A GENERALIZATION OF PRIME IDEALS IN RINGS

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Introduction

In [2], van der Walt has defined s-prime ideals in noncommutative rings and obtained analogous results of McCoy [1] for s-prime ideals. In the present paper, we shall give a generalized concept of prime ideals, called f-prime ideals, by using some family of ideals, and obtain analogous results in [2]. If our family of ideals is, in particular, the set of principal ideals of the ring, the f-prime ideals coincide with the prime ideals and conversely. In addition, if we take multiplicatively closed systems as kernels, the f-prime ideals coincide with the s-prime ideals.

1. f-prime ideals and the f-radical of an ideal

Let \( R \) be an arbitrary (associative) ring. Throughout this paper, the term "ideals" will always mean "two-sided ideals in \( R \)."

For each element \( a \) of \( R \), we shall associate an ideal \( f(a) \) which is uniquely determined by \( a \) and satisfies the following conditions:

(I) \( a \in f(a) \), and

(II) \( x \in f(a) + A \Rightarrow f(x) \subseteq f(a) + A \) for any ideal \( A \).

The principal ideal \( (a) \) generated by \( a \) is an example of the \( f(a) \), and this is the case of [2]. Moreover there are other interesting examples of the \( f(a) \). For example, let \( Q \) be any subset of \( R \). If we define, for each element \( a \) of \( R \), \( f(a) = (a, Q) \), the ideal generated by \( a \) and \( Q \), then it is easy to see that \( f(a) \) satisfies the above conditions. If, in particular, \( Q \) is the empty set, then the \( f(a) \) coincides with the principal ideal \( (a) \).

REMARK. As is easily seen, the following four conditions are equivalent:

(i) For any element \( a \) of \( R \), \( f(a) = (a) \),

(ii) \( f(0) = 0 \),

(iii) For any ideal \( A \), \( x \in A \Rightarrow f(x) \subseteq A \),

(iv) For any element \( a \) of \( R \), \( x \in (a) \Rightarrow f(x) \subseteq (a) \).

DEFINITION 1.1. A subset \( S \) of \( R \) is called an f-system if \( S \) contains an
An ideal $P$ is said to be \textit{f-prime} if its complement $C(P)$ in $R$ is an \textit{f-system}.

$R$ is evidently an \textit{f-prime} ideal. Obviously an \textit{s-prime} ideal in the sense of [2] is a prime ideal in the sense of [1], and it follows from Lemma 1.4 below that if we assume $f(a) = (a)$ for every element $a$ in $R$, then prime ideals are nothing but \textit{f-prime} ideals. But it can be shown that this is not always true with a suitable choice of $f(a)$.

\textbf{Example 1.3.} Consider the ring $\mathbb{Z}$ of integers. Let $P$ be the ideal $(p^2)$ and let $S^*$ be the $m$-system $\{q, q', q'^2, \ldots\}$, where $p$ and $q$ are different prime numbers. If we put $f(a) = (a, q)$ for each element $a$ in $\mathbb{Z}$, then the complement $C(P)$ of $P$ in $\mathbb{Z}$ is an \textit{f-system} with kernel $S^*$. Hence $P$ is an \textit{f-prime} ideal, but not a prime ideal. This also shows that an \textit{f-prime} ideal need not be an \textit{s-prime} ideal, in general.

\textbf{Lemma 1.4.} For any \textit{f-prime} ideal $P$, 
\[ f(a_1)f(a_2)\cdots f(a_n) \subseteq P \Rightarrow a_i \in P \text{ for some } i. \]

Proof. It is evident from the definition of \textit{f-systems}.

\textbf{Lemma 1.5.} Let $S(S^*)$ be an \textit{f-system} in $R$, and let $A$ be an ideal in $R$ which does not meet $S$. Then $A$ is contained in a maximal ideal $P$ (in the class of all ideals, each of which does not meet $S$). The ideal $P$ is necessarily an \textit{f-prime} ideal.

Proof. If $S$ is empty, the assertion is trivial, and so suppose that $S$ is not empty. The existence of $P$ follows from Zorn's lemma. We now show that $C(P)$ is an \textit{f-system} with kernel $S^* + P$. For any element $a$ of $C(P)$, the maximal property of $P$ implies that $f(a) + P$ contains an element $s$ of $S$, and thus we can choose an element $s^* = f(s) \cap S^*$. Since $f(s)$ is contained in $f(a) + P$, we can write $s^* = a' + p$ where $a'$ in $f(a)$ and $p$ in $P$. Then $a' = s^* - p$ is contained in $f(a) \cap (S^* + P)$, which completes the proof of the lemma.

\textbf{Definition 1.6.} The \textit{f-radical} $r(A)$ of an ideal $A$ will be defined to be the set of all elements $a$ of $R$ with the property that every \textit{f-system} which contains $a$ contains an element of $A$.

\textbf{Theorem 1.7.} The \textit{f-radical} of an ideal $A$ is the intersection of all the \textit{f-prime} ideals containing $A$. 
Proof. We show that if \( P \) is an \( f \)-prime ideal containing \( A \), then \( r(A) \) is contained in \( P \). For suppose that \( r(A) \) is not contained in \( P \). Then there exists an element \( x \) in \( r(A) \) not in \( P \). Since \( C(P) \) is an \( f \)-system, \( C(P) \cap A \neq \phi \). But this contradicts the fact that \( A \) is contained in \( P \). Hence \( r(A) \) is contained in the intersection of all \( f \)-prime ideals which contain \( A \).

Conversely, let \( a \) be an element of \( R \), but not in \( r(A) \). Then there exists an \( f \)-system \( S(S^*) \) which contains \( a \) but does not meet \( A \). There exists, by Lemma 1.5, an \( f \)-prime ideal \( P \) which contains \( A \) and does not meet \( S \). Hence, \( P \) does not contain \( a \) and \( a \) can not be in the intersection of all \( f \)-prime ideals containing \( A \). This completes the proof.

**Corollary 1.8.** The \( f \)-radical of an ideal is an ideal.

Now, let \( S(S^*) \) be an \( f \)-system in \( R \) and let \( A \) be an ideal which does not meet \( S \). It follows from Zorn's lemma that there exists a maximal \( m \)-system \( S^*_f \) which contains \( S^* \) and does not meet \( A \). Let us consider the set \( S_i = \{ x \in R \mid f(x) \cap S^*_f \neq \phi \} \cap C(A) \). Then \( S_i \) is an \( f \)-system with kernel \( S^*_f \) and does not meet \( A \). According to Lemma 1.5, there exists an \( f \)-prime ideal \( P \) which contains \( A \) and does not meet \( S_i \). As is seen in the proof of Lemma 1.5, \( C(P) \) is an \( f \)-system with kernel \( S^*_f + P \), and the maximal property of \( S^*_f \) implies that \( S^*_f + P = S^*_f \). Hence we have \( C(P) = S_i \) by the definition of \( S_i \).

In view of this we make the following definition:

**Definition 1.9.** An \( f \)-prime ideal \( P \) is said to be a minimal \( f \)-prime ideal belonging to an ideal \( A \) if \( P \) contains \( A \) and there exists a kernel \( S^* \) for the \( f \)-system \( C(P) \) such that \( S^* \) is a maximal \( m \)-system which does not meet \( A \).

It follows from the above consideration that any \( f \)-prime ideal \( P \) containing \( A \) contains a minimal \( f \)-prime ideal belonging to \( A \). From Theorem 1.7, we can conclude the following:

**Theorem 1.10.** The \( f \)-radical of an ideal \( A \) coincides with the intersection of all minimal \( f \)-prime ideals belonging to \( A \).

2. Elements \( f \)-related to an ideal

We now make the following definition:

**Definition 2.1.** An element \( a \) of \( R \) is said to be (left-)\( f \)-related to an ideal \( A \) if, for every element \( a' \) in \( f(a) \), there exists an element \( c \) not in \( A \) such that \( a'c \) is in \( A \). An ideal \( B \) is said to be (left-)\( f \)-related to \( A \) if every element of \( B \) is \( f \)-related to \( A \). Elements and ideals not \( f \)-related to \( A \) is called (left-)\( f \)-unrelated to \( A \).

Elements and ideals right-\( f \)-related to \( A \) can be similarly defined, but the right hand definitions and theorems will be omitted.
Proposition 2.2. Let $A$ be an ideal. Then the set $S$ consisting of all elements of $R$ which are $f$-unrelated to $A$ is an $f$-system.

Proof. For every element $a$ in $S$, we can choose an element $a^*$ in $f(a)$ such that, for every element $c$ not in $A$, $a^*c$ is not in $A$. The set $S^*$ which consists of all such elements $a^*$ is multiplicatively closed and hence $S$ is an $f$-system with kernel $S^*$.

It is natural to consider that every element of $R$ is $f$-related to $R$. Furthermore we shall now assume, in this section, the following condition:

$(\alpha)$ Each ideal $A$ is $f$-related to itself.

It may be remarked that $(\alpha)$ can be stated in the following convenient form:

$(\alpha')$ 0 is $f$-related to each ideal $A$.

For suppose that 0 is $f$-related to $A$. Let $a$ be any element in $A$. Then $a$ is in $A+f(0)$ and hence $f(a)$ is contained in $A+f(0)$. For any element $a'$ in $f(a)$, there exist $a''$ in $A$ and $b''$ in $f(0)$ such that $a'=a''+b''$. Since 0 is $f$-related to $A$, we can choose an element $c$ not in $A$ such that $b''c$ is in $A$. Therefore, $a'c=a''c+b''c$ is in $A$ and this means that $A$ is $f$-related to itself.

Clearly, $(\alpha)$ is fulfilled in case $f(a)=(a)$ for every element $a$ in $R$. And, it can be proved that, whenever $R$ has no right zero-divisors, $R$ satisfies $(\alpha)$ if and only if $f(a)=(a)$ for every element $a$ in $R$. But, in case of general rings, this need not be true as is seen from the following example.

Example 2.3. Consider a simple module $M$ such that $m_1m_2=0$ for any two elements $m_1$ and $m_2$ in $M$. Let $K$ be a field and let $R$ be the direct sum of $M$ and $K$ as modules. Then $R$ can be made into a commutative ring by defining as

$$(m_1+k_1)(m_2+k_2) = k_1k_2,$$

where $m_1$, $m_2$ in $M$ and $k_1$, $k_2$ in $K$. As is easily seen, the ideals in $R$ are $R$, $M$, $K$ and $(0)$. If we define $f(a)=(a, M)$ for every element $a$ in $R$, then $R$ satisfies $(\alpha)$, but $f(a)$ does not coincide with $(a)$, since $f(0)=M \neq (0)$.

Proposition 2.4. Let $A$ be an ideal. Then the $f$-radical $r(A)$ of $A$ is $f$-related to $A$.

Proof. Let $S$ be as in Proposition 2.2. If $r(A)$ contains an element $f$-unrelated to $A$, then, by the definition of the radical, we have $S \cap A=\phi$, a contradiction.

It follows from this proof, in terms of relatedness, that the assumption $(\alpha)$ can be also restated as follows: for any ideal $A$, the $f$-radical of $A$ is $f$-related to $A$.

Let $A$ be an ideal and let $S$ be the $f$-system consisting of all elements $f$-
unrelated to $A$. Then $S$ does not meet the ideal $(0)$, and hence, by Lemma 1.5, there exists a maximal ideal (in the class of all ideals, each of) which does not meet $S$, or equivalently, a maximal ideal (each of) which is $f$-related to $A$. Each such maximal ideal is necessarily an $f$-prime ideal. In view of this, we put the following:

**Definition 2.5.** A maximal ideal in the class of all ideals, each of which is $f$-related to an ideal $A$, is called a maximal $f$-prime ideal belonging to $A$.

**Proposition 2.6.** Let $A$ be an ideal. Then $A$ is contained in every maximal $f$-prime ideal belonging to $A$.

Proof. Let $P$ be any maximal $f$-prime ideal belonging to $A$. Then it is sufficient to show that $A+P$ is $f$-related to $A$. Let $a+p$ be any element in $A+P$, where $a$ in $A$ and $p$ in $P$. Since $a+p$ is in $A+f(p)$, $f(a+p)$ is contained in $A+f(p)$, and hence each element $a'$ in $f(a+p)$ can be written as $a'=a''+p''$, where $a''$ in $A$ and $p''$ in $f(p)$. We can choose an element $c$ not in $A$ such that $p''c$ is in $A$. Then $a'c=a''c+p''c$ is contained in $A$, which completes the proof.

Since any $f$-prime ideal containing $A$ contains a minimal $f$-prime ideal belonging to $A$, it follows from Proposition 2.6 that every maximal $f$-prime ideal belonging to $A$ necessarily contains a minimal $f$-prime ideal belonging to $A$. The converse is also true in case of [1], but we can provide an example to show that this need not be true in our case.

**Example 2.7.** Let us consider the ideal $A=(xy)$ in the ring $K[x, y]$ of polynomials in two non-commutative indeterminates $x$ and $y$ over a field $K$. If we define $f(a)=(a)$ for every element $a$ in $K[x, y]$, then the assumption $(\alpha)$ is satisfied and $A$ is $f$-related to itself. Hence we can consider the maximal $f$-prime ideal belonging to $A$. As is easily seen, the ideal $(y)$ is a minimal $f$-prime ideal belonging to $A$, but it is $f$-unrelated to $A$. Thus, $(y)$ is not contained by any maximal $f$-prime ideal belonging to $A$.

**Proposition 2.8.** Let $A$ be an ideal. Then every element or ideal which is $f$-related to $A$ is contained in a maximal $f$-prime ideal belonging to $A$.

Proof. Obviously, an element $a$ is $f$-related to $A$ if and only if $f(a)$ is $f$-related to $A$. So we shall prove the only case of an ideal which is $f$-related to $A$. Let $B$ be such an ideal, and let $S$ be the $f$-system consisting of all elements of $R$ which are $f$-unrelated to $A$. Then $B$ does not meet $S$ and hence, by Lemma 1.5, $B$ is contained in a maximal $f$-prime ideal $P$ belonging to $A$.

It follows from this proposition that the ideals of $R$ which are $f$-related to $A$ are spread over the maximal $f$-prime ideals belonging to $A$.

**Definition 2.9.** Let $A$ be an ideal and let $b$ be an element in $R$. The (left-)
f-quotient $A:b$ of $A$ by $b$ will be defined to be the set of all elements $x$ of $R$ such that $f(b)f(x)$ is contained in $A$. Moreover, for any ideal $B$, the (left-)f-quotient of $A$ by $B$ will be defined as $\cap_{b\in B} (A:b)$, and denoted by $A:B$.

From this definition, we have

1. $A' \subseteq A'' \Rightarrow A':b \subseteq A'' : b$ and $A' : B \subseteq A'' : B$,
2. $B' \subseteq B'' \Rightarrow A : B' \subseteq A : B''$,
3. $(A' \cap A'') : b = (A' : b) \cap (A' : b)$ and $(A' \cap A'') : B = (A' : B) \cap (A' : B)$.

We note that $A:b$ may be empty. However, if it is not, it is an ideal containing $A$. To see this, take an arbitrary element $x+a$ in $(A:b)+A$, where $x$ in $A:b$ and $a$ in $A$. Then $x+a$ is contained in $f(x)+A$, and so is $f(x+a)$. Hence $f(b)f(x+a)$ is contained in $A$. That is, $(A:b)+A$ is contained in $A:b$.

**Definition 2.10.** Let $A$ be an ideal, and let $P$ be any maximal f-prime ideal belonging to $A$. The principal f-component $A_P$ of $A$ determined by $P$ will be defined as follows:

$$A_P = \begin{cases} \cup_{s \in P}(A:s) & \text{(if } P \neq R) \\ A & \text{(if } P = R) \end{cases}$$

For $P \neq R$, the principal f-component $A_P$ may be empty in certain cases. In case $f(a) = (a)$ for every $a$ in $R$ it is not empty, but, as is seen from Example 2.3, there exists a ring in which $(a)$ is satisfied, and $f(a)$ need not be $(a)$, and $A_P$ is not empty for all $A$ and $P \neq R$.

So we shall assume, in the rest of this paper, the following condition:

$(\beta)$ For any ideal $A$ and ideal $B$ not contained in $r(A)$, we have $A:B=\phi$.

For any maximal f-prime ideal $P$ belonging to $A$, it follows from Proposition 2.6 that $P$ contains $A$, and hence $r(A)$ is contained in $P$. If $s$ is not in $P$, then $s$ does not contained in $r(A)$. Hence, from the assumption $(\beta)$, $A:s=\phi$ and therefore we have $A_P=\phi$.

We now show that $A_P$ is an ideal containing $A$. If $P=R$, the assertion is trivial. Let $P \neq R$ and let $x$, $y$ be any two elements of $A_P$. Then there exist $s$ and $t$ in $C(P)$ such that both $f(s)f(x)$ and $f(t)f(y)$ are contained in $A$. Take two elements $s^* \in S^* \cap f(s)$ and $t^* \in S^* \cap f(t)$, where $S^*$ is a kernel of $C(P)$. Since $S^*$ is an m-system, $s^*zt^*$ is in $S^*$ (whence is in $C(P)$) for some $z$ in $R$. Thus $s^*zt^* \subseteq f(s) \cap f(t)$, $f(s^*zt^*) \subseteq f(s) \cap f(t)$. Hence $f(s^*zt^*)f(x+y) \subseteq (f(s) \cap f(t))(f(x) + f(y)) \subseteq f(s)f(x)+f(t)f(y) \subseteq A$.

Now let $x=x'+x''$ be any element in $A_P+A$, where $x'$ in $A_P$ and $x''$ in $A$. Then $f(s)f(x')$ is contained in $A$ for some $s$ in $C(P)$. Since $x$ is in $f(x')+A$, $f(x)$ is contained in $f(x')+A$, and hence we have $f(s)f(x) \subseteq f(s)f(x') + f(s)A \subseteq A$. Thus $x$ is in $A_P$ and $A$ is contained in $A_P$.

For any maximal f-prime ideal $P$ belonging to $A$, since $A \subseteq A_P \subseteq P$, $A_P=R$ if and only if $A=R$. Furthermore, if $P$ is the only maximal f-prime ideal belong-
Proposition 2.11. Let $A$ be an ideal, and let $P$ be any maximal $f$-prime ideal belonging to $A$. Then the principal $f$-component $A_P$ is contained in every ideal $D$ such that $A$ is contained in $D$ and that any element of $C(P)$ are $f$-unrelated to $D$.

Proof. If $P=R$, the assertion is trivial. Let $P \neq R$ and let $D$ be any ideal such that $A$ is contained in $D$ and that any element of $C(P)$ are $f$-unrelated to $D$. If $x$ is an arbitrary element of $A_P$, then there exists an element $s$ in $C(P)$ such that $f(s)f(x) \subseteq A$. Since $s$ is $f$-unrelated to $D$, we can choose an element $s^*$ in $f(s)$ such that $s^*c \in D$ implies $c \in D$. $s^*x$ is in $D$ and hence $x$ is in $D$.

We note from Proposition 2.8 that any element of $C(P)$ are $f$-unrelated to $D$ if and only if any maximal $f$-prime ideal belonging to $D$ are contained in $P$.

Theorem 2.12. Any ideal $A$ is represented as the intersection of all its principal $f$-components $A_P$.

Proof. Since $A$ is contained in every principal $f$-component of $A$, it is also contained in their intersection. To prove the converse, let $a$ be an arbitrary element of the intersection of all principal $f$-components $A_P$. For any maximal $f$-prime ideal $P$ belonging to $A$, $f(s)f(a) \subseteq A$ for some $s$ in $S=C(P)$. Consider the ideal $B$ which consists of all elements $b$ of $R$ such that $f(b)f(a) \subseteq A$. Then $B$ is not contained in $P$, and hence according to Proposition 2.8, $B$ cannot be $f$-related to $A$. This means that $B$ contains at least one element $b$ which is $f$-unrelated to $A$. Since $f(b)f(a)$ is in $A$, the $f$-unrelatedness of $b$ implies that $a$ is in $A$. The theorem is therefore established.

Remark. It is natural to define a (left-)f-primal ideal as follows: an ideal $A$ is said to be (left-)f-primal, if the set $X$ of the elements, each of which is (left-)f-related to $A$, forms an ideal. If $A$ is f-primal, $X$ is called the (left-)adjoint of $A$. Then we can prove that the principal $f$-component of $A$ determined by the maximal $f$-prime ideal $P$ is contained in the intersection of all $f$-primal ideals $A_\lambda$ such that (1) $A_\lambda$ contains $A$, and (2) the adjoint of $A_\lambda$ is contained in $P$.

3. $f$-primary decompositions

In this section, we shall consider $f$-primary decompositions of ideals on the analogy of the primary decompositions of ideals in a commutative Noetherian ring. For this purpose, we assume besides $(\beta)$, throughout this section, the following condition:

$(\gamma)$ If $S$ is an $f$-system with kernel $S^*$, and if for any ideal $A$, $S \cap A$ is not empty, then so is $S^* \cap A$. 
Clearly, this assumption is satisfied in case \( f(a) = (a) \) for every element \( a \) in \( R \). But, for a suitable choice of \( f(a) \), this is not always satisfied as is seen from the following example:

**Example 3.1.** As is seen from Example 1.3, for the ideal \( P = (p^i) \) in the ring \( \mathbb{Z} \) of integers, its complement \( S = C(P) \) is an \( f \)-system with kernel \( S^* = \{q, q^2, q^3, \ldots\} \), where \( p \) and \( q \) are different prime numbers. Now, let \( A \) be the ideal \( (p) \), then we have \( S \cap A \neq \phi \), though \( S^* \cap A = \phi \).

**Proposition 3.2.** Let \( A \) and \( B \) be any two ideals. Then

1. \( A \subseteq B \Rightarrow r(A) \subseteq r(B) \),
2. \( r(r(A)) = r(A) \),
3. \( r(A \cap B) = r(A) \cap r(B) \).

**Proof.** (1) and (2) follow from the definition of the radical.

It is clear that \( r(A \cap B) \subseteq r(A) \cap r(B) \). Conversely, let \( x \) be any element in \( r(A \cap B) \) and let \( S \) be any \( f \)-system containing \( x \). Then, there exist two elements \( a \) and \( b \) in \( S \cap A \) and \( S \cap B \) respectively. By the assumption (\( \gamma \)), we can choose two elements \( a^* \) and \( b^* \) in \( S^* \cap A \) and \( S^* \cap B \) respectively. Since \( S^* \) is an \( m \)-system, \( a^*b^* \) is in \( S^* \) for some element \( z \) in \( R \). Therefore \( a^*b^* \in S^* \cap (A \cap B) \), and hence \( S \cap (A \cap B) \) is not empty. This means that \( x \) is in \( r(A \cap B) \), which completes the proof of (3).

**Definition 3.3.** An ideal \( Q \) is called (left-)\( f \)-primary, if \( f(a)f(b) \subseteq Q \) implies that \( a \in r(Q) \) or \( b \in Q \).

Let us note that, by Lemma 1.4, \( f \)-prime ideals are always \( f \)-primary ideals. As is easily seen from Definition 3.3, we have

**Proposition 3.4.** If \( Q' \) and \( Q'' \) are \( f \)-primary ideals such that \( r(Q') = r(Q'') \), then \( Q = Q' \cap Q'' \) is also an \( f \)-primary ideal such that \( r(Q) = r(Q') = r(Q'') \).

Another characterization of \( f \)-primary ideals can be given by means of \( f \)-quotients.

**Proposition 3.5.** An ideal \( Q \) is \( f \)-primary if and only if \( Q : B = Q \) for all ideals \( B \) not contained in \( r(Q) \).

**Proof.** Suppose that \( Q \) is \( f \)-primary and that \( B \) is an ideal not contained in \( r(Q) \). We can choose an element \( b \) in \( B \) but not in \( r(Q) \). By the assumption (\( \beta \)), \( Q : b \) is not empty, and for any element \( a \) in \( Q : b \), \( f(b)f(a) \) is contained in \( Q \). Since \( Q \) is \( f \)-primary and \( b \) is not in \( r(Q) \), \( a \) is in \( Q \). Thus \( Q : b \) is contained in \( Q \). This shows that \( Q = Q : B \), because again by (\( \beta \)) \( Q : B \) is an ideal such that \( Q \subseteq Q : B \subseteq Q : b \).

Conversely, suppose that \( f(a)f(b) \) is contained in \( Q \) and that \( a \) is not in
$r(Q)$. Then $f(a)$ is not contained in $r(Q)$, and hence we have $Q:f(a)=Q$. For an arbitrary element $a'$ in $f(a)$, $f(a')f(b) \subseteq f(a)f(b) \subseteq Q$, and thus $b$ is in $Q:f(a)=Q$. This proves that $Q$ is $f$-primary.

If an ideal $A$ can be written as

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n,$$

where each $Q_i$ is an $f$-primary ideal, this will be called an $f$-primary decomposition of $A$, and each $Q_i$ will be called the $f$-primary component of the decomposition. A decomposition in which no $Q_i$ contains the intersection of the remaining $Q_j$ is called irredundant. Moreover, an irredundant $f$-primary decomposition, in which the radicals of the various $f$-primary components are all different, is called a normal decomposition. As is easily seen from Proposition 3.4, each $f$-primary decomposition can be refined into one which is normal.

Besides the assumptions ($\beta$) and ($\gamma$), we assume, in this section, the following condition:

$(\delta)$ For any $f$-primary ideal $Q$, we have $Q:Q=R$.

Evidently, this assumption is satisfied in case $f(a)=(a)$ for every element $a$ in $R$. But, for a suitable choice of $f(a)$, this is not all true.

Example 3.6. As is seen from Example 1.3, the ideal $(p^2)$ is $f$-prime and hence is an $f$-primary ideal in $\mathbb{Z}$. Suppose that the assumption ($\delta$) is satisfied for this $(p^2)$. Then we have $f(p^2) \subseteq (p^2)$ and hence $(p^2)=f(p^2)=(p^2)+(q)$, a contradiction.

Now we shall prove, under the assumptions ($\beta$), ($\gamma$) and ($\delta$), that the number of $f$-primary components and the radicals of $f$-primary components of a normal decomposition of $A$ depend only on $A$ and not on the particular normal decomposition considered. This is a main theorem of this section.

**Theorem 3.7.** Suppose that an ideal $A$ has an $f$-primary decomposition, and let

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_m$$

be two normal decompositions of $A$. Then $n=m$, and it is possible to number the $f$-primary components in such a way that $r(Q_i)=r(Q'_i)$ for $1 \leq i \leq n=m$.

Proof. If $A$ coincides with $R$, the assertion is trivial. We may suppose therefore that $A$ does not coincide with $R$, in which case all the $f$-primary components $Q_1, \ldots, Q_n, Q'_1, \ldots, Q'_m$ are proper ideals. Among the radicals $r(Q_1), \ldots, r(Q_n), r(Q'_1), \ldots, r(Q'_m)$ take one which is maximal in this set, and we may assume that it is $r(Q_1)$. We now prove that $r(Q_1)$ occurs among $r(Q'_1), \ldots, r(Q'_m)$. To prove this it will be enough to show that $Q_1$ is contained in $r(Q'_j)$ for some $j$. 

Suppose that $Q_1$ is not contained in $r(Q'_1)$ for $1 \leq j \leq m$. Then we have, by Proposition 3.5, $Q'_j : Q_j = Q'_j$ for $1 \leq j \leq m$, and consequently

$$A : Q_1 = (Q'_1 \cap \cdots \cap Q'_n) : Q_1 = (Q'_1 : Q_1) \cap \cdots \cap (Q'_n : Q_1) = Q'_1 \cap \cdots \cap Q'_n = A.$$ 

If $n=1$, then, by the assumption (δ), we have

$$R = Q'_1 : Q_1 = A : Q_1 = A,$$

a contradiction. On the other hand, if $n>1$, then we have again by (δ) since $Q_1$ is not contained in $r(Q'_i)$ for $2 \leq i \leq n$. This is a contradiction. Now we may arrange that $Q_1$ and $Q'_j$ so that $r(Q_i) = r(Q_i)$. We shall use an induction on the number $n$ of $f$-primary components. If $n=1$, then $A=Q'_1 = Q'_1 \cap \cdots \cap Q'_n$, and moreover if $m>1$, then $Q_1$ is not contained in $r(Q'_1)$ for $2 \leq j \leq m$. Since

$$R = Q'_1 : Q_1 = (Q'_1 : Q_1) \cap \cdots \cap (Q'_n : Q_1),$$

we have $R = Q'_1 = Q'_1 = \cdots = Q'_n$, by Proposition 3.5, a contradiction. Similarly, $m=1$ implies that $n=1$, and in this case the assertion is trivial.

Let us now assume that $n \leq m$. We shall show that $n=m$ and by a suitable ordering $r(Q_i) = r(Q'_i)$ for $1 \leq i \leq n=m$. Assume that these results are valid for ideals which may be represented by fewer than $n$ $f$-primary components. Put $Q = Q_1 \cap Q'_1$, then by Proposition 3.4, $Q$ is an $f$-primary ideal such that $r(Q) = r(Q'_1) = r(Q'_1)$. Also $Q_i : Q = Q_i$ for $2 \leq i \leq n$, and $Q_i : Q = R$. For the first relation follows from the fact that $Q$ is not contained in $r(Q_i)$, while the second follows from $R = Q_i : Q \subseteq Q'_i : Q$. Consequently $A : Q = Q'_1 \cap \cdots \cap Q'_n$, and an exactly similar argument shows that $A : Q = Q'_1 \cap \cdots \cap Q'_n$. Hence, we have

$$Q'_1 \cap \cdots \cap Q'_n = Q'_1 \cap \cdots \cap Q'_n,$$

and moreover both decompositions are normal. Thus by the induction hypothesis we have $n-1 = m-1$, that is, $n=m$. Furthermore, by a suitable ordering we have $r(Q_i) = r(Q'_i)$ for $2 \leq i \leq n=m$. This completes the proof.

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References

