In the present note we construct a \([p]\)-typical formal group \(F_p\), \(p\) a prime, which is universal for \([p]\)-typical formal groups over arbitrary ground rings; and we study structure of the ground ring of \(F_p\) (Corollary 5). Using \(F_p\) we describe the kernel of the forgetful homomorphism \(S: \Omega^U \to \Omega^{SO}\) of complex structures (Corollary 7).

Our basic reference is [1] and we use the notations of [1] freely.

1. Universal \([p]\)-typical formal group

Let \(U\) be the Lazard ring and \(F_U\) the universal one-dimensional formal group defined over \(U\). As usual we identify \(U\) with the complex cobordism ring \(\Omega^U\). Then \(U\) is graded by non-negative even degrees (or by non-positive even degrees when we regard \(\Omega^U\) as \(U^*(pt)\)).

Let \(p\) be a prime, \(R\) a commutative ring with 1 and \(F\) a (commutative one-dimensional) formal group over \(R\). By the terminology of [1] \(F\) is \([p]\)-typical iff \(f_pF\gamma_0 = 0\), where \(f_p\) is the Frobenius operator and \(\gamma_0\) is the identity curve.

Let \(F\) be \([p]\)-typical and \(u: U \to R\) the unique unitary homomorphism of rings such that \(u_F = F\). By the notation of [1] we put

\[
(f_{p, U\gamma_0})(T) = \sum_{n \geq 1} F_{u(v^{(p)}_{np-1}T^m)}.
\]

Then \(v^{(p)}_{np-1} \in U_{(np-1)}\). Now

\[
u_*(f_{p, U\gamma_0}) = f_{p, \gamma_0} = 0.
\]

Hence

\[
\sum_{n \geq 1} F(u(v^{(p)}_{np-1})T^m) = 0,
\]

and by [1], Proposition 2.10, we obtain

\[
u^{(p)}_{np-1} = 0 \quad \text{for all } n \geq 1.
\]

Let

\[
J_p = (v^{(p)}_{p-1}, v^{(p)}_{2p-1}, \ldots, v^{(p)}_{np-1}, \ldots),
\]
the ideal of $U$ generated by $v_{np^{-1}}$, $n \geq 1$. By (1.1) $u$ factorizes as the composition of the sequence

$$(1.2) \quad U \xrightarrow{\pi_p} U/J_p \xrightarrow{u_p} R.$$  

Define

$$V_{(p)} = U/J_p \quad \text{and} \quad F_{(p)} = \pi_p F_U.$$  

Then $u_p F_{(p)} = F$ and the homomorphism $u_p : V_{(p)} \to R$, $u_p F_{(p)} = F$, is unique by the uniqueness of $u$. And obviously $f_{p, F_{(p)}} = 0$. Thus we obtain

**Proposition 1.** $F_{(p)}$ is a $[p]$-typical formal group over $V_{(p)}$ and universal for $[p]$-typical formal groups.


2. **Structure of $\hat{U}/\hat{J}_p$**

Let

$$\log_U T = \sum_{k \geq 0} m_k T^{k+1}, \quad m_0 = 1,$$

the logarithm of $F_U$ over $U \otimes \mathbb{Q}$ and put

$$\hat{U} = Z[m_1, m_2, \ldots, m_n, \ldots]$$

as in [1]. As usual we can identify $\hat{U}$ with $H_*(MU)$. Then the inclusion map

$$(2.1) \quad U \subset \hat{U}$$

is identified with the Hurewicz homomorphism

$$\pi_*(MU) \to H_*(MU).$$

Let $p$ be a prime and put

$$\hat{J}_p = (v_{p^{-1}}, v_{2p^{-1}}, \ldots, v_{np^{-1}}, \ldots) \hat{U},$$

the ideal of $\hat{U}$ generated by $v_{np^{-1}}$, $n \geq 1$. In this section we observe structure of the quotient ring $\hat{U}/\hat{J}_p$.

Recall the relation (6.2) of [1]:

$$(2.2) \quad pm_{np^{-1}} = v_{np^{-1}} + \sum_{i \leq j < n} m_{j-1} (v_{ij}^{(p)})^i.$$  

This is the basic relation we use here. This shows that
(2.3) \[ pm_{n-1} \in J_p \]
on one hand, and by an induction on \( n \),

(2.4) \[ (1/p)v_{pq-1} \in \mathcal{U} \]
on the other hand.

Let \( p \) and \( q \) be different primes. For each integer \( k \geq 1 \) we have

(2.5) \[ pv_{pq-1} \in J_p \]

Proof by induction on \( k \). By (2.2) we have

\[ qm_{pq-1} = v_{pq-1} + m_{p-1}(v_{q-1})^p. \]

Hence

\[ pv_{pq-1} = qpm_{pq-1} - pm_{p-1}(v_{q-1})^p \in J_p, \]

by (2.3). Thus (2.5) is true for \( k = 1 \). Now assume that \( pv_{pq_j-1} \in J_p \) for \( j < k \). Then by (2.2) we have

\[ pv_{pq_k-1} = pm_{pq_k-1} - \sum_{i=1}^{k-1} pm_{j-1}(v_{q_i-1})^i. \]

\[ pm_{pq_k-1} \in J_p \] by (2.3). For each term under the summation, if \( p \mid j \) then \( pm_{j-1} \in J_p \), and if \( p \not\mid j \) then \( p \mid i \) and \( pv_{q_i-1} \in J_p \) by induction hypothesis. Thus

\[ pv_{pq_k-1} \in J_p, \]

Q.E.D.

Here we recall Milnor basis of \( U \). Let \( s_n \) denote the Chern number corresponding to \( \sum t_i^n \). As is well-known a series of elements \( u_n \in U_{2n}, n \geq 1 \), forms a polynomial basis of \( U \) if it satisfies

\[ s_n(u_n) = q \quad \text{when} \quad n = q^l-1 \quad \text{for some prime} \quad q, \]

\[ = 1 \quad \text{otherwise.} \]

Such a basis is called Milnor basis. We shall choose a Milnor basis in a specific form.

By (2.2) we see that

(2.6) \[ s_{n-1}(v_{nq-1}) = q \]

for any prime \( q \) and \( n \geq 1 \). First we choose

(2.7) \[ u_n = v_{nq-1} \quad \text{when} \quad n = q^l-1, \quad q \text{ a prime.} \]

Now let \( p \) be the specified prime. When \( p \mid n+1 \) and \( n+1 \) is not a power of \( p \), choosing the smallest prime \( q \) dividing \( n+1 \) and differing from \( p \), we can express \( n \) as \( n = pqk-1 \), \( k \) a positive integer. In such a case we put

(2.8) \[ u_{pqk-1} = s_{pqk-1}^{(q)} + tv_{pqk-1}^{(q)}, \]
where $s$ and $t$ are integers such that $sq + tp = 1$. Then

$$s_{pqk-1}(u_{pqk-1}) = 1$$

by (2.6).

For remaining $n$, i.e., $p \nmid n+1$ and $n+1$ is not a prime power, we choose $u_n$ arbitrarily so that $s_n(u_n) = 1$.

Hereafter we use only the above special choice of Milnor basis. First of all we have

(2.9) 

$$pu_{pqk-1} \subseteq \hat{J}_p$$

for elements of type (2.8), which follows from (2.5).

Put

$$m'_n = (1/q)u_n \quad \text{when } n+1 = q^s, \quad q \text{ a prime},$$

$$= u_n \quad \text{when } n+1 \text{ is not a prime power}.$$  

These are well defined elements of $\hat{U}$ by (2.4) and

(2.11)  

$$\hat{U} = Z[m'_1, m'_2, \ldots, m'_s, \ldots]$$

since $s_k(m'_s) = 1$.

For degrees of type (2.8) we observe the elements $pm'_{pqk-1} - v_{pqk-1}^{(p)}$. These belong to $\hat{J}_p$ by (2.9) and are decomposable in $\hat{U}$ since $s_n$-numbers are zero. Thus by induction on $qk$ we can replace the ideal basis elements $v_{pqk-1}^{(p)}$ of $\hat{J}_p$ by $pm'_{pqk-1}$ for such degrees and we obtain

**Proposition 2.** $\hat{J}_p = (pm'_{pm-1}, n \geq 1)$.

**Corollary 3.** $\hat{U}/\hat{J}_p$ is a direct sum of copies of $Z$ and $Z/pZ$ of which each direct summand is generated by a monomial of $m'_s$'s. A monomial is of order $p$ when it contains an element $m'_s$ with $p \mid k+1$ as a factor, and otherwise of infinite order.

### 3. Structure of $V_{(p)}$

Under our special choice of Milnor basis of $U$ we could choose a polynomial basis of $U$ so that its each element is a constant multiple of the corresponding element of the Milnor basis (cf., (2.10)—(2.11)).

**Theorem 4.** $J_p = (u_{p^k-1}, k \geq 1, pu_{np-1}, n \neq p^s)$.

**Proof.** Inductively on $n$ we replace generators $v_{np}^{(p)}$ of $J_p$ by the elements stated in Theorem. Since $u_{p-1} = v_{p-1}^{(p)}$ the replacement is already done for $n = 1$. Assume the replacement is done for $k < n$. When $n = p^s$ it is done already. Suppose $n$ is not a power of $p$. Since $pu_{pn-1} - v_{pn-1}^{(p)}$ is decomposable we can
express it as a polynomial of \( u_k \)'s such that \( 1 \leq k < pn - 1 \), say, \( P \). The polynomial expression \( P' \) of \( P \) in \( \hat{U} \) can be obtained by replacing each monomial in \( P \) by the corresponding monomial of \( m' \), multiplied with a non-zero integer. Now \( pu_{pn-1} - v_{pn-1}^{(n)} \in F_p \) by (2.9). Then by Proposition 2 each summand of \( P' \) belongs to \( F_p \). This implies that each monomial in \( P \) with non-zero coefficient contains a \( u_m \) with \( m = ps - 1 \) as a factor and, when it contains no \( u_m \) with \( m = p^j - 1 \) as a factor, then \( p \) divides its coefficient. Hence each summand of \( P \) belongs to \( F_p \) and \( pu_{pn-1} - v_{pn-1}^{(n)} \in F_p \). Now we can replace \( v_{pn-1}^{(n)} \) by \( pu_{pn-1} \) in the system of generators of \( F_p \).

**Q.E.D.**

**Corollary 5.** \( V_{F_p} \) is a direct sum of copies of \( Z \) and \( Z/pZ \) of which each direct summand is generated by a monomial of \( u_n \)'s such that \( n \neq p^s - 1 \). A monomial is of order \( p \) when it contains an element \( u_n \) with \( p | k + 1 \) as a factor, and otherwise of infinite order.

### 4. The forgetful homomorphism \( \Omega^U_\ast \rightarrow \Omega^{SO}_\ast \)

Let

\[
S: \Omega^U_\ast \rightarrow \Omega^{SO}_\ast \quad \text{and} \quad \Psi: \Omega^{SO}_\ast \rightarrow \mathcal{R}_\ast
\]

be the forgetful homomorphisms of complex structures and orientations respectively. Milnor [2] observed that

\[
(\Psi \circ S)(\Omega^U_\ast) = (\mathcal{R}_\ast)^2,
\]

where \((\mathcal{R}_\ast)^2\) is the subalgebra of \( \mathcal{R}_\ast \) consisting of bordism classes of manifold squares \( N \times N \). Let \( M \) be a weakly complex \( 2n \)-manifold and \( \Psi \circ S(M) = [N \times N] \). Then the Milnor's result shows that

\[
s'_n(M) = s'_n(N) \mod 2,
\]

where \( s'_n \) denotes the Whitney number corresponding to \( \sum t_i^n \). Thus we have a polynomial basis \( \{ x_n, n \neq 2^h - 1 \} \) of \( \mathcal{R}_\ast \) such that

\[
(\Psi \circ S)(u_n) = x_n^2, \quad n \neq 2^h - 1
\]

and \( \Psi \circ S \) induces an isomorphism

\[
(4.1) \quad \Omega^U_\ast/(u^{k-1}_{\neq}, k \geq 1) \otimes \mathbb{Z}/2\mathbb{Z} \simeq (\mathcal{R}_\ast)^2.
\]

As we remarked in [1], §5, the oriented cobordism \( \Omega^\ast(\_\_) \) is complex-oriented, [2]-typical and \( S^*_F U = F_{SO} \). Thus \( S \) factorizes as the composition of the sequence

\[
\Omega^U_\ast = U \xrightarrow{\pi_2} V_{[1]} \xrightarrow{\Phi} \Omega^{SO}_\ast.
\]
By Corollary 5 we have

\[(4.2) \quad V_{[a]} \otimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}[u_n, n \neq 2^k - 1].\]

By (4.1) and (4.2) we see that $\Psi \circ \Phi$ induces the isomorphism

\[(4.3) \quad V_{[a]} \otimes \mathbb{Z}/2\mathbb{Z} \cong (\mathbb{R}_+)^{\mathbb{Z}}.\]

By Corollary 5 we have

\[V_{[a]}/\text{Tors} = Z[u_n, n \geq 1].\]

Then by [3], p. 180, we conclude that

\[(4.4) \quad \Phi/\text{Tors}: V_{[a]}/\text{Tors} \cong \Omega_+^{SO}/\text{Tors}.\]

Finally by (4.3) and (4.4) we obtain

**Theorem 6.** $\Phi: V_{[a]} \rightarrow \Omega_+^{SO}$ is an injection.

**Corollary 7.** $\text{Ker } S = J_z, \text{ Im } S \cong V_{[a]}$.

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**References**

