The purpose of this paper is to discuss an application of the theory of vector bundle valued harmonic forms on a Riemannian manifold to the study of immersions.

Let $M$ be a Riemannian manifold and $E$ a Riemannian vector bundle over $M$. Then we can define in a natural way the Laplacian $\Delta$ operating on $E$-valued differential forms and we can express the scalar product $\langle \theta, \theta \rangle$, where $\theta$ is an $E$-valued $p$-form, in terms of curvature and covariant differentials. Moreover, if $M$ is compact, we obtain, by integrating over $M$, a formula analogous to Bochner's for ordinary (i.e. real valued) differential forms.

Let $f$ be an immersion of $M$ into a Riemannian manifold $M'$. We may regard the second fundamental form $a$ of $(M, f)$ as a Hom $(T(M), N(M))$-valued 1-form. Assuming that $M'$ is of constant sectional curvature, we shall prove that the second fundamental form $a$ is harmonic, i.e. $\Delta a = 0$, if the mean curvature normal of $(M, f)$ is parallel. In particular, if the immersion $f$ is a minimal immersion, then $a$ is harmonic. Conversely, if $M$ is compact and if $a$ is harmonic, then the mean curvature normal is parallel. We obtain from this result together with the formula of Bochner type the results of Simons [5], Chern [1], Nomizu-Smyth [4] and Erbacher [2] proved by them in different ways. In a future paper we shall discuss the case where $M$ is a Kähler manifold.

1. Let $M$ be an $n$-dimensional Riemannian manifold and $E$ a vector bundle over $M$ with a metric along the fibers and a covariant differentiation $D_X$ satisfying

$$X\langle \varphi, \psi \rangle = \langle D_X \varphi, \psi \rangle + \langle \varphi, D_X \psi \rangle$$

for any vector field $X$ and any sections $\varphi$ and $\psi$ of $E$. A vector bundle $E$ with these properties will be called a Riemannian vector bundle.

We shall denote $C^p(E)$ the real vector space of all $E$-valued differential $p$-forms on $M$. We define an operator

$$\partial : C^p(E) \rightarrow C^{p+1}(E), (p = 0, 1, \cdots)$$
by the formula

$$(\partial \theta)(X_1, \ldots, X_p) = \sum_{i=1}^{p+1} (-1)^{i+1} D_X(\theta(X_1, \ldots, \hat{X_i}, \ldots, X_p))$$

$$+ \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_1, \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots, X_p),$$

where $X_i$'s denote vector fields on $M$. The covariant derivative $D_X \theta$ of $\theta \in \Omega^p(E)$ is an $E$-valued $p$-form such that

$$(D_X \theta)(X_1, \ldots, X_p) = D_X(\theta(X_1, \ldots, X_p)) - \sum_{i=1}^p \theta(\nabla_X X_i, \ldots, X_p),$$

where $\nabla_X X_i$ denotes the covariant derivative of the vector field $X_i$ in the Riemannian manifold $M$.

For an $E$-valued 1-form $\theta$ we have the formula

$$(\partial \theta)(X, Y) = (D_X \theta)(Y) - (D_Y \theta)(X)$$

The covariant differential $D \theta$ of $\theta$ is an $E$-valued $(p+1)$-tensor defined by

$$(D \theta)(X_1, \ldots, X_p, X) = (D_X \theta)(X_1, \ldots, X_p).$$

We define an operator

$$\partial^* : C^p(E) \to C^{p-1}(E) \quad (p > 0)$$

as follows. Let $x \in M$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ of $M$ at $x$. For any $p-1$ tangent vectors $u_1, \ldots, u_{p-1}$ at $x$, put

$$(\partial^* \eta)_x(u_1, \ldots, u_{p-1}) = - \sum_{i=1}^n (D_{e_i} \eta)_x(e_k, u_1, \ldots, u_{p-1}),$$

where $(D_{e_i} \eta)_x$ denotes the value of $D_{X} \eta$ at $x$ for any vector field $X$ such that $X_x = e_k$. Then $(\partial^* \eta)_x$ is an alternating $(p-1)$-linear map of $T_x(M)$ into $E_x$, the fiber of $E$ over $x$, and the assignment $x \to (\partial^* \eta)_x$ defines an $E$-valued $(p-1)$-form $\partial^* \theta$. For any $E$-valued 0-form $\theta$, we define $\partial^* \theta = 0$.

The Laplacian $\Box$ for $E$-valued differential forms is defined as

$$\Box = \partial \partial^* + \partial^* \partial.$$

The curvature $\hat{R}$ of the covariant differentiation $D$ in $E$ is a $\text{Hom}(E, E)$-valued 2-forms given by

$$\hat{R}(X, Y) \varphi = D_X(D_Y \varphi) - D_Y(D_X \varphi) - D_{[X,Y]} \varphi$$

for any section $\varphi$ of $E$ and for any vector fields $X$ and $Y$ in $M$. We shall denote by $\langle \theta, \eta \rangle$ the scalar product of two $E$-valued $p$-forms, that is, $\langle \theta, \eta \rangle$ is the smooth function on $M$ given by
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\[ \langle \theta, \eta \rangle(x) = \sum_{i_1, \ldots, i_p} \langle \theta(e_{i_1}, \ldots, e_{i_p}), \eta(e_{i_1}, \ldots, e_{i_p}) \rangle, \]

where \( \{e_1, \ldots, e_n\} \) denotes an orthonormal basis of \( T_x(M) \).

Now we prove the following

**Theorem 1.** Let \( \theta \) be an \( E \)-valued 1-form. Then

\[ \langle \Box \theta, \theta \rangle = \frac{1}{2} \Delta \langle \theta, \theta \rangle + \langle D\theta, D\theta \rangle + A, \]

where \( \Delta \) denotes the Laplacian of the Riemannian manifold \( M \) and \( A \) denotes a smooth function in \( M \) defined as follows:

\[ A(x) = \sum_i \langle (R(e_i, e_i)\theta)(e_i), \theta(e_i) \rangle + \sum_i \langle \theta(S(e_i)), \theta(e_i) \rangle, \]

where \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( T_x(M) \) and \( S \) denotes the endomorphism of \( T_x(M) \) defined by the Ricci tensor \( S \) of \( M \), i.e. \( S(e_i) = \sum_k S_{ki} e_k \).

**Proof.** Fix a point \( x \in M \) and let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_x(M) \). We can choose \( n \) vector fields \( E_1, \ldots, E_n \) in \( M \) such that \( E_i(x) = e_i \) and \( (\nabla_{E_k} E_i)_x = 0 \) for \( i, k = 1, \ldots, n \). Then, because \( \nabla_{e_i} E_i \) are zero for \( i, s = 1, \ldots, n \), we have

\[ \langle \partial^* \partial \theta \rangle(e_i) = - \sum_i (D_{e_i} \partial \theta)(e_s, e_i) = - \sum_i D_{e_i}((\partial \theta)(E_s, E_i)) \]
\[ = - \sum_i D_{e_i}((D_{E_i} \theta)(E_i)) - (D_{E_i} \theta)(E_i)) \]
\[ = \sum_i (D_{E_i} D_{e_i} \theta)(e_i) - \sum_i (D_{E_i} D_{e_i} \theta)(e_i). \]

On the other hand, \( \partial^* \theta = - \sum_i g^{st}((D_{E_i} \theta)(E_s)) \) where \( (g^{st}) \) is the inverse matrix of the matrix \( (g(E_s, E_i)) \), we have

\[ \langle \partial^* \partial \theta \rangle(e_i) = D_{e_i}((\partial^* \theta)(e_i) = - \sum_i (e_i g^{st})(D_{e_i} \theta)(e_s) - \sum_i g^{st} e_i ((D_{E_i} \theta)(E_s)) \]
\[ = - \sum_i e_i ((D_{E_i} \theta)(E_s)) = - \sum_i (D_{E_i} D_{e_i} \theta)(e_s), \]

because \( \nabla_{e_i} E_k = 0 \) at \( x \).

Therefore we obtain

\[ \langle \Box \theta \rangle(e_i) = \sum_i ((D_{E_i} D_{E_i} - D_{E_i} D_{E_i}) \theta)(e_s) - \sum_i (D_{E_i} D_{E_i} \theta)(e_i). \]

Since \( [E_s, E_k] = 0 \) at \( x \), we have

\[ ((D_{E_s} D_{E_i} - D_{E_i} D_{E_s}) \theta)(e_s) = (([D_{E_s}, D_{E_i}]) - D_{[E_s, E_i]} \theta)(e_s) \]
\[ = R(e_s, e_i)(\theta(e_s)) - \theta(R(e_s, e_i)e_s). \]

Therefore
\[ \langle \Box \theta, \theta \rangle = \sum_i \langle \Box \theta(e_i), \theta(e_i) \rangle = \sum_i \langle \mathcal{R}(e_i, e_i) \theta(e_i), \theta(e_i) \rangle + \sum_i \theta(S(e_i), \theta(e_i)) - \sum_i \langle (D_{E_i} D_{E_i} \theta)(e_i), \theta(e_i) \rangle. \]

Now by a local computation we see that

\[ -\sum_i \langle (D_{E_i} D_{E_i} \theta)(e_i), \theta(e_i) \rangle = \langle D\theta, D\theta \rangle(x) + \frac{1}{2} \langle \Delta \langle \theta, \theta \rangle, \theta \rangle(x). \]

Thus we have proved that

\[ \langle \Box \theta, \theta \rangle = \frac{1}{2} \Delta \langle \theta, \theta \rangle + \langle D\theta, D\theta \rangle + A. \]

**Corollary 1.** Let \( \theta \) be an \( E \)-valued 1-form. Assume that \( \Box \theta = 0 \) and \( \Delta \langle \theta, \theta \rangle = 0 \). Then we have \( A \leq 0 \) everywhere on \( M \).

Assume now that \( M \) is compact and oriented. Then we can define the inner product \( \langle \theta, \eta \rangle \) of two \( E \)-valued \( p \)-forms by

\[ (\theta, \eta) = \int_M \langle \theta, \eta \rangle \ast 1. \]

Then we obtain from Theorem 1 the following corollary.

**Corollary 2.** Let \( \theta \) be an \( E \)-valued 1-form such that \( \Box \theta = 0 \). Then we have

\[ (D\theta, D\theta) + \int_M A \ast 1 = 0. \]

If \( A \geq 0 \) everywhere on \( M \), then we have \( A \equiv 0 \) and \( D\theta = 0 \).

We remark that the operator \( \partial^* \) is the adjoint operator of \( \partial \), i.e.

\[ (\partial \theta, \eta) = (\theta, \partial^* \eta) \]

for any \( \theta \in C^p(E) \) and \( \eta \in C^{p+1}(E) \) and hence we have

\[ (\Box \theta, \theta) = (\partial \theta, \partial \theta) + (\partial^* \theta, \partial^* \theta). \]

Therefore, if \( M \) is compact, \( \Box \theta = 0 \) if and only if \( \partial \theta = 0 \) and \( \partial^* \theta = 0 \).

2. Let \( M \) be an \( n \)-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold \( M' \) of dimension \( n + p \). We shall denote by \( N(M) \) and \( \alpha \) the normal bundle and the second fundamental form of \( M \) [3]. The second fundamental form \( \alpha \) is an \( N(M) \)-valued symmetric 2-form on \( M \).

In the following we put
and we interprete \( \alpha \) as an \( E \)-valued 1-form \( \beta \) as follows: For any vector field \( X \) in \( M \), \( \beta(X) \) is a section of \( E \) such that

\[
\beta(X) \cdot Y = \alpha(X, Y)
\]

for all vector field \( Y \) in \( M \). Then we have

\[
\beta(X) \cdot Y = \beta(Y) \cdot X.
\]

We call also \( \beta \) the second fundamental form of \( M \).

A metric along the fibres of \( E \) is defined naturally by the Riemann metrics of \( M \) and \( M' \) and a covariant derivation \( D_X \) in \( E \) is also naturally defined by the covariant differentiation \( \nabla_X \) in \( M \) and \( D_X^\perp \) in \( N(M) \), where for any normal vector \( \xi \) of \( M \), \( D_X^\perp \xi \) is defined as the normal component of \( \nabla_X' \xi \), where \( \nabla_X' \) denote the covariant differentiation in the Riemannian manifold \( M' \) (See [3]).

Let \( \varphi \) be a section of \( E \). We may regard \( \varphi \) as an \( N(M) \)-valued 1-form on \( M \) and we have

\[
(D_X\varphi)(Y) = D_X^\perp(\varphi(Y)) - \varphi(\nabla_X Y),
\]

\[
\langle D_X\varphi, \psi \rangle + \langle \varphi, D_X\psi \rangle = X \langle \varphi, \psi \rangle
\]

for any sections \( \varphi \) and \( \psi \) of \( E \).

The following Proposition 1 may be considered as an interpretation of the equation of Codazzi in our formalism.

**Proposition 1.** Assume that \( M' \) is a Riemannian manifold of constant sectional curvature. Then the second fundamental form \( \beta \) of \( M \) satisfies the equation \( \partial \beta = 0 \).

**Proof.** By a straightforward computation we see that

\[
(\partial \beta(X, Y))(Z) = \{ D_{\nabla^X Y}^\perp(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_Z X) \} - \{ D_{\nabla^X Y}^\perp(\alpha(X, Z)) - \alpha(\nabla^X Y, Z) - \alpha(X, \nabla_Y Z) \}
\]

and the right hand side is 0 by [3, Vol. II, P. 25, Cor. 4.4].

For each normal vector \( \nu \in N_x(M) \) we define an endomorphism \( A_\nu \) of \( T_x(M) \) by the formula

\[
\langle A_\nu(u), v \rangle = \langle \beta(u) v, \nu \rangle
\]

for any tangent vectors \( u, v \in T_x(M) \). The mean curvature normal \( \eta \) of \( M \) is a
normal vector field in $M$ such that
\[
\frac{1}{n} \text{Tr} A_x = \langle \nu, \eta(x) \rangle
\]
for any $\nu \in N_x(M)$ and $x \in M$.

$M$ is said to be \textit{minimal} in $M'$ if the mean curvature normal vanishes at each point, that is, if $\text{Tr} A_x = 0$ for any $\nu \in N_x(M)$ and $x \in M$.

We say that $M$ has a \textit{constant mean curvature} if the mean curvature normal $\eta$ is parallel, that is, $D_\nu \eta = 0$ for any vector field $X$ in $M$.

Let $\nu$ be a normal vector field. Then we have $\text{Tr} A_x = n \langle \nu, \eta \rangle$ and hence $X \cdot \text{Tr} A_x = n \{ \langle D_\nu \nu, \eta \rangle + \langle \nu, D_\nu \eta \rangle \}$. Therefore $M$ has a constant mean curvature, if and only if
\[
X \cdot \text{Tr} A_x = \text{Tr} A_{D_\nu \nu}
\]
for any normal vector field $\nu$ and any vector field $X$ in $M$.

\textbf{Proposition 2.} Let $M'$ be a Riemmanian manifold of constant sectional curvature. Then the second fundamental form $\beta$ of $M$ satisfies the equation $\partial^* \beta = 0$ if and only if $M$ has a constant mean curvature.

\textbf{Proof.} Let $x$ be a point in $M$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x(M)$. Let $E_1, \ldots, E_n$ be vector fields in a neighborhood of $x$ such that $(E_i)_x = e_i$ and $\nabla_{E_i} E_k = 0$ at $x$ for $i, k = 1, \ldots, n$. Let $(g^{st})$ the inverse matrix of the matrix $(\langle E_s, E_t \rangle)$. Then $\partial^* \beta = -g^{st}(D_{E_s} \beta)(E_t)$ and $(\partial^* \beta) \cdot E_k = -g^{st}(D_{E_s} \beta)(E_t) \cdot E_k$.

Since $(D_{E_s} \beta)(E_t) E_k = D_{E_t}^k(\alpha(E_t, E_s)) - \alpha(\nabla_{E_k} E_t, E_s) - \alpha(E_k, \nabla_{E_t} E_s)$ and since $\alpha$ is symmetric, we get $(D_{E_s} \beta)(E_t) E_k = (D_{E_t} \beta)(E_k) E_s$. On the other hand, by Proposition 1, we have $\partial \beta = 0$ and hence $(D_{E_t} \beta)(E_k) = (D_{E_k} \beta)(E_t)$, hence $(D_{E_t} \beta)(E_s) E_k = (D_{E_k} \beta)(E_t) E_s$. Therefore, for any normal vector field $\nu$, we have
\[
\langle (\partial^* \beta) \cdot E_k, \nu \rangle = -g^{st} \langle (D_{E_k} \beta)(E_t) E_s, \nu \rangle = -g^{st} \{ D^k_{E_t} (\alpha(E_t, E_s)), \nu \} - \langle \alpha(\nabla_{E_k} E_t, E_s), \nu \rangle - \langle \alpha(E_t, \nabla_{E_k} E_s), \nu \rangle.
\]

Now
\[
g^{st} \langle D^k_{E_t} (\alpha(E_t, E_s)), \nu \rangle = g^{st} \langle E_k, \alpha(E_t, E_s), \nu \rangle - \langle \alpha(E_t, E_s), D^k_{E_t} \nu \rangle = E_k(g^{st} \langle \alpha(E_t, E_s), \nu \rangle - (E_k g^{st}) \langle \alpha(E_t, E_s), \nu \rangle - g^{st} \langle \alpha(E_t, E_s), D^k_{E_t} \nu \rangle = E_k(T_r A_{\nu}) - T_r A_{D^k_{E_t} \nu} - E_k g^{st} \langle \alpha(E_t, E_s), \nu \rangle.
\]

1) We omit here the summation signs.
Since \( \nabla_{E_k} E_i = 0 \) at \( x \), we have \( E_k E_i = 0 \) at \( x \). Therefore we get from the above that

\[
\langle (\partial^* \beta) E_k, \nu \rangle(x) = \text{Tr} A_{D_k} - E_k(\text{Tr} A_i)
\]

at \( x \) for \( k = 1, 2, \ldots, n \) and hence for any vector field \( X \) we have \( \langle (\partial^* \beta) X, \nu \rangle(x) = \text{Tr} A_{D_X} - X(\text{Tr} A_i) \) at \( x \). Since \( x \) is an arbitrary point of \( M \) and \( \nu \) is an arbitrary normal vector field, we see from the above equation that \( \partial^* \beta = 0 \) if and only if \( M \) has a constant mean curvature.

From Propositions 1 and 2 we get the following

**Theorem 2.** Let \( M \) be a Riemannian manifold immersed isometrically into a Riemannian manifold \( M' \) of constant sectional curvature. Let \( \beta \) be the second fundamental form of \( M \) regarded as a Hom \((T(M), N(M))\)-valued 1-form. Then \( \beta \) satisfies the equation \( \Box \beta = 0 \), if \( M \) has a constant mean curvature. Conversely, if \( M \) is compact and orientable and \( \Box \beta = 0 \), then \( M \) has a constant mean curvature.

3. We shall discuss in this section some applications of Theorems 1 and 2. Let \( M \) be a Riemannian manifold immersed isometrically into a Riemannian manifold \( M' \) of constant sectional curvature \( c \). Let \( x \in M \) and let \( \{e_1, \ldots, e_n\} \) and \( \{\nu_1, \ldots, \nu_p\} \) be orthonormal bases of \( T_x(M) \) and \( N_x(M) \) respectively. We shall denote by \( A_a(a = 1, 2, \ldots, p) \) the endomorphism of \( T_x(M) \) defined by

\[
\langle A_a u, v \rangle = \langle \beta(u), v \rangle
\]

and put \( A_a e_i = \sum_j (A_a)_i^j e_j \). Then we have the following Gauss equation:

\[
(3.1) \quad R_{klij} = c \delta_{k} \delta_{ij} \delta_{li} + \sum_a \{(A_a)_l^i - (A_a)_{l}^i (A_a)_i^l\},
\]

where \( R_{klij} \) denote the components of the curvature tensor with respect to the basis \( \{e_1, \ldots, e_n\} \) of \( T_x(M) \). Then the endomorphism \( S \) of \( T_x(M) \) defined by

\[
S(e_j) = \sum_j S_{ij} e_j = \sum_k R_{hkij} e_j
\]

is of the form

\[
(3.2) \quad S = c(n-1)I + \sum_a (\text{Tr} A_a) A_a - \sum_a A_a^2,
\]

where \( I \) denotes the identity endomorphism of \( T_x(M) \).

Let \( K \) be the scalar curvature of \( M \). Then

\[
K(x) = \text{Tr} S = c(n-1)n + \sum_a (\text{Tr} A_a)^2 - \sum_a \text{Tr} A_a^2.
\]

The value \( \eta(x) \) at \( x \) of the mean curvature normal \( \eta \) is given by \( \eta(x) = \frac{1}{n} \sum_a \text{Tr} A_a \cdot \nu_a \) and hence \( n^2 \langle \eta, \eta \rangle(x) = \sum (\text{Tr} A_a)^2 \). Analogously we have \( \langle \beta, \beta \rangle(x) = \sum \text{Tr} A_a^2 \). Hence we get

\[
(3.3) \quad K = c(n-1)n + n^2 \langle \eta, \eta \rangle - \langle \beta, \beta \rangle,
\]
where \( \beta \) and \( \eta \) denotes the second fundamental form and the mean curvature normal of \( M \) respectively. For any Riemannian vector bundle \( E \) over \( M \) we have defined the endomorphism \( \hat{R}(u, v) \) of the fiber \( E_x \), where \( u, v \in T_x(M) \). Let \( E = \text{Hom}(T(M), N(M)) \) and let \( \varphi \in E_x \). Then \( \hat{R}(u, v) \varphi \) is an element of \( E_x = \text{Hom}(T_x(M), N_x(M)) \) such that

\[
(\hat{R}(u, v)\varphi)(w) = R^{1}(u, v)(\varphi)-\varphi(R(u, v)w),
\]

where \( u, v, w \in T_x(M) \) and \( R^{1} \) denotes the curvature of the Riemannian vector bundle \( N(M) \).

Let \( v \) be a normal vector of \( M \) at \( x \) and let \( N \) be a normal vector field such that \( N_x = v \). Let \( X \) and \( Y \) be vector fields in \( M \) such that \( X_x = u \) and \( Y_x = v \). Then we have

\[
R^{1}(u, v)v = (D^{\bot}_{\hat{v}}D^{\bot}_{\hat{v}} - D^{\bot}_{\hat{v}}D^{\bot}_{\hat{v}} - D^{\bot}_{\{X, Y\}}) N
\]
at \( x \).

Denote by \( \nabla' \) the covariant derivation in the ambiant space \( M' \). Then we have

\[
\nabla'X'Y = \nabla_XY + \alpha(X, Y), \quad \nabla'X'N = -A_N(X) + D_XN.
\]

We see from these two equations that the normal component \( (R'(X, Y)N)^{1}\) of \( R' \) \( (X, Y)N \), where \( R' \) denotes the curvature tensor of \( M_2 \), is equal to \( R^{1}(X, Y)N - \alpha(A_N(X), X) + \alpha(A_N(Y), Y) \). Since \( M' \) is of constant curvature \( R'(X, Y)N = \varepsilon\{X, Y\}X - \langle N, X \rangle Y \) = 0 and hence we get \( R^{1}(X, Y)N = -\alpha(A_N(X), Y) + \alpha(A_N(Y), X) \). Thus we have

\[
R^{1}(u, v)v = -\alpha(A_u, v) + \alpha(A_v, u).
\]

In particular

\[
R^{1}(u, v)v = -\alpha(A_u, v) + \alpha(A_v, u).
\]

Since \( \alpha(A_u, v) = \sum_b \langle A(A_u, v)\rangle v_b = \sum_v (A(A_u, v)\rangle v_b \)
and \( \alpha(u, A_v) = \sum_b \langle A_u, A_v\rangle v_b = \sum \langle A_u, A_v\rangle v_b \)
we get

\[
R^{1}(u, v)v = \sum_b \langle [A_u, A_v]u, v\rangle v_b.
\]

Now by Theorem 1, we have

\[
\langle \square \beta, \beta \rangle = \frac{1}{2} + \Delta\langle \beta, \beta \rangle + \langle D\beta, D\beta \rangle + A.
\]
where

\[(3.6) \quad A(x) = \sum_{i,j} \langle R(e_j,e_i)\beta(e_j), \beta(e_i) \rangle + \sum_i \langle \beta(S(e_i), \beta(e_i) \rangle.
\]

Now

\[
\sum_i \langle \beta(S(e_i), \beta(e_i) \rangle = \sum_{i,j} \langle \alpha(S(e_i), e_j), \alpha(e_i, e_j) \rangle
\]

\[
= \sum_{i,j} \langle A_a(S(e_i)), e_j \rangle \langle A_a(e_i), e_j \rangle = \sum_a \text{Tr}(SA_a^2)
\]

and by (3.2) we get

\[(3.7) \quad \sum_i \langle \beta(S(e_i), \beta(e_i) \rangle = c(n-1) \sum_a \text{Tr} A_a^2 + \sum_{a,b} \text{Tr} A_a \cdot \text{Tr}(A_a A_b) - \sum_{a,b} \text{Tr}(A_a^2 A_b).
\]

On the other hand,

\[
\sum_{i,j} \langle R(e_j,e_i)\beta(e_j), \beta(e_i) \rangle
\]

\[
= \sum_{i,j,k} \langle R^k(e_j,e_i)\alpha(e_j,e_k), \alpha(e_i,e_k) \rangle - \sum_{i,j,k} \langle \alpha(e_j,R(e_j,e_i)e_k), \alpha(e_i,e_k) \rangle
\]

\[
= \sum_{i,j,k} \sum_{e,e} \langle A_a e_j, e_b \rangle \langle A_a e_i, e_b \rangle \langle R^k(e_j,e_i)\nu_a \nu_b \rangle
\]

\[
- \sum_{i,j,k} \sum_{a,b} \langle A_a e_j, R(e_j,e_i)e_k \rangle \langle A_a e_i, e_b \rangle
\]

and by (3.5), the first term equals \( \sum_{a,b} \text{Tr}(A_a A_b[A_a, A_b]) = - \sum_{a,b} \text{Tr}(A_a^2 A_b^2) + \sum_{a,b} \text{Tr}(A_a A_b)^2 \) and by the Gauss equation (3.1) the second term equals \(-c \sum_a (\text{Tr } A_a)^2 \) and the third term equals \( -c \sum_{a,b} \text{Tr}(A_a^2 A_b)^2 - \sum_a (\text{Tr}(A_a A_b))^2 + \sum_a \text{Tr}(A_a A_b)^2 \).

Therefore we have

\[(3.8) \quad \sum_{i,j} \langle R(e_j,e_i)\beta(e_j), \beta(e_i) \rangle
\]

\[
= c \sum_a \text{Tr} A_a^2 - c \sum_a (\text{Tr } A_a)^2 - \sum_{a,b} \text{Tr}(A_a^2 A_b)^2 - \sum_a (\text{Tr}(A_a A_b))^2 + 2 \sum_{a,b} \text{Tr}(A_a A_b)^2
\]

Then we get from (3.6), (3.7) and (3.8) that

\[(3.9) \quad A(x) = cn \sum_a \text{Tr} A_a^2 - c \sum_a (\text{Tr } A_a)^2 - \sum_{a,b} (\text{Tr}(A_a A_b))^2
\]

\[+ \sum_{a,b} \text{Tr} A_a \cdot \text{Tr}(A_a A_b) + \sum_{a,b} \text{Tr}[A_a, A_b]^2.
\]

Now let \( \lambda_1^{(a)}, \ldots, \lambda_n^{(a)} \) be eigen-values of \( A_a \) and let \( \{e_1^{(a)}, \ldots, e_n^{(a)}\} \) be an orthonormal basis of \( T_x(M) \) such that \( A_a e_i^{(a)} = \lambda_i^{(a)} e_i^{(a)}(i=1,\ldots,n, a = 1,\ldots,p) \).

We shall denote by \( K_{ij}^{(a)} \) the sectional curvature for the 2-plane spanned by \( e_i^{(a)} \) and \( e_j^{(a)} \), \( i \neq j \).

We show that
We write $A(x)$ in the following form:

$$A(x) = B(x) + \sum_{a,b} \text{Tr} A_a \cdot \text{Tr}(A_a A_b) - \sum_{a,b} (\text{Tr}(A_a A_b))^2 + \sum_{a,b} \text{Tr}[A_a, A_b]^2,$$

where

$$B(x) = \sum_{a} \{ c \text{Tr} A_a^2 - c(\text{Tr} A_a)^2 - (\text{Tr} A_a^2)^2 + \text{Tr} A_a \cdot \text{Tr}(A_a^2) \}.$$ 

Now by a lemma of Nomizu-Smyth [4] we have

$$cn \text{Tr} A_a^2 - c(\text{Tr} A_a)^2 - (\text{Tr} A_a^2)^2 + \text{Tr} A_a \cdot \text{Tr}(A_a)^2 = \sum_{i<j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (c + \lambda_i^{(a)} \lambda_j^{(a)})$$

for each $a$. Now fix an index $a$ and let

$$A_b e_i^{(a)} = \sum_j (A_b)_{ij} e_j^{(a)} \quad (b=1,2,\cdots,p)$$

Then we have $(A_a)_{ij} = \delta_{ij} \lambda_j^{(a)}$ and hence

$$(A_a A_a)_{ij} = \lambda_i^{(a)} \lambda_j^{(a)}, \quad (A_a A_a)_{ij} = (A_b)_{ij} \lambda_j^{(a)}.$$ 

By the equation of Gauss we have

$$K_{ij}^{(a)} = R(e_i^{(a)}, e_j^{(a)}, e_i^{(a)}, e_j^{(a)})$$

$$= c + \sum_b (A_b)_{ij} \lambda_i^{(a)} + \sum_b (A_b)_{ij} \lambda_j^{(a)}.$$ 

Hence we have

$$(\lambda_i^{(a)} \lambda_j^{(a)} - \lambda_i^{(a)} \lambda_j^{(a)})(c + \lambda_i^{(a)} \lambda_j^{(a)})$$

$$= (\lambda_i^{(a)} \lambda_j^{(a)} - \lambda_i^{(a)} \lambda_j^{(a)}) K_{ij}^{(a)} + \sum_b (\lambda_i^{(a)} - \lambda_j^{(a)}) (A_b)_{ij} (A_b)_{ij}$$

$$- \sum_b (\lambda_i^{(a)} - \lambda_j^{(a)}) (A_b)_{ij} (A_b)_{ij}.$$ 

This equality holds also for $i = j$ trivially if we define $K_{ii}^{(a)} = 0$.

Then by (3.14)

$$\sum_{i<j} (\lambda_i^{(a)} - \lambda_j^{(a)}) (c + \lambda_i^{(a)} \lambda_j^{(a)}) = \frac{1}{2} \sum_{i,j} (\lambda_i^{(a)} - \lambda_j^{(a)}) (c + \lambda_i^{(a)} \lambda_j^{(a)})$$

$$= \sum_{i<j} (\lambda_i^{(a)} - \lambda_j^{(a)}) K_{ij}^{(a)} - \frac{1}{2} \sum_b \sum_{i,j} (\lambda_i^{(a)} - \lambda_j^{(a)}) (A_b)_{ij} (A_b)_{ij}.$$


\[
- \frac{1}{2} \sum_{i \neq j} \sum_{k} (\lambda^o_i)^2 (A_a)_i^k \sum_{j} (A_b)_j^k - 2 \sum_{i} \lambda^o_i (A_a)_i^k \sum_{j} \lambda^o_j (A_b)_j^k \\
+ \sum_{i} (A_a)_i^k \sum_{j} \lambda^o_j (A_b)_j^k 
\]
\[
= \sum_{i < j} (\lambda^o_i - \lambda^o_j)^2 K_{ij}^o - \frac{1}{2} \sum_k \text{Tr}[A_a, A_b]_k^2 \\
- \sum_{i \neq j} (\text{Tr} A_b \cdot \text{Tr} (A_a A_a^o A_b) - (\text{Tr}(A_a A_a^o))_k^2). 
\]

Then we obtain from (3.11), (3.12) and (3.13) the equality (3.10).

Now we cite the following two lemmas from [1].

**Lemma 1.** Let \( A \) and \( B \) be symmetric \( n \times n \) matrices. Then

\[
\text{Tr}[A, B]^2 \geq -2\text{Tr} A^2 \cdot \text{Tr} B^2, 
\]

and the equality holds for non-zero matrices \( A \) and \( B \) if and only if \( A \) and \( B \) can be transformed simultaneously by an orthogonal matrix into scalar multiple of \( A \) and \( B \) respectively, where

(3.15)

\[
\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. 
\]

**Lemma 2.** Let \( A_a, A_b \) and \( A_c \) be \( n \times n \) symmetric matrices and if

\[
\text{Tr}[A_a, A_b]^2 \geq -2\text{Tr} A_a^2 \cdot \text{Tr} A_b^2
\]

for \( 1 \leq a < b \leq 3 \), then at least one of the matrices \( A_a \) must be zero.

By Lemma 1, we have

\[
\frac{1}{2} \sum_a \text{Tr}[A_a, A_b]^2 \geq - \sum_a \text{Tr} A_a^2 \cdot \text{Tr} A_b^2 = -2 \sum_{a < b} \text{Tr} A_a^2 \cdot \text{Tr} A_b^2. 
\]

Put \( S_a = \text{Tr} A_a^2 \). Then \( \sum_a S_a = \langle \beta, \beta \rangle(x) \).

Since

\[
0 \leq \sum_{a < b} (S_a - S_b)^2 = \sum_{a < b} (S_a^2 + S_b^2) - 2 \sum_{a < b} S_a S_b \\
= (p - 1) \sum_a S_a^2 - 2 \sum_{c < d} S_c S_d \\
= (p - 1) \langle \sum_a S_a \rangle^2 - 2 \sum_a S_a S_b - 2 \sum_a S_a S_b \\
= (p - 1) \langle \beta, \beta \rangle(x) - 2p \sum_a S_a S_b
\]

we have
where
\[ (3.16) \quad \sum_{i \neq j} \left( \sum_{a=1}^{p} A_{a} \right)^{2} \geq \left( \frac{p-1}{p} \right) \langle \beta, \beta \rangle^2(x) \]
and the equality holds if and only if \( \sum_{a=1}^{p} A_{a} = \sum_{b=1}^{p} A_{b} \) for \( a,b = 1, \ldots, p \) and either \( A_{a} \) are all zero except possibly one of them or \( A_{a} \) are all zero except two of them, say \( A_{1} \) and \( A_{2} \), and they can be transformed simultaneously by an orthogonal matrix into scalar multiple of the matrices of the form (3.15). Thus we obtain from (3.10) the inequality

\[ (3.17) \quad A(x) \geq \sum_{i<j} \left( \lambda_{i}^{(a)} - \lambda_{j}^{(a)} \right)^2 K_{ij}^{(a)} - \left( \frac{p-1}{p} \right) \langle \beta, \beta \rangle^2(x) \]

Assume now that the scalar curvatures of \( M \) are bounded below by a positive constant \( d \). Then

\[ \sum_{i<j} \left( \lambda_{i}^{(a)} - \lambda_{j}^{(a)} \right)^2 K_{ij}^{(a)} \geq d \sum_{i<j} \left( \lambda_{i}^{(a)} - \lambda_{j}^{(a)} \right)^2 \]

and

\[ \sum_{i<j} \left( \lambda_{i}^{(a)} - \lambda_{j}^{(a)} \right)^2 = (n-1) \sum_{i} A_{i}^2 - 2 \sum_{i,j} \lambda_{i}^{(a)} \lambda_{j}^{(a)} \]

and hence

\[ \sum_{i<j} \left( \lambda_{i}^{(a)} - \lambda_{j}^{(a)} \right)^2 = n \langle \beta, \beta \rangle(x) - n \langle \eta, \eta \rangle(x) \]

where \( \eta \) denotes the mean curvature normal of \( M \). Thus we get the following inequality

\[ (3.17) \quad A \geq \left( d n - \frac{p-1}{p} \langle \beta, \beta \rangle \right) \langle \beta, \beta \rangle - d n^2 \langle \eta, \eta \rangle \]
at each point of \( M \).

We obtain from Corollaries 1 and 2 of Theorem 1 and Theorem 2 the following

**Theorem 3.** Let \( M \) be an \( n \)-dimensional, Riemannian manifold with sectional curvatures bounded below by a positive constant \( d \). Assume that \( M \) is immersed in a Riemannian manifold \( M' \) of constant sectional curvature of dimension \( n+p \) and that \( M \) has a constant mean curvature. Then, if \( M \) is compact and orientable or if the length of the second fundamental form \( \beta \) of \( M \) is constant, then we have
(3.18) \[ 0 \geq A \geq \left\{dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle - d n^2 \langle \eta, \eta \rangle \]

at each point of \( M \), where \( \eta \) denotes the mean curvature normal of \( M \) which is parallel and \( \langle \eta, \eta \rangle \) is a constant.

Now assume \( M \) is compact and oriented and let \( k = \langle \eta, \eta \rangle \). Then integrating both sides of the inequality (3.18) we obtain

\[
dn^2 k \int_M 1 \geq \int_M \left\{dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle 1
\]

and we have the equality here if and only if

\[
dn^2 k = \left\{dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle
\]

and this implies also that \( A = 0 \) and that \( \beta \) is parallel by Theorem 1. Then \( \langle \beta, \beta \rangle \) must satisfy the quadratic equation \((p-1)x^2 - p \ dn x + p n^2 k = 0\) and since the discriminant of this equation should be positive we should have the inequality

\[
d \geq \frac{4k(p-1)}{p}.
\]

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Bibliography


