ON THE CAUCHY PROBLEM FOR PARABOLIC PSEUDO-DIFFERENTIAL EQUATIONS

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1. Introduction

In the recent paper [8] S. Kaplan has obtained an analogue of Gårding's inequality for parabolic differential operators and applied it to a Hilbert space treatment of the Cauchy problem. D. Ellis [3] has extended those results to higher order parabolic differential operators (see also [4]). On the other hand in [13] the author has studied a Hilbert space treatment of the Cauchy problem for parabolic pseudo-differential equations and generalized the results of S. Kaplan [8].

In the present paper we shall study the Cauchy problem for higher order parabolic pseudo-differential equations of the form

\[ Lu = D_t^l u(t, x) + \sum_{j=1}^{k} p_j(t, x, D_x) D_t^{l-j} u(t, x) = f(t, x) \]

where \( p_j(t, x, \xi) \) are symbols of the class \( S^{m,j}_{0,0} \) introduced in [11] and [12]. We need not assume that the basic weight function \( \lambda(\xi) \) tends to infinity as \( |\xi| \to \infty \). Therefore the theory can be applied to more general classes of operators (including difference operators) than the class of usual parabolic differential operators.

In section 2 we give definitions and lemmas for pseudo-differential operators. In section 3 the algebras and \( L^2 \)-theory are stated. The \( L^2 \)-continuity of pseudo-differential operators has been studied in many papers (see for example, Calderón and Vaillancourt [1], [2], Hörmander [7] and Kumano-go [10]. In the present paper the \( L^2 \)-continuity theorem by Calderón and Vaillancourt in [1] plays an essential role. In section 4 we define the space \( H_{r,s}(\Omega) \) which is needed to study the Cauchy problem. In section 5 we derive energy inequalities for the parabolic system which is associated with a higher order parabolic pseudo-differential operator. These energy inequalities are very similar to those of D. Ellis [3] and [4]. To obtain the energy inequalities the idea of double symbols of pseudo-differential operators is very important. In section 6, using the results in section 4 and 5, we discuss a Hilbert space treatment of Cauchy problem for parabolic systems. In section 7 finally we state the main results for the Cauchy problem for higher order parabolic pseudo-differential equations.
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2. Definitions and lemmas

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi-integer of \( \alpha_j \geq 0, j = 1, \ldots, n \). We put \( |\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1! \cdots \alpha_n! \) and \( \partial^\alpha = (\partial/\partial \xi_1)^{\alpha_1} \cdots (\partial/\partial \xi_n)^{\alpha_n} \).

**Definition 2.1.** Let \( \lambda(\xi) \) be a real valued \( C^\infty \) function defined on the \( n \)-dimensional real space \( \mathbb{R}^n \). We say that \( \lambda(\xi) \) is a basic weight function when \( \lambda(\xi) \) satisfies that

\[
\begin{align*}
(2.1) & \quad \lambda(\xi) \geq 1, \\
(2.2) & \quad |\partial^\alpha \lambda(\xi)| \leq C^|\alpha| \lambda(\xi)^{1-|\alpha|} \text{ for any } \alpha,
\end{align*}
\]

(see [9] and [13]).

We can see that the function \( \phi = (1 + |\xi|^2)^{1/2} = (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2} \) is a basic weight function.

The following lemma was proved in [13].

**Lemma 2.2.** Let \( \lambda(\xi) \) be a basic weight function and \( \delta \) and \( m \) be real numbers satisfying \( 0 \leq \delta < 1 \). Then we have

\[
\begin{align*}
(2.3) & \quad \lambda(\xi) \leq C_1 \phi(\xi), \\
(2.4) & \quad \lambda(\xi + \eta) \leq \lambda(\xi) + C_2 \lambda(\xi) \phi(\eta), \\
(2.5) & \quad C_\delta^{-1} \lambda(\xi) \leq \lambda(\xi + \lambda(\xi)^m \phi(\sigma)) \leq C_\delta \lambda(\xi)
\end{align*}
\]

for any \( \sigma \in \mathbb{R}^n \) satisfying \( |\sigma| \leq 1 \),

\[
\lambda(\xi + \eta)^m \leq C_m \lambda(\xi)^m \phi(\eta)^{m|\eta|},
\]

where \( C_1, C_2, C_\delta \) and \( C_m \) are positive constants which are independent of \( \xi, \eta \) and \( \sigma \).

Throughout this paper the letter \( C \) with or without indices will denote positive constants not necessarily the same at each occurrence.

**Lemma 2.3.** Let \( \lambda_0(\xi) \) be a real valued \( C^1 \) function such that \( \lambda_0(\xi) \geq c_0 \) for some positive constant \( c_0 \) and \( \partial^\alpha \lambda_0(\xi) \) \((j=1, \ldots, n)\) are bounded. Then there exists a basic weight function \( \lambda(\xi) \) which satisfies that

\[
(2.7) \quad c_1 \lambda_0(\xi) \leq \lambda(\xi) \leq c_2 \lambda_0(\xi)
\]

for some positive constants \( c_1 \) and \( c_2 \).

**Proof.** By assumptions for \( \lambda_0(\xi) \) we have \( |\lambda_0(\xi) - \lambda_0(\eta)| \leq C |\xi - \eta| \), so taking \( \epsilon_0 = \frac{1}{2C} \) it holds that \( (1/2) \lambda_0(\xi) \leq \lambda_0(\eta) \leq 2 \lambda_0(\xi) \) for \( |\xi - \eta| \leq \epsilon_0 \lambda_0(\eta) \).
Let $\varphi(\eta) \in C_0(R^n)$ satisfy that $\int_{R^n} \varphi(\eta) d\eta = 1$, $0 \leq \varphi(\eta) \leq C$, $\text{supp} \ \varphi \subset \{ \eta; |\eta| \leq \varepsilon_0 \}$ and $\varphi(\eta) \geq C \varepsilon_0^1 > 0$ for $|\eta| \leq \varepsilon_0/2$. Then the function $\lambda(\xi) = \int_{R^n} \varphi((\xi - \eta)/\lambda(\eta)) \varphi(\eta) d\eta$ is a basic weight function and satisfies the inequality (2.7). In fact,

$$\partial_\xi^a \lambda(\xi) = \int_{R^n} \varphi(\alpha)((\xi - \eta)/\lambda(\eta)) \lambda(\eta)^{-n+1-|a|} d\eta$$

where $\varphi(\alpha)(\eta) = \partial_\alpha \varphi(\eta)$, so

$$|\partial_\xi^a \lambda(\xi)| \leq C_\alpha \int_{|\xi - \eta| \leq \varepsilon_0 C \lambda(\xi)^{-1} \varepsilon_0^{1/4} \lambda(\xi)^{-n+1-|a|} d\xi$$

$$\leq C_\alpha \int_{|\xi - \eta| \leq \varepsilon_0 C \lambda(\xi)^{-1} \varepsilon_0^{1/4} \lambda(\xi)^{-n+1} d\xi \leq C_\alpha C \lambda(\xi)^{-|a|}$$

$$\lambda(\xi) = c_\alpha \int_{|\xi - \eta| \leq \varepsilon_0 C \lambda(\xi)^{-1} \varepsilon_0^{1/4} \lambda(\xi)^{-n+1} d\xi$$

$$\leq \left( \frac{c_\alpha}{C_\alpha} \right) \int_{|\xi - \eta| \leq \varepsilon_0 C \lambda(\xi)^{-1} \varepsilon_0^{1/4} \lambda(\xi)^{-n+1} d\xi$$

$$\leq C \int_{R^n} \varphi((\xi - \eta)/\lambda(\eta)) \lambda(\eta)^{-n+1} d\eta = C \lambda(\xi)$$

$$\leq C' \int_{|\xi - \eta| \leq \varepsilon_0 C \lambda(\xi)^{-1} \varepsilon_0^{1/4} \lambda(\xi)^{-n+1} d\xi$$

$$\leq C' \int_{|\xi - \eta| \leq \varepsilon_0 C \lambda(\xi)^{-1} \varepsilon_0^{1/4} \lambda(\xi)^{-n+1} d\xi = C' \lambda(\xi).$$

By these inequalities we obtain Lemma 2.3. Q.E.D.

Let $B(R^n) = \{ f(x) \in C^m(R^n); \ |\partial_\alpha^a f(x)| \leq C_\alpha \text{ for any } \alpha \}$, $S = S_n(R^n) = \{ f(x) \in C^m(R^n); \lim_{|x| \to \infty} |x|^m |\partial_\alpha^a f(x)| = 0 \text{ for any } \alpha \text{ and real number } m \}$ and let $S'$ denote the dual space of $S$.

**Definition 2.4.** Let $\lambda(\xi)$ be a basic weight function.

(i) We say that $p(x, \xi)$ belongs to $S_{\alpha, \lambda}^m$ when $p(x, \xi) \lambda(\xi)^{-m} \in B(R^{2n})$.

(ii) We say that $p(x, \xi, x')$ belongs to $S_{\alpha, \lambda}^m$ when $p(x, \xi, x') \lambda(\xi)^{-m} \in B(R^{3n})$.

(iii) We say that $p(x, \xi, x', x'')$ belongs to $S_{\alpha, \lambda}^m$ when $p(x, \xi, x', x'') \lambda(\xi)^{-m} \in B(R^{4n})$.

(iv) We set $S_{\alpha, \lambda}^m = \bigcup_{-m \leq m \leq m} S_{\alpha, \lambda}^m$ and $S_{\alpha, \lambda}^m = \bigcap_{-m \leq m \leq m} S_{\alpha, \lambda}^m$.

(v) Let $\lambda(\xi)$ and $\lambda'(\xi)$ be basic weight functions. Then we say that $p(x, \xi, x', x'')$ belongs to $S_{\alpha, \lambda}^m$ when $p(x, \xi, x', x') \lambda(\xi)^{-m} \lambda'(\xi)^{-m'} \in B(R^{4n})$.

We use the notation: $D_\xi^a = (-i)^{|a|} (\partial/\partial x_i)^{a_1} \cdots (\partial/\partial x_n)^{a_n}$ for any $\alpha$. Then we set $p_{(\alpha, \beta)}(x, \xi) = D_{\alpha}^{\alpha_1} D_{\beta}^{\beta_1} p(x, \theta, \xi)$, $p_{(\alpha, \beta)}(x, \xi) = D_{\alpha}^{\alpha_1} D_{\beta}^{\beta_1} p(x, \theta, \xi)$ and $p_{(\alpha, \beta)}(x, \xi, x', \xi') = D_{\alpha}^{\alpha_1} D_{\beta}^{\beta_1} D_{\alpha'}^{\alpha'_1} p(x, \xi, x', \xi')$ for any $\alpha, \beta, \alpha'$ and $\beta'$. We can see that
\[ (i) \quad p(x, \xi) \in S_{0, \lambda}^m \text{ if and only if } |p(x, \xi)| \leq C_{\alpha, \beta \lambda}(\xi)^m \text{ for any } \alpha \text{ and } \beta, \\
(ii) \quad p(x, \xi, x') \in S_{0, \lambda}^m \text{ if and only if } |p(x, \xi, x')| \leq C_{\alpha, \beta \lambda}(\xi)^m \text{ for any } \alpha, \beta \text{ and } \beta', \\
(iii) \quad p(x, \xi, x', \xi') \in S_{0, \lambda}^{m'} \text{ if and only if } |p(x, \xi, x', \xi')| \leq C_{\alpha, \beta, \lambda}(\xi)^m \lambda'(\xi') \text{ for any } m, \alpha, \beta \text{ and } \beta', \\
(iv) \quad \text{when } m_1 \geq m_2, \text{ it holds that } S_{0, \lambda}^{m_1} \supset S_{0, \lambda}^{m_2}. \]

In this paper we write \( \int f(x)dx \) for \( \int f(x)dx + d\xi \) for \( (2\pi)^{-n} d\xi \).

**DEFINITION 2.5.** (i) For \( p(x, \xi) \in S_{0, \lambda}^m \), we define the pseudo-differential operator \( p(X, D_x) \) by
\[
(2.8) \quad p(X, D_x)u(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{u}(\xi) d\xi \quad \text{for } u \in S, \text{ where } \hat{u}(\xi) \text{ denote the Fourier transform} \]
\[
\int e^{-ix'\cdot\xi'} u(x')dx' \text{ of } u(x) \text{ and } x' = x_1 + \cdots + x_n. \]

(ii) For \( p(x, \xi, x') \in S_{0, \lambda}^m \), we define the operator \( p(X, D_x, X') \) by
\[
(2.9) \quad p(X, D_x, X')u(x) = \int \int e^{i(x-x')\cdot\xi} p(x, \xi, x') u(x') dx' \cdot d\xi \quad \text{for } u \in S, \text{ where } dx' \cdot d\xi \text{ means the integration in } \xi \text{ follows the integration in } x'. \]

(iii) For \( p(x, \xi, x', \xi') \in S_{0, \lambda}^{m'} \) or \( S_{0, \lambda}^{m''} \), we define the operator \( p(X, D_x, X', D_{x'}) \) by
\[
(2.10) \quad p(X, D_x, X', D_{x'})u(x) = \int \int \int e^{i(x-x')\cdot\xi + ix'\cdot\xi'} p(x, \xi, x', \xi') \hat{u}(\xi') d\xi' \cdot dx' \cdot d\xi \quad \text{for } u \in S. \]

We can see that the above operators \( p(X, D_x) \) and \( p(X, D_x, X') \) are continuous linear operators from \( S(R^n) \) to \( S(R^n) \). We say that the functions \( p(x, \xi) \), \( p(x, \xi, x') \) and \( p(x, \xi, x', \xi') \) are symbols of the pseudo-differential operators \( p(X, D_x) \), \( p(X, D_x, X') \) and \( p(X, D_x, X', D_{x'}) \) respectively and in particular \( p(x, \xi, x', \xi') \) is often called a double symbol.

**DEFINITION 2.6.** Let \( \lambda(\xi) \) be a basic weight function and \( s \) be a real number. We define a Sobolev space \( H_s \) by
\[
H_s = H_{s, \lambda} = \{ u \in S'; \ u(\xi) \in L^1_{loc}(R^n), \ \lambda(\xi)\hat{u}(\xi) \in L^1(R^n) \}. \]

We can see that \( H_{s, \lambda} \) is a Hilbert space with inner product
\[
(2.11) \quad (u, v)_{s, \lambda} = (u, v)_{s, \lambda} = \int \lambda(\xi)^s \hat{u}(\xi) \hat{v}(\xi) d\xi \]
and the set \( S = S(R^n) \) is a dense subset of \( H_{s, \lambda} \).

For \( s = 0 \), \( H_{0, \lambda} = L^1(R^n) \). When \( s_1 \leq s_2 \leq s_3 \), for any \( \varepsilon > 0 \) there exists a constant \( C = C_{s_1, s_2, s_3, \varepsilon} \) such that
\[
(2.12) \quad ||u||_{s_2}^2 \leq \varepsilon ||u||_{s_3}^2 + C ||u||_{s_1}^2 \quad \text{for any } u \in S, \]

where \( ||u||_s = \sqrt{(u, u)_{s, \lambda}} \) (see [13]).
When \(P(x, \xi) = \left( p_{i,j}(x, \xi) \right)\) is a \(k \times k\) matrix function, we say that \(P(x, \xi)\) belongs to \(S^m_{\alpha, \lambda}\) if all the elements \(p_{i,j}(x, \xi)\) belong to \(S^m_{\alpha, \lambda}\) in the sense of Definition 2.4 (i). By the same way we define \(P(x, \xi, x') \in S^m_{\alpha, \lambda, \lambda'}\) and \(P(x, \xi, x', \xi') \in S^m_{\alpha, \lambda, m'}\). For \(P(x, \xi) = \left( p_{i,j}(x, \xi) \right) \in S^m_{\alpha, \lambda}\), we define the pseudo-differential operator \(P(X, D_x)\) by

\[
P(X, D_x)U(x) = \int e^{ix \cdot \xi} P(x, \xi) \hat{U}(\xi) d\xi,
\]

where \(U(x) = \left( u_0(x), \ldots, u_k(x) \right) \in \{S\}^k\) and \(P(x, \xi) \hat{U}(\xi) = \left( \sum_{i=1}^{k} p_{i,1}(x, \xi) \hat{u}_i(\xi), \ldots, \sum_{i=1}^{k} p_{i,k}(x, \xi) \hat{u}_i(\xi) \right)\).

By the same way we can define the operators \(P(X, D_x, X')\) and \(P(X, D_x, X', D_{x'})\).

**Remark 2.7.** With the aid of Lemma 2.3, we can see that
(i) for any basic weight functions \(\lambda_1(\xi)\) and \(\lambda_2(\xi)\), there exists a basic weight function \(\lambda(\xi)\) such that \(c_1 \lambda(\xi) \leq \lambda_1(\xi) + \lambda_2(\xi) \leq c_2 \lambda(\xi)\),
(ii) for any basic weight function \(\lambda(\tau, \xi)\) in \(R^n\) and real number \(m \geq 1\), there exists a basic weight function \(\lambda_1(\tau, \xi)\) in \(R^{n+1}\) such that \(c_1 \lambda_1(\tau, \xi) \leq (\tau^2 + \lambda(\xi)^{2m})^{1/2m} \leq c_2 \lambda_1(\tau, \xi)\) (see [12] and [13]).

The fact of Remark 2.7 (ii) is important to define the spaces which are necessary to study the Cauchy problem for parabolic pseudo-differential equations.

**Remark 2.8.** From the definition of basic weight functions, if \(\lambda(\xi)\) is a basic weight function in \(R^n\), \(\lambda(\xi)\) is also a basic weight function in \(R^{n+1}\).

### 3. Properties of pseudo-differential operators

All the theorems and corollaries of this section are stated in [12] and [13], so we omit the proofs.

**Theorem 3.1.** Let \(\lambda(\xi)\) and \(\lambda'(\xi)\) be basic weight functions and let \(p(x, \xi, x', \xi') \in S^{m, m'}_{\alpha, \lambda, \lambda'}\). Then there exists a function \(p_L(x, \xi)\) such that

\[
p_L(x, \xi) \lambda(\xi)^{-m} \lambda'(\xi)^{-m'} \in B(R^n)
\]

and

\[
p_L(X, D_x)u = p(X, D_x, X', D_x)u \quad \text{for any } u \in S.
\]

**Corollary 3.2.** (i) Let \(p_1(x, \xi) \in S_{\alpha, \lambda}\) and \(p_2(x, \xi) \in S_{\alpha, \lambda'}\). Then there exists a function \(p_L(x, \xi)\) such that

\[
p_L(x, \xi) \lambda(\xi)^{-m} \lambda'(\xi)^{-m'} \in B(R^n)
\]

and

\[
p_L(X, D_x)u = p_1(X, D_x) \cdot p_2(X, D_x)u \quad \text{for any } u \in S.
\]

(ii) For \(p(x, \xi) \in S_{\alpha, \lambda}\), there exists a symbol \(p^*(x, \xi) \in S_{\alpha, \lambda}\) such that
(\rho(X, D_s)u, v) = (u, \rho^*(X, D_s)v) \text{ for any } u, v \in S.

When \lambda(\xi) = \lambda'(\xi), the assertions in Corollary 3.2 mean that the class of pseudo-differential operators defined by the symbols in \(S_{\delta, \lambda}^m\) forms an algebra.

**Theorem 3.3.** Let \(0 < \delta \leq 1\) and \(p(x, \xi, x', \xi') \in S_{\delta, \lambda}^{m, m'}\). We assume that 
\[ \partial_{\xi_j}p(x, \xi, x', \xi') \in S_{\delta, \lambda}^{m, -m'}. \]
Then for \(p_L(x, \xi)\) in Theorem 3.1 and \(p_\lambda(x, \xi) = p(x, \xi, x, \xi)\), it holds that

\[ (3.5) \quad \{p_L(x, \xi) - p_\lambda(x, \xi)\} \lambda(\xi)^{-m+\delta} \lambda'(\xi)^{-m'} \in B(R^{2m}). \]

**Corollary 3.4.** (i) Let \(p_1(x, \xi) \in S_{\delta, \lambda}^m\) and \(p_2(x, \xi) \in S_{\delta, \lambda}^{m', n}\). Assume that 
\[ \partial_{\xi_j}p_1(x, \xi) \in S_{\delta, \lambda}^{m-\delta}(j = 1, \ldots, n) \text{ for some } \delta \in (0, 1]. \]
Then

\[ (3.6) \quad \{p_L(x, \xi) - p_1(x, \xi) - p_2(x, \xi)\} \lambda(\xi)^{-m+\delta} \lambda'(\xi)^{-m'} \in B(R^{2m}), \]
where \(p_L(x, \xi)\) is the function defined in Corollary 3.2.

(ii) Assume that \(p(x, \xi) \in S_{\delta, \lambda}^{m, m} \text{ and } \partial_{\xi_j}p(x, \xi) \in S_{\delta, \lambda}^{m, -m} \). Then for \(p_\lambda(x, \xi)\) in Corollary 3.2 (ii) we have

\[ (3.7) \quad \{p_\lambda(x, \xi) - p_\lambda(x, \xi)\} \in S_{\delta, \lambda}^{m, -m}. \]

**Corollary 3.5.** For \(p(x, \xi) \in S_{\delta, \lambda}^m\), there exists a symbol \(p_{L, m'}(x, \xi)\) such that

\[ (3.8) \quad \{p_{L, m'}(x, \xi) - p(x, \xi)\lambda(\xi)^{m'}\} \lambda'(\xi)^{-m+\delta} \lambda(\xi)^{-m} \in B(R^{2m}), \]

\[ (3.9) \quad p_{L, m'}(X, D_s)u = \lambda'(D_s)^{m'}p(x, D_s)u \text{ for any } u \in S. \]

**Corollary 3.6.** Let \(p_1(x, \xi) \in S_{\delta, \lambda}^m\) and \(p_2(x, \xi) \in S_{\delta, \lambda}^{m', n}\). Assume that 
\[ \partial_{\xi_j}p_1(x, \xi) \in S_{\delta, \lambda}^{m-\delta}(j = 1, \ldots, n) \text{ and } \partial_{\xi_j}p_2(x, \xi) \in S_{\delta, \lambda}^{m', -m}(j = 1, \ldots, n). \]
Then there exists a symbol \(p(x, \xi)\) \(S_{\delta, \lambda}^{m, m'}\) such that

\[ (3.10) \quad p(X, D_s)u = [p_1(X, D_s), p_2(X, D_s)]u \]

\[ = \{p_1(X, D_s) \cdot p_2(X, D_s) - p_2(X, D_s) \cdot p_1(X, D_s)\}u \]

for any \(u \in S\).

The following \(L^p\)-estimate was proved in [1].

**Lemma 3.7.** Let \(p(x, \xi) \in S_{\delta, \lambda}^m\). Then it holds that

\[ (3.11) \quad \|p(X, D_s)u\|_p \leq C\|u\|_p \text{ for any } u \in S, \]
where \(C = C_\rho = C_{\delta, \lambda} = \sum_{|\alpha| + |\beta| + N} \sup_{(x, \xi) \in \Omega} |p^{(\alpha)}(x, \xi)| \) for some positive integer \(N\).

Using Corollary 3.2 (i) and Lemma 3.7 we have

**Theorem 3.8.** Let \(s\) be an arbitrary real number and \(p(x, \xi) \in S_{\delta, \lambda}^m\). Then it holds that

\[ (3.12) \quad \|p(X, D_s)u\|_{L^p} \leq C\|u\|_{L^p} \text{ for any } u \in S. \]

**Corollary 3.9.** When \(p(x, \xi) \in S_{\delta, \lambda}^m\), we have
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\[(3.13) \quad |(p(X, D_x) u, u)_\alpha| \leq C |u|_{m/2, \lambda}^2 \quad \text{for any } u \in S.
\]

For any \( p(x, \xi) \in S_{0, \lambda}^m \) we denote \( |p|_m = \sup_{(\xi, \xi)} |p(x, \xi)| \lambda^{-m} |\).

Using the Friedrichs approximation (see [5], [10] and [13]) we have,

**Theorem 3.10.** Assume that \( 0 < \delta \leq 1 \) and \( p^{\alpha}(x, \xi) \in S_{0, \lambda}^{m-\delta|\alpha|} \) for \( |\alpha| \leq 1 \).

Then we have

\[(3.14) \quad |Re (p(X, D_x) u, u)_\alpha| \leq |Re p|_m |u|_{m/2, \lambda}^2 + C |u|_{m-\delta/2, \lambda}^2 \quad \text{for any } u \in S.
\]

**Corollary 3.11.** Assume that \( p^{\alpha}(x, \xi) \in S_{0, \lambda}^{m-|\alpha|} \) for \( |\alpha| \leq 1 \), then we have

\[(3.15) \quad |(p(X, D_x) u, u)_\alpha| \leq |p|_m^2 |u|_{m+2, \lambda}^2 + C |u|_{m+2-\delta/4, \lambda}^2 \quad \text{for any } u \in S.
\]

We note that all the theorems and corollaries of this section except for Corollary 3.6 remain valid when the symbols of operators are \( k \times k \) matrix functions. But in the case of matrix symbols we must replace \( |Re p|_m \) in (3.14) and \( |p|_m^2 \) in (3.15) by \( k |Re p|_m \) and \( k |p|_m^2 \) respectively, where we mean that for \( p(x, \xi) = (p_{i,j}(x, \xi)) \in S_{0, \lambda}^m \), \( Re p = \frac{1}{2} \{ p(x, \xi) + p(x, \xi)^* \} \) and \( |p|_m = \sup_{(i,j)=1} |Re p_{i,j}(x, \xi)|^{1/2} \).

In the case of matrix symbols, Corollary 3.6 holds if matrix \( p(x, \xi) \) commutes with \( p(x, \xi) \).

By virtue of Corollary 3.2 (ii), we can define the pseudo-differential operators on the space \( S' \) by \( \langle p(X, D_x) u, v \rangle = \langle u, p^*(X, D_x) v \rangle \) for \( u \in S' \) and \( v \in S \). Then inequalities (3.11), (3.12), (3.13), (3.14) and (3.15) hold for functions in \( H_{s, \lambda} \) spaces.

4. Spaces \( H_{r,s}(\Omega) \)

In what follows we fix a basic weight function \( \lambda(\xi) \) in \( R^n \) and a real number \( m \geq 1 \). By Remark 2.7 (ii), there exists a basic weight function \( \Lambda_4(\tau, \xi) \) in \( R^{n+1} \) such that \( c_4 \Lambda_4(\tau, \xi) \leq (\tau^2 + \Lambda(\xi)^{-1})^{1/2} \leq c_5 \Lambda_4(\tau, \xi) \).

**Definition 4.1.** For any real numbers \( r \) and \( s \), we define the space \( H_{r,s} \) by \( H_{r,s} = \{ u \in S'(R^{n+1}); \hat{u}(\tau, \xi) \in L_{1oo}(R^{n+1}), \lambda_4(\tau, \xi)^{1/2} \hat{u}(\tau, \xi) \in L^2(R^{n+1}) \} \) where \( \hat{u}(\tau, \xi) \) is the Fourier transform of \( u(t, x) dt dx \) of \( u(t, x) \).

The space \( H_{r,s} \) is a Hilbert space with inner product

\[ (u, v)_{r,s} = \int \lambda_4(\tau, \xi)^{1/2} \lambda_4(\tau, \xi)^{1/2} \hat{u}(\tau, \xi) \hat{v}(\tau, \xi) d\tau d\xi.
\]

We can see that \( S(R^{n+1}) \) is a dense subset of \( H_{r,s} \).

For \( -\infty \leq a < b \leq +\infty \), we set \( \Omega = \Lambda_{a,b} = \{(t, x) \in R^{n+1}; a < t < b, x \in R^n \} \).

**Definition 4.2.** (i) \( H_{r,s}(\Omega) = \{ u \in D'(\Omega); v|_{a} = u \} \) for some \( v \in H_{r,s} \}, \)

where \( v|_{a} = u \) means that the restriction of \( v \) to \( \Omega \) coincides with \( u \) and \( D'(\Omega) \)
denote the space of distributions on $\Omega$.

(ii) For any closed set $K$ in $\mathbb{R}^{n+1}$, we set $H_{0,r,s}(K) = \{ u \in H_{r,s} ; \text{supp } u \subseteq K \}$. 

(iii) For any open set $G$ in $\mathbb{R}^{n+1}$, we set $C^\infty(G) = \{ \varphi \in C^\infty(\mathbb{R}^{n+1}) ; \varphi \in C^\infty(\mathbb{R}^{n+1}) \}$.

For $u \in H_{0,r,s}(\Omega)$ we define the norm of $u$ by $\| u \|_{r,s,\Omega} = \inf \{ \| v \|_{r,s} ; v \in H_{r,s}, v|_{\Omega} = u \}$ where $\| v \|_{r,s} = \sqrt{\langle v, v \rangle_{r,s}}$. The space $H_{0,r,s}(\Omega)$ is a Banach space with norm $\| u \|_{r,s,\Omega}$. We can see that $H_{0,r,s}(K)$ is a closed subspace of $H_{r,s}$.

Using a similar method in [6], [8] and [11], we can see that for any $r$ and $s$, the set $C^\infty_0(\Omega)$ is dense in $H_{r,s}(\Omega)$, $C^\infty_0(\Omega)$ is dense in $H_{0,r,s}(\Omega)$ and $C^\infty_0(\Omega^c)$ is dense in $H_{0,r,s}(\Omega^c)$, where $\Omega^c$ means the complement of $\Omega$.

The following lemmas are stated in [13] and can be proved by the similar methods to those in [8] and [11].

**Lemma 4.3.** Assume that $u \in H_{r,s+m}(\Omega)$ and $\frac{\partial}{\partial t} u \in H_{r,s}(\Omega)$, Then $u \in H_{r+m,s}(\Omega)$ and

$$(4.2) \quad \| u \|_{r+m,s,\Omega} \leq C \left( \| u \|_{r,s+m,\Omega} + \| \frac{\partial}{\partial t} u \|_{r,s,\Omega} \right).$$

**Lemma 4.4.** Assume that $2r > m$ and $-\infty < a < b \leq \infty$.

(i) We can define the trace operator $\gamma_a : H_{r,s}(\Omega) \rightarrow H_{r+s-m/2,\lambda}$ such that $(\gamma_a u)(x) = u(a, x)$ for $u(t, x) \in S(\mathbb{R}^{n+1})$ and

$$(4.3) \quad \| \gamma_a u \|_{r+s-m/2,\lambda} \leq C \| u \|_{r,s,\Omega}.$$  

(ii) There exists a bounded linear operator $\gamma_a : H_{r+s-m/2,\lambda} \rightarrow H_{r,s}(\Omega)$ such that $\gamma_a \gamma_a u = u$ for $u \in H_{r+s-m/2,\lambda}$.

**Lemma 4.5.** Assume that $|r| < m/2$. We put

$$(4.4) \quad H_a \varphi(t, x) = \begin{cases} \varphi(t, x) & \text{for } t \geq a, \\ 0 & \text{for } t < a, \end{cases}$$

for $\varphi(t, x) \in S(\mathbb{R}^{n+1})$, then it holds that $\| H_a \varphi \|_{r,s,\Omega} \leq C \| \varphi \|_{r,s,\Omega}$. That is, the operator $H_a$ can be extended to a bounded linear operator on $H_{r,s}$ and the range of $H_a$ is $H_{0,r,s}(\Omega_{a,\infty})$.

When a function $p(t, x, \xi)$ satisfies that $|\partial_t^j \partial_x^\alpha \partial_\xi^\beta p(t, x, \xi)| \leq C_{j, \alpha, \beta, \lambda}(\xi)^j$ for any $j, \alpha$ and $\beta$, we write $p(t, x, \xi) \in S_{0,\lambda}^j$, by the same notation as in Definition 2.4. For $u(t, x) \in S(\mathbb{R}^{n+1})$, we define

$$(4.5) \quad p(t, X, D_x)u(t, x) = \int e^{ix(t-\tau-x)\xi} p(t, x, \xi) \tilde{u}(\tau, \xi) d\tau d\xi = \int e^{ix(t-\tau)} p(t, x, \xi) \tilde{u}(\tau, \xi) d\xi,$$

where $\tilde{u}(t, \xi) = \int e^{-ix(t-\tau)} u(t, \xi) dx$.

**Proposition 4.6.** Let $r$ and $s$ be arbitrary real numbers. For $p(t, x, \xi) \in S_{0,\lambda}^j$, it holds that

$$(4.6) \quad \| p(t, X, D_x)u \|_{r,s} \leq C \| u \|_{r,s+t} \text{ for } u \in S(\mathbb{R}^{n+1}).$$
Proof. By the definitions,
\[ \|p(t, X, D_x)u\|_{r,s} = \|\lambda_i(D, D_x)\lambda(D_x)^s \cdot p(t, X, D_x)u\|_{L^2(R^{n+1})}, \]
where \( \lambda_i(D, D_x)v = \int e^{(\tau + \xi \cdot x)}\lambda_i(\tau, \xi) \theta(\tau, \xi) d\tau d\xi. \)

Using Theorem 3.1 and Corollary 3.2 (i) we can write
\[ \lambda_i(D, D_x)^s \cdot p(t, X, D_x)u(t, x) = p_r(t, X, D_x)u(t, x) \]
where \( p_r(t, x, \tau, \xi)\lambda_i(\tau, \xi)^{-s} \lambda(\xi)^{-s-1} \in B(R^{n+1}) \).

From Lemma 3.7, we have
\[ \|p(t, X, D_x)u\|_{r,s} = \|p_r(t, X, D_x)\lambda(D_x)^s u\|_{L^2(R^{n+1})}; \leq C(\|\lambda_i(D, D_x)^s \cdot p(t, X, D_x)u\|_{L^2(R^{n+1})}) \]
\[ = C\|u\|_{r,s+1}. \]

Q.E.D.

By Proposition 4.6, the pseudo-differential operator \( p(t, X, D_x) \) with symbol \( p(t, x, \xi) \in S^0_{\rho, \lambda} \) can be extended to a bounded linear operator from \( H^{r,s}_{\rho,\lambda} \) to \( H^{r,s} \). In the above proof we used the fact that when \( \lambda(\xi) \) is a basic weight function in \( R^n \), \( \xi \) is also a basic weight function in \( R^{n+1} \).

For any \( u \in H^{r,s}_{\rho,\lambda} \), we take a sequence \( \{u_j\}_{j=1}^\infty \) in \( C^\infty_0(\Omega) \) such that \( u_j \rightarrow u \) in \( H^{r,s} \). Then by Proposition 4.6, \( p(t, X, D_x)u_j \rightarrow p(t, X, D_x)u \) in \( H^{r,s-1} \). Therefore, we have \( p(t, X, D_x)u \in H^{r,s+1}_{\rho,\lambda}(\Omega) \). This fact permits us to extend the operator \( p(t, X, D_x) \) from \( H^{r,s}_{\rho,\lambda}(\Omega) \) to \( H^{r,s+1}_{\rho,\lambda}(\Omega) \). Indeed, let \( u \in H^{r,s}_{\rho,\lambda}(\Omega) \), \( v_1|_\Omega = v_2|_\Omega = u \) and \( v_1, v_2 \in H^{r,s} \). Since \( v_1 - v_2 \in H^{r,s}_{\rho,\lambda}(\Omega^s) \), we have \( p(t, X, D_x) (v_1 - v_2) \in H^{r,s+1}_{\rho,\lambda}(\Omega^s) \). So we define \( p(t, X, D_x)u \) by \( p(t, X, D_x)u = p(t, X, D_x)v|_\Omega \) for \( v \in H^{r,s}_{\rho,\lambda} \) such that \( v|_\Omega = u \). Furthermore, we have
\[ \|p(t, X, D_x)u\|_{r,s+1,\Omega} = \inf \{\|v\|_{r,s+1,\Omega}; v|_\Omega = p(t, X, D_x)u, v \in H^{r,s}_{\rho,\lambda}\} \leq \|p(t, X, D_x)u\|_{r,s,\Omega}. \]

Thus we can extend the operator \( p(t, X, D_x) \) to a bounded linear operator from \( H^{r,s}_{\rho,\lambda}(\Omega) \) to \( H^{r,s+1}_{\rho,\lambda}(\Omega) \).

For \( \varphi(t, x), \psi(t, x) \in C^\infty_0(R^{n+1}) \), we write \( [\varphi, \psi] = \int_{R^{n+1}} \varphi(t, x) \psi(t, x) dtdx. \)

Then we can see that \( \|\varphi\|_{r,s} = \sup \{|[\varphi, \psi]|_{r,s} ; \psi \neq 0, \psi \in C^\infty_0(R^{n+1})\}. \)

Thus, \( H^{r,s}_{\rho,\lambda} \) and \( H^{r,s}_{\rho,\lambda} \) are dual Hilbert spaces and the form \([\cdot, \cdot]\) can be extended to a sesqui-linear form defined on \( H^{r,s}_{\rho,\lambda} \times H^{r,s}_{\rho,\lambda} \).

Let \( \{\xi_i(t, x)\}_{i=1}^{\infty} \) be a sequence of \( C^\infty_0(R^{n+1}) \) and \( \{\psi_i(\xi)\}_{i=1}^{\infty} \) a sequence of \( C^\infty_0(R^{n+1}) \) functions satisfying the following conditions:
(i) \( \sum \xi_i(t, x)^2 = 1, \sum \psi_i(\xi)^2 = 1, \)
(ii) \( \sum |\partial_{\xi}^{\alpha} \xi_i(t, x)| \leq C_{l,\alpha}, \sum |\partial_{\xi}^{\alpha} \psi_i(\xi)| \leq C_\alpha \) for any \( l \) and \( \alpha, \)
(iii) there exists a positive integer \( N \) such that for any \( (t, x) \in R^{n+1} \), the
number of \(\text{supp } \xi_i\) containing \((t, x)\) is at most \(N\) and for any \(\xi \in \mathbb{R}^n\), the number of \(\text{supp } \psi_j\) containing \(\xi\) is at most \(N\).

Let \(\{c_{ij}\}_{i,j=1}^\infty\) be a bounded sequence of complex numbers.

Then,
\[
\sum [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), c_{ij} \xi_i(t, x) \psi_j(D_x) \psi(t, x)] \\
= \sum [\psi_j(D_x) [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), \psi(t, x)] \\
= [\sum \psi_j(D_x) [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), \psi(t, x)].
\]

By assumptions of \(\{c_{ij}\}, \{\xi_j(t, x)\}\) and \(\{\psi_j(\xi)\}\), we can consider the operator \(\sum \psi_j(D_x) [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), \psi(t, x)]\) as a pseudo-differential operator with a double symbol \(\sum \psi_j(\xi) [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), \psi(t, x)] \in S^0_\lambda,\)

Hence we have
\[
\sum [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), c_{ij} \xi_i(t, x) \psi_j(D_x) \psi(t, x)] \\
\leq C \|\varphi\|_{r,s} \|\psi\|_{r,-s}.
\]

From this inequality we obtain the following proposition.

**Proposition 4.7.** The form \(\sum [c_{ij} \xi_i(t, x) \psi_j(D_x) \varphi(t, x), c_{ij} \xi_i(t, x) \psi_j(D_x) \psi(t, x)]\) for \(\varphi, \psi \in C^0_c(\mathbb{R}^n)\) can be extended uniquely to a continuous sesquilinear form defined on \(H_{r,s} \times H_{r,-s}\).

Using Lemma 4.5 and Proposition 4.7, we obtain the similar proposition to Proposition 7 in [3].

**Proposition 4.8.** Let \(\{c_{ij}\}, \{\xi_j(t, x)\}\) and \(\{\psi_j(\xi)\}\) satisfy the above conditions. Let \(r_1, r_2, \text{ and } r_2\) be real numbers satisfying that \(r_1 + r_2 \geq 0, r_1 + r_2 < s_1 + s_2 \geq 0, \text{ and } \min (r_1, r_2) > -m/2\) and let \(-\infty \leq a < b \leq +\infty\).

Then the form
\[
\sum \int_a^b (c_{ij} \xi_i(t) \psi_j(D_x) \varphi(t), c_{ij} \xi_i(t) \psi_j(D_x) \psi(t))_n dt
\]
for \(\varphi(t, x), \psi(t, x) \in C^0_c(\Omega)\) can be extended uniquely to a continuous sesquilinear form on \(H_{r_1,s_1}(\Omega) \times H_{r_2,s_2}(\Omega)\).

5. Parabolic operators and energy inequalities

Consider the operator \(L = D_t + \sum_{j=1}^k \delta_j(t, X, D_x) D_t^{-j}\) where \(D_t = (-i) \partial / \partial t\).

We assume that the operator \(L\) satisfies the following conditions:

(i) we can write \(L = L_0 + L_1\) where \(L_0 = D_t + \sum_{j=1}^k \delta_j(t, X, D_x) D_t^{-j}\) and \(L_1 = \sum_{j=1}^k \delta_j(t, X, D_x) D_t^{-j}\),

(ii) \(\delta_j(t, x, \xi) \in S^{\sigma_\lambda}_\lambda (j=1, \cdots, k)\),

(iii) for some \(0 < \delta < 1, \partial_t \psi_j(t, x, \xi) \in S^{\sigma_\lambda}_{\lambda-1}\) (\(i=1, \cdots, n, j=1, \cdots, k\)) and \(\psi_j(t, x, \xi) \in S^{\sigma_\lambda}_{\lambda-1}\),
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(iv) roots \( p_{x}(t, x, \xi), \ldots, p_{k}(t, x, \xi) \) of the equation \( \sigma(L)=\tau^k+\sum_{j=1}^{k} p_{j}(t, x, \xi) \tau^{k-j}=0 \) satisfy the inequalities \( \operatorname{Im} p_{j}(t, x, \xi) \geq c_{0}|\lambda(\xi)|^{m} \) \( (j=1, \ldots, k) \) where \( c_{0} \) is a positive constant.

We can consider the operator \( L \) as an extended form for higher order parabolic differential operators.

For any \( u \in S(R^{n+1}) \), we put \( u_{j}=\lambda(D_{x})^{m(k-j)}D_{t}^{-j}u \) for \( j=1, \ldots, k \), and \( U=t(u, \ldots, u_{k}) \). Then we have \( D_{t}u_{j}=\lambda(D_{x})^{m}u_{j+1} \) for \( j=1, \ldots, k-1 \) and \( D_{t}u_{k}=D_{t}^{k}u=Lu-\sum_{j=1}^{k} q_{j}(t, x, D_{x})D_{t}^{k-j}u=Lu-\sum_{j=1}^{k} p_{k-1-j+1} \)

\( (t, x, D_{x})\lambda(D_{x})^{m}u_{j}-\sum_{j=1}^{k} q_{j-1-j+1}(t, x, D_{x})u_{j} \) where \( p_{k-1-j+1}(t, x, \xi)=p_{0}^{j-1-j+1}(t, x, \xi) \)

\( \lambda(\xi)^{-m(k-j+1)} \in S_{0, l}^{b} \) and \( q_{j-1-j+1}(t, x, \xi)=q_{k-j+1}(t, x, \xi)\lambda(\xi)^{-m(k-j)} \in S_{0, l}^{b} \).

Thus we can write

\[
D_{t}U = h(t, X, D_{x}) \cdot \lambda(D_{x})^{m}U + \frac{1}{i} J(t, X, D_{x}) U + (Lu)e_{k}
\]

where \( e_{k} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \), \( h(t, x, \xi) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ -p_{k-1} & -p_{k-2} & \cdots & \cdots & -p_{1} \end{pmatrix} \)

and \( J(t, x, \xi) = \begin{pmatrix} 0 \\ -iq_{1} \cdot \cdots \cdot -iq_{l} \end{pmatrix} \)

Thus, \( \partial/\partial t U=H \cdot \lambda(D_{x})^{m}U+JU+iu(Lu)e_{k} \) and \( H=ih(t, X, D_{x}) \). We put \( R=\partial/\partial t - H \cdot \lambda(D_{x})^{m}-J \).

From the assumptions of operator \( L \), we have

(i) \( \sigma(H)=ih(t, x, \xi) \subseteq S_{0, l}^{b}, \partial_{\xi} \sigma(H) \subseteq S_{0, l}^{b} \) \( (j=1, \ldots, n) \) and \( \sigma(J)=J(t, x, \xi) \in S_{0, l}^{b} \),

(ii) the eigenvalues of \( \sigma(H) \) are contained in a fixed compact subset of the set \( \{x \in C; \operatorname{Re} x \leq -c_{0}\} \).

For a matrix \( A=(a_{ij}) \) we denote \( |A| = \{\sum|a_{ij}|^{2}\}^{1/2} \).

The following lemma is shown in [3].

**Lemma 5.1.** For any \( (t, x, \xi) \), there exists a \( k \times k \) matrix \( N(t, x, \xi) \) such that

(i) \( |N(t, x, \xi)| + |N(t, x, \xi)^{-1}| \leq C, \)

(ii) \( \operatorname{Re} (N(t, x, \xi)^{-1}H(t, x, \xi)N(t, x, \xi)^{\xi}, \xi) \leq -\frac{c_{0}}{4} |\xi|^{2} \) for any \( \xi = (\xi_{1}, \ldots, \xi_{k}) \in C^{k} \),

where the constant \( C \) is independent of \( (t, x, \xi) \).

**Lemma 5.2.** We fix an arbitrary point \( (t_{0}, x_{0}, \xi_{0}) \) and put \( N_{0}=N(t_{0}, x_{0}, \xi_{0}), H_{0}=H(t_{0}, x_{0}, \xi_{0}) \) and \( R_{0}=\partial/\partial t - H_{0} \lambda(D_{x})^{m}-J \). Then we have

\[
c_{l}||U(b)||_{H_{0}}^{2} - c_{l}||U(a)||_{H_{0}}^{2} + \mu_{l} \int_{a}^{b} ||U(t)||_{H_{0}}^{2} dt
\]
for any \( U \subset \{ S(R^{n+1}) \}^k \), where \( c_1, c_2, \mu, \) and \( \mu_2 \) are constants which are independent of \((t_0, x_0, \xi_0)\) and

\[
\|U(t)\|^2 = \int \lambda(\xi)^{2\mu} |\hat{U}(t, \xi)|^2 d\xi.
\]

Proof. Since \( H_0 \) and \( N_0 \) are constant matrices, we can write

\[
\text{Re}(N_0^{-1}R_0U, N_0^{-1}U)_0 = \text{Re}\left(N_0^{-1}\frac{\partial U}{\partial t}, N_0^{-1}U\right)_0 - \text{Re}(N_0^{-1}H_0\lambda(D_\alpha)^mU, N_0^{-1}U)_0
\]

Putting \( N_0^{-1}\lambda(D_\alpha)^mU = V \), we have

\[
\text{Re}\int_a^b (N_0^{-1}R_0U, N_0^{-1}U)_0 dt \geq \frac{1}{2} \|N_0^{-1}U(b)\|^2 - \frac{1}{2} \|N_0^{-1}U(a)\|^2
\]

\[
- \text{Re}\int_a^b (N_0^{-1}H_0N_0V, V)_0 dt - C \int_a^b \|JU\|\|U\|_{m-\delta/2} dt.
\]

By Theorem 3.8, it holds that

\[
\|JU\|\|U\|_{m-\delta/2} \leq C \|U\|_{m-\delta/2}.
\]

Using Lemma 5.1,

\[
\text{Re}(N_0^{-1}H_0N_0V, V)_0 = \text{Re}\int N_0^{-1}H_0N_0 \hat{V}(t, \xi) \cdot \hat{V}(t, \xi) d\xi
\]

\[
\leq \frac{c_2}{4} \int |\hat{V}(t, \xi)|^2 d\xi \leq -\mu_1' \int \lambda(\xi)^{2\mu} |\hat{U}(t, \xi)|^2 d\xi = -\mu_1' \|U(t)\|^{2\mu/2}.
\]

Hence we have

\[
\text{Re}\int_a^b (N_0^{-1}R_0U, N_0^{-1}U)_0 dt \geq c_1 \|U(b)\|^2 - c_2 \|U(a)\|^2
\]

\[
+ (\mu_1' - \varepsilon) \int_a^b \|U(t)\|^{2\mu/2} dt - C \int_a^b \|U(t)\|^{2\mu/2} dt
\]

for any \( \varepsilon > 0 \). Taking \( \varepsilon = \mu_1'/2 \), we obtain (5.1). Q.E.D.

To obtain the similar energy inequalities to those of [3] or [4], we use the partition of unity of the space \( R^{n+1}_{\delta_0} \) and \( R^*_{\delta} \). Let \( \varepsilon \) be a sufficiently small positive number which will be determined later.

Let \( \xi(t, x) \in C^\infty(S(R^{n+1})) \) satisfy \( 0 \leq \xi(t, x) \leq 1 \), \( \text{supp} \xi \subset \{(t, x) ; \ |t| < 1, \ |x_j| < 1 \}

\]

\[
\text{let} \ g = (g_0, g', \ldots, g_n) \text{ and } h = (h_0, h') \text{ denote } (n+1)\text{-tuples of integers.}
\]

We put \( \xi_g(t, x) = \sum \xi \left( \frac{1}{\varepsilon} t - g_0, \frac{1}{\varepsilon} x - g' \right) \left\{ \varepsilon \right\}^\frac{\varepsilon}{\delta} .
\]
Enumerating the points \( \{ \xi_g \} \) and the corresponding functions \( \{ \zeta_g \} \) in some order, we denote them by \( (t_1, x_1), (t_2, x_2), \ldots \) and \( \xi_1, \xi_2, \ldots \).

Then we have,

(i) \( \sum_i \zeta_i(t, x)^2 = 1 \),

(ii) \( \sum_l | \partial_l \zeta_i(t, x) | \leq C_i, \alpha \) for any \( l \) and \( \alpha \),

(iii) the supp \( \zeta_i \) overlap in such a way that each fixed point in \( \mathbb{R}^{n+1} \) is contained in at most \( 2^{n+1} \) distinct ones of them,

(iv) \( |H(t, x, \xi) - H(t, x, \xi)| \leq C \{ |t - t_i| + |x - x_i| \} \leq C_i \varepsilon \) for any \( (t, x) \in \text{supp} \zeta_i \). 

We take the set \( \{ g_{i,j} \} \) of points in \( \mathbb{R}^n \) as follows:

(i) \( g_{1,0} = 0 \),

(ii) \( g_{i+1,j} = g_{i,j} \) for \( i = j \),

(iii) when \( 0 < |(3^n - 1)| \leq j \leq (l+1)(3^n - 1) \), \( l = 0, 1, \ldots \), writing \( g_{i,j} = (a_{i,j}, a_{i,j+1}, \ldots, a_{i,n}) \), \( a_{i,j} = 2(3^n - 1) \) or \( a_{i,j} = -2(3^n - 1) \), \( i = 1, \ldots, n \). We put \( \tilde{a}_{i,j} = 2 \) and \( \tilde{a}_{i,j} = 2(3^n - 1) \) for \( 0 < |(3^n - 1)| \leq j \leq (l+1)(3^n - 1) \), \( l = 0, 1, \ldots \). We put \( \Delta_{i,j} = \{ \xi \in \mathbb{R}^n; \quad |\xi_i - a_{i,j}| \leq \frac{1}{2} \tilde{a}_{i,j}, i = 1, \ldots, n \} \) for \( g_{i,j} = (a_{i,j}, \ldots, a_n) \).

Then it holds that \( \mathbb{R}^n = \bigcup_{j=0}^{\infty} \Delta_{i,j} \), \( \bigcup_{j=0}^{\infty} \partial \Delta_{i,j} \) is a set of measure zero and for almost everywhere \( \xi \in \mathbb{R}^n \), there is a number \( j \) uniquely such that \( \xi \in \Delta_{i,j} \).

Enumerating the cubes which satisfy \( a_{i,j} \leq \varepsilon \lambda(\tilde{g}_{i,j}) \delta_1 \), we denote them by \( \Delta_{1,1}, \Delta_{1,2}, \ldots \) and their centers and the lengths of sides by \( g_{1,1}, g_{1,2}, \ldots \) and \( a_{1,1}, a_{1,2}, \ldots \) respectively.

Similarly we write \( \Delta'_{1,1}, \Delta'_{1,2}, \ldots, g'_{1,1}, g'_{1,2}, \ldots \) and \( a'_{1,1}, a'_{1,2}, \ldots \) for the cubes satisfying \( \tilde{a}_{1,j} > \varepsilon \lambda(\tilde{g}_{1,j}) \delta_1 \).

We divide each \( \Delta'_{1,1} \) into \( 2^n \) congruent cubes and enumerate such cubes in some order: \( \Delta_{2,1}, \Delta_{2,2}, \ldots \). We denote the center and length of side of each cube \( \Delta_{2,j} \) by \( g_{2,j} \) and \( \tilde{a}_{2,j} \) respectively.

By the same way as above we write \( \{ \Delta_{2,j} \} j = \{ \Delta_{3,j} \} j, \{ g_{2,j} \} j = \{ g_{3,j} \} j \), and \( \{ \tilde{a}_{2,j} \} j = \{ a_{2,j} \} j \) if \( \tilde{a}_{2,j} \leq \varepsilon \lambda(\tilde{g}_{2,j}) \delta_1 \) and \( \{ \Delta_{3,j} \} j = \{ \Delta'_{3,j} \} j \) if \( \tilde{a}_{3,j} > \varepsilon \lambda(\tilde{g}_{3,j}) \delta_1 \).

Repeating this process, we obtain cubes \( \{ \Delta_{i,j} \} j = \{ \Delta_{i+1,j} \} j \) with centers \( g_{i,j} \) and lengths of sides \( a_{i,j} \).

**Lemma 5.3.** (i) \( \mathbb{R}^n = \bigcup_{i,j} \Delta_{i,j} \)

(ii) for sufficiently small \( \varepsilon > 0 \), \( \{ \Delta_{i,j} \} = \{ \Delta'_{i,j} \} \)

(iii) for sufficiently small \( \varepsilon > 0 \), we have \( c_0 \varepsilon \lambda(\tilde{g}_{i,j}) \delta_1 \leq a_{i,j} \leq \varepsilon \lambda(\tilde{g}_{i,j}) \delta_1 \) (\( 0 < c_0 < 1 \)).

Proof of (i). We note that \( \mathbb{R}^n = \bigcup_{j=0}^{\infty} \Delta_{i,j} \). Assume that there exists a point \( \xi \in \mathbb{R}^n \) such that for any \( i, j \), \( \xi \not\in \Delta'_{i,j} \), for some \( j \). Then by the definition of
\[ \Delta_{i,j}; \quad |\xi_i - a_i'| \leq \frac{1}{2} a_i'; i = 1, \ldots, n, \quad a_i', j > \varepsilon \lambda(g_{i,j})^b_i \geq \xi \text{ and } a_i', j = \frac{1}{2} a_i' \text{ for some } j, \text{ here } \xi = (\xi_1, \ldots, \xi_n) \text{ and } g_{i,j}' = (a_i', \ldots, a_n'). \]

Taking sufficiently large \( l \), we have a contradiction. Hence for any \( \xi \in \mathbb{R}^n \), there exists \( l \) and \( j \), such that \( \xi \in \Delta_{i,j}. \)

Proof of (ii). Taking \( \varepsilon > 0 \) sufficiently small, we have \( \varepsilon \lambda(0)^b < 2 = a_{i,0}, \) hence \( \Delta_{1,0} \subseteq \{\Delta_{i,j}\}. \) For any \( j \geq 1 \), by definitions, \( 2 \leq a_{i,j} = |\tilde{g}_{i,j}| \leq \sqrt{n} a_{i,j}. \)

By Lemma 2.2 (2.3), \( \lambda(\tilde{g}_{i,j})^b_i \leq C_i |\tilde{g}_{i,j}|^b_i \leq C_i |\tilde{g}_{i,j}| \leq 2C_i |\tilde{g}_{i,j}| \leq (2C_1 \sqrt{n}) a_{i,j}. \)

Hence, taking \( 0 < \varepsilon < (2C_1 \sqrt{n})^{-1} \), we have \( \varepsilon \lambda(\tilde{g}_{i,j})^b_i < a_{i,j}. \) This means \( \Delta_{i,j} \subseteq \{\Delta_{i,j}\}. \)

Proof of (iii). By definitions we have \( a_{i,j} \leq \varepsilon \lambda(g_{i,j})^b_i. \) By virtue of Lemma 2.2, we can take \( \varepsilon > 0 \) sufficiently small such that

\[ (5.2) \quad \frac{3}{4} \lambda(\xi) \leq \lambda(\eta) \leq \frac{4}{3} \lambda(\xi) \text{ for } |\xi - \eta| \leq 2\sqrt{n} \varepsilon \lambda(\xi)^b_i. \]

By definitions and (ii), \( \Delta_{i,j} \subseteq \Delta_{i-1,j}. \) Then we have \( a_{i,j} > \frac{1}{2} a_{i-1,j} > \frac{1}{2} \varepsilon \lambda(g_{i-1,j})^b_i. \)

Since \( g_{i-1,j} \subseteq \Delta_{i,j}, \quad |g_{i-1,j} - g_{i,j}| \leq \frac{1}{2} \sqrt{n} a_{i,j} \leq \frac{1}{2} \sqrt{n} \varepsilon \lambda(g_{i,j})^b_i \)

\( \leq 2\sqrt{n} \varepsilon \lambda(g_{i,j})^b_i. \) Hence, we have \( a_{i,j} > \frac{1}{2} \varepsilon \left(\frac{3}{4}\right)^b_i \lambda(g_{i,j})^b_i. \) Q.E.D.

We put \( \Delta^*_{i,j} = \{\xi; |\xi_i - a_i| \leq \frac{5}{9} a_{i,j}, i = 1, \ldots, n\} \) where \( g_{i,j} = (a_i, \ldots, a_n). \)

It is clear that \( \Delta_{i,j} \subseteq \Delta^*_{i,j}. \)

**Lemma 5.4.** We take \( \varepsilon > 0 \) sufficiently small so that Lemma 5.3 (ii) and the inequality (5.2) hold. Then if \( \Delta^*_{i,j} \cap \Delta_{i,j} \neq \emptyset, \) it holds that \( \frac{1}{3} a_{i,j} \leq a_{i,j} < 3 a_{i,j}. \)

Proof. Assume that \( \Delta^*_{i,j} \cap \Delta_{i,j} \neq \emptyset \) and \( a_{i,j} < \frac{1}{3} a_{i,j}. \) By definitions and Lemma 5.3 (ii), \( \Delta_{i,j} \subseteq \Delta_{i-1,j} \) for some \( \Delta_{i-1,j}. \) Taking \( \xi \in \Delta^*_{i,j} \cap \Delta_{i,j} \) we have

\[ |g_{i,j} - g_{i-1,j}^{'''}| \leq |g_{i,j} - \xi| + |\xi - g_{i-1,j}^{'''}| \leq \frac{5}{9} \sqrt{n} a_{i,j} + \frac{1}{2} \sqrt{n} a_{i-1,j}^{'''} \]

\[ = \frac{5}{9} \sqrt{n} a_{i,j} + \frac{8}{9} \sqrt{n} a_{i,j} \leq 2\sqrt{n} \varepsilon \lambda(g_{i,j})^b_i. \]

From (5.2) we have \( \varepsilon \lambda(g_{i-1,j}^{''''}) = 2 a_{i,j} < \frac{2}{3} a_{i,j} \leq \frac{2}{3} \varepsilon \lambda(g_{i,j})^b_i \leq \frac{2}{3} \left(\frac{4}{3}\right)^b_i \varepsilon \lambda(g_{i-1,j}^{''''})^b_i \leq \varepsilon \lambda(g_{i-1,j}^{''''})^b_i. \)

This contradicts to the definition of \( \Delta_{i-1,j}^{''''}. \)
Hence we have $a_{l', j'} \geq \frac{1}{3} a_{l, j}$.

By the same way we can prove that $a_{l', j'} \leq 3 a_{l, j}$.

We denote the volume of cube $\Delta$ by $|\Delta|$.

**Lemma 5.5.** There is a positive integer $M$ such that for any $l, j$, the number of cubes $\Delta^*_{l', j'}$ which satisfy $\Delta_{l, j} \cap \Delta^*_{l', j'} \neq \emptyset$ is at most $M$.

**Proof.** By Lemma 5.4, we have,

$$\bigcup_{l', j'} \Delta^*_{l', j'} \subset \{ \xi; |\xi_i - a_i| \leq 4 a_{l, j} \}$$

where $g_{l, j} = (a_1, \ldots, a_n)$ and the union is taken for the cubes satisfying $\Delta^*_{l', j'} \cap \Delta_{l, j} \neq \emptyset$.

We write the number of such cubes by $M_0$.

Consider the number $M_1$ of cubes which satisfy $|\Delta| \geq \left( \frac{1}{3} a_{l, j} \right)^n$ and $\Delta \subset \{ \xi; |\xi_i - a_i| \leq 4 a_{l, j}, i = 1, \ldots, n \}$.

Then we have,

$$M_1 \left( \frac{1}{3} a_{l, j} \right)^n \leq (8 a_{l, j})^n,$$

hence, $M_1 \leq 24^n$.

Using Lemma 5.4, we obtain $M_0 \leq M_1 \leq 24^n$.

Rearranging $\{\Delta_{l, j}\}$, $\{g_{l, j}\}$ and $\{a_{l, j}\}$, we denote them by $\{\Delta_j\}_{j=1}^n$, $\{g_j\}_{j=1}^n$ and $\{a_j\}_{j=1}^n$.

Let $\psi(\xi) \in C^\infty_0(\mathbb{R}^n)$ satisfy that $\psi(\xi) = 1$ for $|\xi_i| \leq \frac{1}{2} (i = 1, \ldots, n)$ and $0 \leq \psi(\xi) \leq 1$ and supp $\psi(\xi) \subset \{ \xi; |\xi_i| \leq \frac{5}{9}, i = 1, \ldots, n \}$.

We put $\psi_j(\xi) = \psi \left( \frac{\xi - g_j}{a_j} \right)$, $\check{\psi}(\xi) = \left\{ \sum \psi_j(\xi)^2 \right\}^{1/2}$ and $\varphi_j(\xi) = \psi_j(\xi) / \check{\psi}(\xi)$.

**Theorem 5.6.** For sufficiently small $\varepsilon > 0$, we have,

(i) $\varphi_j(\xi) \in C^\infty_0(\mathbb{R}^n)$, $0 \leq \varphi_j(\xi) \leq 1$,

(ii) $\sum_j \varphi_j(\xi)^2 = 1$,

(iii) $\sum_j |\partial^\alpha_j \varphi_j(\xi)| \leq C_{\alpha, \lambda}(\xi)^{-\varepsilon_1^{|\alpha|}}$ for any $\alpha$,

(iv) there exists a positive integer $M$ such that each $\xi \in \mathbb{R}^n$ is contained in the supports of at most $M$ of $\{\varphi_j\}$.

**Proof.** We put $\Delta^*_j = \{ \xi; |\xi_i - b_i| \leq \frac{5}{9} a_j (i = 1, \ldots, n) \}$ here $g_j = (b_1, \ldots, b_n)$.

Then by definitions supp $\varphi_j \subset \Delta^*_j$ and $\varphi_j(\xi) = 1$ for $\xi \in \Delta_j$.

Using Lemma 5.3 (i) and Lemma 5.5, $\check{\psi}(\xi)$ is well-defined and $1 \leq \check{\psi}(\xi) \leq M$.

Therefore from the definitions of $\varphi_j(\xi)$, we obtain (i), (ii) and (iv).

Since $\partial^\alpha_j \varphi_j(\xi) = \psi(\xi) \left( \frac{\xi - g_j}{a_j} \right)^{-|\alpha|}$, using Lemma 5.3 (iii) and (5.2) we have,
\[ |\partial_n^{\alpha} \psi_j(\xi)| \leq |\psi_0(\xi - g_j)| a_j^{-|\alpha|} \leq C_\alpha(\xi, g_j)^{-|\alpha|} |\lambda(g_j)|^{-\frac{|\alpha|}{2}} \]
\[ \leq C_1 \alpha, \lambda(\xi)^{-\frac{|\alpha|}{2}} \text{ for any } \alpha. \]

Hence \[ |\partial_n^{\alpha} \psi(\xi)| \leq C_1 \alpha, M_\alpha(\xi)^{-\frac{|\alpha|}{2}} \text{ for any } \alpha. \]

Using these inequalities we obtain (iii). Q.E.D.

We can see that for any \((t, x) \in \mathbb{R}^n\) and \(\xi \in \text{supp } \varphi_j\),

\[ |H(t, x, \xi) - H(t, x, g_j)| \leq C |\xi - g_j| \sup_{0 \leq t \leq 1} \lambda(g_j + s(\xi - g_j))^{-\frac{1}{2}} \leq C \varepsilon. \]

Taking \(\varepsilon > 0\) sufficiently small, we have the following Theorem.

**Theorem 5.7.** We put \(N_i = N(t_i, x_i, g_i)\). There exist positive constants \(c_1, c_2, \mu_1\) and \(\mu_2\) such that

\[ (5.4) \quad c_1 ||U(b)||_2^2 - c_2 ||U(a)||_2^2 + \mu_1 \int_a^b ||U(t)||_2^2 dt \]
\[ - \mu_2 \int_a^b ||U(t)||_2^2 dt \leq \text{Re} \int_a^b (N_1) R_{ij} U, N_1 \zeta U dt. \]

for any \(U \in \{S(\mathbb{R}^{n+1})\}^k\) where \(\Phi_j = \varphi_j(D_x)\).

**Proof.** We put \(H_{ij} = H(t_i, x_i, \xi_j)\) and \(R_{ij} = \partial/\partial t - H_{ij} \lambda(D_x)^m\). By Lemma 5.2, there exist positive constants \(c_1, c_2, \mu_1\) and \(\mu_2\) such that

\[ c_1 ||U(b)||_2^2 - c_2 ||U(a)||_2^2 + \mu_1 \int_a^b ||U(t)||_2^2 dt \]
\[ - \mu_2 \int_a^b ||U(t)||_2^2 dt \leq \text{Re} \int_a^b (N_1) R_{ij} U, N_1 \zeta U dt. \]

Hence we have

\[ c_1 ||\zeta_i(b)\Phi_j U(b)||_2^2 - c_2 ||\zeta_i(a)\Phi_j U(a)||_2^2 \]
\[ + \mu_1 \int_a^b ||\zeta_i(t)\Phi_j U(t)||_2^2 dt - \mu_2 \int_a^b ||\zeta_i(t)\Phi_j U(t)||_2^2 dt \]
\[ \leq \text{Re} \int_a^b (N_1) R_{ij} \zeta_i \Phi_j U, N_1 \zeta_i \Phi_j U dt. \]

We can see that

\[ \sum_{i,j} \zeta_i \Phi_j U(t)||_2^2 = \sum_{i,j} \text{Re} (\zeta_i \lambda(D_x)^{2s} \zeta_i \Phi_j U, \Phi_j U) \]
\[ = \text{Re} \sum_{i,j} (\sum_{i,j} \zeta_i(t, X) \cdot \lambda(D_x)^{2s} \zeta_i(t, X') \Phi_j U, \Phi_j U) \]

Since \(\sum_{i,j} \zeta_i(t, x) \lambda(\xi)^{2s} \zeta_i(t, x') \in S_{\alpha, \lambda}^m\), from Theorem 3.3, we can write

\[ \sum_{i,j} \zeta_i(t, X) \cdot \lambda(D_x)^{2s} \zeta_i(t, X') = \lambda(D_x)^{2s} I + p'(t, X, D_x) \]

where \(p'(t, x, \xi) \in S_{\alpha, \lambda}^{m-1}\). Hence we obtain
\[
\sum_{i,j} \| \xi_i \Phi_j U(t) \|_2^2 \leq \| U(t) \|_2^2 + C \| U(t) \|_{-1/2}^2
\]

in particular,

\[
\sum_{i,j} \| \xi_i \Phi_j U(t) \|_2^2 = \| U(t) \|_2^2.
\]

By (5.5), \[\sum_{i,j} (N_{-1}^{-1} \xi_i \Phi_j JU, N_{-1}^{-1} \xi_i \Phi_j U)_0 \]

\[\leq C \sum_{i,j} \| \xi_i \Phi_j JU(t) \|_{-(m-\epsilon_2)/2} \| \xi_i \Phi_j U(t) \|_{(m-\epsilon_2)/2} \]

\[\leq C \sum_{i,j} \{ \| \xi_i \Phi_j JU(t) \|_{-(m-\epsilon_2)/2} + \| \xi_i \Phi_j U(t) \|_{-(m-\epsilon_2)/2} \}
\]

\[\leq C \{ \| JU \|_{-(m-\epsilon_2)/2} + \| U(t) \|_{-(m-\epsilon_2)/2} \}
\]

\[\leq C \| U(t) \|_{-(m-\epsilon_2)/2}^2
\]

\[\leq \frac{\mu_1}{N_0} \| U(t) \|_{m/2}^2 + C_{N_0} \| U(t) \|_2^2,
\]

for any positive number \( N_0 \).

By (5.5), \[\sum_{i,j} \| \xi_i \Phi_j U(t) \|_{m/2}^2 \]

\[\geq \left( 1 - \frac{1}{N_0} \right) \| U(t) \|_{m/2}^2 - C_{N_0} \| U(t) \|_2^2.
\]

Hence we get the inequality

\[
(5.7) \quad c_1 \| U(b) \|_2^2 - c_2 \| U(a) \|_2^2 + \left( 1 - \frac{2}{N_0} \right) \mu_1 \int_a^b \| U(t) \|_{m/2}^2 dt
\]

\[-c_{N_0} \int_a^b \| U(t) \|_2^2 dt
\]

\[\leq \text{Re} \int_a^b \sum_{i,j} (N_{-1}^{-1} R_i \xi_i \Phi_j U, N_{-1}^{-1} \xi_i \Phi_j U)_0 dt
\]

\[-\text{Re} \int_a^b \sum_{i,j} \{ (N_{-1}^{-1} \xi_i \Phi_j JU, N_{-1}^{-1} \xi_i \Phi_j U)_0 + A_{ij} \} dt.
\]

The right hand side of this inequality can be written in the form:

\[\text{Re} \int_a^b \sum_{i,j} \{ (N_{-1}^{-1} \xi_i \Phi_j R U, N_{-1}^{-1} \xi_i \Phi_j U)_0 + A_{ij} \} dt.
\]

where

\[A_{ij} = (N_{-1}^{-1} \left( \frac{\partial}{\partial t} \xi_i \right) \Phi_j U, N_{-1}^{-1} \xi_i \Phi_j U)_0
\]

\[+ (N_{-1}^{-1} \xi_i \Phi_j H \lambda(D_2)^m U, N_{-1}^{-1} \xi_i \Phi_j U)_0
\]

\[-(N_{-1}^{-1} H \lambda(D_2)^m \xi_i \Phi_j U, N_{-1}^{-1} \xi_i \Phi_j U)_0
\]

\[+ (N_{-1}^{-1} \xi_i \left( H - H_{ij} \right) \lambda(D_2)^m \Phi_j U, N_{-1}^{-1} \xi_i \Phi_j U)_0
\]

\[= I_{ij} + II_{ij} + III_{ij} + IV_{ij}.
\]

We can see that
By Theorem 5.3, we get \([\Phi_j, H]\) = p_j^*(t, X, D_x) where \(p_j^*(t, x, \xi) \in \mathcal{S}_{\alpha, \lambda}^{m-1}\).

Thus,
\[
|\sum_{i,j} II_{ij}| \leq \left| \sum_{j} \left( (\sum_{i} (N^*)^{-1} N_{ij}^*) [\Phi_j, H] \lambda(D_x)^m U, \Phi_j U \right) \right|.
\]

where \(\xi^{(2)} = \sum_{i} (N^*)^{-1} \xi_i (t, x)^2\). Since
\[
\sum_{i} \phi_j (\xi_i (t, x)^2) p_j^*(t, x', \xi') \lambda(\xi') \in \mathcal{S}_{\alpha, \lambda}^{m-2},
\]
we have
\[
(5.9) \quad |\sum_{i,j} II_{ij}| \leq |(p(t, X, D_x) U, U)|
\]
\[
\leq \|p(t, X, D_x) U\|_{m-(\alpha_0/2)} \|U\|_{m-(\alpha_0/2)}
\]
\[
\leq C_{\epsilon} \|U\|_{m-\alpha_0/2}^2 \leq \frac{\mu_1}{N_0} \|U(t)\|_{m/2}^2 + C_{N_0, \epsilon} \|U(t)\|_5^2
\]
where \(p(t, x, \xi) \in \mathcal{S}_{\alpha, \lambda}^{m-1}\).

By the similar way, we can obtain
\[
\sum_{i,j} III_{ij} = (p^*(t, X, D_x) U, U_0)
\]
where \(p^*(t, x, \xi) \in \mathcal{S}_{\alpha, \lambda}^{m-1}\). Hence we get
\[
(5.10) \quad |\sum_{i,j} III_{ij}| \leq C_{\epsilon} \|U\|_{m-(\alpha_0/2)}^2 \leq \frac{\mu_1}{N_0} \|U(t)\|_{m/2}^2 + C_{N_0, \epsilon} \|U(t)\|_5^2.
\]

To estimate the term \(\sum_{i,j} IV_{ij}\), we write
\[
IV_{ij} = (\sum_{i} \phi_j \lambda(D_x)^m U, \phi_j U)
\]

where \(p(t, x, \xi) \in \mathcal{S}_{\alpha, \lambda}^{m-1}\), and
\[
(5.11) \quad |\sum_{i,j} (C_{ij} + D_{ij})| = |(p(t, X, D_x) U, U)| \leq C_{\epsilon} \|U\|_{m-(\alpha_0/2)}^2
\]
\[
\leq \frac{\mu_1}{N_0} \|U(t)\|_{m/2}^2 + C_{N_0, \epsilon} \|U(t)\|_5^2
\]

where \(p(t, x, \xi) \in \mathcal{S}_{\alpha, \lambda}^{m-1}\). Furthermore we have
\[
(5.12) \quad |\sum_{i,j} E_{ij}| = |(p^*(t, X, D_x) U, U)| \leq C_{\epsilon} \|U\|_{m-(\alpha_0/2)}^2
\]
\[
\leq \frac{\mu_1}{N_0} \|U(t)\|_{m/2}^2 + C_{N_0, \epsilon} \|U(t)\|_5^2,
\]
where \(p(t, x, \xi) \in \mathcal{S}_{\alpha, \lambda}^{m-2}\). Furthermore we have
\[
|\sum_{i,j} B_{ij}| \leq C_{\epsilon} \sum_{i,j} \|\xi_i \lambda(\phi) \| \|U\|_5^2
\]
\[
+ \frac{1}{N_0} \sum_{i,j} \|\phi_j \lambda(D_x)^m U\|_5^2
\]
where the constant $C_0$ is independent of $N_0$ and $\varepsilon$.

Using Theorem 3.3 and Corollary 3.4 (i), (ii), we obtain

$$
\sum_{i,j} \| \xi_i \{ \mathbf{H} - H_{ij} \} \Phi_j \lambda(D_x)^m \mathbf{U} \|_0^2
$$

$$
= \langle \mathbf{p}'(t, X, D_x) \lambda(D_x)^m \mathbf{U}, \lambda(D_x)^m \mathbf{U} \rangle_0
$$

$$
+ \langle \mathbf{p}'(t, X, D_x) \lambda(D_x)^m \mathbf{U}, \lambda(D_x)^m \mathbf{U} \rangle_0,
$$

where $p(t, x, \xi) = \sum_{i,j} \xi_i(t, x) \{ \mathbf{H}(t, x, \xi) - H_{ij} \}^{*}$

$$
\times \{ \mathbf{H}(t, x, \xi) - H_{ij} \} \Phi_j(\xi)^2
$$

and $p(t, x, \xi) \in S_{0, \mathbb{H}}^2$.

By the assumptions of $\mathbf{H}$, $\xi_i$, $\Phi_j$ and $H_{ij}$,

$$
|p'(t, x, \xi)| \leq \sum_{i,j} |\xi_i(t, x)|^2 |\Phi_j(\xi)|^2 \{ C_1 + C_2 \} \xi^2 \leq C_3 \xi,
$$

where $C_1$ is the constant in (iv) of the definition of $\{ \xi_i \}$ and $C_2$ is the one in (5.3), and $\partial_\xi \mathbf{p}'(t, x, \xi) \in S_{0, \mathbb{H}}$ $i = 1, \ldots, n$.

Hence by Theorem 3.10, we have

$$
\| (\mathbf{p}'(t, X, D_x) \lambda(D_x)^m \mathbf{U}, \lambda(D_x)^m \mathbf{U})_0 \|
$$

$$
\leq C_3 \varepsilon \| \mathbf{U}(t) \|_{m/2}^2 + C_3 \| \lambda(D_x)^m \mathbf{U} \|_{m - b/2}^2.
$$

Therefore,

$$
\sum_{i,j} \| \xi_i \{ \mathbf{H} - H_{ij} \} \Phi_j \lambda(D_x)^m \mathbf{U} \|_0^2
$$

$$
\leq C_3 \varepsilon \| \mathbf{U}(t) \|_{m/2}^2 + C_3 \| \lambda(D_x)^m \mathbf{U} \|_{m - b/2}^2 + C_3 \| \mathbf{U}(t) \|_{m - b/2}^2.
$$

Thus we obtain

(5.13) \[ \| \sum_{i,j} B_{ij} \| \leq \{ C_1 C_3 N_0 \} \varepsilon \| \mathbf{U}(t) \|_{m/2}^2 + \frac{C_0}{N_0} \| \mathbf{U}(t) \|_{m/2}^2 \]

$$
+ \frac{C_3 N_0 \varepsilon}{N_0} \| \mathbf{U}(t) \|_{m - b/2}^2 + \frac{C_3 \mu_1}{N_0} \| \mathbf{U}(t) \|_{m/2}^2 + C_3 N_0 \varepsilon \| \mathbf{U}(t) \|_0^2.
$$

By virtue of the inequalities (5.7) \sim (5.13), we obtain

(5.14) \[ c_1 \| U(b) \|_{\mathbb{L}^2} \leq c_2 \| U(a) \|_{\mathbb{L}^2} + \left\{ \frac{1}{N_0} \mu_1 - C_0 C_3 N_0 \varepsilon - \frac{C_0}{N_0} \right\} \]

$$
\times \int_a^b \| \mathbf{U}(t) \|_{m/2}^2 dt - C_3 N_0 \varepsilon \int_a^b \| \mathbf{U}(t) \|_0^2 dt
$$

$$
\leq \text{Re} \int_a^b \sum_{i,j} (N_i \Phi_j(D_x) \mathbf{U}, N_i \Phi_j(\xi))_{0} dt.
$$

Taking $\varepsilon \leq \frac{\mu_1}{N_0^3 C_0 C_4}$ and $N_0$ sufficiently large so that $\mu_1 - \frac{7 \mu_1 + C_0}{N_0} \geq \frac{\mu_1}{2}$,

we complete the proof. Q.E.D.

Let $r$ and $s$ be real numbers satisfying $r > m/2$ and let $-\infty \leq a < b \leq +\infty$.

**Theorem 5.8.** For sufficiently small $\varepsilon$ there exist positive constants $c_1$, $c_2$, $\mu_1$ and $\mu_2$ such that
\begin{align}
(5.15) \quad & c_1\|U(b)\|_2^2 - c_2\|U(a)\|_2^2 + \mu_1\int_a^b \|U(t)\|_{p+m/2}^2 dt \\
& - \mu_2 \int_a^b \|U(t)\|_{p}^2 dt \\
& \leq \text{Re} \sum_{i,j} \int_a^b (N_i^j \xi_i \Phi_j \lambda(D_x)^{p} R U, N_i^j \xi_i \Phi_j \lambda(D_x)^{p} U) dt
\end{align}

for any \( U \in \{H_{r,s}(\Omega)\}^k \), where \( p = r + s - m/2 \) and \( U(t) = \gamma_t U \), and \( \gamma_t \) is the trace operator defined in Lemma 4.4.

Proof. At first we assume \( r + s - m/2 = p = 0 \), then by Theorem 5.7, the inequality (5.14) holds for \( U \in \{S(R^{m+1})\}^k \). Since \( R : \{H_{r,s}(\Omega)\}^k \rightarrow \{H_{r,s}(\Omega)\}^k \) is a continuous linear operator, the form

\begin{align}
\int_a^b \sum_{i,j} (N_i^j \xi_i \Phi_j R U, N_i^j \xi_i \Phi_j V) dt
\end{align}

is a continuous sesquilinear form defined on \( \{H_{r,s}(\Omega)\}^k \times \{H_{r,s}(\Omega)\}^k \), because of Proposition 4.8. Using the continuity of the trace operator \( \gamma_t \), we obtain the theorem for \( p = 0 \).

Let \( r + s - m/2 = p \). We have that \( R \lambda(D_x)^{p} = \lambda(D_x)^{p} R + [\{R \lambda(D_x)^{p} - \lambda(D_x)^{p} R\} = \lambda(D_x)^{p} R + [\lambda(D_x)^{p}, H] \lambda(D_x)^{m} + [\lambda(D_x)^{p}, J] \). By assumptions of \( H \) and \( J \), we have \( [\lambda(D_x)^{p}, H] \lambda(D_x)^{m} = p'(t, X, D_x) \) and \( [\lambda(D_x)^{p}, J] = p'(t, x, \xi) \) where \( p'(t, x, \xi) \) and \( p'(t, x, \xi) \) belong to \( S_{\xi}^{m+\beta-\delta} \).

Thus we have

\begin{align}
|\text{Re} \sum_{i,j} \int_a^b (N_i^j \xi_i \Phi_j [\lambda(D_x)^{p}, H] \lambda(D_x)^{m} U, N_i^j \xi_i \Phi_j \lambda(D_x)^{p} U) dt| \\
& \leq C \int_a^b \|U(t)\|_{p+m/2}^2 dt + C_{N_0, \xi} \int_a^b \|U(t)\|_{p+m/2}^2 dt
\end{align}

for any \( U \in \{H_{r,s}(\Omega)\}^k \). Similarly,

\begin{align}
|\text{Re} \sum_{i,j} \int_a^b (N_i^j \xi_i \Phi_j [\lambda(D_x)^{p}, J] U, N_i^j \xi_i \Phi_j \lambda(D_x)^{p} U) dt| \\
& \leq \frac{1}{N_0} \int_a^b \|U(t)\|_{p+m/2}^2 dt + C_{N_0, \xi} \int_a^b \|U(t)\|_{p+m/2}^2 dt
\end{align}

for any \( U \in \{H_{r,s}(\Omega)\}^k \).

Taking \( N_0 \) sufficiently large and using (5.4) for \( \lambda(D_x)^{p} U \) in place of \( U \) we obtain the theorem.

Q.E.D.

6. The Cauchy problem for the operator \( R \)

In the proof of Lemma 4 in [3] (p. 193) replacing \( |\xi|^{2k} \) by \( \lambda(\xi)^m \), we have the following lemma.
Lemma 6.1. We fix an arbitrary point \((t_0, x_0, \xi_0)\), and put \(H_0 = H(t_0, x_0, \xi_0)\) and \(R_0 = \partial/\partial t - H_0 \lambda(D_x)^m\). Then there exists \(C > 0\) such that

\[
(6.1) \int_{\mathbb{R}^{n+1}_+} \left( \tau^2 + \lambda(\xi) \cdot \eta + \eta^2 \right) |\tilde{U}(\tau, \xi)|^2 d\tau d\xi \\
\leq C \left\| (R_0 + \eta I) U \right\|_{\mu, 0}^2
\]

for any \(\eta > 0\) and \(U \in \{S(\mathbb{R}^{n+1})\}^k\), where \(I\) is the \(k \times k\) identity matrix and \(C\) is a constant independent of \((t_0, x_0, \xi_0)\).

Theorem 6.2. There exist constants \(C_1, C_2 > 0\) such that

\[
(6.2) \int_{\mathbb{R}^{n+1}_+} \left( \tau^2 + \lambda(\xi) \cdot \eta + \eta^2 \right) |\tilde{U}(\tau, \xi)|^2 d\tau d\xi \\
\leq C_1 \left\| (R_0 + \eta I) U \right\|_{\mu, 0}^2 + C_2 \left\| U \right\|_{\mu, 0}^2
\]

for any \(U \in \{S(\mathbb{R}^{n+1})\}^k\).

Proof. For sufficiently small \(\varepsilon > 0\), we take \(\{\xi_i\}, \{\varphi_j\},\) as in Section 5 and put \(H_{ij} = H(t_i, x_i, \xi_j)\). By Lemma 6.1, we have

\[
\sum_{i,j} \int_{\mathbb{R}^{n+1}_+} \left( \tau^2 + \lambda(\xi) \cdot \eta + \eta^2 \right) |\tilde{U}(\tau, \xi)|^2 d\tau d\xi \\
\leq C \left\| (R_{ij} + \eta I) U \right\|_{\mu, 0}^2
\]

for any \(U \in \{S(\mathbb{R}^{n+1})\}^k\), where \(R_{ij} = \partial/\partial t - H(t_i, x_i, \xi_j) \lambda(D_x)^m\).

Taking \(\xi_i(t, x) \varphi_j(D_x) U(t, x)\) in place of \(U(t, x)\), we have

\[
\sum_{i,j} \int_{\mathbb{R}^{n+1}_+} \left( \tau^2 + \lambda(\xi) \cdot \eta + \eta^2 \right) |\tilde{U}(\tau, \xi)|^2 d\tau d\xi \\
\leq C \left\| (R_{ij} + \eta I) \xi_i \varphi_j U \right\|_{\mu, 0}^2.
\]

Now we shall estimate various error terms to obtain (6.2). At first,

\[
\sum_{i,j} \int_{\mathbb{R}^{n+1}_+} \left( \tau^2 + \lambda(\xi) \cdot \eta + \eta^2 \right) |\tilde{U}(\tau, \xi)|^2 d\tau d\xi \\
= \sum_{i,j} \int \left| \frac{\partial}{\partial t} \left\{ \xi_i \varphi_j U(t, x) \right\} \right|^2 dt dx \\
\geq \int \left| \frac{\partial}{\partial t} U(t, x) \right|^2 dt dx - C \int \left| U(t, x) \right|^2 dt dx \\
= \int \tau^2 |\tilde{U}(\tau, \xi)|^2 d\tau d\xi - C \left\| U \right\|_{\mu, 0}^2.
\]

By the same way as in Section 5, we have

\[
\sum_{i,j} \int \lambda(\xi) \cdot \eta + \eta^2 |\tilde{U}(\tau, \xi)|^2 d\tau d\xi \\
= \sum_{i,j} \left\| \lambda(D_x)^m \{\xi_i \varphi_j U\} \right\|_{\mu, 0}^2 \\
= \text{Re}(\sum_{i,j} \Phi_j \xi_i \lambda(D_x)^m \Phi_j U, U) \\
= \text{Re}(p'(t, X, D_x) U, U),
\]

where \(p'(t, x, \xi) = \lambda(\xi) \cdot I + p'(t, x, \xi)\) and \(p'(t, x, \xi) \in S_{\mu, \alpha}^{2m-\delta_1}\).
So we get,
\[
\sum_{i,j} \int \lambda(\xi)^{2m} |\xi_{\tilde{U}}(\tau, \xi)|^2 \, d\tau d\xi
\geq \int \lambda(\xi)^{2m} |\tilde{U}(\tau, \xi)|^2 \, d\tau d\xi - C ||U||^2_{\delta, m-\delta/2}.
\]

We can see easily that
\[
\sum_{i,j} \int \eta^2 |\xi_{\tilde{U}}(\tau, \xi)|^2 \, d\tau d\xi = \eta^2 \int |\tilde{U}(\tau, \xi)|^2 \, d\tau d\xi.
\]

Now we can write,
\[
\sum_{i,j} \frac{1}{2} \Phi_j U(\tau, \xi) \xi_{\tilde{U}}(\tau, \xi) \, d\tau d\xi
\leq C \sum_{i,j} \frac{1}{2} \Phi_j U(\tau, \xi) \xi_{\tilde{U}}(\tau, \xi) \, d\tau d\xi + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + ||\Phi_j U(\tau, \xi)||^2_{\delta, 0}.
\]

Using the method as in the proof of Theorem 5.7, we have
\[
\sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} \leq 2 \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + ||\Phi_j U(\tau, \xi)||^2_{\delta, 0} + ||\Phi_j (H-H_{ij}) \lambda(D_x)^m U||^2_{\delta, 0} + ||\Phi_j J U||^2_{\delta, 0} + 2 ||J U||^2_{\delta, 0}.
\]

and we have,
\[
\sum_{i,j} ||(R_{ij}, \xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} \leq 2 \sum_{i,j} \frac{1}{2} \xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + ||\Phi_j U||^2_{\delta, 0} + ||H_{ij}(\partial_\xi D_x)^m U||^2_{\delta, 0} + \text{Constant} ||U||^2_{\delta, m-\delta/2}.
\]

Summerizing these inequalities, we have,
\[
\left\{\tau^2 + \lambda(\xi)^{2m} + \eta^2\right\} |\tilde{U}(\tau, \xi)|^2 \, d\tau d\xi
\leq C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + \text{Constant} ||U||^2_{\delta, m-\delta/2}.
\]

Hence, taking \(\varepsilon\) sufficiently small, we get,
\[
\left\{\tau^2 + \lambda(\xi)^{2m} + \eta^2\right\} |\tilde{U}(\tau, \xi)|^2 \, d\tau d\xi
\leq C ||(R+\eta I) U||^2_{\delta, 0} + C \sum_{i,j} ||\xi_{\tilde{U}}(\tau, \xi)||^2_{\delta, 0} + \text{Constant} ||U||^2_{\delta, m-\delta/2}.
\]
Thus we obtain (6.2) for some constants $C_1, C_2 > 0$. Q.E.D.

**Theorem 6.3.** For any real numbers $r$ and $s$, there exist positive constants $\eta_0$ and $c_0$ such that for any $\eta > \eta_0$, it holds that

\[(6.3) \quad c_0\|U\|_{r+m_s} \leq \|(R + \eta I)U\|_{r,s} \leq C_0\|U\|_{r+m_s} \text{ for any } U \in \{H_{r+m_s}\},\]

for some positive constant $C_0$.

Proof. The inequality $\|(R + \eta I)U\|_{r,s} \leq C_0\|U\|_{r+m_s}$ is clear. Because $\sigma(R) = i\tau I - H(t, x, \xi)\lambda(\xi)^m - J(t, x, \xi) \in S^m_{\alpha}(\tau, \xi)$, so

\[
\|(R + \eta I)U\|_{r,s} \leq \eta\|U\|_{r+m_s} + \|RU\|_{r,s} = \gamma U_{r+m_s} + \|RU\|_{r,s} 
\]

and by Corollary 3.2 (i), we can write $\lambda(D_t)^\alpha \lambda(D_t, D_x)^y R = p'(t, X, D_t, D_x)$ where $p'(t, x, \tau, \xi)\lambda(\xi)^m = \lambda(\tau, \xi)^{-r-m} \in B(R^{n+1})$.

Hence, $\|\lambda(D_x)^\alpha \lambda(D_t, D_x)^y RU\|_{r,s} = \|p'(t, X, D_t, D_x)U\|

\|C\|\|U\|_{r+m,s}.

Thus we get $\|(R + \eta I)U\|_{r,s} \leq (C + \eta\|U\|_{r+m,s}$, for any $U \in \{S(R^{n+1})\}$.

For any $U \in \{S(R^{n+1})\}$,

\[
\|(R + \eta I)U\|_{r,s}^2 = \|(\lambda(D_x)^\alpha \lambda(D_t, D_x)^y (R + \eta I)U\|_{r,s}^2, \n\]

\[
\leq \frac{1}{2} \|(R + \eta I)\lambda(D_x)^\alpha \lambda(D_t, D_x)^y (R + \eta I)U\|_{r,s}^2 
\]

\[-2\|(R, \lambda(D_x)^\alpha \lambda(D_t, D_x)^y)U\|_{r,s}^2.\]

Now from Theorem 6.2, we have

\[
\|(R + \eta I)\cdot \lambda(D_x)^\alpha \lambda(D_t, D_x)^y U\|_{r,s}^2 
\]

\[
\geq \frac{c}{2} (\tau^2 + \lambda(\xi)^m \eta^\gamma \lambda(\xi)^m \lambda(\tau, \xi)^{-r} \|U(\tau, \xi)\|^2 d\tau d\xi - C\|U\|_{r,s}^2 
\]

\[
\geq \frac{c}{2} \|U\|_{r+m,s}^2 + (\gamma^2 - C)\|U\|_{r,s}^2.\]

Using Corollary 3.4 (i), we get

\[
\|(R, \lambda(D_x)^\alpha \lambda(D_t, D_x)^y)U\|_{r,s}^2 = \|p'(t, X, D_t, D_x)U\|_{r,s}^2, \n\]

where $p'(t, x, \tau, \xi)\lambda(\xi)^m = \lambda(\tau, \xi)^{-r} \in B(R^{n+1})$.

So, $\|(R, \lambda(D_x)^\alpha \lambda(D_t, D_x)^y)U\|_{r,s}^2 \leq C\|U\|_{r,s}^2 
\]

\[
\leq \|U\|_{r,s}^2 + \|U\|_{r+m,s}^2 \leq \|U\|_{r,s}^2 + C\|U\|_{r+m,s}^2.\]

for any $\varepsilon_0 > 0$. Thus, we obtain,

\[
\|(R + \eta I)U\|_{r,s}^2 \leq \left(\frac{1}{2} C - 2\varepsilon_0\right)\|U\|_{r+m,s}^2 + \left(\frac{1}{2} \gamma^2 - C - C\varepsilon_0\right)\|U\|_{r,s}^2.\]
Taking $\varepsilon_0$ sufficiently small and $\eta_0$ such that $\frac{1}{2} \eta_0^2 - C - C_0 = 0$, we have (6.3) for any $U \in \{S(R^{n+1})\}^k$. Hence we have the theorem. Q.E.D.

Let $R^*$ be the formal adjoint operator of $R$, then we have

$$R^* = -\frac{\partial}{\partial t} - \{H \cdot \lambda(D_x)^m\}^* - J$$

$$= -\frac{\partial}{\partial t} - H^* \cdot \lambda(D_x)^m - J,$$

where $\sigma(J_i) = J_i(t, x, \xi) \in S^{m-\lambda}_0$ and $\sigma(H^*) = H(t, x, \xi)^* = \frac{\partial}{\partial \xi} H(t, x, \xi)$. In fact, by Corollary 3.2 (ii) and Corollary 3.4 (ii), we have that

$$\sigma\{\{H \cdot \lambda(D_x)^m\}^*\} = H(t, x, \xi)^* \lambda(\xi)^m \in S^{m-\lambda}_0$$

and $\sigma(J^*) = J^*(t, X, \xi) \in S^{m-\lambda}_0$.

Hence we can write,

$$R^* = -\frac{\partial}{\partial t} - H^* \cdot \lambda(D_x)^m - J.$$

Using the same way as the proof of Theorem 6.2 and Theorem 6.3, we have that for any real $r$ and $s$, there exist constant $\eta_0$ and $c_0$ such that for any $\eta > \eta_0$ it holds that

$$c_0 ||u||_{r+m,s}^2 \leq ||(R^* + \eta I) u||_{r,s} \leq C_0 ||u||_{r+m,s}$$

for any $U \in \{H_{r+m,s}\}^k$.

Using (6.3) and (6.4), we have,

**Corollary 6.4.** For any real numbers $r$ and $s$, there exists positive constant $\eta_0$ such that for any $\eta > \eta_0$, $R^* + \eta I$ is a topological isomorphism of $\{H_{r,s}\}^k$ onto $\{H_{r-m,s}\}^k$. (See Theorem 2 in [8]).

Using Theorem 5.8 and Corollary 6.4, we have

**Theorem 6.5.** For any real numbers $r$, $s$ and $a$, there exists $\eta_0$ such that for any $\eta > \eta_0$, $R^* + \eta I$ is an isomorphism of $\{H_{0,r,s}(\Omega_{a,b})\}^k$ onto $\{H_{0,m,s}(\Omega_{a,b})\}^k$.

**Theorem 6.6.** Let real numbers $r, s, a$ and $b$ satisfy $r > \frac{m}{2}$ and $-\infty < a < b < \infty$. Then the mapping $U \mapsto <RU, \gamma_a U>$ is a topological isomorphism of $\{H_{r,s}(\Omega_{a,b})\}^k$ onto $\{H_{r-m,s}(\Omega_{a,b})\}^k$.

This theorem can be shown by using Lemma, 4.3, 4.4, 4.5 and Theorem 6.5 (See [8] and [13]).

**7. Cauchy problem for operator $L$**

Let real numbers $r, s, a$ and $b$ satisfy $r > (k-1/2)m$ and $-\infty < a < b < +\infty$, and let $\Sigma = \Sigma_{a,b}$.

Then we have the following main theorems.

**Theorem 7.1.** The mapping $u \mapsto <Lu, \gamma_a u, \gamma_a \frac{\partial}{\partial t} u, \ldots, \gamma_a \left(\frac{\partial}{\partial t}\right)^k u>$
is a one to one mapping from $H_{r,s}(\Omega)$ into $H_{r-m(k-1)/2}(\Omega)$. 

Proof. We can see that

$$
(7.1) \quad \sum_{i,j} \int_a^b (N_i^j \xi_i \Phi_j \lambda(D_x)^p U, N_i^j \xi_i \Phi_j \lambda(D_x)^p U) dt 
\leq C \int_a^b \| \xi_i \Phi_j \lambda(D_x)^p U \|_p^2 dt = C \int_a^b \| U(t) \|_p^2 dt .
$$

By Theorem 5.8 and (7.1), it holds that for any $\eta > 0$,

$$
\begin{align*}
&c_1 \| U(b) \|_p^2 - c_2 \| U(a) \|_p^2 + \mu_1 \int_a^b \| U(t) \|_{p+m/2}^2 dt \\
&+ c(\eta - \mu_2) \int_a^b \| U(t) \|_p^2 dt
\end{align*}
$$

for any $U \in \{H_{r-m(k-1)/2}(\Omega)\}^k$, where $\rho = r + s - (k - 1/2)m$.

Since $-\infty < a < b < +\infty$, $e^{-\eta t} U \in \{H_{r,s}(\Omega)\}^k$ for any $U \in \{H_{r,s}(\Omega)\}^k$.

For each $u \in H_{r,s}(\Omega)$, let $U = \left( \begin{array}{c} u_j \\ u_k \end{array} \right)$ where $u_j = \lambda(D_x)^{m(k-j)} u$. Then $U \in \{H_{r-m(k-1)/2}(\Omega)\}^k$ and $RU \in \{H_{r-mk,s}(\Omega)\}^k$. In the above inequality, replacing $U$ by $e^{-\eta t} U$ and putting $Lu = f \in H_{r-mk,s}(\Omega)$, we have

$$
(7.2) \quad c_1 e^{-\eta b} \| U(b) \|_p^2 - c_2 e^{-\eta a} \| U(a) \|_p^2 \\
+ \mu_1 e^{-\eta b} \int_a^b \| U(t) \|_{p+m/2}^2 dt + c(\eta - \mu_2) e^{-\eta b} \int_a^b \| U(t) \|_p^2 dt \\
\leq \sum_{i,j} \int_a^b e^{-\eta t} (N_i^j \xi_i \Phi_j \lambda(D_x)^p U, N_i^j \xi_i \Phi_j \lambda(D_x)^p U) dt
$$

for $\eta > \mu_2$. Assume that $Lu = f = 0$. Then,

$$
\begin{align*}
&c_1 e^{-\eta b} \| U(b) \|_p^2 - c_2 e^{-\eta a} \| U(a) \|_p^2 \\
+ \mu_1 e^{-\eta b} \int_a^b \| U(t) \|_{p+m/2}^2 dt + c(\eta - \mu_2) e^{-\eta b} \int_a^b \| U(t) \|_p^2 dt \\
\leq 0 .
\end{align*}
$$

If $\gamma_a u = 0, \gamma_a \frac{\partial}{\partial t} u = 0, \cdots, \gamma_a \left( \frac{\partial}{\partial t} \right)^{k-1} u = 0$, we can see that $U(a) = 0$.

Thus we have

$$
\begin{align*}
&c_1 e^{-\eta b} \| U(b) \|_p^2 + \mu_1 e^{-\eta b} \int_a^b \| U(t) \|_{p+m/2}^2 dt \\
+ c(\eta - \mu_2) e^{-\eta b} \int_a^b \| U(t) \|_p^2 dt \leq 0 .
\end{align*}
$$

This inequality means $U = 0$ and therefore $u = 0$. Q.E.D.

**Theorem 7.2.** Under the same assumptions as Theorem 7.1, the mapping
\( u \otimes u' \otimes <Lu, \gamma_au, \gamma_a \frac{\partial}{\partial t} u, \ldots, \gamma_a \left( \frac{\partial}{\partial t} \right)^{k-1} u > \) is a topological isomorphism from 
\( H_{r,s}(\Omega) \) onto 
\( H_{r-mk,s}(\Omega) \oplus H_{r+s-m/2} \oplus H_{r+s-3m/2} \oplus \cdots \oplus H_{r+s-(k-1/2)m} \).

Proof. We denote 
\( \mathcal{L}u = <Lu, \gamma_au, \gamma_a \frac{\partial}{\partial t} u, \ldots, \gamma_a \left( \frac{\partial}{\partial t} \right)^{k-1} u > . \) By Theorem 7.1, the operator \( \mathcal{L} \) is a one to one mapping from 
\( H_{r,s}(\Omega) \) to 
\( H_{r-mk,s}(\Omega) \oplus H_{r+s-m/2} \oplus \cdots \oplus H_{r+s-(k-1/2)m} \).

So we have only to show that \( \mathcal{L} \) is an onto mapping, due to the open mapping theorem. But the fact that \( \mathcal{L} \) is onto can be shown by the same way as the proof of Theorem 8 in [3]. In this case we use the argument on Theorem 4.16 in [13], in place of Theorem 9 of [8].

Q.E.D.

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References


