CLASSIFICATION OF HOMOGENEOUS WILLMORE SURFACES IN S^n

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Abstract

In this note we consider homogeneous Willmore surfaces in S^{n+2}. The main result is that a homogeneous Willmore two-sphere is conformally equivalent to a homogeneous minimal two-sphere in S^{n+2}, i.e., either a round two-sphere or one of the Borůvka-Veronese 2-spheres in S^{2m}. This entails a classification of all Willmore $\mathbb{C}P^1$ in S^{2m}. As a second main result we show that there exists no homogeneous Willmore upper-half plane in S^{n+2} and we give, in terms of special constant potentials, a simple loop group characterization of all homogeneous surfaces which have an abelian transitive group.

1. Introduction

Homogeneous Willmore surfaces are the simplest ones among all Willmore surfaces. Ejiri [13] constructed the first homogeneous Willmore torus which is not conformally equivalent to any minimal surface in space forms. In [17], homogeneous minimal tori in S^{2n+1} and in $\mathbb{C}P^n$ are discussed systematically. So far there has not been any systematic discussion of homogeneous Willmore surfaces. Very recently homogeneous Willmore 2-spheres have been studied by Pedit, Ma and Wang in terms of a variational method [19]. In this note we provide a description of all homogeneous Willmore surfaces in spheres in terms of the loop group theory developed in [9]. As an application, we derive a classification of homogeneous Willmore 2-spheres in S^{n+2}. They turn out to be exactly the only homogeneous minimal 2-spheres. Using the loop group theory, it is also easy to derive a characterization of homogeneous Willmore complex planes (including tori) in terms of potentials. Note there exist examples of homogeneous Willmore tori and planes which are not minimal in any space form, which is different from the 2-sphere. See e.g. [9], [13] and [18]. Moreover, we also show that there exists no homogeneous Willmore upper half plane.

The paper is organized as follows: Section 2 is a collection of basic results on homogeneous Riemann surfaces. Then we recall some basic results on the loop group theory of Willmore surfaces in Section 3. In Section 4 we derive a description of homogeneous Willmore planes with abelian group action. Section 5 provides a non-existence theorem of homogeneous Willmore upper half plane. Then we end this paper with a proof of the classification of homogeneous Willmore 2-spheres in spheres.

The present paper was basically finished in 2012. It was finalized after the second named author had written separately a note on a partial result of this paper.

2. Basic Results

2.1. Introductory definitions and results. The notion of a homogeneous Willmore surface can be given in several ways. We adopt here the most direct one. Note that for a Riemannian manifold we denote by $Conf(M)$ the group of conformal diffeomorphisms.

Definition 2.1. *Let* $X = H/H_0$ *be a connected homogeneous Riemann surface with a connected Lie group H and a closed Lie subgroup H_0 of bi-holomorphic maps of X and let* $y : X \to S^m$ *be a Willmore immersion. Then y is called "homogeneous" relative to H, if there exists a (continuous) homomorphism $R : H \to Conf(S^m)$ such that $y(h.p) = R(h)y(p)$ for all $p \in X$ and all $h \in H$.*

Since $X = H/H_0$ is a two-dimensional real manifold with a Riemannian metric g and a transitive conformal action of the connected group H, we can assume that either $X \cong \mathbb{R}P^2$ or $X = H/H_0$ is a Riemann surface and H a transitive group of bi-holomorphic maps.

From the classification of homogeneous Riemann surfaces we obtain: (See Theorem V.4.1 of [15])

(1) The universal cover $\tilde{X}$ of X is either the unit sphere S^2, the unit disk $\mathbb{E}$ (which we will frequently replace by the bi-holomorphically equivalent upper half-plane $\mathbb{H}$), or the whole complex plane.
(2) The cylinder $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
(3) The tori $\mathbb{T} \cong \mathbb{C}/\mathcal{L}$, where $\mathcal{L}$ is a rank two lattice in $\mathbb{C}$.

It turns out that for each of the spaces listed above any (connected) transitive Lie group of bi-holomorphic automorphisms contains a transitive subgroup of a special type. Below we list these groups. The following result is well known (It can be derived from Theorem V.4.1 of [15]).

Theorem 2.2. *Let X denote any of the homogeneous Riemann surfaces listed above.*

(1) *If $X \cong S^2$, then each transitive Lie group of bi-holomorphic maps contains a conjugate of $SU(2)/\{\pm I\}$.*
(2) *If $X \cong \mathbb{H}$, then each transitive Lie group of bi-holomorphic maps contains a conjugate of the group Δ of real upper triangular matrices of determinant 1 and positive diagonal elements.*
(3) *If $X \cong \mathbb{C}$, then each transitive Lie group of bi-holomorphic maps contains the group of all translations.*
(4) *If $X = \mathbb{C}^*$, then each transitive Lie group of bi-holomorphic automorphisms contains $\mathbb{C}^*$ acting by multiplication.*
(5) *If $X \cong \mathbb{T} \cong \mathbb{C}/\mathcal{L}$, then each transitive Lie group of bi-holomorphic maps contains the group of all translations.*

The original question of dealing with conformal transformations has turned into a question of dealing with bi-holomorphic automorphisms of Riemann surfaces. In particular, if $X = H/H_0$ is a Riemann surface on which the connected group H acts bi-holomorphically and transitively and $y : X \to S^m$ is a homogeneous Willmore surface relative to H, then the natural projection $\tilde{\pi} : \tilde{M} \to M$ yields a Willmore immersion $\tilde{y} : \tilde{M} \to S^m$, given by $\tilde{y} = y \circ \tilde{\pi}$. Moreover, there exists some connected transitive group $\tilde{H}$ of bi-holomorphic automorphisms of $\tilde{X}$ which acts transitively on $\tilde{X}$ and satisfies

$$\tilde{y}(\tilde{\gamma}.z) = R(\tilde{\gamma})\tilde{y}(z) \text{ for all } z \in \tilde{M} \text{ and all } \tilde{\gamma} \in \tilde{H}.$$

As a consequence we obtain:

Theorem 2.3. *If* $y : X = H/H_0 \to S^m$ *is a homogeneous Willmore surface different from* S^2 *and* $\tilde{\pi}$ *the natural projection from the universal cover* $\tilde{X}$ *to* X, *then the lift* $\tilde{y} = y \circ \tilde{\pi}$ *is a homogeneous Willmore surface defined on* $\tilde{M}$. *In particular,* $\tilde{y}$ *satisfies*

$$(2.1) \qquad \tilde{y}(\tilde{\gamma}.z) = R(\tilde{\gamma})\tilde{y}(z) \text{ for all } z \in \tilde{M} \text{ and all } \tilde{\gamma} \in \tilde{H},$$

and R *is a homomorphism of Lie groups.*

3. The loop group formalism

From now on, let $y : X \to S^{n+2}$ denote a homogeneous Willmore surface. Then with $\tilde{X}$ denoting the universal cover of X we obtain a homogeneous Willmore immersion $\tilde{y} : \tilde{X} \to S^{n+2}$. We recall briefly from [9] the basic facts about the loop group approach to Willmore immersions.

The oriented conformal Gauss map of a conformal immersion y maps each point p of X to the oriented mean curvature 2-sphere of y at p. In this way, one obtains a map $f : \tilde{X} \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$. Note that a well-known fact [1, 3, 14] states that y is Willmore if and only if f is a harmonic map. We refer to [5, 9] for more details.

If $\tilde{X}$ is non-compact, then there exists a smooth extended frame $F(z, \bar{z}, \lambda)$, the frame of the conformal Gauss map f of $\tilde{y}$. Moreover, $f : \tilde{X} \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$ is an associated family of conformally harmonic maps, where $f = \pi(F)$ for some frame $F : \tilde{X} \to SO^+(1, n + 3)$ of f. Here $\pi : SO^+(1, n + 3) \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$ is the standard projection. Every harmonic map from $\tilde{X}$ to some symmetric space can be derived from some holomorphic potential as well as from some normalized potential (for notation see [9]).

In the case of $X = S^2$ there exists some meromorphic potential, a meromorphic one form on S^2, with the appropriate behaviour in λ which induces the conformal Gauss map f associated with the Willmore immersion y. Moreover, the monodromy at each pole of the potential is trivial ($= \pm I$). For more details concerning this case, see [9, 10].

So far we have only discussed potentials for Willmore immersions from X (and its universal cover $\tilde{X}$) to S^{n+2}. The assumed homogeneity imposes additional restrictions. We have seen above that there exists some subgroup H of the group of all conformal transformations of $\tilde{X}$ which acts transitively on $\tilde{X}$ and for which we have:

$$(3.1) \qquad R_h y(z) = y(h(z)), \quad \text{for all} \quad z \in \tilde{X}, \ h \in H,$$

where R_h is a conformal transformation of S^{n+2}.

This induces (see e.g. [7, 8, 11]) quite directly from the basic definitions the relation

$$(3.2) \qquad \chi(h, \lambda)F(z, \bar{z}, \lambda) = F(h(z), \overline{h(z)}, \lambda)k(z, \bar{z}), \quad \text{for all} \quad z \in \tilde{X},$$

where $h \in H$ is arbitrary and k is a λ−independent matrix function with values in the stabilizer group $K = SO^+(1, 3) \times SO(n)$. Here $\chi(h, \lambda)$ is the monodromy matrix depending on h and λ, which appears due to the path dependence of the lift F.

On the level of potentials we consider on the universal cover $\tilde{X}$ of X a potential which is

the pull-back of some one-form defined on X. Then the fundamental group $\pi_1(X)$ acts on $\tilde{X}$ and satisfies

$$(3.3) \qquad \gamma^*\eta = \eta \quad \text{for all} \quad \gamma \in \pi_1(X).$$

In the orientable case, this has some restrictive meaning only in the case of cylinders and tori. Later a similar formula will apply to the only non-orientable homogeneous case.

We will now continue to discuss the homogeneous Willmore surfaces according to what universal cover they have.

4. The case of $\tilde{X} = \mathbb{C}$

In this section we deal with the cases (3) and (5) above. Let H be a connected Lie subgroup of the group of bi-holomorphic automorphisms of $\mathbb{C}$. Then Theorem 2.2 implies that in this case there always exists a connected abelian transitive Lie group.

Theorem 4.1.

(1) *If $y : \mathbb{C} \to S^{n+2}$ is a homogeneous Willmore surface relative to a group H of translations, then the Maurer-Cartan form α of y can be assumed to be of the form $\alpha = \mathcal{A}(\lambda)\mathrm{d}u + \mathcal{B}(\lambda)\mathrm{d}v$, where $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are independent of u and v (with $z = u + iv$) and satisfy*

$$(4.1) \qquad [\mathcal{A}(\lambda), \mathcal{B}(\lambda)] = 0.$$

(2) *Conversely, if $\alpha = \mathcal{A}(\lambda)\mathrm{d}u + \mathcal{B}(\lambda)\mathrm{d}v$ has the same form as the Maurer-Cartan form in Proposition 2.2 of [9] for each $\lambda \in S^1$, with $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ being constant in (u, v) and satisfying (4.1), then*

$$(4.2) \qquad F(z, \bar{z}, \lambda) = e^{u\mathcal{A}(\lambda)+v\mathcal{B}(\lambda)}$$

is the extended frame of a homogeneous Willmore immersion $y : \mathbb{C} \to S^{n+2}$. In particular, y can be read off from $F(z, \bar{z}, \lambda)$ by Proposition 2.2 of [9].

Proof. One can apply Proposition 2.2 of [9] and obtain that the Maurer-Cartan form α of an extended frame $F(z, \lambda)$ of y satisfying $F(0, \lambda) = I$ is of the form $\alpha = \mathcal{A}(\lambda)\mathrm{d}u + \mathcal{B}(\lambda)\mathrm{d}v$. The integrability condition then yields that $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ commute. The property of being equivariant relative to all translations implies $F(z + z_0, \lambda)|_{\lambda=1} = R(z_0)F(z, \lambda)|_{\lambda=1}$ for any $z_0 \in \mathbb{C}$. As a consequence, one has

$$F(u + vi, \lambda)|_{\lambda=1} = \exp(uE_1 + vE_2)F(0, \lambda)_{\lambda=1} = \exp(uE_1 + vE_2)$$

for some E_1 and E_2. Hence $\mathcal{A}(\lambda)|_{\lambda=1} = E_1$ and $\mathcal{B}(\lambda)|_{\lambda=1} = E_2$. In particular, $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are constant in z and $\bar{z}$.

The converse part is also a straightforward application of Proposition 2.2 of [9]. From the form of F stated in (4.2) above it follows that y is homogeneous. $\qquad\square$

In the case just discussed one only needs to quote a result which involves the loop group technique for Willmore surfaces. For the other cases we will need to involve the loop group technique in more detail.

5. The case of $\tilde{X} = \mathbb{H}$

In this case, we can assume w.l.g. that the group H contains at least the group

$$(5.1) \qquad H_1 = \left\{ \gamma = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{R}^+, b \in \mathbb{R} \right\}, \text{ with } \gamma.z = \frac{az+b}{0z+a^{-1}} = a^2 z + ab.$$

So we have

$$(5.2) \qquad\qquad y(\gamma.z) = \chi(\gamma)y(z)$$

for γ. Set

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have

$$y(e^{t\sigma_3}.z) = e^{tB}y(z) \quad \text{and} \quad y(e^{sv}.z) = e^{sD}y(z).$$

Here B and D are the images of σ_3 and v by the monodromy representation χ respectively. Note: we obtain

$$(5.3) \qquad\qquad [B, D] = 2D.$$

It is straightforward to compute

$$(e^{1/2 \ln(v)\sigma_3} e^{v^{-1}uv}).i = z.$$

Altogether we thus obtain

$$(5.4) \qquad y(z) = y(u+iv) = e^{1/2 \ln(v)B} e^{v^{-1}uD} y(i) = e^{uD} e^{1/2 \ln(v)B} y(i).$$

Similarly we obtain for the frame (and equally well for the extended frame):

$$(5.5) \qquad F(z) = F(u+iv) = e^{1/2 \ln(v)B} e^{v^{-1}uD} F(i)k(u,v) = e^{uD} e^{1/2 \ln(v)B} F(i)\hat{k}(u,v).$$

Now we apply what was mentioned at the end of the last section. It is well known that one can classify, up to conjugation, which choices there exist for B and D. In formula (7) of [20], it is shown that each non-abelian maximal solvable subalgebra of $so(1, p)$ is of the form $\mathfrak{s} = \mathbb{R}S \oplus \mathcal{N}$ with S a semisimple matrix and $\mathcal{N}$ an abelian subalgebra of nilpotent matrices of dimension $p-1$ such that $[S, N] = N$ for all $N \in \mathcal{N}$. Considering

$$(5.6) \qquad\qquad S = \left\{ \begin{pmatrix} 0 & a & X^t \\ a & 0 & -X^t \\ X & X & 0 \end{pmatrix} \right\},$$

it is easy to verify the properties listed above for a maximal solvable subalgebra of $so(1, p)$. Therefore S is maximal solvable.

It follows that up to a conformal transformation of our original Willmore immersion we can assume that the monodromy group of our homogeneous Willmore upper half-plane has the form (5.6). In particular, D has the form

$$\begin{pmatrix} 0 & 0 & X^t \\ 0 & 0 & -X^t \\ X & X & 0 \end{pmatrix}.$$

On the other hand since $\exp(tv)$ is a translation along the v−axis, we can view y as an equivariant surface along v direction. So the corresponding monodromy gives a Delauney type matrix [4, 12]. But a Delauney type matrix can not be of the above form, since it is derived by the Maurer-Cartan form of $F(z, \lambda)$ [12]. Altogether we obtain

Theorem 5.1. *There exists no homogeneous Willmore immersion from $\mathbb{H}$ to S^{n+2}.*

6. The case of $X = S^2$

In this section, we will show that a homogeneous Willmore 2-sphere is conformally equivalent to either the round 2-sphere or one of the Borůvka-Veronese 2-spheres in S^{2m}. The main idea is to use the representations of $\mathfrak{so}(3)$ in $\mathfrak{so}^+(1, n + 3)$ and loop group theory to show that a representation is irreducible and hence the orbit gives one of the Borůvka-Veronese 2-spheres.

We introduce our main result in Section 6.1. Then we consider the monodromy matrices of S^2 and their representations in Section 6.2. Using these matrices, in Section 6.3 we obtain the normalized potentials of the homogeneous Willmore 2-spheres as well as a property of the matrices. With these preparations, we can show in Section 6.4 that the representation is necessarily irreducible and obtain the proof of our main result. Finally we also consider the antipodal symmetries of the surfaces and show whether they will reduce to $\mathbb{R}P^2$ or not.

6.1. The main result.

Theorem 6.1. *Let $y : M \to S^{n+2}$ be a homogeneous and full Willmore S^2, $n \geq 1$. Then $n = 2m - 2$ and y is conformally equivalent to the full Borůvka-Veronese 2-sphere in S^{2m}.*

The theorem is a corollary of the following result and the classical results of Borůvka [2] and Calabi [6] describing homogeneous minimal two-spheres in S^{n+2}.

Proposition 6.2. *Each full homogeneous Willmore two-sphere is conformally equivalent to a minimal two-sphere in S^{n+2}.*

In Section 6.4 we will provide a proof of Proposition 6.2. Before proceeding, let's first explain the idea of the proof. Since the surface is homogeneous, it can be viewed as an orbit of a group $SO(3) \subset SO^+(1, n + 3)$. This induces a representation of $\mathfrak{so}(3)$ in $\mathfrak{so}^+(1, n + 3)$. Using this representation, we can describe the Maurer-Cartan form of the surface y. The Willmore property allows us to introduce the loop parameter λ into the Maurer-Cartan form. Moreover, we can obtain the normalized potential of y. Finally with the help of the loop parameter, we can show that the representation is irreducible and hence we can determine the form of the normalized potential of y. Then by Theorem 1.1 of [21], the surface is minimal in S^{n+2}.

6.2. The monodromy matrices. To begin with, let us first recall some basic results concerning $SO(3)$. Let T_j be a basis of $\mathfrak{so}(3)$ which satisfies the commutation relations (see [16])

(6.1) $[T_3, T_2] = -T_1, \quad [T_3, T_1] = T_2, \quad [T_1, T_2] = T_3.$

Actually we assume (setting $S^2 = \mathbb{C} \cup \{\infty\}$):

(1) The matrix T_1 generates rotations about $1 \in \mathbb{C}$: $e^{tT_1}.z = \frac{z\cos(\frac{1}{2}t) - i\sin(\frac{1}{2}t)}{-zi\sin(\frac{1}{2}t) + \cos(\frac{1}{2}t)}$;

(2) The matrix T_2 generates rotations about $i \in \mathbb{C}$: $e^{tT_2}.z = \frac{z\cos(\frac{1}{2}t) - \sin(\frac{1}{2}t)}{z\sin(\frac{1}{2}t) + \cos(\frac{1}{2}t)}$;

(3) The matrix T_3 generates rotations about $0 \in \mathbb{C}$: $e^{tT_3}.z = e^{-it}z$.

If we have a symmetry $y(g.z) = Uy(z)$ of some Willmore surface into S^{n+2} with $U \in SO(n+3)$, then the corresponding matrix in $SO^+(1, n+3)$, acting on the Grassmannian, the frames etc, is naturally of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}.$$

For a homogeneous Willmore S^2, we can assume that the group acting on the Grassmannian is contained in the natural $SO(n+3)$ as above. So the monodromy representation $\chi_U(g, \lambda = 1)$ takes values in $SO(n+3)$. Considering the representation $\chi_U(g, \lambda)$ of $SO(3)$ in $\Lambda SO^+(1, n+3)_\sigma$ one can decompose it into irreducible ones.

It is well known that all irreducible representations of $SO(3)$ act on odd-dimensional spaces and are uniquely determined by the eigenvalues of the image $\chi_U(T_3, \lambda)$ of $T_3 = \frac{1}{2}diag(-i, i) \in \mathfrak{su}(2)$ under χ_U (See e.g, Theorem 4.12 of [16]). These eigenvalues all are of the form ik, where $k \in \mathbb{Z}$. Moreover, the irreducible summands can be read off from the multiplicities of the eigenvalues of $\chi_U(T_3, \lambda)$. As a consequence, we have

Corollary 6.3. *The monodromy representation of $SO(3)$ is irreducible on $\mathbb{R}^{n+3}$ if and only if the 0-eigenvalue of $\chi_U(T_3, \lambda)$, acting on $\mathbb{R}^{n+4}$, has multiplicity 2.*

Let $y : S^2 \to S^{n+2}$ be a homogeneous Willmore two-sphere. Then the approach of [9] requires to consider an extended frame, from which y can be reproduced again. If $F_U(z, \lambda)$ denotes the extended frame associated with the representation χ_U above, then it turns out to be useful to also consider the extended frame $\hat{F}(z, \lambda) = F_U(0, \lambda)^{-1}F_U(z, \lambda)$. This yields the representation

$$\hat{\chi}(g, \lambda) = F_U(0, \lambda)^{-1}\chi_U(g, \lambda)F_U(0, \lambda)$$

of $SO(3)$ in $\Lambda SO^+(1, n+3)_\sigma$. Note that the image of $SO(3)$ of χ is again compact, but not necessarily contained in $\Lambda SO(n+3)_\sigma$. But all statements about eigenvalues and irreducible representations made above still hold in the present situation as well.

Under the monodromy representation $\hat{\chi}(\cdot, \lambda)$ relative to $\hat{F}$ we define (using, by abuse of notation, $\hat{\chi}$ for $d\hat{\chi}$):

(6.2) $\hat{\chi}(T_1, \lambda) = A_1(\lambda), \ \hat{\chi}(T_2, \lambda) = A_2(\lambda)$ an $\hat{\chi}(T_3, \lambda) = A_3$.

Note that by [12] (also see below at the end of the proof of the next proposition), A_3 does not depend on λ and is contained in $\mathfrak{k}$ and in $\mathfrak{so}(n+3)$. More precisely, using also the notation

(6.3) $$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

we will show

Proposition 6.4. *The matrices $A_j(\lambda)$ only contain the powers λ^{-1}, λ^0, and λ^1. Moreover, the matrices $A_1(\lambda)$ and $A_2(\lambda)$ have the form of a generator of a translationally equivariant Willmore surface and A_3 has the diagonal block form $A_3 = (0, S, A_{32})$, with the 2×2 matrices 0 and S and the $n \times n$ matrix A_{32}.*

Proof. Let's consider the matrix $A_1(\lambda)$. We remove 1 from $\mathbb{C}$ and consider the universal cover $\tilde{M} \cong \mathbb{C}$ of $M = \mathbb{C} \setminus \{1\}$. We can do this such that the covering map maps 0 to 0 and that the action of the one parameter group e^{tT_1} lifted to $\tilde{M}$ acts by translations parallel to the real axis. Lifting the extended frame to the universal cover we observe that it attains the value I at 0, whence the monodromy is of the form $e^{t\tilde{A}_1(\lambda)}$ and by [4] it follows that $\tilde{A}_1(\lambda)$ has the form as claimed. It is easy to see that actually $\tilde{A}_1(\lambda) = A_1(\lambda)$ holds. The claim for $A_2(\lambda)$ follows verbatim. Let's consider finally A_3. The relation

$$(6.4) \qquad \hat{F}(e^{it}z, \lambda) = \hat{F}(e^{-itT_3}z, \lambda) = e^{-tA_3(\lambda)} \hat{F}(z, \lambda) k_3(t, z)$$

and the property $\hat{F}(0, \lambda) = I$ imply

$$(6.5) \qquad I = e^{-tA_3(\lambda)} k_3(t, 0).$$

From this we infer that $A_3(\lambda)$ actually is independent of λ and contained in $\mathfrak{k}$. From (4.7) of [11] and the last equation we know how the first 4×4 block of A_3 looks like. Now it is straightforward to verify that A_3 has the form stated. □

6.3. An Application of Wu's Formula. Using the transformation properties of the one-parameter groups generated by the T_j we introduce a new frame $\tilde{F}$ which permits to apply Wu's formula easily. We have $y(e^{i\theta}r) = y(e^{-\theta T_3}.z) = e^{-\theta A_3}.y(r)$. Moreover, considering the rotation about $i \in \mathbb{C} \subset S^2$, we have $y(r) = e^{-2\arctan(r)A_2}.y(0)$ and altogether we have the formula

$$(6.6) \qquad y(e^{i\theta}r) = e^{-\theta A_3} e^{-2\arctan(r)A_2}.y(0).$$

For the frame $\hat{F}$, which satisfies $\hat{F}(z = 0) = I$, we obtain

$$(6.7) \qquad \hat{F}(e^{i\theta}r) = e^{-\theta A_3} e^{-2\arctan(r)A_2} k_2(0, r) k_3(\theta, r),$$

where k_2 and k_3 are the factors occurring in formula (4.7) of [11]. These equations also hold after introducing λ (note that there is no λ inside the k's !). Since $k_2(0, r) k_3(\theta, r) \in SO^+(1, 3) \times SO(n)$, we consider a new extended frame of y:

$$(6.8) \qquad \tilde{F}(e^{i\theta}r, \lambda) = e^{-\theta A_3} e^{-2\arctan(r)A_2(\lambda)} e^{\theta A_3}.$$

Note that $\tilde{F}$ and $\hat{F}$ have the same monodromy representation, since they only differ by a gauge in K. Moreover, $\tilde{F}$ is a priori only defined, where the polar representation for z is valid. However, since A_3 is independent of λ and has purely imaginary eigenvalues with integer imaginary parts, the frame $\tilde{F}$ actually is defined on $\mathbb{C}^*$. The following proposition shows it is in fact defined on $\mathbb{C}$.

Proposition 6.5. *The frame $\tilde{F}$ has the form (where $z = u + iv$)*

$$(6.9) \qquad \tilde{F}(z, \lambda) = \tilde{F}(e^{i\theta}r, \lambda) = e^{-\theta A_3} e^{-2\arctan(r)A_2(\lambda)} e^{\theta A_3} = e^{\frac{-2\arctan(r)}{r}(uA_2(\lambda) + vA_1(\lambda))}.$$

In particular, $\tilde{F}$ is real analytic at $z = 0$ and we have $\tilde{F}(0, \lambda) = I$. As a consequence, $\tilde{F}$ and

$\hat{F}$ *have the same normalized potential and generate the same Willmore surface.*

Proof. In view of (6.8) it suffices to consider

$$E = e^{-\theta A_3} \arctan(r) A_2(\lambda) e^{\theta A_3} = \arctan(r) exp(-\theta ad(A_3)(A_2)).$$

But by (6.1) this unravels to

$$E = \arctan(r)(A_2 \cos\theta + A_1 \sin\theta).$$

Now in the upper half-plane the angle θ of an element of $\mathbb{C}$ in the range $(-\pi, \pi)$ is given by $\arccos(u/r)$ and by $-\arccos(u/r)$ in the lower half-plane. Hence $E = \arctan(r)(\frac{u}{r}A_2 + \frac{\sqrt{r^2-u^2}}{r}A_1) = \frac{\arctan(r)}{r}(uA_2 + vA_1)$ for $u + iv$ in the upper half-plane and $E = \arctan(r)(\frac{u}{r}A_2 - \frac{\sqrt{r^2-u^2}}{r}A_1) = \frac{\arctan(r)}{r}(uA_2 + vA_1)$ for $u + iv$ in the lower half-plane. As a consequence, E is well defined on $\mathbb{C}$ and real-analytic there. Finally, it is clear now that $E(z = 0) = 0$, whence $\tilde{F}(0, \lambda) = I$. □

As a consequence of (6.9) we can compute the normalized potential of the given Willmore immersion by using Wu's Formula [22]. In the statement of loc.cit one considers the Maurer-Cartan form of some frame for the given harmonic map (in our case the conformal Gauss map of the given Willmore surface). Then, at least in some neighbourhood of the base point, say, $z = 0$, one can set $\bar{z} = 0$, since the frame is real-analytic. From the resulting differential one-form one computes the normalized potential.

Actually, since the frame $\tilde{F}$ is real-analytic near the origin, one can set $\bar{z} = 0$ already directly in the frame. Doing this we obtain (by using (6.9))

$$\tilde{F}(z, \bar{z} = 0, \lambda) = e^{-zA_2(\lambda)+izA_1(\lambda)},$$

since the real analytic function $\arctan(r)/r$ attains the value 1 at $r = 0$. Now, the Maurer-Cartan form of $\tilde{F}(z, \bar{z} = 0, \lambda)$ is

(6.10) $$hol(\alpha) = (-A_2(\lambda) + iA_1(\lambda))dz.$$

In view of Proposition 6.4 we can write

(6.11) $$A_1(\lambda) = \lambda^{-1}H_1 + H_0 + \lambda\bar{H}_1 \text{ and } A_2(\lambda) = \lambda^{-1}L_1 + L_0 + \lambda\bar{L}_1.$$

From this we derive

Theorem 6.6. *We have*

(6.12) $$L_1 = -iH_1,$$

and the normalized potential ξ of the given homogeneous Willmore surface is

(6.13) $$\xi(z)dz = e^{z\beta_0}\beta_1 e^{-z\beta_0} = e^{zad(\beta_0)}\beta_1,$$

where $\beta_0 = -L_0 + iH_0$ and $\beta_1 = \lambda^{-1}(-L_1 + iH_1)$.

Proof. The first claim follows from the fact that $hol(\alpha)$ does not contain the power $\lambda = \lambda^1$, as stated in the proof of "Wu's Formula", Theorem 2.1 of [22]. From the proof of Theorem 2.1 of [22] it is also clear that we need to consider $\beta_1 = \lambda^{-1}(-L_1 + iH_1)$ and $\beta_0 = -L_0 + iH_0$ and solve the ode $d\psi_0 = \psi_0\beta_0$ with $\psi_0(0) = I$. Since this solution is $\psi_0(z) = e^{z\beta_0}$ the claim

follows directly from Theorem 2.1 of [22].   □

6.4. Uniqueness of the monodromy representation. First we describe A_3 explicitly in view of Proposition 6.4.

Lemma 6.7. *We have (up to a conjugation in K) that A_3 is a diagonal block matrix*

$$(6.14) \qquad\qquad A_3 = diag\{0_{2\times 2}, S, \mathbb{S}\}, \ with \ \mathbb{S} = (S_0, S_1, \dots),$$

where $S_j = j \begin{pmatrix} S & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & S \end{pmatrix}_{2n_j \times 2n_j}$ *and* $0 \leq j \leq m$. *See* (6.3) *for the definition of S.*

6.4.1. **Irreducibility of the monodromy representation.** By Corollary 6.3, the monodromy representation is irreducible if and only if the S_0−term does not show up in (6.14). Substituting (6.11) and (6.12) into the commutation properties $[A_3, A_2(\lambda)] = -A_1(\lambda)$, $[A_3, A_1(\lambda)] = A_2(\lambda)$, $[A_1(\lambda), A_2(\lambda)] = A_3$, we obtain

Lemma 6.8.
(1) $[A_3, L_1] = -iL_1$, $[A_3, [A_3, L_1]] = -L_1$;
(2) $[A_3, -L_0 + iH_0] = -i(-L_0 + iH_0)$, $[A_3, [A_3, -L_0 + iH_0]] = -(-L_0 + iH_0)$;
(3) $[H_0, L_0] + 2i[L_1, \bar{L}_1] = A_3$;
(4) $[\bar{L}_1, -L_0 + iH_0] = 0$.

We decompose L_1 and $-L_0 + iH_0$ according to (6.14)

$$L_1 = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} \text{ with } B_1 = \begin{pmatrix} B_{10} & \cdots & B_{1m} \\ B_{20} & \cdots & B_{2m} \end{pmatrix}$$

and

$$-L_0 + iH_0 = \begin{pmatrix} \hat{R}_{11} & \hat{R}_{00} & 0 & \cdots & 0 \\ -\hat{R}_{00}^t I_{1,1} & \hat{R}_{22} & 0 & \cdots & 0 \\ 0 & 0 & R_{00} & \cdots & R_{0m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -R_{0m}^t & \cdots & R_{mm} \end{pmatrix}.$$

Set

$$Q_1 = \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \text{ and } e_0 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

By using (1) and (2) of Lemma 6.8 and elementary computations, we derive

Lemma 6.9.
(1) $B_{1j} = 0$ *for all* $j \neq 1$ *and* $B_{11} = \begin{pmatrix} a_{11}Q_1 & \cdots & a_{1n_1}Q_1 \end{pmatrix}$;
(2) $B_{2j} = 0$ *for all* $j \neq 0, 2$, *and*

$$B_{20} = \begin{pmatrix} b_{11}e_0 & \hat{b}_{11}e_0 & \cdots & b_{1n_0}e_0 & \hat{b}_{1n_0}e_0 \end{pmatrix}, \ B_{22} = \begin{pmatrix} c_{11}Q_2 & \cdots & c_{1n_2}Q_2 \end{pmatrix};$$

(3) $\hat{R}_{11} = \hat{R}_{22} = 0$ *and* $\hat{R}_{00} = \begin{pmatrix} a & -ia \\ c & -ic \end{pmatrix}$;

(4) *In the (skew-symmetric) block of matrices R_{jk} all matrices, except the ones next to the diagonal, vanish. The matrices $R_{j,j+1}$, $j = 0, \ldots, m-1$ just right of the diagonal satisfy $R_{j,j+1} = \left(q_{kl}^{(j)} Q_2 \right)_{1 \leq k \leq n_j, 1 \leq l \leq n_{j+1}}$.*

Lemma 6.10. *If $B_{20} = 0$, then $A_1(\lambda)$, $A_2(\lambda)$ and A_3 define an irreducible representation.*

Proof. In this case, by (3) of Lemma 6.8, we have $R_{01}\bar{R}_{01}^t - \bar{R}_{01}R_{01}^t = 0$. As a consequence we have $\sum_{l=1}^{n_0} |q_{kl}^{(0)}|^2 = 0$, for all $k = 1, \cdots, n_0$. That is, $R_{01} = 0$. So all $A_1(\lambda)$, $A_2(\lambda)$ and A_3 are of the form

$$\begin{pmatrix} \cdots & 0 & \cdots \\ 0 & 0 & 0 \\ \cdots & 0 & \cdots \end{pmatrix}.$$

As a consequence, the representation is in a smaller space, contradicting the fullness of the surface. Hence A_3 has no S_0 term, i.e., A_3 has exactly two 0−eigenvalues and by Corollary 6.3 the representation is irreducible. $\qquad\square$

So we only need to show

Lemma 6.11. $B_{20} = 0$.

Proof. Suppose $B_{20} \neq 0$. Recall that for Willmore surfaces one has $B_1^t I_{1,3} B_1 = 0$. As a consequence

$$B_{22}^t B_{20} = \left(c_{1j} b_{1k} e_0 \right)_{1 \leq j \leq n_2, 1 \leq k \leq n_0} = 0.$$

Since $B_{20} \neq 0$, one of the b_{1k} satisfies $b_{1k} \neq 0$ and hence $c_{1j} = 0$ for all j, whence, $B_{22} = 0$.

Moreover, by (4) of Lemma 6.3, we have

$$-\hat{R}_{00}^t I_{1,1} \bar{B}_{11} + \bar{B}_{20} R_{01} = 0 \text{ and } \bar{B}_{11} R_{01}^t + \hat{R}_{00} \bar{B}_{20} = 0.$$

The last equation reads
(6.15)
$$\left(\sum_{j=1}^{n_1} 2a_{1j} q_{1j}^{(0)} Q_1 \quad \cdots \quad \sum_{j=1}^{n_1} 2a_{1j} q_{n_0 j}^{(0)} Q_1 \right) + 2 \begin{pmatrix} a\bar{b}_{11} & a\bar{\hat{b}}_{11} & \cdots & a\bar{b}_{1n_0} & a\bar{\hat{b}}_{1n_0} \\ c\bar{b}_{11} & c\bar{\hat{b}}_{11} & \cdots & c\bar{b}_{1n_0} & c\bar{\hat{b}}_{1n_0} \end{pmatrix} = 0.$$

Next we claim that $|a| < |c|$ holds. As a consequence, (6.15) can not holds, since one of the b_{1k} satisfies $b_{1k} \neq 0$.

Now let's prove the claim. Consider the left upper 4×4 diagonal block of the equation (3) of Lemma 6.8: $[H_0, L_0] + 2i[L_1, \bar{L}_1] = A_3$. We obtain that

$$|c|^2 - |a|^2 = 1 + 2\sum_j (|b_{1j}|^2 + |\hat{b}_{1j}|^2 + 2|c_{1j}|^2),$$

from which we see that $|c|^2 - |a|^2 \geq 1 > 0$.

The above contradiction indicates $B_{20} = 0$, which finishes the proof. $\qquad\square$

Proof of Proposition 6.2. From the proof of the above lemma we see that $|c| > |a|$ and $a/c \in \mathbb{R}$. Consider the hyperbolic rotation $T_t = \text{diag}\{T_{1t}, I\}$ with $T_{1t} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ and $a\cosh t + c\sinh t = 0$. The new representation $T_1 A_j T_1^{-1}$ has the same form as A_j except

that now the new $\tilde{a}$ satisfies $\tilde{a} = 0$. So w.l.g. we can assume that $a = 0$. Now we see that all of $A_1(\lambda)$, $A_2(\lambda)$ and A_3 take values in $\mathfrak{so}(n + 3)$ and as a consequence, the normalized potential (6.13) takes values in $\mathfrak{so}(n + 3)$. By Theorem 1.1 of [21], y is a minimal surface in S^{n+2}. $\square$

6.5. On homongeneous Willmore $\mathbb{R}P^2$. Finally we consider the case where a homogeneous Willmore immersion from the two-sphere descends to a map defined on $\mathbb{R}P^2$, i.e., where the Willmore two-sphere is invariant under the fixed point free antiholomorphic involution $\mu(z) = -\frac{1}{\bar{z}}$. By (6.16), we obtain

$$(6.16) \qquad y(\mu(z)) = y(-e^{i\theta}\frac{1}{r}) = e^{-\theta A_3}e^{2\arctan(\frac{1}{r})A_2}.y(0) = e^{-\theta A_3}e^{-2\arctan(r)A_2}e^{\pi A_2}.y(0).$$

Here we have used the fact that $\arctan(\frac{1}{r}) + \arctan(r) = \frac{\pi}{2}$.

Let's consider next the action $e^{\pi A_2}.y(0)$. It is a rotation of $y(0)$ by degree $m\pi$. Here m is the degree of the representation and hence half of the dimension of S^{2m} if y is full in S^{2m}. Obviously, $e^{\pi A_2}.y(0) = y(0)$ if $m = 2\tilde{m}$ and $e^{\pi A_2}.y(0) = -y(0)$ if $m = 2\tilde{m} + 1$.

If $e^{\pi A_2}.y(0) = y(0)$, then we have $y(\mu(z)) = y(z)$. If $e^{\pi A_2}.y(0) = -y(0)$, we have $y(\mu(z)) = -y(z)$.

Summing up we obtain:

Corollary 6.12. *Let y be a homogeneous Willmore two-sphere.*

(1) *If y is full in $S^{4\tilde{m}}$, it descends to a Willmore immersion from $\mathbb{R}P^2$.*

(2) *The full homogeneous Willmore two-sphere in $S^{4\tilde{m}-2}$ has an antipodal symmetry*

$$y(-\frac{1}{\bar{z}}) = -y(z).$$

REMARK 6.13. This can also be read off from the spherical harmonics functions. If the degree of these function is even, then the surface reduces to an immersion from $\mathbb{R}P^2$. Otherwise the surface has an antipodal symmetry as above. We believe this result is well-known to experts. But so far we did not find a reference in the literature.

NOTE ADDED IN PROOF. Since the acceptance of this paper we have split reference [9] into three papers. The two papers containing relevant information for the present paper are:

J. Dorfmeister and P. Wang: Weierstrass-Kenmotsu representation of Willmore surfaces in spheres, to appear in Nagoya Mat. J, doi:10.1017/nmj.2020.6.

J. Dorfmeister and P. Wang: Willmore surfaces in spheres: the DPW approach via the conformal Gauss map, Abh. Math. Semin. Univ. Hambg. 89 (2019), 77–103.

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