ON A CONSTRUCTION OF RECURRENT
MARKOV CHAINS

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Let $S$ be a denumerable (possibly finite) set and $B$ the space of all real valued and bounded functions defined on $S$. For a given measure $\mu$, strictly positive at each point of $S$, we shall denote by $N(\mu)$ the collection of functions $f$ such that the support of $f$ is finite and $\langle \mu, f \rangle = \sum_{x \in S} \mu(x)f(x) = 0$. A linear operator $R$ from $N(\mu)$ to $B$ is said to satisfy the semi-complete maximum principle if it has the following property:

\[(S.C.M) \quad \text{For any } f \in N(\mu), \text{ if } Rf \leq m \text{ on the set } \{f > 0\}, \text{ then } Rf \leq m \text{ everywhere, where } m \text{ is a real constant.}\]

We know that if $R$ is a weak potential operator for a recurrent semi-group $(P_t)_{t \geq 0}$ with an invariant measure $\mu$, it satisfies this maximum principle [7, p. 337]. In this work we shall consider the converse problem: Given a measure $\mu$ and a linear operator $R$ satisfying (S.C.M), can we find a recurrent semi-group $(P_t)_{t \geq 0}$ which has $\mu$ as an invariant measure and $R$ as a weak potential operator?

If $\mu$ is bounded, this problem has an affirmative answer, which will be stated in section 2. However, if $\mu$ is unbounded, there are several cases, for example, some operators are weak potential operators for transient semi-groups with invariant measure $\mu$ and others are never weak potential operators for any Markov semi-group with invariant measure $\mu$. We shall give such examples in section 3. The appropriate conditions under which the problem is solved are not known yet. In section 1 we shall study, for later use, another type of maximum principle which is satisfied by weak potential operators (weak inverses in Orey [10]) for recurrent Markov chains with discrete parameters.

1. Potential operators satisfying the reinforced semi-complete maximum principle

Throughout this work notations and terminology are mainly taken from [7]. We shall denote the collection of all non-empty finite subsets of $S$ by $\mathcal{K}$. Further, for each $E \in \mathcal{K}$, we shall use the following notations:
The function restricted to $E$.

The measure restricted to $E$.

The space of all functions $f_E$.

The space of functions of $N(μ)$ with supports in $E$.

For any function $f$ on $S$, $f^+ = \sup (f, 0)$ and $f^- = \sup (-f, 0)$. The indicator function of a set $E$ will be denoted by $χ_E$.

A linear operator $G$ from $N(μ)$ to $B$ is said to satisfy the reinforced semi-complete maximum principle if it has the following property:

(R.S.C.M) For any function $f \in N(μ)$, if $Gf ≤ m$ on the set $\{f > 0\}$, then $Gf \leq m - f^-$ everywhere, where $m$ is a real constant.

Let $G$ be a linear operator from $N(μ)$ to $B$ satisfying (R.S.C.M).

**Lemma 1.** $G$ is non-singular in the sense: If $f$ is a non-zero element of $N(μ)$, then $Gf$ is never equal to a constant on the support of $f$. So that $Gf = 0$ implies $f = 0$.

Proof. Let $f$ be a non-zero element of $N(μ)$ and $Gf = m$ on the support of $f$, where $m$ is a constant. From (R.S.C.M) it follows that $Gf \leq m - f^-$ everywhere and hence, $m = Gf \leq m - f^-$ on the set $\{f < 0\}$. Therefore $f^- = 0$. Similarly we have $f^+ = 0$, for, $-m = G(-f) \leq -m - (-f)^- = -m - f^+$ on the set $\{f > 0\}$. Thus $f = 0$, which is a contradiction.

**Lemma 2.** There is a family of (signed) measures $(λ^E)_{E \in K}$ on $S$ such that:

(i) the support of each $λ^E$ is contained in $E$, (ii) $\langle λ^E, 1 \rangle = 1$ and (iii) $\langle λ^E, Gf \rangle = 0$ for all $f \in N^E$. Such a family is unique.

Proof. Let $E \in K$ and the number of elements of $E$ be $n$. Then the linear dimensions of $B_E$ and $N^E$ are equal to $n$ and $n - 1$ respectively. Let us define a linear operator $G^E$ from $N^E$ to $B_E$ by

(1.1) $G^Ef = (Gf)_E$ for $f \in N^E$.

From Lemma 1 it follows that if $G^Ef = 0$, then $f = 0$ and that $1_E$, the restriction of the function $1$ to $E$, does not belong to the range $G^E(N^E)$. Therefore, since $\dim G^E(N^E) = \dim N^E = n - 1$ and $1_E \in G^E(N^E)$, we can find exactly one linear functional $l_E$ on $B_E$ such that $l_E(1_E) = 1$ if and only if $g_E \in G^E(N^E)$ and $l_E(1_E) = 1$. Thus if we define the measure $λ^E$ by $λ^E(y) = l_E(χ_{\{y\}})$ for $y \in E$ and $λ^E(y) = 0$ for $y \in S \setminus E$, the family $(λ^E)_{E \in K}$ is the desired one. The uniqueness of $(λ^E)_{E \in K}$ is obvious from the above proof.

Let $g \in B$ and $E \in K$. If we put $h_E = (g - \langle λ^E, g \rangle)_E$, then $l_E(h_E) = \langle λ^E, g \rangle - \langle λ^E, g \rangle = 0$, so that we can find unique $f^E \in N^E$ such that $h_E = G^Ef^E$. Now
let us define the mappings $H^E$ and $\Pi^E$ from $B$ to $B$ by

(1.2) \quad H^E g = Gf^E + \langle \lambda^E, g \rangle

and

(1.3) \quad \Pi^E g = Gf^E + \langle \lambda^E, g \rangle - f^E = H^E g - f^E

respectively. Obviously, $H^E$ and $\Pi^E$ are linear and $H^E g = \Pi^E g$ on $S \setminus E$.

**Lemma 3.** (i) If $g \geq 0$ on $E$, then $H^E g \geq 0$ and $\Pi^E g \geq 0$ everywhere. (ii) $H^E 1 = 1$ and $\Pi^E 1 = 1$. (iii) If $E, F \subseteq \mathcal{K}$ and $E \subseteq F$, then $H^F H^E g = H^E g$ and $\Pi^F H^E g = \Pi^E g$.

Proof. Let $g \geq 0$ on $E$ and $H^E g = Gf^E + \langle \lambda^E, g \rangle$ where $f^E \in N^E$. Since $Gf^E + \langle \lambda^E, g \rangle = g$ on $E$, $Gf^E \geq -\langle \lambda^E, g \rangle$ on the support of $f^E$. Therefore, using (R.S.C.M), we have

\[ Gf^E \geq -\langle \lambda^E, g \rangle + (f^E)^+ \]

everywhere, so that

\[ H^E g = Gf^E + \langle \lambda^E, g \rangle \geq (f^E)^+ \geq 0 \]

and

\[ \Pi^E g = Gf^E + \langle \lambda^E, g \rangle - f^E \geq Gf^E + \langle \lambda^E, g \rangle - (f^E)^+ \geq 0 \]

everywhere. Thus, the assertion (i) is true. Next, if $H^E 1 = Gf^E + \langle \lambda^E, 1 \rangle$, then $f^E = 0$ by Lemma 1. Therefore $H^E 1 = H^E 1 = 1$, which implies (ii). Finally, let $E \subseteq F$ and let

\[ h = H^E g = Gf^E + \langle \lambda^E, g \rangle \quad (f^E \in N^E) \]

\[ H^F h = Gf^F + \langle \lambda^F, h \rangle \quad (f^F \in N^F). \]

Since $H^F h = h$ on $F$, we have

\[ Gf^F + \langle \lambda^F, h \rangle = Gf^E + \langle \lambda^E, g \rangle \]
on $F$. Therefore

\[ G(f^F - f^E) = \langle \lambda^E, g \rangle - \langle \lambda^F, h \rangle = \text{const.} \]
on the support of $f^F - f^E$. Using Lemma 1, we have $f^F = f^E$ and $\langle \lambda^E, g \rangle = \langle \lambda^F, h \rangle$, which implies $H^F H^E g = H^E g$ and that

\[ H^F H^E g = H^F h - f^F = h - f^E = \Pi^E g. \]

Thus the assertion (iii) was proved.
From this lemma we can see that $H^E$ and $H^E$ are Markov kernels on $S$ and that for each $x \in S$ the supports of measures $H^E(x, \cdot)$ and $H^E(x, \cdot)$ are contained in $E$.

**Corollary.** If $E, F \in \mathcal{K}$, $E \subseteq F$ and $g$ is a non-negative function on $S$ with support in $E$, then $H^E g \geq H^F g$ everywhere.

For, \[ H^E g(x) = H^F H^E g(x) \]
\[ = \sum_{y \in E} H^F(x, y) H^E g(y) + \sum_{y \in S \setminus E} H^E(x, y) H^E g(y) \]
\[ \geq \sum_{y \in E} H^F(x, y) H^E g(y) \]
\[ = H^F g(x) \]
for all $x \in S$.

**Theorem 1.** Let $\mu$ be a bounded measure which is strictly positive everywhere and $G$ a linear operator from $\mathcal{N}(\mu)$ to $\mathcal{B}$ satisfying the reinforced semi-complete maximum principle. Then there is a kernel $P$ on $S$ such that

\[
\begin{align*}
(1.4) \quad P \geq 0 \quad \text{and} \quad P1 = 1, \\
(1.5) \quad \mu P = \mu, \\
(1.6) \quad (I - P)Gf = f \quad \text{for all} \quad f \in \mathcal{N}(\mu).
\end{align*}
\]

Such a kernel is unique.\(^1\)

Further, $P$ is irreducible recurrent in the sense:

\[
\sum_{n=0}^{\infty} P^n(x, y) = \infty \quad \text{for all} \quad (x, y) \in S \times S.
\]

**Proof.** Let $(E_n)_{n \geq 1}$ be an increasing sequence of $\mathcal{K}$ with the union $S$ and $x, y \in S$. Then, there is some $n$ such that $y \in E_k$ for all $k \geq n$. So that, by Corollary of Lemma 2, we have

\[ H^E_n(x, y) \geq H^E_{n+1}(x, y) \geq \cdots \geq 0. \]

Therefore the limit,

\[
P(x, y) = \lim_{n \to \infty} H^E_n(x, y),
\]
exists for any $(x, y) \in S \times S$. We shall prove the kernel $P$ defined by (1.8) has

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\(^1\) Precisely speaking, a Markov kernel satisfying (1.6), if it exists, is unique, even if $\mu$ is unbounded. We can see this in the proof of the theorem. Similar circumstance occurs in Lemma 5 and Theorem 2 in the next section.
all the properties stated in the theorem. Since $\Pi E_n$ are Markov kernels, $P$ is obviously sub-Markov kernel, that is, $P \geq 0$ and $P \leq 1$, by Fatou's inequality. From the definition of the kernel $H E$, we can find $f E_n \in N E_n$ such that

$$H E_n(x, y) = G f E_n(x) + \lambda E_n(x).$$

Since,

$$\Pi E_n(x, y) = H E_n(x, y) - f E_n(x),$$

we have,

$$\sum_{x \in E_n} \mu(x) \Pi E_n(x, y) = \sum_{x \in E_n} \mu(x) H E_n(x, y) - \sum_{x \in E_n} \mu(x) f E_n(x) = \mu(y),$$

whenever $y \in E_n$. On the other hand, since $0 \leq \chi E_n(x) \Pi E_n(x, y) \leq 1$, \lim $n \chi E_n(x) \Pi E_n(x, y) = P(x, y)$ and $\mu$ is a bounded measure, we have

$$\mu P(y) = \sum_{x \in S} \mu(x) \lim_{n \to \infty} \chi E_n(x) \Pi E_n(x, y) = \lim_{n \to \infty} \sum_{x \in S} \mu(x) \chi E_n(x) \Pi E_n(x, y) = \mu(y)$$

for all $y \in S$. Thus (1.5) was proved. From (1.5) it follows that $\langle \mu, P1 \rangle = \langle \mu P, 1 \rangle = \langle \mu, 1 \rangle$. Since $0 \leq P1 \leq 1$, we have $P1 = 1$ almost everywhere with respect to $\mu$. However, since $\mu$ is strictly positive everywhere, we have $P1 = 1$. That is, (1.4) is true. Let $f \in N(\mu)$ and $g = Gf + \|Gf\|$, where $\| \|$ denotes the uniform norm in $B$. If $n$ is so large that the support of $f$ is contained in $E_n$, we have

$$\Pi E_n g(x) = \Pi E_n Gf(x) + \|Gf\| = Gf(x) - f(x) + \|Gf\|$$

for all $x \in S$ and hence, noting that $g \geq 0$, we have

$$P g(x) \leq \lim \inf_{n \to \infty} \Pi E_n g(x) = Gf(x) - f(x) + \|Gf\|,$$

which implies

$$P Gf \leq Gf - f.$$

Similarly, by replacing $f$ to $-f$ in (1.9), we have $P Gf \geq Gf - f$, so that $P Gf = Gf - f$ which proves (1.6). If $\bar{P}$ is any kernel satisfying (1.4) and (1.6), then for any $g \in B$

$$\bar{P} g = \lim_{n \to \infty} \bar{P} H E_n g = \lim_{n \to \infty} \bar{P} (G f E_n + \langle \lambda E_n, g \rangle) = \lim_{n \to \infty} (G f E_n - f E_n + \langle \lambda E_n, g \rangle) = \lim_{n \to \infty} P H E_n g = P g,$$

as required.
where $H^{*} = G_{f^{*}} + \langle \lambda^{*}, g \rangle$ and $f^{*} \in N^{*}$. Thus the uniqueness of $P$ is proved. Finally we shall prove (1.7). If there is some $y \in S$ such that

$$\sum_{n=0}^{\infty} P^n(y, y) < \infty,$$

then

$$\sum_{n=0}^{\infty} P^n(x, y) \leq \sum_{n=0}^{\infty} P^n(y, y) < \infty$$

for all $x \in S$. Consequently $\lim_{n \to \infty} P^n(x, y) = 0$ for all $x \in S$.

Therefore, using (1.5), we have

$$\mu(y) = \sum_{n=0}^{\infty} \mu(x) (\lim_{n \to \infty} P^n(x, y)) = 0,$$

which contradicts the assumption that $\mu$ is strictly positive everywhere. Thus (1.7) is true when $x = y$. To show (1.7) in the case $x \neq y$, it is sufficient that we prove there is some $n$ such that $P^n(x, y) > 0$. Let us introduce the function $e_y$ in $N(\mu)$ by

$$e_y(z) = \begin{cases} 1 & z = x \\ -\mu(x)/\mu(y) & z = y \\ 0 & \text{otherwise} \end{cases}$$

If $P^n(x, y) = 0$ for all $n \geq 0$, we have

$$\sum_{k=0}^{n} P^k(x, x) = \sum_{k=0}^{n} P^k e_y(x)$$

$$= Ge_y(x) - P^{n+1} Ge_y(x)$$

$$= [Ge_y(x) - Ge_y(y)] - P^{n+1}[Ge_y - Ge_y(y)](x)$$

$$\leq Ge_y(x) - Ge_y(y),$$

because $Ge_y \geq Ge_y(y)$ everywhere. Consequently we have

$$\sum_{k=0}^{n} P^k(x, x) \leq Ge_y(x) - Ge_y(y) < \infty$$

which is a contradiction. Thus the theorem was proved.

In the proof of this theorem, we have used essentially the boundedness of the measure $\mu$. Examples of operators $G$ for unbounded measures will be given and discussed in section 3.

2. The potential operators satisfying the semi-complete maximum principle.

Let $\mu$ be a measure on $S$, strictly positive everywhere, and $R$ a linear operator from $N(\mu)$ to $B$ which satisfies the semi-complete maximum principle.
In this section we shall assume always that $\mu$ is bounded. For each positive number $\alpha$, we put $G_\alpha = I + \alpha R$, where $I$ is the identity operator. Evidently $G_\alpha$ is a linear operator from $N(\mu)$ to $B$.

**Lemma 4.** $G_\alpha$ satisfies the reinforced semi-complete maximum principle.

Proof. Let $G_\alpha f \leq m$ on the set $\{f > 0\}$, where $m$ is a real constant. Then $\alpha Rf \leq G_\alpha f \leq m$ on the set $\{f > 0\}$, so that $\alpha Rf \leq m$ everywhere by (S. C. M). Therefore $-f^- + \alpha Rf \leq m - f^-$ everywhere. Hence we have $G_\alpha f = -f^- + \alpha Rf \leq m - f^-$ on the set $\{f \leq 0\}$, which implies $G_\alpha f \leq m - f^-$ everywhere.

Since $G_\alpha$ satisfies (R. S. C. M), we can apply Theorem 1 to $G_\alpha$, so that there is a kernel $Q_\alpha$ on $S$ which has all the properties in Theorem 1. Put $R_\alpha = Q_\alpha/\alpha$, then

**Lemma 5.** The family of kernels $(R_\alpha)_{\alpha > 0}$ satisfies the following conditions:

(2.1) \(\alpha R_\alpha \geq 0\) and $\alpha R_\alpha 1 = 1$,

(2.2) \(\alpha \mu R_\alpha = \mu\),

(2.3) \(R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0\),

(2.4) \((I - \alpha R_\alpha) Rf = R_\alpha f\) for all $f \in N(\mu)$.

Such a family is unique.

Further

(2.5) \(\lim_{\alpha \to 0} R_\alpha(x, y) = \infty\) for all $(x, y) \in S \times S$.

Proof. (2.1), (2.2) and (2.4) are the same as (1.4), (1.5) and (1.6) of Theorem 1 respectively and the uniqueness of such a family is a consequence of Theorem 1, too. So we have only to prove (2.3) and (2.5). Let us denote by $(\lambda^E_\beta)_{E \in K}$ the family of measures satisfying (i), (ii) and (iii) of Lemma 2 for $G_\alpha$ and by $H^E_\alpha$ the kernel defined by (1.2) with respect to $G_\alpha$ and $\lambda^E_\alpha$. If $g \in B$ and $H^E_\beta g = G^E_\beta f + \langle \lambda^E_\beta, g \rangle$, where $f^E \in N^E$, then, noting the relation

\[ H^E_\beta g = G^E_\beta f + (\beta - \alpha) R^E f + \langle \lambda^E_\beta, g \rangle, \]

we have

\[ R_\alpha H^E_\beta g = R^E f + (\beta - \alpha) R_\alpha R^E f + \langle \lambda^E_\beta, g \rangle/\alpha. \]

Since

\[ R_\beta H^E_\beta g = R^E f + \langle \lambda^E_\beta, g \rangle/\beta, \]

we have

\[ R_\alpha H^E_\beta g - R_\beta H^E_\beta g = (\beta - \alpha) [R_\alpha R^E f + \langle \lambda^E_\beta, g \rangle/\alpha \beta]. \]
We can easily verify that the last term is equal to $(\beta - \alpha) R_\alpha R_\beta H^g_\beta g$, so that

\begin{equation}
R_\alpha H^g_\beta g - R_\beta H^g_\beta g = (\beta - \alpha) R_\alpha R_\beta H^g_\beta g
\end{equation}

for all $g \in B$, $E \in \mathcal{K}$ and $\alpha, \beta > 0$. Let $(E_n)_{n \geq 1}$ be an increasing sequence of sets in $\mathcal{K}$ with the union $S$. Since $||H^g_\beta g|| \leq ||g||$ and $\lim_n H^g_\beta g(x) = g(x)$ for all $x \in S$, we have

\begin{align*}
R_\alpha g - R_\beta g &= \lim_{n \to \infty} [R_\alpha H^g_\beta g - R_\beta H^g_\beta g] \\
&= (\beta - \alpha) \lim_{n \to \infty} R_\alpha R_\beta H^g_\beta g \\
&= (\beta - \alpha) R_\alpha R_\beta g ,
\end{align*}

which proves (2.4). Finally we shall prove (2.5). First we prove the inequality

\begin{equation}
R_\alpha(x, y) \leq R_\alpha(y, y) .
\end{equation}

Since $\beta R_{\alpha + \beta}$ is a sub-Markov kernel on $S$ and $I + \beta R_\alpha = \sum_{\alpha = 0}^{\infty} (\beta R_{\alpha + \beta})^\alpha$, we have

\begin{equation}
I(x, y) + \beta R_\alpha(x, y) \leq I(y, y) + \beta R_\alpha(y, y)
\end{equation}

for all $(x, y) \in S \times S$. Hence, dividing both side of (2.8) by $\beta$, and letting $\beta \to \infty$, we obtain (2.7). If there is some $y \in S$ such that $\lim_{\alpha \to 0} R_\alpha(y, y) < \infty$, then $\lim_{\alpha \to 0} \alpha R_\alpha(x, y) = 0$ for all $x \in S$ by (2.7). Therefore

\begin{equation}
\mu(y) = \lim_{\alpha \to 0} \alpha \mu R_\alpha(y) = \mu (\lim_{\alpha \to 0} \alpha R_\alpha)(y) = 0 ,
\end{equation}

which is a contradiction. Thus (2.5) is true when $x = y$. Let $r_\beta(x) = R_\beta(x, y)/R_\beta(y, y)$ and $r(x) = \liminf_{\beta \to 0} r_\beta(x)$. From (2.7) it follows that $0 \leq r(x) \leq 1$ for all $x \in S$. Since the resolvent equation (2.3) implies

\begin{equation}
\alpha R_\alpha r_\beta(x) = \beta R_\alpha r_\beta(x) + r_\beta(x) - R_\alpha(x, y)/R_\beta(y, y)
\end{equation}

and since

\begin{align*}
0 &\leq R_\alpha(x, y)/R_\beta(y, y) \leq 1/\alpha R_\beta(y, y) , \\
0 &\leq \beta R_\alpha r_\beta(x) \leq \beta/\alpha ,
\end{align*}

we have

\begin{equation}
\alpha R_\alpha r(x) \leq \liminf_{\beta \to 0} \alpha R_\alpha r_\beta(x) \leq \liminf_{\beta \to 0} r_\beta(x) = r(x)
\end{equation}

for all $x \in S$, which implies the function $r$ is excessive with respect to the kernel $Q_\alpha = \alpha R_\alpha$. By Theorem 1, $Q_\alpha$ is irreducible recurrent, so that $r$ should be a constant function, which is proved in [5, p. 226]. Since $r(y) = 1$, we have

\begin{equation}
r(x) = \lim_{\alpha \to 0} R_\alpha(x, y)/R_\alpha(y, y) = 1
\end{equation}
for all $x \in S$, which implies $\lim_{a \to 0} R_\alpha(x, y) = \infty$ for all $(x, y) \in S \times S$. Thus the theorem was proved.

Using (2.9), we can obtain easily the following corollaries:

**Corollary 1.**  \[ \lim_{a \to 0} \alpha R_\alpha(x, y) = \mu(y) \langle \mu, 1 \rangle \] for all $(x, y) \in S \times S$.

**Corollary 2.** For each $f \in N(\mu)$ there exists the limit $R_\alpha f = \lim_{a \to 0} R_\alpha f$ and

\[
R_\alpha f = Rf - \langle \mu, Rf \rangle \langle \mu, 1 \rangle \quad \text{for all } f \in N(\mu)
\]
and hence, the linear operator $R_\alpha$ satisfies (S.C.M), too.

Let $a \in S$ and define the function $f_a$ by

\[
f_a(x) = \begin{cases} 
1 & x = y \\
-\mu(y) / \mu(a) & x = a \\
0 & \text{otherwise}.
\end{cases}
\]

If we put $aR(x, y) = Rf_a(x) - Rf_a(a)$, then $aR$ is a non-negative kernel on $S$ with $aR(a, y) = aR(x, a) = 0$ for all $x, y \in S$.

**Corollary 3.**  Put

\[
aR_\alpha(x, y) = R_\alpha(x, y) - R_\alpha(x, a)R_\alpha(a, y)/R_\alpha(a, a)
\]
then $(aR_\alpha)_{a \to 0}$ is a sub-Markov resolvent with $\lim_{a \to 0} aR_\alpha = aR$.

The meaning of these corollaries will be made clear later.

**Theorem 2.** Let $\mu$ be a bounded measure on $S$, strictly positive everywhere, and $R$ a linear operator from $N(\mu)$ to $B$ which satisfies the semi-complete maximum principle. Then there exists a family of kernels $(P_t)_{t > 0}$ such that:

\[ (2.9) \quad P_t \geq 0 \quad \text{and} \quad P_t 1 = 1 \quad \text{for all} \quad t > 0. \]
\[ (2.10) \quad P_t P_s = P_{t+s} \quad \text{for all} \quad s, t > 0. \]
\[ (2.11) \quad \mu P_t = \mu \quad \text{for all} \quad t > 0. \]
\[ (2.12) \quad \text{The functions } t \mapsto P_t(x, y) \text{ are continuous in the open interval } (0, \infty) \text{ for all } (x, y) \in S \times S. \]
\[ (2.13) \quad (I-P_t)Rf(x) = \int_0^t P_s f(x) \, ds \quad \text{for all} \quad f \in N(\mu), x \in S \text{ and } t > 0. \]

Such a family is unique.

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2) If a linear operator $R$ from $N(\mu)$ to $B$ satisfies (2.13) for a Markov semi-group $(P_t)_{t > 0}$, it will be called a weak potential operator for $(P_t)_{t > 0}$. 
Further \((P_t)_{t>0}\) is irreducible recurrent in the sense:

\begin{equation}
\int_0^\infty P_t(x, y) dt = \infty \quad \text{for all } (x, y) \in S \times S .
\end{equation}

Proof. Let \((R_\alpha)_{\alpha>0}\) be the family constructed in Lemma 6. Since it satisfies (2.1) and (2.3), using the result of Reuter [12], we can find \((P_t)_{t>0}\) which satisfies (2.9), (2.10), (2.11) and

\begin{equation}
R_\alpha(x, y) = \int_0^\infty e^{-\alpha t} P_t(x, y) dt \quad \text{for all } (x, y) \in S \times S .
\end{equation}

Since the functions \(t \to \mu P_t(y)\) are continuous in \((0, \infty)\) and

\[ \int_0^\infty e^{-\alpha t} \mu P_t(y) dt = \mu R_\alpha(y)/\alpha = \int_0^\infty e^{-\alpha t} \mu(y) dt , \]

we have (2.11) by the uniqueness of the inverse Laplace transform. We remark here that, for any \(f \in B\) and \(x \in S\), the function \(t \to P_t f(x)\) is continuous in \((0, \infty)\). In fact, if \(0 \leq f \leq 1\), the functions \(t \to P_t f(x)\) and \(t \to P_t (1-f)(x) = 1 - P_t f(x)\) are lower-semi-continuous in \((0, \infty)\) and hence, the function \(t \to P_t f(x)\) is continuous in \((0, \infty)\). The general case is reduced to this case by the usual procedure. From this remark we know that the both sides of (2.13) are continuous with respect to \(t\) in \((0, \infty)\). Since the Laplace transform of (2.13) is equal to (2.4), (2.13) is true by the property of the Laplace transform. Similarly the uniqueness of \((P_t)_{t>0}\) is followed from Lemma 6 and the uniqueness of the inverse Laplace transform. Relation (2.14) is evident by

\[ \int_0^\infty P_t(x, y) dt = \lim_{\alpha \to 0} R_\alpha(x, y) = \infty . \]

Thus the theorem was proved.

Corollary 1 of Lemma 6 implies the ergodic property of \((P_t)_{t>0}\); \(\lim_{t \to \infty} P_t(x, y) = \mu(y)/\langle \mu, 1 \rangle\), and Corollary 2 implies the normality of \((P_t)_{t>0}\); for any \(f \in N(\mu)\) and \(x \in S\), there exists the limit; \(R_\alpha f(x) = \lim_{t \to \infty} \int_0^t P_t f(x) ds\), and which satisfies the equation (2.13), too.

Now we discuss the continuity of \((P_t)_{t>0}\) at \(t=0\).

**Theorem 3.** Under the same conditions of Theorem 1, the relation

\begin{equation}
\lim_{t \to 0} P_t(x, y) = I(x, y) \quad \text{for all } (x, y) \in S \times S
\end{equation}

holds if and only if \(R\) is non-singular.

Proof. First let us assume that \((P_t)_{t>0}\) satisfies (2.16). Let \(f\) be a non-
zero element of \( N(\mu) \) and \( Rf = m \) on the support of \( f \), where \( m \) is a constant. Since \( R \) satisfies (S.C.M), \( Rf = m \) everywhere, so that \( \int_0^t P_s f(x) ds = 0 \) for all \( x \in S \). Therefore, from (2.15) it follows that

\[
f(x) = \lim_{t \to 0} \left[ \int_0^t P_s f(x) ds \right]/t = 0
\]

for all \( x \in S \), which is a contradiction. Therefore if \( f \) is a non-zero element of \( N(\mu) \), \( Rf \) is never equal to a constant on the support of \( f \), which is the meaning of that \( R \) is non-singular. Conversely we assume that \( R \) is non-singular. In this case we can define a family of measures \( (\lambda^E)_{E \in \mathcal{K}} \) and a family of Markov kernels \( (H^E)_{E \in \mathcal{K}} \) corresponding to \( R \) in the same way as stated in Lemma 2 and Lemma 3 of section 1 respectively. Let \( (E_n)_{n \geq 1} \) be an increasing sequence of \( \mathcal{K} \) with the union \( S \) and further let \( g = \chi_{(y)} \) and

\[
H^{E_n}g = Rf^{E_n} + \langle \lambda^{E_n}, g \rangle,
\]

where \( f^{E_n} \in N^{E_n} \). Then, using (2.9) and (2.13), we have

\[
P_t H^{E_n}g = P_t Rf^{E_n} + \langle \lambda^{E_n}, g \rangle
\]

\[
= Rf^{E_n} - \int_0^t P_s f^{E_n} ds + \langle \lambda^{E_n}, g \rangle
\]

\[
= H^{E_n}g - \int_0^t P_s f^{E_n} ds
\]

for each \( n \) and \( t > 0 \). On the other hand, we know that, for each \((x, y) \in S \times S\), there exists the limit

\[
W(x, y) = \lim_{t \to 0} P_t (x, y)
\]

and the kernel \( W \) is a sub-Markov kernel with \( W^2 = W \) [1, p. 118]. Therefore, using Fatou's inequality, we have

\[
WH^{E_n}g(x) \leq \liminf_{t \to 0} [H^{E_n}g(x) - \int_0^t P_s f^{E_n}(x) ds]
\]

\[
= H^{E_n}g(x)
\]

for each \( n \) and \( x \in S \). Noting that \( 0 \leq H^{E_n}g \leq 1 \) and \( \lim_{n} H^{E_n}g(x) = \chi_{(y)}(x) = I(x, y) \) for all \( x \in S \), we have from (2.19)

\[
W(x, y) \leq I(x, y) \quad \text{for all } (x, y) \in S \times S
\]

Thus \( W(x, y) = w(x) I(x, y) \), where \( w \) is a function on \( S \) which takes only two values 0 or 1, for \( W^2 = W \). However, since
\[
\mu(y)w(y) = \mu W(y) = \lim_{t \to 0} \mu P_t(y) = \mu(y)
\]
for all \(y \in S\) and since \(\mu\) is strictly positive everywhere, we have \(w = 1\) on \(S\). Therefore,
\[
I = W = \lim_{t \to 0} P_t.
\]
Thus the theorem was proved.

Now the meaning of Corollary 3 of Lemma 5 is the following. Assume that \(R\) is non-singular, then the corresponding semi-group \((P_t)_{t \geq 0}\) in Theorem 3 is continuous at \(t=0\). In this case we can find a Markov process \(X = (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, \theta_t, (P_x)_{x \in S})\) with an enlarged state space \(\bar{S}\) such that
\[
P_x(X_t = y) = P_t(x, y) \quad \text{for all } (x, y) \in S \times S \text{ and } t > 0
\]
(for precise definitions, see [7]). For any \(a \in S\), if we define the family of kernels \(\left(^aP_t\right)_{t \geq 0}\) by
\[
^aP_t(x, y) = P_x(X_t = y, t < T^a) \quad \text{for } (x, y) \in S \times S,
\]
where \(T^a\) denotes the first hitting time of the set \(\{a\}\), then \(\left(^aP_t\right)_{t \geq 0}\) is a sub-Markov semi-group which is continuous at \(t=0\). Corollary 3 shows that \(\left(^aP_t\right)_{t \geq 0}\) is transient and its potential kernel is \(^aR\).

3. Examples

In this section we shall give examples of operators satisfying (R.S.C.M) with unbounded measures. Since (R.S.C.M) implies (S.C.M), these are also examples of non-singular operators satisfying (S.C.M).

**Example 1.** Let \(S\) be the set of all integers and \(\mu(x) = 1\) for all \(x \in S\). Define a linear operator \(G\) by
\[
Gf(x) = -\sum_{y \in S} |y-x| f(y) \quad \text{for } f \in \mathcal{N}(\mu).
\]
Then, by simple calculations, we have the following formulae;

\[
\begin{align*}
Gf(x) &= Gf(x-1) + 2 \sum_{y \geq x} f(y), \\
Gf(x) &= Gf(x+1) + 2 \sum_{x \geq y} f(y), \\
Gf(x) &= \frac{1}{2} [Gf(x-1) + Gf((x+1)] + f(x)
\end{align*}
\]
for all \(x \in S\). If the support of \(f\) is contained in \(\{a, a+1, \ldots, b\}\), by (3.3) and (3.2), \(Gf(x) = Gf(a)\) for \(x < a\) and \(Gf(x) = Gf(b)\) for \(x > b\), respectively. Therefore \(Gf\) is bounded on \(S\), that is, \(G\) maps \(\mathcal{N}(\mu)\) into \(B\). To show that \(G\) satisfies
(R.S.C.M) we assume $Gf \leq m$ on the set $\{f > 0\} = \{a_1, a_2, \ldots, a_p\}$, where $a_1 < a_2 < \cdots < a_p$. For each $x < a_1$, using (3.3), we have $Gf(x) \leq Gf(x+1)+f(x)$ and $Gf(x+1) \leq Gf(a_1)$, so that $Gf(x) \leq Gf(a_1)+f(x) \leq m-f^-(x)$. Similarly, for each $x > a_p$, using (3.2), we have $Gf(x) \leq m-f^-(x)$. For $a_k < x < a_{k+1}$, using (3.4), we have $Gf(x) \leq \sup (Gf(a_k), Gf(a_{k+1}))+f(x) \leq m-f^-(x)$, $k=1, 2, \ldots, p-1$. Therefore $Gf \leq m-f^-$ everywhere, so that $G$ satisfies (R.S.C.M). Let us introduce a Markov kernel $P$ on $S$ by

$$P(x, y) = \begin{cases} 1/2 & y = x \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

then $\mu$ is an invariant measure for $P$ and the relation (1.6) of Theorem 1 holds for all $f \in N(\mu)$, for, (1.6) is equivalent to (3.4) in this case. Further, since $P$ is the transition function of (simple) symmetric random walk of dimension one, it is irreducible recurrent. Thus, Theorem 1 is valid for $G$, though $\mu$ is unbounded. If we define the Markov semi-group $(P_t)_{t>0}$ by $P_t = e^{tP}$, that is,

$$P_t(x, y) = e^{-t} \sum_{n=0}^{\infty} \frac{1}{n!} (tP)^n(x, y) \quad \text{for} \quad (x, y) \in S \times S,$$

it has an invariant measure $\mu$ and a weak potential operator $G$. Obviously $(P_t)_{t>0}$ is irreducible recurrent, so that Theorem 2 is valid for $G$, too.

**Example 2.** Let $S$ and $\mu$ be the same in Example 1. Define a linear operator $G$ from $N(\mu)$ to $B$ by

$$Gf(x) = \sum_{y \geq x} f(y) \quad \text{for all} \quad f \in N(\mu).$$

To show that $G$ satisfies (R.S.C.M) we assume that $Gf \leq m$ on the set $\{f > 0\} = \{a_1, a_2, \ldots, a_p\}$, where $a_1 < a_2 < \cdots < a_p$. Since $0 \leq Gf(a_i) \leq m$, $m$ should be non-negative. If $a_{k-1} < x < a_k$,

$$Gf(x) = Gf(x+1)+f(x) \leq Gf(a_k)+f(x) \leq m-f^-(x), \quad k=1, 2, \cdots, p \quad (\text{we regard} \ a_0 \ \text{as} \ -\infty).$$

If $x > a_p$,

$$Gf(x) = Gf(x+1)+f(x) \leq \sup (Gf(a_p), 0)+f(x) \leq m-f^-(x).$$

Consequently $Gf \leq m-f^-$ everywhere, which shows that $G$ satisfies (R.S.C.M).

Let us now define a Markov kernel $P$ on $S$ by

$$P(x, y) = \begin{cases} 1 & y = x+1 \\ 0 & \text{otherwise} \end{cases}$$
Obviously, $P$ has $\mu$ as an invariant measure and satisfies the relation (1.6) of Theorem 1. However, since $\sum_{n=0}^{\infty} P^n(x, y) = 0$ or $-1$ according as $x > y$ or $x \leq y$, $P$ is not irreducible recurrent. If we define a Markov semi-group $(P_t)_{t \geq 0}$ by $P_t = e^{t(P-I)}$, it has $\mu$ as an invariant measure and $G$ as a weak potential operator. But it is transient in the sense:

$$\int_0^\infty P_t(x, y) dt < \infty \quad \text{for all } (x, y) \in S \times S.$$ 

**Example 3.** Let $S = \{0, 1, \cdots\}$ and $\mu(x) = 1$ for all $x \in S$. Define a linear operator $G$ from $N(\mu)$ to $B$ by

$$(3.6) \quad Gf(x) = \sum_{y \geq x} f(y) \quad \text{for all } f \in N(\mu).$$

That $G$ satisfies (R.S.C.M) is proved in the same way as stated in Example 2. Let us introduce a Markov kernel $P$ on $S$ by

$$P(x, y) = \begin{cases} 1 & y = x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $P$ satisfies (1.6) of Theorem 1. Since a Markov kernel satisfying (1.6) is unique, $P$ is only such a kernel. However, the relation $1 = \mu(0) > \mu P(0) = 0$, shows that $\mu$ is not an invariant measure for $P$. If a Markov semi-group $(P_t)_{t \geq 0}$ with a weak potential operator $G$ exists, it should be equal to that defined by $P_t = e^{t(P-I)}$. Since $\mu$ is not an invariant measure for $(P_t)_{t \geq 0}$, there is never Markov semi-group which has $\mu$ as an invariant measure and $G$ as a weak potential operator.

Finally we notice some remarks on our problem. We shall assume again that $S$ is any denumerable set and $\mu$ is any measure on $S$, strictly positive everywhere. Let $R$ be a non-singular operator from $N(\mu)$ to $B$ satisfying (R. C. M.), for example, an operator satisfying (R. S. C. M.). Take a function $g$ on $S$ which is strictly positive everywhere and $\langle \mu, g \rangle < \infty$. Define a measure $\tilde{\mu}$ on $S$ by $\tilde{\mu}(x) = g(x) \mu(x)$ for all $x \in S$. Then, $f \in N(\tilde{\mu})$ if and only if $gf \in N(\mu)$, so that we may define a linear operator $\tilde{R}$ from $N(\tilde{\mu})$ to $B$ by $\tilde{R}f = R(gf)$. We can easily verify that $\tilde{R}$ is also a non-singular operator satisfying (R. C. M.). Since $\tilde{\mu}$ is bounded, by Theorem 2 and 3, we can find a Markov semi-group $(\tilde{P}_t)_{t \geq 0}$ which is continuous at $t = 0$ and has $\tilde{\mu}$ and $\tilde{R}$ as its own invariant measure and weak potential operator, respectively. Let $\tilde{X} = (\Omega, \mathcal{F}, (\tilde{X}_t)_{t \geq 0}, (\tilde{\theta}_t)_{t \geq 0}, (P_t)_{x \in S})$ be a Markov process with a state space $\tilde{S}$, some metric completion of $S$, such that

$$\tilde{P}_t(x, y) = P_x(\tilde{X}_t = y) \quad \text{for all } (x, y) \in S \times S.$$ 

Let us introduce an additive functional $(A_t)_{t \geq 0}$ for $\tilde{X}$ by
\[ A_t = \begin{cases} \int_0^t [1/g(X_s)] \, ds & \text{for } t < T \\ \infty & \text{for } t \geq T \end{cases} \]

where \( T = \sup \{ t : \int_0^t [1/g(X_s)] \, ds < \infty \} \). Further we put \( C_t = s \) if and only if \( A_s = t \) for \( s \in [0, T) \). If we denote \( X_t = \bar{X}_{C_t} \) and \( \theta_t = \bar{\theta}_{C_t} \), \( X = (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (\theta_t)_{t \geq 0}, (P_x)_{x \in S}) \) is a Markov process with a state space \( \bar{S} \), too. Using properties of \( \bar{X} \), we can prove that a family of kernels \((P_t)_{t \geq 0}\) on \( S \) defined by:

\[ P_t(x, y) = P_{x}(X_t = y) \]

for all \((x, y) \in S \times S\), is a sub-Markov semi-group on \( S \), continuous at \( t = 0 \). If the condition;

\[ (3.7) \quad P_x(T = \infty) = 1 \]

is satisfied, we can prove that \((P_t)_{t \geq 0}\) is an irreducible recurrent Markov semi-group with an invariant measure \( \mu \) and a weak potential operator \( R \). In Example 1, condition (3.7) is true, however, in Example 2 and 3, (3.7) is not true. Unwillingly, we could not express these facts as analytic conditions on \( R \).

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**Bibliography**


