\sigma^2}+\sum_{n_2=1}^{\infty}\mathrm{e}^{-2\pi i n_2\sigma^2}=\frac{\mathrm{e}^{2\pi i\sigma^2-\varepsilon_+}}{1-\mathrm{e}^{2\pi i\sigma^2-\varepsilon_+}}-\frac{1}{1-\mathrm{e}^{2\pi i\sigma^2-\varepsilon_-}}$$
$$=-1\qquad\text{modulo }\ \delta(\sigma^2)\,. \tag{G.5}$$

$$\therefore\qquad F(\sigma^2)=(2\pi i)^2\left(-\frac{1}{2}(\sigma^2)^2+A(\sigma^2)+B\right)\,. \tag{G.6}$$

Using

$$A=F'(0)=0\,,\qquad B=F(0)=2\sum_{n_2=1}^{\infty}\frac{1}{n_2^2}=2\zeta(2)=2\cdot\frac{\pi^2}{6}\,, \tag{G.7}$$

we obtain

$$\frac{1}{\tau_2}\sum_{\substack{n_2=-\infty\\ n_2\neq 0}}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}|n_2|^2}\mathrm{e}^{2\pi i n_2\sigma^2}=\frac{1}{\tau_2}\frac{(\tau-\bar{\tau})^2}{4(2\pi)^2}(2\pi i)^2\left(-\frac{1}{2}(\sigma^2)^2+2\cdot\frac{\pi^2}{6}\right)$$
$$=-\frac{1}{2}\frac{(\mathrm{Im}z)^2}{\tau_2}+2\tau_2\cdot\frac{\pi^2}{6}\,. \tag{G.8}$$

As a result of eqs. (G.1), (G.3) and (G.8),

$$G_{++}(z,\bar{z}|0,0)=\frac{2}{4(2\pi)}\left[\ln|\zeta-1|^2+\sum_{m=1}^{\infty}\ln|1-\zeta q^m|^2\left|1-\frac{q^m}{\zeta}\right|^2\right]$$
$$-\frac{1}{2}\frac{(\mathrm{Im}z)^2}{\tau_2}+2\tau_2\cdot\frac{\pi^2}{6}\,. \tag{G.9}$$

Due to

$$
\begin{aligned}
&\prod_{m=1}^{\infty}(1-\zeta q^m)\left(1-\frac{q^m}{\zeta}\right)\\
&=\frac{iq^{\frac{1}{8}}\mathrm{e}^{\pi i z}(1-\mathrm{e}^{-2\pi i z})\prod_{n=1}^{\infty}(1-q^n)}{iq^{\frac{1}{8}}\mathrm{e}^{\pi i z}(1-\mathrm{e}^{-2\pi i z})\prod_{n=1}^{\infty}(1-q^n)}\frac{(-2\pi)\left\{q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\right\}^3}{(-2\pi)\left\{q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\right\}^3}\\
&\qquad\times\prod_{m=1}^{\infty}\left(1-\mathrm{e}^{2\pi i z}q^m\right)\left(1-\mathrm{e}^{-2\pi i z}q^m\right)\\
&\overset{\text{eqs.}(B.31),(B.42)}{=}\frac{(2\pi i)\prod_{n=1}^{\infty}(1-q^n)^2}{\mathrm{e}^{\pi i z}(1-\mathrm{e}^{-2\pi i z})}\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(0)}\,,
\end{aligned}
\tag{G.10}
$$

the terms in the box brackets of eq. (G.9) can be recast as follows:

$$
\begin{aligned}
&\ln|\zeta-1|^2+\sum_{m=1}^{\infty}\ln|1-\zeta q^m|^2\left|1-\frac{q^m}{\zeta}\right|^2\\
&\overset{\text{eq.}(G.10)}{=}\ln\left|\mathrm{e}^{2\pi i z}-1\right|^2\left|\frac{(2\pi i)\prod_{n=1}^{\infty}(1-q^n)^2}{\mathrm{e}^{\pi i z}(1-\mathrm{e}^{-2\pi i z})}\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(0)}\right|^2\\
&=\ln\left|\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(0)}\right|^2+2\sum_{n=1}^{\infty}\ln|1-q^n|^2-2\pi(\mathrm{Im}\,z)+2\ln(2\pi)\,.
\end{aligned}
\tag{G.11}
$$

Hence we obtain

$$
\begin{aligned}
&G_{++}(z,\bar{z}|0,0)\\
&\overset{\text{eqs.}(G.9),(G.11)}{=}\frac{1}{2\pi}\ln\left|\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(0)}\right|-\frac{1}{2}\frac{(\mathrm{Im}\,z)^2}{\tau_2}\\
&\qquad+\left[\frac{1}{2\pi}2\sum_{n=1}^{\infty}\ln|1-q^n|-\frac{1}{2}(\mathrm{Im}\,z)+\frac{1}{2\pi}\ln(2\pi)+2\tau_2\cdot\frac{\pi^2}{6}\right]\,.
\end{aligned}
\tag{G.12}
$$

The terms in the brackets on the last line vanish when acting $\Delta=4\partial_z\partial_{\bar{z}}$.

G.2 $G_{+-}(z,\bar{z}|0,0) = G\left[\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}\right](z,\bar{z}|0,0)$

$$
\begin{aligned}
&G_{+-}(z,\bar{z}|0,0)\\
&\equiv G\left[\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}\right](z,\bar{z}|0,0)\\
&= \frac{1}{\tau_2}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}\left|\left(n_2+\frac{1}{2}\right)-n_1\tau\right|^2}\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}\\
&= \frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}\left|\left(n_2+\frac{1}{2}\right)-n_1\tau\right|^2}\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}\\
&\qquad +\frac{1}{\tau_2}\sum_{n_2=-\infty}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}\left|n_2+\frac{1}{2}\right|^2}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}\ .
\end{aligned}
\tag{G.13}
$$

The first term in the last line of eq. (G.13) is

$$
\begin{aligned}
&\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar\tau)^2}\left|\left(n_2+\frac{1}{2}\right)-n_1\tau\right|^2}\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}\\
&=\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{\tau-\bar\tau}{4(2\pi)^2}\frac{1}{n_1}\left\{\frac{1}{\left(n_2+\frac{1}{2}\right)-n_1\tau}-\frac{1}{\left(n_2+\frac{1}{2}\right)-n_1\bar\tau}\right\}\\
&\qquad\qquad\times\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}\\
&=\frac{1}{\tau_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0}}^{\infty}\frac{\tau-\bar\tau}{4(2\pi)^2}\frac{1}{n_1}\left[\frac{2\pi i}{1-\mathrm{e}^{-2\pi i\frac{1}{2}}q^{n_1}}\int_0^1 \mathrm{d}\sigma\mathrm{e}^{-2\pi i\left\{\left(n_2+\frac{1}{2}\right)-n_1\tau\right\}\sigma}\right.\\
&\qquad\qquad\left.-\frac{2\pi i}{1-\mathrm{e}^{2\pi i\frac{1}{2}}\bar q^{-n_1}}\int_0^1 \mathrm{d}\sigma\mathrm{e}^{-2\pi i\left\{\left(n_2+\frac{1}{2}\right)-n_1\bar\tau\right\}\sigma}\right]\mathrm{e}^{2\pi i n_1\sigma^1}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}\\
&=\frac{-2}{4(2\pi)}\sum_{\substack{n_1=-\infty\\ n_1\neq 0}}^{\infty}\frac{1}{n_1}\left[\frac{\int_0^1\mathrm{d}\sigma\left(\delta(\sigma^2-\sigma)\right)\mathrm{e}^{2\pi i\frac{1}{2}(\sigma^2-\sigma)}\mathrm{e}^{2\pi i n_1(\sigma^1+\tau\sigma)}}{1-(-q^{n_1})}\right.\\
&\qquad\qquad\left.-\frac{\int_0^1\mathrm{d}\sigma\left(\delta(\sigma^2-\sigma)\right)\mathrm{e}^{2\pi i\frac{1}{2}(\sigma^2-\sigma)}\mathrm{e}^{2\pi i n_1(\sigma^1+\bar\tau\sigma)}}{1-(-\bar q^{-n_1})}\right]\\
&=\frac{-2}{4(2\pi)}\sum_{\substack{n_1=-\infty\\ n_1\neq 0}}^{\infty}\frac{1}{n_1}\left[\frac{\zeta^{n_1}}{1-(-q^{n_1})}-\frac{\bar\zeta^{-n_1}}{1-(-\bar q^{-n_1})}\right]\\
&=\frac{-2}{4(2\pi)}\\
&\times\sum_{n_1=1}^{\infty}\left[\frac{1}{n_1}\frac{\zeta^{n_1}}{1-(-q^{n_1})}-\frac{1}{n_1}\frac{\left(\frac{\bar q}{\bar\zeta}\right)^{n_1}}{1-(-\bar q^{n_1})}-\frac{1}{n_1}\frac{\left(\frac{q}{\zeta}\right)^{n_1}}{1-(-q^{n_1})}+\frac{1}{n_1}\frac{\bar\zeta^{+n_1}}{1-(-\bar q^{+n_1})}\right]\\
&\overset{\text{eq.}(G.15)}{=}\frac{-2}{4(2\pi)}\sum_{m=0}^{\infty}\left[(-1)^{m+1}\ln(1-\zeta q^m)-(-1)^{m+1}\ln\left(1-\frac{\bar q}{\bar\zeta}\bar q^m\right)\right.\\
&\qquad\qquad\left.-(-1)^{m+1}\ln\left(1-\frac{q}{\zeta}q^m\right)+(-1)^{m+1}\ln(1-\bar\zeta\bar q^m)\right]\\
&=\frac{2}{4(2\pi)}\left[\ln|1-\zeta|^2+\sum_{m=1}^{\infty}(-1)^m\ln|1-\zeta q^m|^2\left|1-\frac{q^m}{\zeta}\right|^2\right], \qquad\qquad (G.14)
\end{aligned}
$$

where we have used

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{x^n}{1-(-y^n)}=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=0}^{\infty}x^n(-y^n)^m=\sum_{m=0}^{\infty}(-1)^m\sum_{n=1}^{\infty}\frac{1}{n}(xy^m)^n$$
$$=\sum_{m=0}^{\infty}(-1)^m\{-\ln(1-xy^m)\}=\sum_{m=0}^{\infty}(-1)^{m+1}\ln(1-xy^m)\,. \tag{G.15}$$

The second term in eq. (G.13) is

$$\frac{1}{\tau_2}\sum_{n_2=-\infty}^{\infty}\frac{1}{\frac{4(2\pi)^2}{(\tau-\bar\tau)^2}\left|n_2+\frac{1}{2}\right|^2}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}$$
$$=\frac{1}{\tau_2}\frac{(\tau-\bar\tau)^2}{4(2\pi)^2}\left(\sum_{n_2=0}^{\infty}\frac{1}{\left(n_2+\frac{1}{2}\right)^2}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}+\sum_{n_2=1}^{\infty}\frac{1}{\left(n_2-\frac{1}{2}\right)^2}\mathrm{e}^{-2\pi i\left(n_2-\frac{1}{2}\right)\sigma^2}\right)$$
$$\equiv\frac{1}{\tau_2}\frac{(\tau-\bar\tau)^2}{4(2\pi)^2}\tilde F(\sigma^2)\,. \tag{G.16}$$

Then

$$\frac{1}{(2\pi)^2}\frac{\mathrm{d}^2\tilde F(\sigma^2)}{\mathrm{d}(\sigma^2)^2}$$
$$=\sum_{n_2=0}^{\infty}\mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2}+\sum_{n_2=1}^{\infty}\mathrm{e}^{-2\pi i\left(n_2-\frac{1}{2}\right)\sigma^2}$$
$$=\mathrm{e}^{2\pi i\frac{1}{2}\sigma^2}\left(\frac{1}{1-\mathrm{e}^{2\pi i\sigma^2-\varepsilon_+}}+\frac{\mathrm{e}^{-2\pi i\sigma^2-\varepsilon_-}}{1-\mathrm{e}^{-2\pi i\sigma^2-\varepsilon_-}}\right)=0 \qquad \text{modulo } \delta(\sigma^2)\,. \tag{G.17}$$

$$\therefore \qquad \tilde F(\sigma^2)=(2\pi)^2\left(A\sigma^2+B\right) \tag{G.18}$$

and

$$B = \tilde F(0)=\sum_{n_2=0}^{\infty}\frac{1}{\left(n_2+\frac{1}{2}\right)^2}+\sum_{n_2=1}^{\infty}\frac{1}{\left(n_2-\frac{1}{2}\right)^2}$$
$$\overset{\text{eq.}(B.5)}{=} \zeta\left(2,\frac{1}{2}\right)+\left\{\zeta\left(2,-\frac{1}{2}\right)-\frac{1}{\left(0-\frac{1}{2}\right)^2}\right\}$$
$$\overset{\text{eqs.}(B.4),(B.6)}{=} \frac{\pi^2}{2}+\left\{\left(4+\frac{\pi^2}{2}\right)-4\right\}=2\cdot\frac{\pi^2}{2}=\pi^2\,. \tag{G.19}$$

Using the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a}\coth(a\pi)\,, \tag{G.20}$$

$$\begin{aligned} A &= \tilde{F}'(0) = \sum_{n_2=0}^{\infty} \frac{2\pi i}{n_2+\frac{1}{2}} + \sum_{n_2=1}^{\infty} \frac{-2\pi i}{n_2-\frac{1}{2}} \\ &= 2\pi i\left[\frac{1}{0+\frac{1}{2}} + \sum_{n_2=1}^{\infty}\left\{\frac{1}{n_2+\frac{1}{2}} - \frac{1}{n_2-\frac{1}{2}}\right\}\right] \\ &= 2\pi i\left[2 - \sum_{n_2=1}^{\infty} \frac{1}{n_2^2+\left(\frac{i}{2}\right)^2}\right] \\ &\overset{\text{eq.}(G.20)}{=} 2\pi i\left[2 - \left\{-\frac{1}{2\left(\frac{i}{2}\right)^2} + \frac{\pi}{2\frac{i}{2}}\coth\left(\frac{i}{2}\pi\right)\right\}\right] = 0\,. \end{aligned} \tag{G.21}$$

$$\therefore \qquad \tilde{F}(\sigma^2) = (2\pi)^2\pi^2\,. \tag{G.22}$$

Therefore

$$\begin{aligned} \frac{1}{\tau_2}\sum_{n_2=-\infty}^{\infty} \frac{1}{\frac{4(2\pi)^2}{(\tau-\bar{\tau})^2}\left|n_2+\frac{1}{2}\right|^2} \mathrm{e}^{2\pi i\left(n_2+\frac{1}{2}\right)\sigma^2} &\equiv \frac{1}{\tau_2}\frac{(\tau-\bar{\tau})^2}{4(2\pi)^2}\tilde{F}(\sigma^2) \\ &= \frac{1}{\tau_2}\frac{(2i\tau_2)^2}{4(2\pi)^2}(2\pi)^2\pi^2 \\ &= -\pi^2\tau_2\,. \end{aligned} \tag{G.23}$$

Finally,

$$\begin{aligned} &G_{+-}(z,\bar{z}|0,0) \\ &\overset{\text{eqs.}(G.14),(G.23)}{=} \frac{1}{2\pi}\left[\ln|1-\zeta| + \sum_{m=1}^{\infty}(-1)^m \ln|1-\zeta q^m|\left|1-\frac{q^m}{\zeta}\right|\right] - \pi^2\tau_2\,. \end{aligned} \tag{G.24}$$

H Supertorus Green function and supersphere Green function

Since

$$\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\tau)} \overset{z\sim 0}{\sim} z\,, \qquad \ln\left|1-\mathrm{e}^{2\pi i z}\right| \overset{z\sim 0}{\sim} \ln|z| \tag{H.1}$$

$$G_{+\pm}(z,\bar{z}|0,0) \overset{z\sim 0}{\sim} \frac{1}{2\pi}\ln|z| \tag{H.2}$$

and

$$\mathcal{S}_{\nu_{\mathrm{f}}}(z,\bar{z}|0,0) \overset{z\sim 0}{\sim} \frac{i}{\pi}\frac{1}{z}\,, \qquad \overline{\mathcal{S}}_{\nu_{\mathrm{f}}}(z,\bar{z}|0,0) \overset{\bar{z}\sim 0}{\sim} -\frac{i}{\pi}\frac{1}{\bar{z}}\,, \tag{H.3}$$

where $\nu_{\mathrm{f}} = (--), (-+), (+-)$. Using eqs. (H.2) and (H.3),

$$\begin{aligned}
&\mathbf{G}^{\text{supertorus}}_{\substack{+\pm\\ \nu_{\mathrm{f}}}}(z_I,\bar{z}_I|z_J,\bar{z}_J)\\
&\equiv G_{+\pm}(z_I,\bar{z}_I|z_J,\bar{z}_J) + \frac{\theta_I\theta_J}{4}\mathcal{S}_{\nu_{\mathrm{f}}}(z_I,\bar{z}_I|z_J,\bar{z}_J) - \frac{\bar{\theta}_I\bar{\theta}_J}{4}\overline{\mathcal{S}}_{\nu_{\mathrm{f}}}(z_I,\bar{z}_I|z_J,\bar{z}_J)\\
&\overset{z_I\sim z_J}{\sim} \frac{1}{2\pi}\ln|z_I-z_J| + \frac{\theta_I\theta_J}{4}\left(\frac{i}{\pi}\frac{1}{z_I-z_J}\right) - \frac{\bar{\theta}_I\bar{\theta}_J}{4}\left(-\frac{i}{\pi}\frac{1}{\bar{z}_I-\bar{z}_J}\right)\\
&= \frac{1}{4\pi}\ln(z_I-z_J+i\theta_I\theta_J) + \frac{1}{4\pi}\ln(\bar{z}_I-\bar{z}_J+i\bar{\theta}_I\bar{\theta}_J)\\
&= \frac{1}{2\pi}\ln|z_I-z_J+i\theta_I\theta_J| = \mathbf{G}^{\text{supersphere}}(z_I,\bar{z}_I|z_J,\bar{z}_J)\,.
\end{aligned} \tag{H.4}$$

I Supplement to $N^{IJ}_{+\pm}, B^{IJ}_{\nu_{\mathrm{f}}}, C^{IJ}_{+\pm}, E^{IJ}_{+\pm}$

I.1 Properties under $I \leftrightarrow J$

Here we check the properties under $I \leftrightarrow J$.

Due to the even/odd properties for theta functions,

$$\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)} = \frac{-\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)} \tag{I.1}$$

$$\mathcal{S}_{\nu_{\rm f}}(-z) = \frac{i}{\pi}\frac{\vartheta_{\nu_{\rm f}}(-z)}{\vartheta_{\nu_{\rm f}}(0)}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z)} = \frac{i}{\pi}\left(\frac{+\vartheta_{\nu_{\rm f}}(z)}{\vartheta_{\nu_{\rm f}}(0)}\right)\left(\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0)}{-\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}\right) = -\mathcal{S}_{\nu_{\rm f}}(z). \tag{I.2}$$

In addition, by the fact that the derivative of a even/odd function becomes odd/even,

$$\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z)} = \frac{+\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}{-\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)} = -\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)} \tag{I.3}$$

$$\frac{\vartheta''\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z)} = \frac{-\vartheta''\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}{-\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)} = +\frac{\vartheta''\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)}\,. \tag{I.4}$$

Using these,

$$\begin{aligned}
\pi N_{++}^{JI} &\overset{\text{eqs.}(I.1),(I.2)}{=} \ln\left|-\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(0\,\middle|\,\frac{i\tau_2}{2}\right)}\right| + \frac{\pi}{2}\frac{(z_I-z_J)^2}{\tau_2} = \pi N_{++}^{IJ} \\
B_{\nu_{\rm f}}^{JI} &\overset{\text{eq.}(I.2)}{=} \frac{1}{2}\frac{\pi}{i}(-1)\mathcal{S}_{\nu_{\rm f}}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right) = -B_{\nu_{\rm f}}^{IJ} \\
C_{++}^{JI} &\overset{\text{eq.}(I.3)}{=} \frac{1}{2}(-1)\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\middle|\frac{i\tau_2}{2}\right)} + (-1)\pi\frac{z_I-z_J}{\tau_2} = -C_{++}^{IJ} \\
E_{++}^{JI} &\overset{\text{eqs.}(I.3),\,(I.4)}{=} \frac{1}{4}\left\{+\frac{\vartheta''\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)} - \left(-\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\frac{z_I}{2}-\frac{z_J}{2}\,\middle|\,\frac{i\tau_2}{2}\right)}\right)^2\right\} + \frac{\pi}{\tau_2} \\
&= E_{++}^{IJ}\,.
\end{aligned} \tag{I.5}$$

From eq. (12.10),

$$
\begin{aligned}
\pi N^{JI}_{+-} &= +\pi N^{IJ}_{+-} \\
B^{JI}_{\nu_{\mathrm{f}}} &= -B^{IJ}_{\nu_{\mathrm{f}}} \\
C^{JI}_{+-} &= -C^{IJ}_{+-} \\
E^{JI}_{+-} &= +E^{IJ}_{+-} \,.
\end{aligned}
\tag{I.6}
$$

I.2 Singularity at $z_I \sim z_J$

Let us look at a singularity of $\pi N^{IJ}_{+\pm}$, $B^{IJ}_{\nu_{\mathrm{f}}}$, $C^{IJ}_{+\pm}$ and $E^{IJ}_{+\pm}$ at $z_I \sim z_J$.

By eq. (H.1),

$$
\begin{aligned}
\pi N^{IJ}_{+\pm} &\overset{z_J\sim z_I}{\sim} \ln\left|\frac{z_I}{2} - \frac{z_I}{2}\right| \\
B^{IJ}_{\nu_{\mathrm{f}}} &\overset{z_J\sim z_I}{\sim} \frac{1}{2}\frac{1}{\frac{z_I}{2} - \frac{z_I}{2}} \\
C^{IJ}_{+\pm} &\overset{z_J\sim z_I}{\sim} \frac{1}{2}\frac{1}{\frac{z_I}{2} - \frac{z_I}{2}} \\
E^{IJ}_{+\pm} &\overset{z_J\sim z_I}{\sim} -\frac{1}{4}\frac{1}{\left(\frac{z_I}{2} - \frac{z_I}{2}\right)^2} \,,
\end{aligned}
\tag{I.7}
$$

where we have also used

$$
\begin{aligned}
&\lim_{z\to 0}\frac{\vartheta''\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)} \\
&\overset{\mathrm{L'Hopital's\ rule}}{=} \lim_{z\to 0}\frac{\vartheta'''\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)} \overset{\mathrm{eq.}(B.41)}{=} \lim_{z\to 0}\frac{\frac{\partial}{\partial z}\left\{4\pi i\frac{\partial}{\partial\tau}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)\right\}}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)} \\
&= 4\pi i\lim_{z\to 0}\frac{\frac{\partial}{\partial\tau}\left\{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)\right\}}{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)} = 4\pi i\frac{\frac{\partial}{\partial\tau}\left\{\lim_{z\to 0}\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)\right\}}{\lim_{z\to 0}\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)} \\
&= 4\pi i\frac{\frac{\partial}{\partial\tau}\left[-2\pi\{\eta(\tau)\}^3\right]}{-2\pi\{\eta(\tau)\}^3} = 4\pi i\frac{\frac{\partial}{\partial\tau}\left[\{\eta(\tau)\}^3\right]}{\{\eta(\tau)\}^3} \\
&= 4\pi i\frac{\partial}{\partial\tau}\ln\{\eta(\tau)\}^3 = 3\cdot 4\pi i\frac{\partial}{\partial\tau}\ln\eta(\tau)
\end{aligned}
\tag{I.8}
$$

to evaluate E^{IJ}_{++}.

I.3 Eq. (12.8) at $z_I \sim z_J$ in case of maximal supersymmetry

Let us check that eq. (12.8) reduces to that of [32] at $z_I \sim z_J$ in case of maximal supersymmetry.

According to eq. (I.7),

$$
\begin{aligned}
&\exp\left[2\alpha' \sum_{1\le I<J\le N} k_I\cdot k_J \pi N^{IJ}_{++}\right] \\
&\exp\Bigg[2\alpha' \sum_{1\le I<J\le N} \big\{ i k_I\cdot k_J \theta_I\theta_J B^{IJ}_{\nu_{\mathrm{f}}} \\
&\qquad +(k_I\cdot\eta_J\theta_I - k_J\cdot\eta_I\theta_J)B^{IJ}_{\nu_{\mathrm{f}}} + (k_J\cdot\eta_I\theta_I - k_I\cdot\eta_J\theta_J)C^{IJ}_{++} \\
&\qquad -i\eta_I\cdot\eta_J B^{IJ}_{\nu_{\mathrm{f}}} + \eta_I\cdot\eta_J\theta_I\theta_J E^{IJ}_{++}\big\}\Bigg] \\
&\overset{z_I\sim z_J}{\sim} \exp\left[2\alpha' \sum_{1\le I<J\le N} k_I\cdot k_J \ln\left|\frac{z_I}{2}-\frac{z_J}{2}\right|\right] \\
&\exp\Bigg[2\alpha' \sum_{1\le I<J\le N} \Bigg\{ i k_I\cdot k_J\theta_I\theta_J \frac{1}{2}\frac{1}{\frac{z_I}{2}-\frac{z_J}{2}} \\
&\qquad + \left((k_I\cdot\eta_J\theta_I - k_J\cdot\eta_I\theta_J)\frac{1}{2}\frac{1}{\frac{z_I}{2}-\frac{z_J}{2}} + (k_J\cdot\eta_I\theta_I - k_I\cdot\eta_J\theta_J)\frac{1}{2}\frac{1}{\frac{z_I}{2}-\frac{z_J}{2}} \right) \\
&\qquad -i\eta_I\cdot\eta_J\frac{1}{2}\frac{1}{\frac{z_I}{2}-\frac{z_J}{2}} + \eta_I\cdot\eta_J\theta_I\theta_J\frac{(-1)}{4}\frac{1}{\left(\frac{z_I}{2}-\frac{z_J}{2}\right)^2}\Bigg\}\Bigg] \\
&= \prod_{1\le I<J\le N} \left|\frac{z_I}{2}-\frac{z_J}{2}\right|^{2\alpha' k_I\cdot k_J} \\
&\quad \exp\Bigg[\sum_{1\le I<J\le N}\Bigg\{ 2\alpha' i k_I\cdot k_J\theta_I\theta_J\frac{1}{2}\frac{1}{\frac{z_I}{2}-\frac{z_J}{2}} + 2\alpha'(k_I\cdot\eta_J + k_J\cdot\eta_I)\frac{1}{2}\frac{\theta_I-\theta_J}{\frac{z_I}{2}-\frac{z_J}{2}} \\
&\qquad -2\alpha' i\eta_I\cdot\eta_J\frac{1}{2}\frac{1}{\frac{z_I}{2}-\frac{z_J}{2}} - 2\alpha'\eta_I\cdot\eta_J\theta_I\theta_J\frac{1}{4}\frac{1}{\left(\frac{z_I}{2}-\frac{z_J}{2}\right)^2}\Bigg\}\Bigg]\ . \qquad \text{(I.9)}
\end{aligned}
$$

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