ON COMPLEX COBORDISM GROUPS OF CLASSIFYING SPACES FOR DIHEDRAL GROUPS

MASAYOSHI KAMATA

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1. Introduction

Let $G=H \cdot \Gamma$ be a semi-direct product of a finite group $H$ by a finite group $\Gamma$, $X$ a compact $G$-manifold which induces by restriction a principal $H$-manifold and $Y$ a principal $\Gamma$-manifold. Then we have a principal $G$-space $X \times Y$ with a $G$-action defined by $h \gamma(x, y) = (h \gamma x, \gamma y)$, $h \gamma \in H \cdot \Gamma$. The equivariant map $i: X \to X \times Y$ defined by $i(x) = (x, y_0)$, induces a homomorphism

$$i^*: U^*((X \times Y)/G) \to U^*(X/H).$$

We can define a $\Gamma$-action over $U^*(X/H)$ corresponding to a $\Gamma$-action over the complex bordism group of unitary $G$-manifolds defined by (1.3) of [7]. The action is denoted by $x^\gamma$, $x \in U^*(X/H)$, $\gamma \in \Gamma$.

In this paper, we define a homomorphism

$$i_*: U^*(X/H) \to U^*((X \times Y)/G)$$

and obtain the following.

**Theorem 1.1.** For $x \in U^*(X/H)$, $i^* i_*(x) = \sum_{\gamma \in \Gamma} x^\gamma$.

Let $D_p(m, n)$ be the orbit manifold of $S^{2m+1} \times S^n$ by the dihedral group $D_p$ whose action is given in [7]. Making use of Theorem 1.1 and the Atiyah-Hirzebruch spectral sequence of the complex cobordism group, we have the following.

**Theorem 1.2.** Suppose that $p$ is an odd prime. There exists an isomorphism

$$U^m(D_p(2k+1, 4k+3)) \cong U^m(L^k(\mathbb{R}^{2k+1}(p)))^Z_2 \oplus U^m(RP^{k+1}) \oplus U^{2m-3k-6},$$

where $L^k(p) = S^{2l+1}/Z_p$ is a $(2l+1)$-dimensional lens space, $RP^s$ is an $s$-dimensional real projective space and $U^*(\ )^Z_2$ is the subgroup consisting of the elements which are fixed under the $Z_2$-action.

Let $BZ_p$ be a classifying space for $Z_p$. There exists an isomorphism

$$U^*(BZ_p) \cong U^*[X]/([p]_F(X)), U^*(\ ) = \sum U^{2i}(\ ) [8].$$

Consider the $Z_2$-action on $U^*(BZ_p)$ defined by
where \( t \) is a generator of \( \mathbb{Z}_2 \). We use Milnor's short exact sequence \([10]\) and Theorem 1.2 to compute the complex cobordism group of a classifying space for the dihedral group \( D_p \).

**Theorem 1.3.** Suppose that \( p \) is an odd prime. There exist isomorphisms

\[
\tilde{U}^{2m}(BD_p) \cong \tilde{U}^{2m}(BZ_p)^{Z_2} \oplus \tilde{U}^{2m}(BZ_2)
\]

and

\[
\tilde{U}^{2m+1}(BD_p) \cong 0.
\]

Making use of the Conner and Floyd isomorphism

\[
K(X) \cong \tilde{U}^{2m}(X) \otimes U^* Z
\]

and Theorem 1.2, we can deduce the structure of the \( K \)-group of \( D_p(2k+1, 4k+3) \) which is also obtained in \([5]\) and \([6]\).

2. **The homomorphism** \( i^*: U^*(X/H) \to U^*((X \times Y)/G) \)

By a \( G \)-manifold we mean a \( C^\infty \)-manifold which can be embedded equivariantly in some Euclidean \( G \)-space \([11]\). Let \( M \) and \( X \) be \( G \)-manifolds. By a complex orientation of a \( G \)-map \( f: M \to X \) we mean an equivalence class of factorizations

\[
Z \xrightarrow{i} E \xrightarrow{p} X
\]

where \( p: E \to X \) is a complex \( G \)-vector bundle over \( X \) and where \( i \) is an equivariant \( G \)-embedding endowed with a complex structure compatible with the \( G \)-action on its normal bundle \( v_i \). As Quillen \([12]\) we can define equivariantly a cobordant relation joining such proper complex oriented \( G \)-maps for a \( G \)-manifold \( X \). We denote by \( U^m_G(X) \) the set of cobordism classes of proper complex oriented \( G \)-maps of dimension \(-m\). Assume that \( X \) is a principal \( G \)-manifold which is a \( G \)-manifold such that no element of the group other than the identity has a fixed point \([2]\). Then the complex cobordism group \( U^m_G(X) \) is isomorphic to \( U^m(X/G) \) by sending the equivariant cobordism class \([Z \xrightarrow{i} E \xrightarrow{p} X]_G\) to \([Z/G \xrightarrow{i'} E/G \xrightarrow{p'} X/G]\), where \( i' \) and \( p' \) are quotient maps.

From now on, we suppose that \( G \) is a semi-direct product \( H \cdot \Gamma \) of a finite group \( H \) by a finite group \( \Gamma \) and that \( X \) is a \( G \)-manifold whose action restricted to \( H \) is free and \( Y \) is a principal \( \Gamma \)-manifold. The element \( \gamma \) of \( \Gamma \) acts on the group \( H \) by the inner automorphisms \( h' = \gamma^{-1} h \gamma \) and the group operation of \( H \cdot \Gamma \) is given by
The map \( i: X \to X \times Y, i(x) = (x, y_0) \), is an equivariant map. Then, there exists a composition homomorphism

\[
i^*: U^*((X \times Y)/G) \to U^*((X \times Y)/H) \to U^*(X/H)
\]

where \( r^* \) sends an equivariant cobordism class \([Z \to E \to X]_G\) to the class \([Z \to E \to X]_H\) obtained by restriction of the group action and \( i^*_H \) is the quotient map of \( i \). Suppose that \( X \) is a compact principal \( G \)-manifold, \( G = H \Gamma \). Let \([Z \to E \to X]_H\) be an element of \( U_H^0(X) \) represented by an \( H \)-equivariant factorization. Since \( q: X \to X/H \) is a principal bundle, a functor \( q^* \) from the category of vector bundles and homomorphisms over \( X/H \) to the category of \( H \)-vector bundles and \( H \)-homomorphisms over \( X \) is an equivalence [1]. There exists an \( H \)-complex vector bundle \( F \) over \( X \) such that \( E \oplus F = X \times C^n \) where \( H \) acts on \( X \times C^n \) by the rule \( h(x, z) = (hx, z) \). Therefore,

\[
[Z \to E \to X]_H = [Z \to X \times C^n \to X]_H
\]

as equivariant cobordism classes, where \( i(z) = (i(z), 0) \) and \( \tilde{p}(x, z) = x \). We form the quotient space \( G \times HZ \). The group \( G \) acts on \( G \times HZ \) by \( g(ghx, y) = (ghx, g^\gamma y) \). We have then the equivariant embedding

\[
i_1: G \times HZ \times Y \to X \times C^n \times Y \times V
\]

\[
i_1(h\gamma \times HZ, y) = (h\gamma i(z), y, e(\gamma))
\]

where \( G \times HZ \times Y \) is a \( G \)-space by \( h\gamma(g \times HZ, y) = (h\gamma g \times HZ, \gamma y) \), \( V \) is a complex Euclidean \( \Gamma \)-space, for example a regular representation space of \( \Gamma \), \( X \times C^n \times Y \times V \) is a \( G \)-space by \( h\gamma(x, z, y, v) = (h\gamma x, z, \gamma y, \gamma v) \) and \( e: \Gamma \to V \) is a \( \Gamma \)-equivariant embedding.

**Lemma 2.1.** If the normal bundle \( v \) of \( i: Z \to X \times C^n \) has a complex structure compatible with the \( H \)-action, then the normal bundle \( v_1 \) of \( i_1: G \times HZ \times Y \to X \times C^n \times Y \times V \) has a complex structure compatible with the \( G \)-action.

**Proof.** Let \( J: v \to v \) be a complex structure compatible with \( H \)-action, that is, \( hJ = Jh \). We may consider that \( X \) and \( Y \) are embedded in a Euclidean \( G \)-space \( V_x \) and a Euclidean \( \Gamma \)-space \( V_y \) respectively and that each element of \( G \) operates on \( V_x \times C^n \times V_y \times V \) as an orthogonal linear transformation. The total space of the normal bundle \( v_1 \) is described as follows:

\[
E(v_1) = \{(i_1(h\gamma \times HZ, y), (h\gamma w, v)): w \text{ is a vector of a fiber of } v \text{ over } i(z) \text{ and } v \in V\}.
\]

We put
The homomorphism \( J \) is a complex structure of the bundle \( \nu \), q.e.d.

From Lemma 2.1, we have a factorization

\[
G \times H Z \times Y \xrightarrow{i} X \times C^n \times Y \times V \xrightarrow{p_1} X \times Y,
\]

\[ p_1(x, z, y, v) = (x, y), \] which is a complex orientation of a map \( p_1 \cdot i \). We set

\[
i_* [Z \xrightarrow{i} E \xrightarrow{p} X]_H = \left[ G \times H Z \times Y \xrightarrow{i} X \times C^n \times Y \times V \xrightarrow{p_1} X \times Y \right]_G.
\]

This defines a \( U^* \)-module homomorphism

\[
i_*: U^*(X/H) \to U^*((X \times Y)/G)
\]

of degree 0.

We define a \( \Gamma \)-action on \( U^*(X/H) \): We take an equivariant cobordism class \( [Z \xrightarrow{i} X \times C^n \xrightarrow{p} X]_H \subseteq U^*_H(X) = U^*((X/H), with an H-action \( \phi: H \times Z \to Z \). Let \( Z' \) be a copy of \( Z \) whose action \( \phi': H \times Z \to Z \) is given by

\[
\phi'(h, z) = \phi(h', z)
\]

and \( \iota': Z' \to X \times C^n \) be an equivariant \( H \)-map given by

\[
\iota'(z) = \gamma \iota(z).
\]

Denote by \( v \) the normal bundle of \( i: Z \to X \times C^n \) and \( v_x \) the fiber over \( x \). The total space \( E \) of the normal bundle \( v' \) of \( \iota': Z' \to X \times C^n \) is

\[
E = \{ (\iota'(z), \gamma v) : v \text{ is a vector in the fiber } v_{\iota(z)} \}.
\]

Let \( J: v \to v \) be a complex structure compatible with the \( H \)-action. Then, a bundle map \( J': E \to E, J'(\iota'(z), w) = (\iota'(z), \gamma J' \gamma^{-1} w) \), is a complex structure of \( v' \) compatible with the \( H \)-action. We set

\[
[Z \xrightarrow{i} X \times C^n \xrightarrow{p} X]_H = [Z' \xrightarrow{i'} X \times C^n \xrightarrow{p} X]_H.
\]

Proof of Theorem 1.1.

We recall that

\[
i_* [Z \xrightarrow{i} X \times C^n \xrightarrow{p} X]_H = [G \times H Z \times Y \xrightarrow{i} X \times C^n \times Y \times V \xrightarrow{p_1} X \times Y]_G.
\]

Consider the map \( j: X \times C^n \times V \to X \times C^n \times Y \times V, j(x, z, v) = (x, z, y_v, v) \). The map \( j \) is an \( H \)-map and transversally regular on \( i_* (G \times H Z \times Y) \).

Let \( \Gamma \) be the set consisting of \( \gamma_1, \gamma_2, \ldots, \gamma_m \). It follows that

\[
j^{-1}(i_* (G \times H Z \times Y)) = \bigcup \Gamma Z_k
\]
where $Z_k=\{(h\gamma_k i(z), e(\gamma_k)): h\in H, z\in Z\} \subset X \times C^n \times V$. Clearly, $Z_k$ is equivariantly diffeomorphic to $Z \gamma_k$ and $[Z_k \xrightarrow{i_k} X \times C^n \times V \xrightarrow{\tilde{p}} X]_H = [Z \xrightarrow{i} X \times C^n \xrightarrow{\tilde{p}} X]_H$, where $i_k$ is an inclusion. Therefore, we have $i^*i^*[Z \xrightarrow{i} X \times C^n \xrightarrow{\tilde{p}} X]_H$. q.e.d.

3. The structure of $\tilde{U}^{2m}(D_p(2k+1, 4k+3))$

In [7], the manifold $D_p(l, n) = (S^{2l+1} \times S^n)/D_p$ was useful to determine the structure of complex bordism group of principal dihedral group $D_p$-actions. In this section, we determine the additive structure of $\tilde{U}^{2m}(D_p(2k+1, 4k+3))$.

Consider an action of the dihedral group $D_p = \mathbb{Z}_p \times \mathbb{Z}_2$ over $S^{2l+1} \times S^n$ given by

$$(g^it)(z, x) = (\rho^i e^i(z), (-1)^jx), \quad \rho = \exp 2\pi \sqrt{-1}/p$$

where $g$ is a generator of order $p$ and $t$ is the generator of order 2 and $c(z)$ is the conjugation operator. The manifold $D_p(l, n)$ is the orbit space. This manifold is an example of manifolds described in §2. We take a $\mathbb{Z}_p$-space $S^{2l+1}$ with $g \cdot z = pz$ ($z \in S^{2l+1}$, $g$ is a generator of $\mathbb{Z}_p$), a $\mathbb{Z}_2$-sphere $S^n$ with $t \cdot x = (-1)x$ ($x \in S^n$, $t$ is the generator of $\mathbb{Z}_2$) and a $D_p$-space $S^{2l+1} \times S^n$ with the $D_p$-action given by (1). Then, there are equivariant maps

$$i: S^{2l+1} \to S^{2l+1} \times S^n \quad i(z) = (z, (1, 0, \ldots, 0))$$

$$j: S^n \to S^{2l+1} \times S^n \quad j(x) = ((1, 0, \ldots, 0), x)$$

and

$$p: S^{2l+1} \times S^n \to S^n \quad p(z, x) = x$$

with respect to inclusions $i: \mathbb{Z}_p \to D_p$, $j: \mathbb{Z}_2 \to D_p$ and a projection $p: D_p \to \mathbb{Z}_2$ respectively. Denote by $U^*(S^{2l+1}/\mathbb{Z}_p)$ the subgroup consisting of elements fixed under the $\mathbb{Z}_p$-action on $U^*(S^{2l+1}/\mathbb{Z}_p)$ described in §2. Then we have the following.

**Proposition 3.1.** If $p$ is an odd prime, the homomorphism $\Phi: \tilde{U}^{2m}(S^{2l+1}/\mathbb{Z}_p)^{\mathbb{Z}_2} \oplus \tilde{U}^{2m}(S^n/\mathbb{Z}_2) \to \tilde{U}^{2m}(D_p(l, n))$ given by $\Phi(x, y) = i_*(x) + p^*(y)$ is injective.

**Proof.** We remark that $\tilde{U}^{2m}(S^{2l+1}/\mathbb{Z}_p)$ is a $p$-group and $\tilde{U}^{2m}(S^n/\mathbb{Z}_2)$ is a 2-group. Hence, $i^*p^* = 0$. Since $j^*p^* = 1$ and from Theorem 1.1 $i^*i_*(x) = 2x$, $\Phi$ is injective. q.e.d.

Denote by $L^l(p)$ a $(2l+1)$-dimensional lens space. The manifold $D_p(l, n)$ is homeomorphic to the orbit space of $L^l(p) \times S^n$ by a $\mathbb{Z}_2$-action $t([z], x) = ([cz], -x), t \in \mathbb{Z}_2$ the generator. Let $C_i$ and $D_j$ be the standard cells of $L^l(p)$ and $S^n$ respectively. The images $(C_i, D_j)$ of the $C_i \times D_j$ by the quotient map $L^l(p) \times S^n \to D_p(l, n)$ give a cellular decomposition of $D_p(l, n)$. Denote by $(c^i, d^j)$ the dual
Then we have the following coboundary relations
\[ \delta(c^{i+1}, d') = \{(-1)^i + (-1)^j\} \delta(c^{i+1}, d^{j+1}) + p(c^{2i+2}, d^j) \]
\[ \delta(c^i, d') = \{(-1)^i + (-1)^{j+1}\} (c^{2i}, d^{j+1}) \]
Therefore, we have the following.

**Proposition 3.2.** The integral cohomology group \( H^*(D_\rho(l, n); Z) \) is a direct sum of the following groups

(i) case l: even and n: even
   - a free group generated by \((c^{2i+1}, d^n)\), torsion groups generated by the \((c^0, d^{2j})\) and the \((c^{2i+1}, d^{2j-1})\) whose orders are 2 and torsion groups generated by the \((c^{2i}, d^n)\) and the \((c^{2i-2}, d^n)\) whose orders are \(p\),

(ii) case l: even and n: odd
   - a free group generated by \((c^0, d^n)\), torsion groups generated by the \((c^0, d^{2j})\) and the \((c^{2i+1}, d^{2j-1})\) whose orders are 2 and torsion groups generated by the \((c^{2i}, d^n)\) and the \((c^{2i-2}, d^n)\) whose orders are \(p\),

(iii) case l: odd and n: even
   - a free group generated by \((c^{2i+1}, d^n)\), torsion groups generated by the \((c^0, d^{2j})\) and the \((c^{2i+1}, d^{2j})\) whose orders are 2 and torsion groups generated by the \((c^{2i}, d^n)\) and the \((c^{2i-2}, d^n)\) whose orders are \(p\),

(iv) case l: odd and n: odd
   - free groups generated by \((c^0, d^n), (c^{2i+1}, d^n)\) and \((c^{2i+1}, d^n)\), torsion groups generated by the \((c^0, d^{2j})\) and the \((c^{2i+1}, d^{2j})\) whose orders are 2 and torsion groups generated by the \((c^{2i}, d^n)\) and the \((c^{2i}, d^n)\) whose orders are \(p\),

where \(0 \leq 2j \leq n\) and \(0 \leq 2i \leq l\).

Let \(Y_k\) be the \((8k+5)\)-skeleton of \(D_\rho(2k+1, 4k+3)\). Denote by \((E^r_{\ast; \ast}(X), d^r_{\ast; \ast})\) the Atiyah-Hirzebruch spectral sequence for \(U^*(X)\).

**Lemma 3.3.** If \(s \equiv 8k+6\) then an inclusion \(\iota: Y_k \to D_\rho(2k+1, 4k+3)\) induces the isomorphism for any \(r\)
\[ E^r_{\ast; \ast}(Y_k) \cong E^r_{\ast; \ast}(D_\rho(2k+1, 4k+3)) . \]

Proof. Using Proposition 3.2, it follows that \(\iota^*: E^r_{\ast; \ast}(D_\rho(2k+1, 4k+3)) \to E^r_{\ast; \ast}(Y_k)\) is isomorphic if \(s \equiv 8k+6\). We note that the images of the differentials \(d^r_{\ast; \ast}\) for any \(r\) are torsion groups [4]. By induction on \(r\) we have the lemma. q.e.d.

**Proposition 3.4.** There exists a short exact sequence
\[ 0 \to U^{2m-2k-6} \to \bar{U}^{2m}(D_\rho(2k+1, 4k+3)) \to \bar{U}^{2m}(Y_k) \to 0. \]

Proof. Consider the exact sequence of complex cobordism groups for a pair \((D_\rho(2k+1, 4k+3), Y_k)\):
... → \bar{U}^*(D_p(2k+1, 4k+3)) → \bar{U}^*(Y_h) → \bar{U}^{*+1}(D_p(2k+1, 4k+3)/Y_h) →

From Lemma 3.3 \*\*: \bar{U}^i(D_p(2k+1, 4k+3)) → \bar{U}^i(Y_h) is isomorphic for i odd. Since \bar{H}^i(D_p(2k+1, 4k+3)/Y_h; Z) = 0 if i \neq 8k+6 and \bar{H}^{8k+6}(D_p(2k+1, 4k+3)/Y_h; Z) \approx Z, we have that \bar{U}^{2m}(D_p(2k+1, 4k+3)/Y_h) \approx U^{2m-8k-6}. q.e.d.

We investigate the Thom homomorphism \( \mu: U^*(X) \to H^*(X) \) which is the edge homomorphism of the spectral sequence associated with \( U^*(X) \). Let \( X \) be an orientable manifold. We take an element \([M \xrightarrow{id} X \xrightarrow{id} X] \in U^*(X)\) which is represented by an inclusion map \( M \xrightarrow{id} X \) with the normal bundle \( \nu \) equipped with a complex structure. Denote by \( N(\nu) \) the tubular neighborhood of \( M \), and we have a canonical map \( j: (X, \phi) \to (X, \{\text{Int } N(\nu)\})^\times \). Then, we can describe the Thom homomorphism as \( \mu[M \xrightarrow{i} X \xrightarrow{id} X] = j^*\tau(\nu) \), \( \tau(\nu) \) is the Thom class of \( \nu \), and

\[
(2) \quad \mu[M \xrightarrow{i} X \xrightarrow{id} X] = D_i^*\sigma(M)
\]

where \( D \) is the Poincaré duality isomorphism \( H_*(M) \approx H^*(M) \) and \( \sigma(M) \) is a fundamental class of \( M \).

We put

\[
L_{h-m} = [S^{4m+3} \xrightarrow{id} S^{4k+3} \xrightarrow{id}]_{Z_p} \subset U^{4k+3}_p(S^{4k+3}),
\]

where \( S^{4k+3} \) and \( S^{4m+3} \) are \( Z_p \)-spaces with canonical action \( g \cdot z = \rho z \) and \( i \) is the canonical inclusion, and

\[
R_{2k+1-n} = [S^{4m+1} \xrightarrow{id} S^{4k+3} \xrightarrow{id}]_{Z_p} \subset U^{4k+2-2m}_p(S^{4k+3})
\]

where \( S^{4m+1} \) and \( S^{4k+3} \) are \( Z_p \)-spaces with the canonical action \( t \cdot x = (-1)x \), and \( i \) is the canonical inclusion.

**Proposition 3.5.** Suppose that \( p \) is an odd prime, then

\[
\mu^i(L_{h-m} + L_{h-m} + L_{h-m}) = a(c^{(h-m)}, d^0), \quad a \equiv 0 \mod p
\]

and

\[
\mu_p^i(R_{2k+1-n}) = (c^0, d^{h+2-2m}).
\]

Proof. The manifold \( D_p(2k+1, 4k+3) \) is orientable. Using Theorem 1.1 and (2), we have the proposition. q.e.d.

**Proof of Theorem 1.2.**

Proposition 3.5 shows that in the Atiyah-Hirzebruch spectral sequence for \( \bar{U}^*(D_p(2k+1, 4k+3)) \), the \( (c^i, d^0) \) and the \( (c^i, d^{ij}) \) are permanent cycles. It is
easy to prove that the spectral sequence is trivial. Therefore it follows from Propositions 3.1 and 3.5 that there exists an isomorphism

\[ \lambda^* + i_* + p^* : \tilde{U}^{2m}(\mathbb{D}(2k+1, 4k+3)/Y_h) \oplus \tilde{U}^{2m}(S^{4k+3}/\mathbb{Z}) \to \tilde{U}^{2m}(\mathbb{D}(2k+1, 4k+3)) \]

where \( \lambda : \mathbb{D}(2k+1, 4k+3) \to \mathbb{D}(2k+1, 4k+3)/Y_h \) is the projection map. q.e.d.

4. \( \mathbb{U}^*(\mathbb{BZ}_p) \), \( p \) an odd prime

The complex cobordism group \( \mathbb{U}^*(L^n(p)) = \mathbb{U}^{ev}(S^{2m+1}/\mathbb{Z}) \) is a \( U^* \)-module with a generating set \( \{ [S^{2k+1} \to S^{2m+1} \to S^{2m+1}]_{\mathbb{Z}_p} \} \); \( \mathbb{Z}_p \)-equivariant cobordism classes which are represented by the canonical equivariant inclusion map \( i(z_0, \ldots, z_n) = (z_0, \ldots, z_n, 0, \ldots, 0), 0 \leq k \leq n-1 \} \).

**Lemma 4.1.** \( \{ i^* \left( [S^{2k+1} \to S^{2m+1} \to S^{2m+1}]_{\mathbb{Z}_p} \right) \} = i^* \left( [S^{2k+1} \to S^{2m+1} \to S^{2m+1}]_{\mathbb{Z}_p} \right) \),

where \( i_n : L^{n-1}(p) \to L^n(p) \) is the inclusion map \( i_n(z_0, \ldots, z_{n-1}) = (z_0, \ldots, z_{n-1}, 0) \).

**Proof.** By the definition of the \( \mathbb{Z}_p \)-action, \( [S^{2k+1} \to S^{2m+1} \to S^{2m+1}]_{\mathbb{Z}_p} = \left( [S^{2k+1}]_{\mathbb{Z}_p} \right)^{i^*} \left( [S^{2m+1}]_{\mathbb{Z}_p} \right)^{i^*} \) where \( i^*(z) = ci(z) \). Let \( H_n : S^{2m-1} \times I \to S^{2m+1} \) be a map defined by

\[ H_n(z_0, \ldots, z_{n-1}, t) = \frac{1}{A} (tz_0, tz_1 + (1-t)z_0, \ldots, tz_{n-1} + (1-t)z_{n-2}, (1-t)z_{n-1}) \]

where \( A \) is the norm of \( (tz_0, tz_1 + (1-t)z_0, \ldots, (1-t)z_{n-1}) \). \( H_n \) is an equivariant \( \mathbb{Z}_p \)-map. Put

\[ j_n(z) = H_n(z, 0) \text{,} \]

then we have that \( j_n^* = i_n^* \). Moreover \( j_n : S^{2m-1} \to S^{2m+1} \) is transverse regular on \( i^*(S^{2k+1}) \). Therefore, we have

\[ j^* \left( [S^{2k+1}]_{\mathbb{Z}_p} \right)^{i^*} \left( [S^{2m+1}]_{\mathbb{Z}_p} \right)^{i^*} = \left( [S^{2k+1}]_{\mathbb{Z}_p} \right)^{i^*} \left( [S^{2m-1}]_{\mathbb{Z}_p} \right)^{i^*} \text{,} \]

q.e.d.

Let \( F(X, Y) \) be the formal group of the complex cobordism theory. Denote by \( [-1]_F(X) \) the element of \( U^*[\![X]\!] \) satisfying \( F(X, [-1]_F(X)) = 0 \) and by \( [k]_F(X) \) the element of \( U^*[\![X]\!] \) defined by the following formulae

\[
\begin{align*}
\{ [1]_F(X) & = X \\
F(X, [k]_F(X)) & = [k+1]_F(X) 
\end{align*}
\]
We define a $\mathbb{Z}_2$-action on $U^*[X]$ by

$$f(X)^t = f([-1]_p(X)).$$

By the definition of the formal group law, it follows immediately that $\{(p)_p(X)\}^t$ and $(X^{n+1})^t$ belong to the ideal $((p)_p(X), X^{n+1})$ generated by $[p]_p(X)$ and $X^{n+1}$ in $U^*[X]$ and thus $\mathbb{Z}_2$ acts on $U^*[X]/(p)_p(X, X^{n+1})$. We can see that the element $[S^{2n-1} \longrightarrow S^{2n+1} \longrightarrow S^{2n+1}]_p$ corresponds to the cobordism 1-st Chern class $c^*(\xi_n)$ of the canonical line bundle $\xi_n$ over $L^n(p)$ and that $[S^{2n-1} \longrightarrow S^{2n+1} \longrightarrow S^{2n+1}]_p$ is the cobordism 1-st Chern class $c_i(\xi_n)$ of the conjugate bundle $\xi_n$. Therefore, we have the following.

**Lemma 4.2.** $U^*(L^n(p)) \simeq \{ U^*[X]/(p)_p(X, X^{n+1}) \} \mathbb{Z}_2$.

**Proof.** From the definition of the multiplication in $U^*(L^n(p))$, we have that for $0 \leq k, l \leq n$

$$[S^{2k+1} \longrightarrow S^{2n+1} \longrightarrow S^{2n+1}]_p$$

Then, it follows immediately that the $\mathbb{Z}_2$-action on $U^*(L^n(p))$ is multiplicative. There exists an isomorphism $U^*(L^n(p)) \simeq U^*[X]/(p)_p(X, X^{n+1})$ which maps $c_i(\xi_n)$ to $X$ [13]. Since $F(c_i(\xi_n), c_i(\xi_n)) = c_i(\xi_n \otimes \xi_n) = 0$, the lemma follows. q.e.d.

Denote by $j_k: D(p)(2k-1, 4k-1) \rightarrow D(p)(2k+1, 4k+3)$ and $j_k: L^{2k-1}(p) \rightarrow L^{2k+1}(p)$ respectively, the maps induced by the inclusions $S^{4k-1} \times S^{4k-1} \subset S^{4k+3} \times S^{4k+3}$ and $S^{4k+1} \subset S^{4k+3}$. The following diagram is commutative

$$
\begin{array}{ccc}
\bar{U}^{2m}(L^{2k+1}(p)) & \xrightarrow{i_k} & \bar{U}^{2m}(D(p)(2k+1, 4k+3)) \\
\downarrow{j_k}^* & & \downarrow{j_k}^* \\
\bar{U}^{2m}(L^{2k-1}(p)) & \xrightarrow{i_k} & \bar{U}^{2m}(D(p)(2k-1, 4k-1)). 
\end{array}
$$

Since the $\mathbb{Z}_2$-action on $U^*(L^n(p))$ and $j_k^*$ are $U^*$-homomorphisms, it follows from Lemma 4.1 that $i_k^*$ induces a homomorphism of inverse systems

$$i_k^*: \{ \bar{U}^{2m}(L^{2k+1}(p)) \mathbb{Z}_2, j_k^* \} \rightarrow \{ \bar{U}^{2m}(D(p)(2k+1, 4k+3)), j_k^* \}.$$ 

Consider the quotient map of $j_k$

$$j_k: D(p)(2k-1, 4k-1)/Y_k \rightarrow D(p)(2k+1, 4k+3)/Y_k,$$

where $Y_k$ is a $(8k+5)$-skeleton of $D(p)(2k+1, 4k+3)$. Maps $\lambda: D(p)(2k+1, 4k+3)$
\( \text{Theorem 1.2} \) states that there exists a short exact sequence

\[
0 \to \lim_1 \tilde{U}^{*(p)}(D_p(2k+1, 4k+3)) \to \tilde{U}^{*(B)_p} \to \lim_1 \tilde{U}^{*(D_p(2k+1, 4k+3))} \to 0 \quad [10].
\]

Using Lemma 4.3 and 4.4, we have \( \tilde{U}^{*(B)_p} = 0 \).

Lemma 4.3 implies that the inverse system \( \{\tilde{U}^{*(D_p(2k+1, 4k+3))}, j^*_k\} \) satisfies the Mittag-Leffler condition. Therefore, we have that

\[
\tilde{U}^{*(B)_p} = \lim_1 \tilde{U}^{*(D_p(2k+1, 4k+3))}.
\]

Using Theorem 1.2 and Lemma 4.2 we complete the proof.
5. The structure of $K(D_p(2k+1, 4k+3))$

In [3], Conner and Floyd gave the isomorphism

(5.1) \[ c : \tilde{K}(X) \cong \tilde{U}^{ev}(X) \otimes_{U^*} Z, \]

which maps $\eta_{n-n}$ to $c_{i}(\eta_{n}) \times 1$. Consider a $Z_{2}$-action on $K(L_{n}(p))$ defined by $\gamma' = \gamma$, $t$ a generator of $Z_{2}$. Since $Z_{2}$-action on $U^{*}(L_{n}(p))$ is multiplicative, we have the commutative diagram

(5.2) \[
\begin{array}{ccc}
\tilde{K}(L_{n}(p)) & \xrightarrow{c} & \tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z \\
t & & \downarrow t \otimes id \\
\tilde{K}(L_{n}(p)) & \xrightarrow{c} & \tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z
\end{array}
\]

Lemma 5.1. \[ \tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z_{2} = \tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z_{2}, \] where $\tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z_{2}$ is an invariant subgroup of $\tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z$ under the $Z_{2}$-action $\cdot t \times U^*id$.

Proof. By the definition of $Z_{2}$-action of $\tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z$, it follows that $\tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z_{2} \subseteq (\tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z)^{Z_{2}}$. Suppose that $x \otimes_{U^*} m = \tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z$ and $x' \otimes_{U^*} m = x \otimes_{U^*} m$. Since $c$ is isomorphic, there exists an element $\eta \in \tilde{K}(L_{n}(p))$ with $c(\eta) = x \otimes_{U^*} m$. By the commutative diagram (5.2),

\[ c(\eta) = c(\eta)' = c(\eta') \quad \text{and} \quad \eta = \eta'. \]

N. Mahammed [9] proved that $\tilde{K}(L_{n}(p)) = Z[\xi_{n}] / (\xi_{n}^{r} - 1, (\xi_{n} - 1)^{w+1})$, $\xi_{n}$ is the canonical line bundle over $L_{n}(p)$. Put $X = c_{i}(\xi_{n})$. Then, the element $c_{i}(\eta)$ is described as a polynomial $f(X)$ with the coefficient in $U^*$. We can see that $c_{i}(\eta) = f([-1], f(X))$. By the observation in Lemma 4.2, it follows that $c_{i}(\eta) \in \tilde{U}^{ev}(L_{n}(p))^{Z_{2}}$. Therefore, we have that if $x \otimes_{U^*} m \in (\tilde{U}^{ev}(L_{n}(p)) \otimes_{U^*} Z)^{Z_{2}}$, then there exists an element $\eta \in \tilde{K}(L_{n}(p))$ such that \[ x \otimes_{U^*} m = c_{i}(\eta) \otimes_{U^*} 1, \quad c_{i}(\eta) \in \tilde{U}^{ev}(L_{n}(p))^{Z_{2}}. \]

q.e.d.

From the isomorphism (5.1), Lemma 5.1 and Theorem 1.2, we have the following.

Theorem 5.2 ([5] and [6]).

\[ \tilde{K}(D_p(2k+1, 4k+3)) \cong Z \oplus \tilde{K}(L_{2k+1}(p))^{Z_{2}} \oplus \tilde{K}(RP^{4k+3}). \]
References


