ON THE FIRST MAIN THEOREM OF HOLOMORPHIC MAPPINGS FROM \( \mathbb{C}^2 \) INTO \( \mathbb{Q}^n_\lambda(\mathbb{C}) \)

YOSHIHIKO SUYAMA

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0. Introduction

Let \( f \) be a holomorphic mapping of a complex line \( \mathbb{C} \) into a complex projective space \( P_n(\mathbb{C}) \) and suppose that the image \( f(\mathbb{C}) \) is not contained in any hyperplane of \( P_n(\mathbb{C}) \). Put \( V[t] = \{ z \in \mathbb{C} : \log |z| < t \} \), and for a hyperplane \( \xi \) in \( P_n(\mathbb{C}) \) let \( n(t, \xi) \) be the number of points in \( V[t] \cap f^{-1}(\xi) \). Let \( \Omega \) be the closed form of degree 2 associated with the Fubini-Study metric on \( P_n(\mathbb{C}) \) and normalized as \( \int_{P_n^*} \Omega^n = 1 \). The counting function \( N(r, \xi) \) and the order function \( T(r) \) being defined by

\[
N(r, \xi) = \int_0^r n(t, \xi) \, dt ,
\]

\[
T(r) = \int_0^r dt \int_{\xi \in P_n} f^* \Omega
\]

respectively, the following equation is known as the First Main Theorem:

\[
N(r, \xi) + (m(r, \xi) - m(0, \xi)) = T(r) ,
\]

where \( m(r, \xi) \) is a non-negative function defined for \( r \in \mathbb{R}^+ \) and hyperplanes \( \xi \) in \( P_n(\mathbb{C}) \). The term \( (m(r, \xi) - m(0, \xi)) \) is called the compensating term. It follows from the equation (0.3) that the image \( f(\mathbb{C}) \) intersects with almost all hyperplanes in \( P_n(\mathbb{C}) \). Furthermore it is known that the number of hyperplanes in general position not intersecting with \( f(\mathbb{C}) \) is at most \( n + 1 \). These results are originally due to Ahlfors, and treated also by H. Wu [6] and S. S. Chern [1] in a modernized form.

Let \( f \) be a holomorphic mapping of \( \mathbb{C}^2 \) into a complex quadratic \( Q_{n-1}(\mathbb{C}) \) \((n \geq 3)\) satisfying certain non-degenerate conditions [§2]. We consider \( Q_{n-1}(\mathbb{C}) \) as a fixed hypersurface in \( P_n(\mathbb{C}) \). We consider a special family of \((n-2)\)-dimensional projective spaces \( P_{n-2}(\mathbb{C}) \) in \( P_n(\mathbb{C}) \) parametrized by a Grassmann manifold \( G(\mathbb{R}) \) of 2-dimensional linear spaces in \( \mathbb{R}^{n+1} \) [§1]. This family determines a family of \((n-3)\)-dimensional complex quadratic \( \xi_\alpha(\alpha \in G(\mathbb{R})) \) in \( Q_{n-1}(\mathbb{C}) \), each of whose elements is a Poincaré dual of the form \( \Omega^\alpha \) in \( Q_{n-1}(\mathbb{C}) \).
In this paper, we shall consider a value distribution problem in two complex variables with respect to the holomorphic mapping $f$ and the family $\{\xi_a\}$. The complex quadratic $\mathbb{Q}_{n-1}(\mathbb{C})$ being a double covering space of $G(R)$, we may take $\mathbb{Q}_{n-1}(\mathbb{C})$ as a parametrizing space of the family $\{\xi_a\}$ in place of $G(R)$. Thus we have a setting similar to the case of holomorphic curves (holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}_n(\mathbb{C})$). Furthermore $\Omega$ is an invariant form on $\mathbb{Q}_{n-1}(\mathbb{C})$ by a certain transformation group [§5]. This fact also plays an important role as in the case of holomorphic curves [§6].

Our main results are as follows: (1) First Main Theorem [§4], (2) the Crofton formula [§6] and (3) the Distribution theorem [§7]. In more detail, put

$$\Delta(r) = \{(z_1, z_2) \in \mathbb{C}^2 : \log|z_i| < r (i = 1, 2)\}$$

and define

$$n(\Delta(r), \alpha) = \sum_{\xi(j) \in \Delta(1), \xi(j) \in \xi_a} n(p_j, \alpha),$$

where $n(p, \alpha)$ is a certain real number [§3] such that $n(p, \alpha)=1$ if $f(\xi)$ intersects transversely with $\xi_a$ at $f(p)$. We also define the following functions:

\begin{align*}
(0.4) & \quad N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \quad \text{(counting function)} \\
(0.5) & \quad T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \quad \text{(order function)}.
\end{align*}

Then our First Main Theorem states:

\begin{align*}
(0.6) & \quad N(r, \alpha) + m(r, \alpha) - m(0, \alpha) = T(r),
\end{align*}

where $m(r, \alpha)$ is a non-negative function defined for $r \in \mathbb{R}^+$ and submainifold $\xi_a$ ($\alpha \in G(R)$) [§4]. The Crofton formula is as follows:

\begin{align*}
(0.7) & \quad \int_{\mathbb{Q}_{n-1}} n(\Delta(t), \alpha) \Omega^{n-1}(\alpha) = 2 \int_{\Delta(t)} f^* \Omega^2.
\end{align*}

Finally the distribution theorem says: The image $f(\xi)$ intersects with almost all submanifolds in $\{\xi_a\}$ ($\alpha \in G(R)$) i.e., we have $\int_W \Omega^{n-1} = 0$ for $W = \{\alpha \in \mathbb{Q}_{n-1} \mid f(\xi) \cap \xi_a = \phi\}$.

We note that W. Stoll [4], P. Griffths and J. King [2] also developed the First Main Theorem in several complex variables. But our setting is different from theirs.

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1. Preliminaries

We shall recall several basic facts about the complex projective space $\mathbb{P}_n(\mathbb{C})$
and the complex quadratic \( Q_{n-1}(C) \) (c.f. [3]), and moreover we shall define a special family of submanifolds in \( Q_{n-1}(C) \). Let \( C^{n+1} \) (resp. \( R^{n+1} \)) be the complex (resp. real) vector space of \((n+1)\) tuples of complex numbers \((z^0, \ldots, z^n)\) (resp. real numbers \((x^0, \ldots, x^n)\)). We define a symmetric bilinear form \((\ , \ )\) on \( C^{n+1} \) by
\[
(Z, W) = z^0w^0 + \cdots + z^n w^n
\]
for \( Z = (z^0, \ldots, z^n) \) and \( W = (w^0, \ldots, w^n) \). For \( Z = (z^0, \ldots, z^n) \) we put \( \bar{Z} = (\overline{z^0}, \ldots, \overline{z^n}) \), where the bar denotes the complex conjugation. A vector \( Z \in C^{n+1} - \{0\} \) is called real if \( \bar{Z} = Z \). We define a hermitian inner product \( \langle \ , \ \rangle \) on \( C^{n+1} \) by
\[
\langle Z, W \rangle = (Z, \bar{W})
\]
for \( Z, W \in C^{n+1} \). We put \( ||Z|| = \langle Z, Z \rangle^{1/2} \). For the complex projective space \( P_n(C) \) of dimension \( n \), we have the natural holomorphic fibering (called the Hopf fibering)
\[
\Pi : C^{n+1} - \{0\} \to P_n(C),
\]
where \( \Pi(Z) \) is the line passing through the origin and \( Z \). We remark that the natural conjugation \( Z \mapsto \bar{Z} \) in \( C^{n+1} - \{0\} \) induces a diffeomorphism \( z \in P_n(C) \mapsto \bar{z} \in P_n(C) \). Let \( \Omega \) be the 2-form of type \((1, 1)\) on \( C^{n+1} - \{0\} \) given by
\[
\Omega = \frac{i}{2\pi} \frac{1}{||Z||^4} \left( \sum_{j} |z^j|^2 \left( \sum_{j} dz^j \wedge \bar{dz}^j \right) - \left( \sum_{j} \bar{z}^j d\bar{z}^j \right) \wedge \left( \sum_{j} z^j dz^j \right) \right).
\]
It is well-known that there exists a unique 2-form \( \Omega \) of type \((1, 1)\) on \( P_n(C) \) such that \( \Pi^* \Omega = \bar{\Omega} \). Then \( \bar{\Omega} \) is the Kähler form associated with the Fubini-Study metric on \( P_n(C) \) and we have
\[
\int_{P_n(C)} \bar{\Omega}^n = 1.
\]
We consider a family of subspaces \( H \) of \( C^{n+1} \) such that \( H \) is of \((n-1)\)-dimension and \( \bar{Z} \in H \) whenever \( Z \in H \). With such an \( H \), we can associate uniquely a real subspace of \( R^{n+1} \) of dimension 2 by
\[
\{ X \in R^{n+1} : \langle X, H \rangle = 0 \}.
\]
We see that this gives a one to one correspondence, and hence the above family of \( H \)’s is parametrized by the Grassmann manifold \( G(R) \) of 2 planes in \( R^{n+1} \). Especially we note that \([H] = \Pi([H - \{0\}])\) is an \((n-2)\)-dimensional projective space in \( P_n(C) \).

On \( P_n(C) \) with homogeneous coordinate \( z^0, \ldots, z^n \) the complex quadratic \( Q_{n-1}(C) \) is a complex hypersurface defined by the equation
\[
(z^0)^2 + \cdots + (z^n)^2 = 0.
\]
Now the unit sphere \( S^{2n+1} \) is a principal fibre bundle over
For a point \( q \in Q_{n-1}(C) \), take a point \( Z \in S^{2n+1} \) such that \( \Pi(Z) = q \). We can write \( Z \) uniquely in the form \( Z = (X + iY)/\sqrt{2} \), where \( X \) and \( Y \) are orthonormal real vectors in \( C^{n+1} \). Conversely if \( Z = (X + iY)/\sqrt{2} \in S^{2n+1} \) for orthonormal real vectors \( X \) and \( Y \), then we have \( \Pi(Z) \in Q_{n-1}(C) \). Therefore we have

\[
S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) = \{ Z = (X + iY)/\sqrt{2} : X \text{ and } Y \text{ are orthonormal real vectors} \}.
\]

The group \( SO(n+1) \), considered as a subgroup of \( U(n+1) \), acts on \( S^{2n+1} \) and leaves the submanifold \( S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) \) invariant. Moreover \( SO(n+1) \) acts transitively on \( S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) \). The isotropy subgroup of \( SO(n+1) \) at \( Z_0 = (1/\sqrt{2}, i/\sqrt{2}, 0, \ldots, 0) \) coincides with the subgroup \( SO(n-1) \) of \( SO(n+1) \). We denote an element \( g \) of \( SO(n+1) \) by

\[
g = (X_0, X_1, \ldots, X_n),
\]

where each \( X_i \) is a column vector. Then, in the space \( SO(n+1)/SO(n-1) \), the coset including \( g = (X_0, X_1, \ldots, X_n) \) can be represented by the first two vectors \( (X_0, X_1) \). Under this identification, we have a diffeomorphism \( i: SO(n+1)/SO(n-1) \to S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) \) defined by

\[
i((X_0, X_1)) = \frac{1}{\sqrt{2}}(X_0 + iX_1).
\]

From now on we also identify \( SO(n+1)/SO(n-1) \) with \( S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) \) by the above diffeomorphism. We denote by \( \Pi_1 \) the projection: \( SO(n+1)/SO(n-1) \to Q_{n-1}(C) \) defined by

\[
\Pi_1((X_0, X_1)) = \Pi((X_0 + iX_1)/\sqrt{2})
\]

for \( (X_0, X_1) \in SO(n+1)/SO(n-1) \). Note that the space \( Q_{n-1}(C) \) also can be identified canonically with \( SO(n+1)/SO(2) \times SO(n-1) \).

To each point \( \alpha = \Pi_1((X_0, X_1)) \in Q_{n-1}(C) \), we assign the 2-dimensional linear space spanned by \( \{X_0, X_1\} \) in \( \mathbb{R}^{n+1} \). Through this assignment, \( Q_{n-1}(C) \) is a double covering space of \( G(\mathbb{R}) \). We see that the function \( |\langle Z, W \rangle|^2 \) on \( S^{2n+1} \times S^{2n+1} \) induces a function \( |\Pi(Z), \Pi(W)|^2 \) on \( P_n(C) \times P_n(C) \). For each \( \alpha \in Q_{n-1}(C) \), we consider a complex submainifold \( \xi_\alpha \) of \( Q_{n-1}(C) \), defined by

\[
\xi_\alpha = \{ \beta \in Q_{n-1}(C) : |\beta, \alpha|^2 + |\beta, \alpha|^2 = 0 \}.
\]

Let \( (X_0, X_1) \in SO(n+1)/SO(n-1) \) and set \( \Pi_1((X_0, X_1)) = \alpha \). Consider the complex subspace \( H \) of \( C^{n+1} \) orthogonal to the vectors \( X_0, X_1 \). We have \( \xi_\alpha = Q_{n-1}(C) \cap [H] \). \([H] \) is a Poincaré dual of the form \( \Omega^2 \) in \( P_n(C) \), and hence \( \xi_\alpha \) is also, in \( Q_{n-1}(C) \), a Poincaré dual of the form \( \Omega^2 \) restricted to \( Q_{n-1}(C) \). Finally we remark that each \( \xi_\alpha \) is a complex quadratic \( Q_{n-1}(C) \) and \( \xi_\alpha = \xi_\alpha^* \).
2. Holomorphic mapping

Let \( f \) be a holomorphic mapping of \( \mathbb{C}^2 \) into \( \mathbb{Q}_{n-1}(\mathbb{C}) \) \((n \geq 3)\). We consider the following two conditions on \( f \).

Condition (A): \( f \) is an immersion.
Condition (B): For each \( \alpha \in \mathbb{Q}_{n-1}(\mathbb{C}) \), the set \( \{ p \in \mathbb{C}^2 : f(p) \in \xi_n \} \) is discrete.

For each point \( p \in \mathbb{C}^2 \), we can take a small neighborhood \( U(p) \) of \( p \) such that there exists a holomorphic lift \( F=(f^0, \ldots, f^n) \) of \( f \) on \( U(p) \) into \( \mathbb{C}^{n+1}-\{0\} \) i.e., \( \Pi F=f \).

**Proposition 2.1.** Condition (A) is equivalent to the following: for each point \( p \) of \( \mathbb{C}^2 \), choose a holomorphic lift \( F=(f^0, \ldots, f^n) \) of \( f \) on a neighborhood \( U \) of \( p \), then we have

\[
\begin{pmatrix}
  f^0, & \cdots, & f^n \\
  \frac{\partial f^0}{\partial w_1}, & \cdots, & \frac{\partial f^n}{\partial w_1} \\
  \frac{\partial f^0}{\partial w_2}, & \cdots, & \frac{\partial f^n}{\partial w_2} \\
  \vdots & & \vdots \\
  \frac{\partial f^0}{\partial w_n}, & \cdots, & \frac{\partial f^n}{\partial w_n}
\end{pmatrix}
\]

\((p) = 3 \),

where \((w_1, w_2)\) is a coordinate system on the neighborhood \( U \).

**Proof.** We identify the real tangent space \( T_Z(\mathbb{C}^{n+1}) \) at a point \( Z \) in \( \mathbb{C}^{n+1} \) with \( \mathbb{C}^{n+1} \) in the usual way. For \( p \), we take \((X_0, X_1, \ldots, X_n) \in \mathrm{SO}(n+1)\) such that \((X_0+ix_1)/\sqrt{2}=(F||F||)(p)\). Then the tangent space \( T_{(X_0+ix_1)/\sqrt{2}}(\mathbb{S}^{2n+1}) \) has a basis \((i(X_0+ix_1), X_0-ix_1, i(X_0-ix_1), X_2, \ldots, X_n, ix_2, \ldots, ix_n)\). Let \( T_{f(p)} \) be the subspace spanned by \((X_2, \ldots, X_n, ix_2, \ldots, ix_n)\). The projection \( \Pi=\Pi_{1}^{\mathbb{S}^{2n+1}}(\mathbb{S}^{2n+1}(\mathbb{C})) \) induces a linear isomorphism \( \Pi_*: T_{f(p)} \to T_{f(p)}(\mathbb{Q}_{n-1}(\mathbb{C})) \) (c.f. [3] p.p. 279). Hence, \( T_{f(p)}(\mathbb{Q}_{n-1}(\mathbb{C})) \) is identified with the subspace of \( \mathbb{C}^{n+1} \) orthogonal to the vectors \((F||F||)(p)\) and \((F||F||)(p)\) with respect to \( \langle \cdot, \cdot \rangle \). Since we have \( \langle F, F \rangle=0 \) on \( U \), we see \( \langle dF, F \rangle=0 \). We have

\[
d(f ||F||) = \frac{1}{||F||} \sum_{j=1}^{2} \left[ \frac{\partial F}{\partial w_j} \frac{\partial}{\partial w_j} - \left( \frac{\partial F}{\partial w_j} , \frac{F}{||F||} \right) \frac{F}{||F||} \right] dw_j
\]

\[
\sum_{j=1}^{2} iF \frac{\partial}{\partial x_j} \left( \frac{1}{||F||} \right) dx_j - \sum_{j=1}^{2} iF \frac{\partial}{\partial x_j} \left( \frac{1}{||F||} \right) dy_j,
\]

where \( w_j=x_j+iy_j \). Therefore we get

\[
df = \sum_{j=1}^{2} \Pi_* \left[ \frac{1}{||F||} \left( \frac{\partial F}{\partial w_j} - \left( \frac{\partial F}{\partial w_j} , \frac{F}{||F||} \right) \frac{F}{||F||} \right) \right] dw_j.
\]

This shows Proposition 2.1. Q.E.D.

We define
(2.4) \[ Q_{n-3}(f(p)^\perp) = \{ \alpha \in Q_{n-1}(C) : |f(p)|^2 + |\alpha|^2 = 0 \}, \]
that is,
\[ Q_{n-3}(f(p)^\perp) = \{ \alpha \in Q_{n-1}(C) : f(p) \in \xi_\alpha \}. \]

Then \( Q_{n-3}(f(p)^\perp) \) can be identified with \( SO(n-1)/SO(2) \times SO(n-3) \) as follows: Choose an element \((X_0, X_1, \ldots, X_n) \in SO(n+1)\) such that \((X_0 + iX_1)/\sqrt{2} = (F/||F||)(p)\). Let \((A_1, A_2) \in SO(n-1)/SO(n-3)\) where \(A_i = (a_{i1}, \ldots, a_{im})^t (i = 2, 3)\). Consider the mapping
\begin{equation}
(A_2, A_3) \to \left( \sum_{i=1}^n a_{i2} X_i, \sum_{i=1}^n a_{i3} X_i \right).
\end{equation}

We see easily that this gives an identification of \( SO(n-1)/SO(2) \times SO(n-3) \) with \( Q_{n-3}(f(p)^\perp) \), which is independent of the choice of lift \( F \).

For \( \alpha \in Q_{n-3}(f(p)^\perp) \) we take \((X_0, X_1) \in SO(n+1)/SO(n-1)\) such that \( \Pi_i ((X_0, X_1)) = \alpha \). Then the following condition is independent of the choice of \((X_0, X_1)\),
\begin{equation}
\begin{align*}
&\langle \partial F/\partial w_1(p), (X_5 + iX_6)/\sqrt{2} \rangle, \langle \partial F/\partial w_2(p), (X_3 + iX_4)/\sqrt{2} \rangle, \\
&\langle \partial F/\partial w_3(p), (X_5 - iX_6)/\sqrt{2} \rangle, \langle \partial F/\partial w_4(p), (X_3 - iX_4)/\sqrt{2} \rangle \pm 0.
\end{align*}
\end{equation}

**Proposition 2.2.** The condition (2.6) holds if and only if \( f \) intersects transversely with \( \xi_\alpha \) at \( f(p) \).

Proof. Put \( (F/||F||)(p) = (X_2 + iX_3)/\sqrt{2} \). Then we take an element \((X_2, X_3, X_4, X_5, \ldots, X_n) \in SO(n+1)\). As in the proof of Proposition 2.1, we see that the tangent space \( T_{f(p)}(Q_{n-1}(C)) \) is spanned by the vectors \( X_0, iX_0, X_1, iX_1, X_2, iX_2, \ldots, X_n, iX_n \) and the tangent space \( T_{f(p)}(\xi_\alpha) \) is spanned by \( X_4, iX_4, \ldots, X_n, iX_n \) through the identification by \( \Pi_i^* : \left( S_{2n+1} \cap \Pi_i^{-1}(Q_{n-1}(C)) \right) \to T_{f(p)}(Q_{n-1}(C)) \). Therefore by (2.3) (or (2.2)) it is sufficient to show that the condition (2.6) is equivalent to rank \( k ((\partial F/\partial w_1)(p), i(\partial F/\partial w_2)(p), (\partial F/\partial w_3)(p), i(\partial F/\partial w_4)(p), X_2, iX_2, \ldots, X_n, iX_n) = 2(n+1) \). Now this can be seen easily.

Q.E.D.

Now we consider the following condition for \( \alpha = \Pi_i((X_0, X_1)) \in Q_{n-3}(f(p)^\perp) \)
\begin{equation}
\begin{align*}
&\langle \partial F/\partial w_1(p), (X_5 + iX_6)/\sqrt{2} \rangle, \langle \partial F/\partial w_2(p), (X_3 + iX_4)/\sqrt{2} \rangle, \\
&\langle \partial F/\partial w_3(p), (X_5 - iX_6)/\sqrt{2} \rangle, \langle \partial F/\partial w_4(p), (X_3 - iX_4)/\sqrt{2} \rangle = 0.
\end{align*}
\end{equation}

Since the vectors \( (\partial F/\partial w_1)(p) \) and \( (\partial F/\partial w_2)(p) \) are linearly independent, the set of elements \( \alpha \in Q_{n-3}(f(p)^\perp) \) satisfying the condition (2.7) has measure zero in \( Q_{n-3}(f(p)^\perp) \).

**Remark 1.** We shall remark here a certain sufficient condition for Condition (B). For \( w \in C \) we put \( C_2^w = \{ (x, w) : x \in C \} \) and \( C_2^w = \{ (w, z) : z \in C \} \).
Assume the following condition (C): none of \( f(C_i) \) \((i=1, 2, w \in C)\) is contained in a hyperplane in \( P_n(C) \). Let \( f(p) \in \xi_n \) and set \( \Pi_i((X_0, X_i)) = \alpha \). We put \( g_i(w_i, w_0) = \langle F, (X_0 + iX_i)\rangle \sqrt{2} \rangle \langle w_i, w_0 \rangle \) and \( g_i(w_i, w_0) = \langle F, (X_0 - iX_i)\rangle \sqrt{2} \rangle \langle w_i, w_0 \rangle \) on \( U(p) \), where \( (w_i, w_0) \) is a coordinate system on \( U(p) \) such that \( w_i(p) = 0 \) \((i=1, 2)\).

Using the Weierstrass’ preparation theorem we have the following representations

\[
\begin{align*}
g_1(w_i, w_0) &= (a_i(w_i) + a_i(w_0)w_i + \cdots + a_i(w_i)w_i^l)h_i(w_i, w_0) \\
g_2(w_i, w_0) &= (b_i(w_i) + b_i(w_0)w_i + \cdots + b_i(w_i)w_i^l)h_i(w_i, w_0)
\end{align*}
\]

where \( a_i(w_i), b_i(w_i) \) and \( h_i(w_i, w_0) \) are holomorphic such that \( a_i(0) = 0 \) for \( 0 < i < l_i \), \( a_i(0) = 0 \) for \( 0 < i < l_i \), \( b_i(0) = 0 \) and \( h_i(w_i, w_0) = 0 \) \((i=1, 2)\). We denote by \( R(w_i) \) the resultant of \( (a_i(w_i) + \cdots + a_i(w_i)w_i^l) \) and \( (b_i(w_i) + \cdots + b_i(w_i)w_i^l) \).

Since the function \( R(w_i) \) is holomorphic, we have that \( R(\beta) = 0 \) or the following (D): the set \( \{w_i: R(w_i) = 0\} \) is discrete. If, under the assumption of (C), \( f \) satisfies (D) for each \( p \in C^2 \) and \( \alpha \in Q_{n-i}(C) \) such that \( f(p) \in \xi_n \), then Condition (B) holds.

3. Certain forms on \( Q_{n-i}(C) - \xi_n \)

We define one 2-form \( \Omega_\alpha \) on \( Q_{n-i}(C) - \xi_n \) by

\[
\Omega_\alpha(\beta) = dd^c \log \left\{ |\beta|, \alpha|, |\beta| \right\} \omega
\]

where \( d^c = \frac{1}{4\pi i}(\partial - \bar{\partial}) \). We choose a unit vector \( Z_\alpha \) such that \( \Pi(Z_\alpha) = \alpha \), and define a mapping \( P_\alpha \) of \( Q_{n-i}(C) - \xi_n \) into \( P_1(C) \) by

\[
P_\alpha(\beta) = \left( 1 \left\{ |\beta|, \alpha|, |\beta| \right\} \left\langle Z_\beta, Z_\alpha \right\rangle \left\langle Z_\beta, \bar{Z}_\alpha \right\rangle \right) \]

where \( Z_\beta \in S^{2n+1} \) such that \( \Pi(Z_\beta) = \beta \), and \( \hat{\Pi} \) is the Hopf fibering \( S^2 \to P_1(C) \). \( P_\alpha \) is well-defined and holomorphic. Let \( \omega \) be the Kahler 2-form associated with the Fubini-Study metric on \( P_1(C) \) and normalized as \( \int_{P_1(C)} \omega = 1 \). Then \( P_\alpha \omega \) is independent of the choice of \( Z_\alpha \). From now on we also denote by \( \Omega \) the restriction of the form \( \Omega \) to \( Q_{n-i}(C) \).

**Lemma 3.1.** We have

\[
\Omega_\alpha = P_\alpha^* \omega - \Omega \quad \text{on} \quad Q_{n-i}(C) - \xi_n.
\]

**Proof.** Let \( \sigma \) be a local holomorphic cross-section of the Hopf fibering \( \Pi: C^{n+1} - \{0\} \to P_n(C) \) defined on an open set \( U \) in \( Q_{n-i}(C) - \xi_n \). Then we have

\[
\Omega_\alpha = dd^c \log \left\{ \left\langle \sigma, Z_\alpha \right\rangle, |\sigma| \right\} + \left\langle \sigma, \bar{Z}_\alpha \right\rangle \right\} \right\}
\]

\[
= dd^c \log \left\{ \left| \left\langle \sigma, Z_\alpha \right\rangle \right|^2 + \left\langle \sigma, \bar{Z}_\alpha \right\rangle \right\} - dd^c \log |\sigma| \right\} \right\}
\]

\[
= P_\alpha^* \omega - \Omega.
\]

Q.E.D.
We define another 2-form $\Omega'_a$ on $Q_{n-1}(C)_-\xi_a$ by

$$\Omega'_a = \Omega + P^*_a \omega \quad \text{on} \quad Q_{n-1}(C)_-\xi_a.$$  

Put

$$\Omega''_a = -\Omega_a \wedge \Omega'_a \quad \text{on} \quad Q_{n-1}(C)_-\xi_a.$$  

By (3.3) and (3.4), we have

$$\Omega''_a = (\Omega - P^*_a \omega) \wedge (\Omega + P^*_a \omega)$$

$$= \Omega^2 - P^*_a (\omega \wedge \omega) = \Omega^2 \quad \text{on} \quad Q_{n-1}(C)_-\xi_a.$$  

Let $f: C^2 \rightarrow Q_{n-1}(C) (n \geq 3)$ be a holomorphic mapping satisfying Conditions (A) and (B) in §2. For a point $p$ in $C^2$, we take a small neighborhood $U(p)$ of $p$ and a coordinate system $(w_1, w_2)$ on it satisfying $w_i(p)=0$ $(i=1, 2)$. Let $F$ be a holomorphic lift of $f$ on $U(p)$ into $C^{n+1}-\{0\}$. Set $f(p) \in \xi_a$. Then we define a real number $n(p, \alpha)$ by

$$n(p, \alpha) = \lim_{t \to 0} \int_{U_t(p)} d^c \log \{ |<F, Z_a>|^2 + |<F, \tilde{Z}_a>|^2 \} \wedge f^*P^*_a \omega,$$

where $U_t(p) = \{(w_1, w_2) \in U(p): |w_1|^2 + |w_2|^2 < \varepsilon^2 \}$ and $\Pi(Z_a) = \alpha$.

**Lemma 3.2.** $n(p, \alpha)$ is well-defined and finite. Especially if $f$ intersects transversely with $\xi_a$ at $f(p)$, then we have $n(p, \alpha) = 1$.

**Proof.** First we choose a local lift $F$ and a local coordinate system $(w_1, w_2)$ such that $w_i(p)=0$. Take two positive real numbers $\varepsilon_1$ and $\varepsilon_2$ such that $U(p) \supset U_{\varepsilon_1}(p) \supset U_{\varepsilon_2}(p)$. Then we have

$$0 = \int_{U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)} f^*P^*_a (\omega \wedge \omega)$$

$$= \int_{\partial U_{\varepsilon_1}(p) - \partial U_{\varepsilon_2}(p)} d^c \log \{ |<F, Z_a>|^2 + |<F, \tilde{Z}_a>|^2 \} \wedge f^*P^*_a \omega.$$  

Therefore we obtain

$$\int_{\partial U_{\varepsilon_1}(p)} d^c \log \{ |<F, Z_a>|^2 + |<F, \tilde{Z}_a>|^2 \} \wedge f^*P^*_a \omega$$

$$= \lim_{t \to 0} \int_{U_t(p)} d^c \log \{ |<F, Z_a>|^2 + |<F, \tilde{Z}_a>|^2 \} \wedge f^*P^*_a \omega.$$  

The left hand-side of the equation (3.8) is finite and hence so is the right side. In the same way, we see that $n(p, \alpha)$ is independent of the choice of a local coordinate system. Now we shall show that $n(p, \alpha)$ is independent of the choice of $F$. Take two holomorphic lift $F_1$ and $F_2$ of $f$. Then there exists a holomorphic function $g$ such that $F_1 = gF_2$ and $g(q) \equiv 0$ at any $q \in U(p)$. We have
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(3.9) \[ d^c \log \{ \langle F_1, Z_a \rangle^2 + \langle F_2, Z_a \rangle^2 \} \]
\[ = d^c \log |g|^2 + d^c \log \{ |\langle F_1, Z_a \rangle|^2 + |\langle F_2, Z_a \rangle|^2 \} \]
\[ = \frac{1}{4 \pi i} [d \log g - d \log g] + d^c \log \{ |\langle F_1, Z_a \rangle|^2 + |\langle F_2, Z_a \rangle|^2 \} . \]

Since the form \( f^*P_a^* \omega \) is closed on \( \partial U_\epsilon(p) \), \( n(p, \alpha) \) is independent of the choice of \( F \).

Next suppose that \( f \) intersects transversely with \( \xi_a \) at \( f(p) \). Then
\[ \langle \partial F/\partial w_1, Z_a \rangle, \langle \partial F/\partial w_2, Z_a \rangle \quad (p \neq 0), \]
and hence we can choose \((w_1, w_2) = (\langle F, Z_a \rangle, \langle F, Z_a \rangle)\) as a coordinate system on \( U(p) \). We have
\[ n(p, \alpha) = \lim_{r \to 0} \int_{s_1^2 + s_2^2 = s^2} d^c \log (|w_1|^2 + |w_2|^2) \wedge f^*P_a^* \omega . \]

Putting \( w_1 = r_1 e^{i \theta_1}, w_2 = r_2 e^{i \theta_2}, r_1 = r \cos t \) and \( r_2 = r \sin t \), we have
\[ d^c \log (r_1^2 + r_2^2) = \frac{1}{2 \pi} \frac{1}{r_1^2 + r_2^2} (r_1^2 d \theta_1 + r_2^2 d \theta_2) , \]
and
\[ f^*P_a^* \omega = \frac{1}{\pi} \frac{1}{(r_1^2 + r_2^2)} (r_1^2 d r_1 \wedge d \theta_1 + r_2^2 d r_2 \wedge d \theta_2 ) \]
\[ - r_1 r_2 (d r_1 \wedge d \theta_2 - d r_2 \wedge d \theta_1) . \]

Thus we see
\[ d^c \log (r_1^2 + r_2^2) \wedge f^*P_a^* \omega = \frac{1}{2 \pi^2} \sin t \cos t \ d \theta_1 \wedge dt \wedge d \theta_2 \]
on \( r = \text{constant} \).

On the sphere \{\((w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 = r^2\}\), \( d \theta_1 \wedge dt \wedge d \theta_2 \) is a positive form. Therefore we have \( n(p, \alpha) = 1 \).

We denote by \((z_1, z_2)\) the standard coordinate system on \( C^2 \). Put \( \Delta(r) = \{ (z_1, z_2) \in C^2 : |z_i| < r (i = 1, 2) \} \).

**Theorem 1.** Let \( f: C^2 \to Q_{n-1}(C) (n \geq 3) \) be a holomorphic mapping satisfying \((A)\) and \((B)\). Suppose \( f(\partial \Delta(r)) \cap \xi_a = \phi \). Then we have

\[ \int_{\Delta(r)} f^* \Omega^2 = n(\Delta(r), \alpha) + \int_{\partial \Delta(r)} d^c [- \log (|f| \alpha^2 + |f| \alpha^2) f^*(\Omega^* + P_a^* \omega)] , \]
where \( n(\Delta(r), \alpha) = \sum_{f(\xi_i) \in \xi_a, \beta_j \in \Delta(r)} n(p_i, \alpha) \).
Proof. By (3.1), Lemma 3.1, (3.5) and (3.5)', we have

\begin{equation}
\int_{\Delta(r)} f^*\Omega^2 = \lim_{\varepsilon \to 0} \int_{\Delta(r) - \sum_{i} U_{\varepsilon}(p_i)} f^*\Omega^2
\end{equation}

\begin{align*}
= \lim_{\varepsilon \to 0} \int_{\Delta(r) - \sum_{i} U_{\varepsilon}(p_i)} -dd^c \cdot \log(|f, \alpha|^2 + |f, \alpha|^2) \wedge f^*(\Omega + P^*_\alpha \omega)
\end{align*}

\begin{align*}
= \lim_{\varepsilon \to 0} \int_{\Delta(r) - \sum_{i} U_{\varepsilon}(p_i)} dd^c \cdot \left[-\log(|f, \alpha|^2 + |f, \alpha|^2) f^*(\Omega + P^*_\alpha \omega)\right],
\end{align*}

where \( U_{\varepsilon}(p_i) \) is such a neighborhood of \( p_i \) as given in the definition \( n(p_i, \alpha) \).

Applying Stokes Theorem to the equation (3.11), we have

\begin{equation}
\int_{\Delta(r)} f^*\Omega^2 = \int_{\partial\Delta(r)} d^c \cdot \left[-\log(|f, \alpha|^2 + |f, \alpha|^2) f^*(\Omega + P^*_\alpha \omega)\right]

- \lim_{\varepsilon \to 0} \sum_{i} \int_{\partial U_{\varepsilon}(p_i)} d^c \cdot \left[\log||F_i||^2 f^*(\Omega + P^*_\alpha \omega)\right]

+ \lim_{\varepsilon \to 0} \sum_{i} \int_{\partial U_{\varepsilon}(p_i)} d^c \cdot \left[\log\{||F_i, Z_\alpha||^2 + ||F_i, \bar{Z}_\alpha||^2\} f^*\Omega\right]

+ \sum_{i} n(p_i, \alpha),
\end{equation}

where \( F_i \) is a holomorphic lift of \( f \) on \( U(p_i) \). We have

\begin{equation}
\lim_{\varepsilon \to 0} \int_{\partial U_{\varepsilon}(p_i)} d^c \cdot \left[\log||F_i||^2 f^*\Omega\right] = \lim_{\varepsilon \to 0} \int_{\partial U_{\varepsilon}(p_i)} f^*\Omega^2 = 0.
\end{equation}

Set \( r^2 = |w^1|^2 + |w^2|^2 \), where \( (w^1, w^2) \) denotes a coordinate system on \( U(p_i) \), we see

\begin{equation}
d^c \log\{||F_i, Z_\alpha||^2 + ||F_i, \bar{Z}_\alpha||^2\} = 0\left(\frac{1}{r^2}\right)(dw^1_i + dw^2_i + dw^1_i + dw^2_i)
\end{equation}

and

\begin{equation}
dd^c \log\{||F_i, Z_\alpha||^2 + ||F_i, \bar{Z}_\alpha||^2\} = 0\left(\frac{1}{r^2}\right)(dw^1_i \wedge dw^1_i + dw^1_i \wedge dw^1_i + dw^2_i \wedge dw^2_i + dw^2_i \wedge dw^2_i).
\end{equation}

Since \( ||F_i|| \) is positive on \( U(p_i) \), we have

\begin{equation}
d^c \log||F_i||^2 = 0(1)(dw^1_i + dw^1_i + dw^2_i + dw^2_i)
\end{equation}

and

\begin{equation}
f^*\Omega = 0(1)(dw^1_i \wedge dw^1_i + dw^2_i \wedge dw^2_i + dw^2_i \wedge dw^2_i \wedge dw^1_i).
\end{equation}

Since the both sides of the equation (3.8) are finite, comparing (3.14) and (3.15) with (3.16) and (3.17), we have

\begin{equation}
\lim_{\varepsilon \to 0} \int_{\partial U_{\varepsilon}(p_i)} d^c \cdot \left[\log||F_i||^2 f^*P^*_\alpha \omega\right] = 0
\end{equation}
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4. First Main Theorem

Let \( f : C^2 \to Q_{n-1}(C) \) be a holomorphic mapping satisfying (A) and (B). For a point \( \alpha \in Q_{n-1}(C) \), we choose two real numbers \( r_1 \) and \( r_2 \) such that \( r_1 > r_2 \), and the image \( f((r_1, 0) \setminus (r_2, 0)) \) does not intersect with \( \xi \).

We see easily \( |f, \alpha|^2 + |\alpha|^2 \leq 1 \) for \( \beta \in \Omega \). Hence \( \psi = -\log (|f, \alpha|^2 + |\alpha|^2) f*(\Omega + P\omega) \) is a positive form (non-negative form, precisely) on \( \Delta(r_1) \setminus \Delta(r_2) \). Putting \( z_j = e^{x_j + iy_j} (j = 1, 2) \), we can write \( \psi \) on \( \Delta(r_1) \setminus (\Delta(r_2) \cup \{(z, 0) \in C^2 \} \cup \{(0, z) \in C^2 \}) \) as follows:

\[
(4.1) \quad \psi = -\log (|f, \alpha|^2 + |\alpha|^2) f*(\Omega + P\omega) = \psi_1 + \psi_2 + \psi_3.
\]

Remark 2. If we write \( \psi \) with the standard coordinate system \((z_1, z_2)\) on \( C^2 \), we see \( \psi_1(z_1, z_2) = \psi_1(z_1, z_2) e^{x_1 + iy_1}, \psi_2(z_1, z_2) = \psi_2(z_1, z_2) e^{x_2 + iy_2} \) and \( \psi_3(z_1, z_2) = e^{x_1 + iy_1} e^{x_2 + iy_2} \) for certain functions \( \psi_i \) \((i = 1, 2, \ldots, 6)\).

Lemma 4.1. We have

\[
(4.2) \quad \psi_1 \geq 0, \psi_2 \geq 0 \quad \text{and} \quad \psi = \psi_3.
\]

Proof. Choosing a holomorphic lift \( f \) on a sufficiently small open set \( U \) in \( \Delta(r_1) \setminus \Delta(r_2) \), we have

\[
(4.3) \quad f* (\Omega + P\omega) = dd^c [\log ||f||^2 + \log (|f, Z_\alpha|^2 + |f, Z_\beta|^2)],
\]

where \( \Pi(Z_\alpha) = \alpha \). Now we obtain

\[
\begin{align*}
(4.4) \quad d &= \sqrt[2]{\sum_{j=1}^{2} \left[ \frac{\partial}{\partial s_j} \frac{\partial}{\partial \theta_j} \right] } \quad \text{on} \ U \setminus \{(0, z) \in C^2 \} \cup \{(z, 0) \in C^2 \},
\end{align*}
\]

where \((e^{x_1 + iy_1}, e^{x_2 + iy_2})\) is the restriction to \( U \) of the standard coordinate system in \( C^2 \). Putting \( g = -\log (|f, Z_\alpha|^2 + |f, Z_\beta|^2) + \log ||f||^2 \), we have

\[
(4.5) \quad dd^c g = \frac{1}{4\pi} \left[ \left( \frac{\partial^2 g}{\partial \theta_j^2} + \frac{\partial^2 g}{\partial s_j^2} \right) ds_j \wedge d\theta_j + \left( \frac{\partial^2 g}{\partial \theta_j \partial \theta_i} + \frac{\partial^2 g}{\partial s_j \partial s_i} \right) ds_j \wedge d\theta_i \right]
\]

Comparing (4.1) with (4.5), we have \( \psi = \psi_3 \).
We shall show $\psi, \psi^0 \geq 0$ and $\psi, \psi^0 \geq 0$.

(4.6) \[ dd^c \log(\sum_j f^j f^j) = \frac{i}{2\pi} \partial \overline{\partial} \log(\sum_j f^j f^j) \]

\[ = \frac{i}{2\pi} \frac{1}{||F||^2} \left[ \left( \sum_j df^j \wedge df^j \right) - \left( \sum_k f^k f^k \right) \wedge \left( \sum_j f^j f^j \right) \right] \]

\[ = \frac{i}{2\pi} \frac{1}{||F||^2} \left[ \left( \frac{\partial F}{\partial z_j} \right)^2 - \left( \frac{\partial F}{\partial z_j} , F \right)^2 \right] dz_j \wedge dz_j \]

\[ + \left( \frac{\partial F}{\partial z_j} \right)^2 - \left( \frac{\partial F}{\partial z_j} , F \right)^2 \right] dz_j \wedge dz_j + \cdots \]

where $F=(f^0, f^1, \ldots, f^n)$. By the Schwartz inequality and the linear independence of vectors $F$ and $\partial F/\partial z_j (j=1, 2)$, we have

\[ ||F||^2 \left| \frac{\partial F}{\partial z_j} \right|^2 \leq \left( \frac{\partial F}{\partial z_j} , F \right)^2, \text{ and } dz_j \wedge dz_j = e^{2s_j}(-2 ds_j \wedge d\theta_j) \]

($j=1, 2$). Thus we have

\[ \frac{1}{\pi} \frac{1}{||F||^2} \left[ \left( \frac{\partial F}{\partial z_j} , F \right)^2 \right] e^{2s_j} > 0 (j = 1, 2) \]

or

(4.7) \[ \frac{1}{\pi} \left( \sum_k f^k f^k \right) \left[ \left( \sum_k \frac{\partial f^k}{\partial z_j} \right)^2 - \left( \sum_k \frac{\partial f^k}{\partial z_j} , f^k \right)^2 \right] e^{2s_j} > 0 (j = 1, 2). \]

As for $dd^c[\log(|\langle F, Z_\alpha \rangle|^2 + |\langle F, Z_\alpha \rangle|^2)]$, putting $f^0=\langle F, Z_\alpha \rangle$, $f^i=\langle F, F_\alpha \rangle$ and $f^i=0 (j=2, \ldots, n)$ in the equation (4.6), we have also the inequality (4.7) (in this case we replace $>$ by $\geq 0$) with respect to the coefficient of $ds_j \wedge d\theta_j (j=1, 2)$.

Q.E.D.

Let $r$ be in $[r_2, r_1]$. We divide $\partial \Delta(r)$ into $\partial \Delta_1(r)$ and $\partial \Delta_2(r)$, where

(4.8) \[ \partial \Delta_i(r) = \{ (z_1, z_2) \in \partial \Delta(r): \log |z_i| = r \} (i = 1, 2). \]

Lemma 4.2. We have

(4.9) \[ \int_{\partial \Delta(r)} d^c \psi = \frac{1}{4\pi} \left[ -\int_{S^1 \times S^1} \psi_1(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right. \]

\[ - \int_{S^1 \times S^1} \psi_1(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \]

\[ + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[ \int_{\partial \Delta_1(r)} \psi_y \wedge d\theta_1 + \int_{\partial \Delta_2(r)} \psi_y \wedge d\theta_2 \right] \]

Proof. First we remark that $d\theta_1 \wedge ds_2 \wedge d\theta_2$ and $d\theta_2 \wedge ds_1 \wedge d\theta_1$ are positive forms on $\partial \Delta_1(r)$ and $\partial \Delta_2(r)$ respectively.
By (4.1) and the preceding remark 2, we have

\[
\int_{\partial \Delta_1(r)} d^c \psi_a = \int_{\partial \Delta_1(r) \cap \{e^{i \theta_1}, 0\}} d^c \psi_a
\]

\[
= \frac{1}{4\pi} \int_{\partial \Delta_1(r) \cap \{e^{i \theta_1}, 0\}} \left[ -\frac{\partial \psi_3}{\partial s_2} + \frac{\partial \psi_4}{\partial s_1} + \frac{\partial \psi_5}{\partial \theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2
\]

\[
= \frac{1}{4\pi} \int_{\partial \Delta_1(r)} \left[ -\frac{\partial \psi_3}{\partial s_2} + \frac{\partial \psi_4}{\partial s_1} + \frac{\partial \psi_5}{\partial \theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2.
\]

Clearly we have

\[
\int_{\partial \Delta_1(r)} \frac{\partial \psi_3}{\partial \theta_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 = 0.
\]

Therefore we obtain

\[
(4.10) \quad \int_{\partial \Delta_1(r)} d^c \psi_a = \frac{1}{4\pi} \int_{\partial \Delta_1(r)} \left[ -\frac{\partial \psi_3}{\partial s_2} + \frac{\partial \psi_4}{\partial s_1} + \frac{\partial \psi_5}{\partial \theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2.
\]

Similarly we obtain

\[
(4.11) \quad \int_{\partial \Delta_2(r)} d^c \psi_a = \frac{1}{4\pi} \int_{\partial \Delta_2(r)} \left[ \frac{\partial \psi_3}{\partial s_2} - \frac{\partial \psi_4}{\partial s_1} \right] d\theta_2 \wedge ds_1 \wedge d\theta_1.
\]

Now we shall consider the equation (4.10). We have

\[
(4.12) \quad \frac{1}{4\pi} \int_{\partial \Delta_1(r)} \frac{\partial \psi_3}{\partial s_2} d\theta_1 \wedge ds_2 \wedge d\theta_2
\]

\[
= \frac{1}{4\pi} \int_{\partial \Delta_1(r)} d(\psi_3 d\theta_2 \wedge d\theta_1)
\]

\[
= \frac{1}{4\pi} \int_{\partial \Delta_1(r) \cap \partial \Delta_2(r)} \psi d\theta_2 \wedge d\theta_1
\]

\[
= \frac{1}{4\pi} \int_{S^1 \times S^1} \psi_3(e^{i \theta_1}, e^{i \theta_2}) d\theta_2 \wedge d\theta_1.
\]

Since we have

\[
\int_{\partial \Delta_1(r)} \psi d\theta_1 \wedge ds_2 \wedge d\theta_2
\]

\[
= \int_{\partial \Delta_1(r)} \left\{ \left( \int_{-\infty}^{r} \psi_3(e^{i \theta_1}, e^{i \theta_2}) dt \right) d\theta_2 \wedge d\theta_1 \right\}
\]

\[
= \int_{S^1 \times S^1} \left( \int_{-\infty}^{r} \psi_3(e^{i \theta_1}, e^{i \theta_2}) dt \right) d\theta_2 \wedge d\theta_1,
\]

we obtain
By (4.10), (4.12) and (4.13), we obtain

\[(4.14)\]
\[
\int_{\Delta(x)} d\psi_a = \frac{1}{4\pi} \int_{S^1 \times S^1} [\psi_3 - \psi_1](e^{r+i\alpha_1}, e^{r+i\alpha_2}) d\theta_2 \wedge d\theta_1
\]
\[
+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\Delta(x)} \psi_a d\theta_1 \wedge ds_1 \wedge d\theta_2.
\]

By the similar argument as we derived (4.14) from (4.10), we derive the following from (4.11)

\[(4.15)\]
\[
\frac{1}{4\pi} \int_{\Delta(x)} d\psi_a = \frac{1}{4\pi} \int_{S^1 \times S^1} [\psi_3 - \psi_1](e^{r+i\alpha_1}, e^{r+i\alpha_2}) d\theta_1 \wedge d\theta_2
\]
\[
+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\Delta(x)} \psi_a d\theta_2 \wedge ds_1 \wedge d\theta_1.
\]

By (4.14), (4.15) and the definition of $\psi_a$ we obtain (4.9). Q.E.D.

**Lemma 4.3.** We have

\[(4.16)\]
\[
\int_{\Delta(x)} f^\ast \Omega^2 = \frac{1}{4\pi} \frac{\partial}{\partial r} \left[ \int_{\Delta(x)} \psi_a \wedge d\theta_1 + \int_{\Delta(x)} \psi_a \wedge d\theta_2 \right] + n(\Delta(x), \alpha).
\]

**Proof.** By Theorem 1 and Lemma 4.2, we have only to prove that

\[
\frac{1}{4\pi} \int_{S^1 \times S^1} [\psi_3 - \psi_1](e^{r+i\alpha_1}, e^{r+i\alpha_2}) d\theta_2 \wedge d\theta_1 = 0.
\]

We define a mapping $h: C^2 \to C^2$ by $h((z_1, z_2)) = (z_2, z_1)$. Then $(f \circ h)$ satisfies Conditions (A) and (B), and we have

\[
(f \circ h, \alpha^2 + |f \circ h, \alpha|^2)(z_1, z_2) = (|f, \alpha|^2 + |f, \alpha|)(z_2, z_1)
\]

and

\[
n_f((z_1, z_2), \alpha) = \lim_{\Delta(x) \to (z_1, z_2)} \frac{d}{dx} \log \left[ \langle F, Z_{\alpha} \rangle + |\langle F, Z_{\alpha} \rangle|^2 \right] \wedge f^\ast \Omega^2
\]
\[
= \lim_{\Delta(x) \to (z_1, z_2)} \frac{d}{dx} \log \left[ \langle F \circ h, Z_{\alpha} \rangle + |\langle F \circ h, Z_{\alpha} \rangle|^2 \right] \wedge (f \circ h)^\ast \Omega^2
\]
\[
= n_{f \circ h}((z_2, z_1), \alpha).
\]
On the other hand, we have from (4.1)

\begin{equation}
(4.17) \quad (h^* \psi_a) = \psi_1 \circ h \, ds_2 \wedge d\theta_2 + \psi_2 \circ h \, ds_2 \wedge d\theta_1 + \psi_3 \circ h \, ds_1 \wedge d\theta_2 \\
+ \psi_4 \circ h \, ds_1 \wedge d\theta_1 + \psi_5 \circ h \, ds_2 \wedge d\theta_2 + \psi_6 \circ h \, ds_2 \wedge ds_1.
\end{equation}

By Theorem 1, (4.14) and (4.15) in Lemma 4.2, comparing (4.1) with (4.17) we have

\begin{equation}
(4.18) \quad \int_{\Delta(r)} f^* \Omega^2 = \int_{\Delta(r)} h^* f^* \Omega^2 = n(\Delta(r), \alpha)
\end{equation}

By Theorem 1, (4.14) and (4.15) in Lemma 4.2, comparing (4.1) with (4.17) we have

\begin{equation}
\psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 = \psi_1 \circ h \, d\theta_2 \wedge ds_2 \wedge d\theta_1
\end{equation}

We see easily

\begin{equation}
\int_{\Delta(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 = \int_{\Delta(r)} \psi_1 \circ h \, d\theta_2 \wedge ds_2 \wedge d\theta_1
\end{equation}

and

\begin{equation}
\int_{\Delta(r)} \psi_1 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 = \int_{\Delta(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_1 \wedge d\theta_2
\end{equation}

Therefore we have only to prove

\begin{equation}
\int_{S^1 \times S^1} ((\psi_1 \circ h - \psi_i)(e^{r+i\alpha}, e^{r+i\beta}) d\theta_1 \wedge d\theta_2 = 0 \quad (i = 1, 4).
\end{equation}

For any \(\alpha, \beta \in [0, 2\pi]\), we have

\begin{equation}
((\psi_1 \circ h - \psi_i)(e^{r+i\alpha}, e^{r+i\beta}) = \psi_1(e^{r+i\beta}, e^{r+i\alpha}) - \psi_i(e^{r+i\alpha}, e^{r+i\beta})
\end{equation}

\begin{equation}
((\psi_1 \circ h - \psi_i)(e^{r+i\beta}, e^{r+i\alpha}) = \psi_1(e^{r+i\alpha}, e^{r+i\beta}) - \psi_i(e^{r+i\beta}, e^{r+i\alpha})
\end{equation}

Thus we obtain

\begin{equation}
((\psi_1 \circ h - \psi_i)(e^{r+i\alpha}, e^{r+i\beta}) = -((\psi_1 \circ h - \psi_i)(e^{r+i\beta}, e^{r+i\alpha})
\end{equation}

Q.E.D.

For the holomorphic mapping \(f: \mathbb{C}^2 \to Q_{n-1}(\mathbb{C})(n \geq 3)\) satisfying Conditions (A) and (B), we put

\[ T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \quad \text{(order function)} \]
(4.19) \[ N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \] (counting function)
\[ m(r, \alpha) = \frac{1}{4\pi} \left[ \int_{\Delta_1(r)} \psi_{a} \wedge d\theta_1 + \int_{\Delta_2(r)} \psi_{a} \wedge d\theta_2 \right]. \]

We need the following lemma, which can be proved in a similar way as ([5] p.p. 502).

**Lemma 4.4.** For any \( \alpha \), \( m(r, \alpha) \) is continuous with respect to \( r \in [0, \infty) \).

**Theorem 2.** We have
\[ T(r) = m(r, \alpha) - m(0, \alpha) + N(r, \alpha) \] for any \( r \geq 0 \), and \( m(r, \alpha) \) is non-negative.

Proof. Integrating the equation in Lemma 4.3 with respect to \( r \in [r_2, r_1] \), we have
\[ \int_{r_2}^{r_1} dr \int_{\Delta(r)} f^* \Omega = \int_{r_2}^{r_1} n(\Delta(r), \alpha) dr + m(r_1, \alpha) - m(r_2, \alpha). \]
By Lemma 4.4 we obtain the equation (4.20). It follows from Lemma 4.1 and Lemma 4.4 that the function \( m(r, \alpha) \) is non-negative. Q.E.D.

**Lemma 4.5.** For any \( r \), \( m(r, \alpha) \) is continuous with respect to \( \alpha \in Q_{n-1}(C) \).

We also omit this proof by the same reason as in Lemma 4.4. (c.f. [5] p.p. 504).

**Theorem 3.** There exists a positive constant \( C \) satisfying
\[ T(r) + C > N(r, \alpha) \] whenever \( r \geq 0 \) and \( \alpha \in Q_{n-1}(C) \).

Proof. By Theorem 2 we have
\[ T(r) + m(0, \alpha) \geq N(r, \alpha) \] for any \( r \geq 0 \).
Therefore by Lemma 4.5 we have the equation (4.21). Q.E.D.

5. Induced form by \( f \)

We denote by \((X_0, X_1, \ldots, X_n)\) an element of \( SO(n+1) \), where \( X_i \)'s(0 ≤ i ≤ n) are column vectors, and we put \( X_i = (x_{i,0}, \ldots, x_{i,n})^t \). The left invariant forms \( \theta_{ij} \) (0 ≤ i, j ≤ n) on \( SO(n+1) \) are defined by the following equation:
\[ \left( \begin{array}{c} dX_0 \\ dX_1 \\ \vdots \\ dX_n \end{array} \right) (X_0, \ldots, X_n) = \left( \begin{array}{c} X_0^t \\ X_1^t \\ \vdots \\ X_n^t \end{array} \right) \left( \begin{array}{c} dX_0 \\ dX_1 \\ \vdots \\ dX_n \end{array} \right) = \left( \begin{array}{c} 0, \theta_{10}, \ldots, \theta_{1m} \\ \vdots \\ \theta_{n0}, \ldots, 0 \end{array} \right), \]
where \( \theta_{ij} = -\theta_{ji} \).

Therefore we have 
\[
\langle dX_i, X_j \rangle = \theta_{ij} \quad \text{i.e.,}
\]
(5.2)
\[
dX_i = \sum_j \theta_{ij} X_j.
\]
Taking its exterior derivative, we see
(5.3)
\[
d\theta = \sum_k \theta_{ik} \wedge \theta_{kl} = -\sum_k \theta_{ik} \wedge \theta_{lk}.
\]
We remark that \( d\theta \) is a 2-form on \( SO(n+1)/SO(n-1) \). Furthermore it is a lift of a 2-form on \( Q_{n-1}(C) \) by \( \Pi_i \). In fact, let \( U \) be an open neighborhood of \( Q_{n-1}(C) \), and \((X_0, X_1)\) be a local cross-section of \( U \) into \( SO(n+1)/SO(n-1) \): \( \Pi_i (\langle X_0, X_1 \rangle) = \)identity on \( U \). We have
(5.4)
\[
\Pi^{-1}(\Pi_i, (X_0, X_1)) = \{(X_0, X_1) \mid \cos \theta, -\sin \theta \colon 0 \leq \theta < 2\pi \}. 
\]
Then we have on \( \Pi^{-1}(U) \),
(5.5)
\[
d\theta_{01} = \langle d(\cos \theta \cdot X_0 + \sin \theta \cdot X_1), (\sin \theta \cdot X_1 + \cos \theta \cdot X_0) \rangle = d\langle dX_0, X_1 \rangle = d\langle dX_0, X_1 \rangle.
\]
Let \( \sigma \) be a local holomorphic cross-section on \( U \) into \( \mathbb{C}^{n+1} \) \( \setminus \{0\} \) with respect to the Hopf fibering: \( \Pi \sigma = \)identity on \( U \). We can write \( \sigma \) in the form \( \sigma = X + iY \) for orthogonal real vectors \( X \) and \( Y \) at each point of \( U \). Then we see
(5.6)
\[
\Omega = d\omega \log \|\sigma\|^2 = -\frac{1}{2\pi} d\langle d(X/\|X\|), Y/\|Y\| \rangle.
\]
Thus, \( d\theta_{01} \) is the lift of \(-2\pi \Omega \) by \( \Pi^* \) i.e.,
(5.7)
\[
\Pi^* \Omega = -\frac{1}{2\pi} d\theta_{01}.
\]

In the equation (5.1) we defined \( \theta_{ij} \)'s and \( \theta_{ij} \)'s \((0 \leq j \leq n)\) as 1-forms on \( SO(n+1) \). They are also regarded as 1-forms on \( SO(n+1)/SO(n-1) \). To prove this fact we shall identify \( SO(n+1)/SO(n-1) \) with \( S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) \). We take a local coordinate \( x = (x^1, \cdots, x^{2n-1}) \) on a small open set \( U \) in \( S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)) \) and write a point \( Z(x) \) of \( U \) in the form \( (X_0(x) + iX_1(x))/\sqrt{2} \), where \( \langle X_0, X_0 \rangle = \langle X_1, X_1 \rangle = 1 \) and \( \langle X_0, X_1 \rangle = 0 \). For each \( x \), extending \( X_0(x) \) and \( X_1(x) \), we take a real orthonormal basis \( X_0(x), \cdots, X_n(x) \) in \( C^{n+1} \) such that \( \langle X_0, \cdots, X_n \rangle (x) \in SO(n+1) \). Then the tangent space \( T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))) \) has a basis \( (iX_0 - X_1)(x), X_0(x), \cdots, X_n(x), iX_1(x), \cdots, iX_1(x) \) (c.f. [3] p.p. 279).

In the equation \( dZ = \sum_{i=1}^{2n+1} \frac{\partial Z}{\partial x^i} \, dx^i \), we see \[
\frac{\partial Z}{\partial x^i} = Z_*(\frac{\partial}{\partial x^i})(1 \leq i \leq 2n-1) \text{ and hence } \frac{\partial Z}{\partial x^i} \text{ 's are tangent vectors of } T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))). \text{ Thus there exists } 1-
forms θ_j's (1≤j≤n) and θ̄_j's (2≤j≤n) on U such that dZ=θ_1(iX_0−X_1)+ \sum_{j=2}^n (θ_j+iθ̄_j)X_j. Comparing this form with (5.2), we have θ_1=θ_1/√2, θ_j=θ_j/√2 (2≤j≤n) and θ̄_j=θ̄_j/√2 (2≤j≤n). Thus we have from (5.2), (5.3) and (5.7)

(5.8) \((Π^*Ω)_{(x_0, x_i)} = \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, X_j \rangle \wedge \langle dX_1, X_j \rangle\),

where \((X_0, X_1, \ldots, X_n)\)∈SO(n+1). For the volume form Ω^{n-1} on \(Q_{n-1}(C)\), we have

(5.9) \((Π^*Ω^{n-1})_{(x_0, x_1)} = \left( \frac{1}{2\pi} \right)^{-1} (n-1)! \langle dX_0, X_2 \rangle \wedge \langle dX_1, X_2 \rangle \wedge \ldots \wedge \langle dX_0, X_n \rangle \wedge \langle dX_1, X_n \rangle\).

We shall obtain a formula for \(f^*Ω^2\) on \(C^2\). Let \(F\) be a holomorphic lift of \(f\) on a neighborhood \(U\) in \(C^2\) by \(Π\). Set \((X_0+ix_1)/\sqrt{2}=F||F||\), where \(X_i\) \((i=0, 1)\) are the orthonormal real vectors. With the coordinate system \((x_1+iy, x_2+iy)\) on \(C^2\), we can write:

\begin{align*}
dX_0 &= ω_1B_zdx_1−λ_3B_ydy_1−λ_4B_ydx_2−λ_2B_ydy_2, \\
dX_1 &= ω_2B_zdx_1+λ_3B_ydy_1+λ_4B_ydx_2+λ_2B_ydy_2,
\end{align*}

where \(B_z\)'s (2≤i≤5) are differentiable vectors satisfying \(⟨B_z, B_z⟩=1\), \(λ_i\)'s (2≤i≤5) are differentiable functions and \(ω_i\)'s (1≤i≤2) are 1-forms on \(U\). Then we take differentiable orthonormal vectors \(B_i (2≤i≤5)\) such that \(B_2=B_3, B_3=α_2B_2+α_2B_3, B_4=β_2B_2+β_3B_4+β_4B_4, B_5=γ_2B_2+γ_3B_4+γ_5B_5\), where \(α_i, β_i, γ_i\) are differentiable functions satisfying \(Σ_2^i=1\), \(Σ_3^i=1\) and \(Σ_4^i=1\). We choose differentiable vectors \(B_1, \ldots, B_n\) on \(U\) such that \((X_0, X_1, B_1, \ldots, B_n)\)∈SO(n+1) at each point of \(U\). By (5.8) we have

(5.11) \(f^*Ω = \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, B_j \rangle \wedge \langle dX_1, B_j \rangle\)

\begin{align*}
&= \frac{1}{2\pi} \left\{ [\lambda_2α_5γ_2−λ_2α_5β_2−λ_3λ_4α_3β_3] (dx_1 \wedge dx_2+dy_1 \wedge dy_2) \\
&\quad +[\lambda_3α_5γ_2]dx_1 \wedge dy_2+[\lambda_4α_5γ_2]dy_1 \wedge dx_2 \right. \\
&\quad \left. +[\lambda_2λ_3γ_2+λ_3λ_4γ_2+λ_2α_5γ_3] (dx_1 \wedge dy_2−dy_1 \wedge dx_2) \right\}. 
\end{align*}

Furthermore we obtain

(5.12) \(f^*Ω^2 = \left( \frac{1}{2\pi} \right)^2 \times 2 \times \{[\lambda_3α_5γ_2] [\lambda_4α_5γ_2]−[\lambda_2λ_3γ_2+λ_3λ_4γ_2+λ_2α_5γ_3]^2\} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2\).
6. Crofton formula

In §3 we have defined \( n(\Delta(r), \alpha) \) for a holomorphic mapping \( f: \mathbb{C}^2 \rightarrow \mathbb{Q}_{n-1}(\mathbb{C}) \) \((n \geq 3)\) satisfying Conditions (A) and (B). Then we have:

**Theorem 4** (Crofton formula). *Let \( D \) be an open set in \( \mathbb{C}^2 \) with compact closure. Then we have*

\[
\int_{\mathbb{Q}_{n-1}(\mathbb{C})} n(D, \xi) d\xi = 2 \int_{\partial D} f^* \Omega^2 ,
\]

where \( d\xi = d\xi = d\alpha = \Omega^{n-1} \).

**Proof.** First we assume that \( D \) is so small that there exists a differentiable lift \( \sigma = (X_0, X_1) \) of \( f \) on \( D: \Sigma f = f \). Let \( q \) be a point in \( D \) and set \( f(q) \in \xi \). For any real orthonormal vectors \( Y_0, Y_1 \) such that \( \Pi_r((Y_0, Y_1)) = \alpha \), we have

\[
\langle X_0(q), Y_0 \rangle = \langle X_1(q), Y_1 \rangle = \langle X_2(q), Y_2 \rangle = 0 .
\]

We set

\[
Q_{n-3}(f(q)) = \{ \alpha \in \mathbb{Q}_{n-1}(\mathbb{C}): f(q) \in \xi \}
\]

\[
f(D) = \{ \alpha \in \mathbb{Q}_{n-1}(\mathbb{C}): f(D) \cap \xi = \phi \} .
\]

and

\[
D' = \Pi_1(f(D))
\]

\[
D'' = \{ (q, a): q \in D, a = (A_2, A_3, \ldots, A_n) \in SO(n-1) \} .
\]

For \( a = (A_2, A_3, \ldots, A_n) \in SO(n-1) \) we write its column vector \( A_i \) as \( A_i = (a_{i2}, \ldots, a_{in}) \). Then we define a mapping \( t: D'' \rightarrow SO(n+1) \) by

\[
t((q, a)) = (B_2, B_3, X_0, X_1, B_4, \ldots, B_n) (q)
\]

\[
\begin{pmatrix}
  a_{22} & a_{23} & 0 & 0 & a_{45} & \ldots & a_{n2} \\
  a_{23} & a_{23} & 0 & 0 & a_{43} & \ldots & a_{n3} \\
  0 & 0 & 1 & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & 1 & 0 & \ldots & 0 \\
  a_{24} & a_{24} & 0 & 0 & a_{44} & \ldots & a_{n4} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{2n} & a_{2n} & 0 & 0 & a_{4n} & \ldots & a_{nn}
\end{pmatrix}
\]

where \( (X_0, X_1, B_2, \ldots, B_n) (q) \) is the one given in §5. Let \( \Pi' \) be the projection \( D \times (SO(n-1)/SO(n-3)) \rightarrow D \times \mathbb{Q}_{n-3}(\mathbb{C}) \) defined by \( \Pi'((q, (A_2, A_3))) = (q, \Pi''((A_2, A_3))) \), where \( \Pi'' \) is the projection with respect to the Hopf fibering \( SO(n-1)/SO(n-3) \rightarrow \mathbb{Q}_{n-3}(\mathbb{C}) \). We consider the following diagram:
\[ D \times (SO(n-1)/SO(n-3)) \xrightarrow{t'} D' \subset SO(n+1)/SO(n-1) \]
\[ D \times Q_{n-3}(C) \xrightarrow{t''} f(D)^{-1} \subset Q_{n-3}(C), \]

where \( t'((q, (A_2, A_3)))=(\Sigma_{i=2}^{n} a_i B_i(q), \Sigma_{i=2}^{n} a_i B_i(q)) \) and \( t'' \) is defined by \( \Pi_1 \circ t' = t'' \circ \Pi' \). Then, in the above diagram, we remark that \( t''((q, Q_{n-3}(C)))=Q_{n-3}(f(q)^{-1}) \) for each \( q \in D \). Putting \( t((q, a))=(X_0, X_1', \ldots, X_n') \), we obtain

\[ (\Pi')^* (t'')^* \Omega^{n-1} = (t')^* (\Pi_1)^* \Omega^{n-1} \]
\[ = \left( \frac{1}{2\pi} \right)^{n-1} (n-1)! \left< \text{d}X_0, X_2' \right> \wedge \left< \text{d}X_1, X_2' \right> \wedge \cdots \wedge \left< \text{d}X_{n-1}, X_2' \right> \wedge \left< \text{d}X_0, X_2' \right> \]
\[ = \left( \frac{1}{2\pi} \right)^{n-1} (n-1)! \times \frac{1}{16} \left< \text{d}(X_0+iX_1), X_2'-iX_2' \right> \wedge \left< \text{d}(X_0+iX_1), X_2'+iX_2' \right> \wedge \left< \text{d}A_3, A_4 \right> \wedge \cdots \wedge \left< \text{d}A_3, A_n \right> \wedge \left< \text{d}A_3, A_n \right> \]
\[ = -\frac{1}{4} \left( \frac{1}{2\pi} \right)^2 (n-1)(n-2) \left| \left< \lambda_2 B_2 + i\lambda_3 B_3, X_0' + iX_1' \right>, \left< \lambda_4 B_4 + i\lambda_5 B_5, X_0' + iX_1' \right> \right|^2 \]
\[ \wedge \left< \lambda_2 B_2 + i\lambda_3 B_3, X_0' - iX_1' \right>, \left< \lambda_4 B_4 + i\lambda_5 B_5, X_0' - iX_1' \right> \wedge \left< \text{d}A_3, A_4 \right> \wedge \cdots \wedge \left< \text{d}A_3, A_n \right> \wedge \left< \text{d}A_3, A_n \right>. \]

We put \( C=\{ \beta \in f(D)^{-1} \mid \text{there exists } \beta' \in (t'')^{-1}(\beta) \text{ such that } (dt'')(\beta') \text{ is singular} \} \). From Sard's Theorem the set \( C \) has measure zero. If we take \( \alpha \in (f(D)^{-1}\setminus C) \), the set \( (t')^{-1}(\alpha) \) consists of finite elements because of the compactness of \( D \) and Condition (B). We denote by \( n_\alpha \) the number of elements \( (t')^{-1}(\alpha) \). Then, for each \( \alpha \in (f(D)^{-1}\setminus C) \) there exists a connected neighborhood \( V \) of \( \alpha \) in \( (f(D)^{-1}\setminus C) \) such that \( (t')^{-1}(V) \) has \( n_\alpha \) connected components and \( t' \) maps each component onto \( V \) diffeomorphically. Let \( \{ V_i \} \) be a locally finite covering of \( (f(D)^{-1}\setminus C) \) by such open sets and \( \{ \phi_i \} \) be a partition of unity subordinated to \( \{ V_i \} \). Now we have

\[ \int_{f(D)^{-1}\setminus C} n_\alpha d\alpha = \sum_i \int_{f(D)^{-1}\setminus C} \phi_i(\alpha) n_\alpha d\alpha \]
\[ = \sum_i \int_{f(D)^{-1}\setminus C} \phi_i(\alpha) d\alpha = \sum_i \int_{(t')^{-1}(V_i)} -(t'')^*(\phi_i(\alpha)) d\alpha \]
\[ = \sum_i \int_{(t')^{-1}(V_i)} -(t'')^* d\alpha = \int_{f(C)^{-1}\setminus C} -(t'')^* d\alpha, \]
where \( C' \) is the set of critical points of \( t'' \). If 

\[
\begin{align*}
t''((q, \xi_j)) &= \alpha \quad \text{and} \\
\left\langle \frac{\partial F}{\partial z_k}, Z_{\alpha} \right\rangle, \left\langle \frac{\partial F}{\partial z_k}, Z_{\alpha} \right\rangle (q) \\
\left\langle \frac{\partial F}{\partial z_k}, Z_{\alpha} \right\rangle, \left\langle \frac{\partial F}{\partial z_k}, Z_{\alpha} \right\rangle
\end{align*}
\]

(which is equal to \( \frac{||F||}{2} \))

\[
\begin{align*}
\langle \lambda_2 \bar{z}_2 + i \lambda_3 \bar{z}_3, Z_{\alpha} \rangle, \langle \lambda_2 \bar{z}_2 + i \lambda_3 \bar{z}_3, Z_{\alpha} \rangle (q) = 0
\end{align*}
\]

for \( \Pi(Z_{\alpha}) = \alpha \), then \( dt''((q, \xi_j)) \) is singular because of (6.7). By Lemma 3.2 we have \( n(D, \alpha) = n_{\alpha} \) on \( f(D) \setminus C \). Therefore we have

\[
(6.9) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha = \frac{1}{4} \left( \frac{1}{2\pi} \right)^2 (n-1) (n-2) \int_D dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2
\]

\[
\times \int_{Q_{n-1}} \left\langle \langle \lambda_2 \bar{z}_2 + i \lambda_3 \bar{z}_3, X_0' + i X_1' \rangle, \langle \lambda_2 \bar{z}_2 + i \lambda_3 \bar{z}_3, X_0' + i X_1' \rangle \right\rangle \Omega^{n-3}.
\]

Next we have the following equation:

\[
(6.10) \quad \int_{Q_{n-1}} \left\langle \langle \lambda_2 \bar{z}_2 + i \lambda_3 \bar{z}_3, X_0' + i X_1' \rangle, \langle \lambda_2 \bar{z}_2 + i \lambda_3 \bar{z}_3, X_0' - i X_1' \rangle \right\rangle \Omega^{n-3}
\]

\[
+ \left( \lambda_2^2 + \lambda_3^2 \alpha \xi_3 \right) \left( \lambda_2^2 \beta_3 + \lambda_3^2 \gamma_3 \right) (q) \int_{Q_{n-1}} \left\langle \langle B_2, X_0' + i X_1' \rangle, \langle B_2, X_0' - i X_1' \rangle \right\rangle \Omega^{n-3}
\]

\[
+ \left( \lambda_2^2 + \lambda_3^2 \alpha \xi_3 \right) \left( \lambda_2^2 \beta_3 + \lambda_3^2 \gamma_3 \right) (q) \int_{Q_{n-1}} \left\langle \langle B_2, X_0' - i X_1' \rangle, \langle B_2, X_0' - i X_1' \rangle \right\rangle \Omega^{n-3}
\]

\[
+ \left( \lambda_2^2 + \lambda_3^2 \alpha \xi_3 \right) \left( \lambda_2^2 \beta_3 + \lambda_3^2 \gamma_3 \right) (q) \int_{Q_{n-1}} \left\langle \langle B_2, X_0' - i X_1' \rangle, \langle B_2, X_0' + i X_1' \rangle \right\rangle \Omega^{n-3}
\]

\[
+ \left( \lambda_2^2 + \lambda_3^2 \alpha \xi_3 \right) \left( \lambda_2^2 \beta_3 + \lambda_3^2 \gamma_3 \right) (q) \int_{Q_{n-1}} \left\langle \langle B_2, X_0' + i X_1' \rangle, \langle B_2, X_0' - i X_1' \rangle \right\rangle \Omega^{n-3}
\]

\[
+ \left( \lambda_2^2 + \lambda_3^2 \alpha \xi_3 \right) \left( \lambda_2^2 \beta_3 + \lambda_3^2 \gamma_3 \right) (q) \int_{Q_{n-1}} \left\langle \langle B_2, X_0' - i X_1' \rangle, \langle B_2, X_0' - i X_1' \rangle \right\rangle \Omega^{n-3}.
\]

In fact, the integral of the other terms which appear at the right hand side of (6.10) turns out to be zero. For example we consider the following integral:
We have

\[ l = \int_{SO(n-1)/SO(n-3)} \frac{1}{2\pi} \left( \frac{1}{2\pi} \right)^{n-2} (n-3)! \, d\theta \wedge \langle dA_2, A_2 \rangle \wedge \langle dA_3, A_3 \rangle \wedge \cdots \wedge \langle dA_{n-2}, A_{n-2} \rangle \wedge \langle dA_n, A_n \rangle, \]

where \(0 ≤ \theta ≤ 2\pi\). For each vector \( A_i = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in})\) we set \( \bar{A}_i = (a_{i1} - a_{i3}, a_{i2}, a_{i3}, \ldots, a_{in})\). This induces a diffeomorphism \( k: SO(n-1) \to SO(n-1)\) by \( k((A_2, A_3, A_4, A_5, \ldots, A_n)) = (A_2, A_3, A_4, A_5, \ldots, A_n)\). Then we have

\[ l = \int_{SO(n-1)/SO(n-3)} \frac{1}{2\pi} \left( \frac{1}{2\pi} \right)^{n-2} (n-3)! \, d\theta \wedge \langle dA_2, A_2 \rangle \wedge \langle dA_3, A_3 \rangle \wedge \cdots \wedge \langle dA_{n-2}, A_{n-2} \rangle \wedge \langle dA_n, A_n \rangle \wedge \langle dA_2, A_2 \rangle \wedge \langle dA_3, A_3 \rangle \wedge \cdots \wedge \langle dA_{n-2}, A_{n-2} \rangle \wedge \langle dA_n, A_n \rangle. \]

Since we have \( \langle dA_i, A_j \rangle = \langle dA_i, A_j \rangle (2 ≤ i ≤ 3, 4 ≤ j ≤ n)\), we obtain \( l = 0\). In the equation (6.10), the integrals

\[ \int_{Q_{\ast}(f(q,r-3))} \left| \begin{array}{c} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \end{array} \right| \Omega^{n-3}, \]

\[ \int_{Q_{\ast}(f(q,r-3))} \left| \begin{array}{c} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \end{array} \right| \Omega^{n-3}, \]

\[ \int_{Q_{\ast}(f(q,r-3))} \left| \begin{array}{c} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \end{array} \right| \Omega^{n-3}, \]

\[ \int_{Q_{\ast}(f(q,r-3))} \left| \begin{array}{c} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \end{array} \right| \Omega^{n-3}, \]

and

\[ \int_{Q_{\ast}(f(q,r-3))} \left| \begin{array}{c} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \end{array} \right| \Omega^{n-3}, \]

are all equal and furthermore its value is independent of \( q \). We denote by \( C_0 \) its common value. Then by (5.12), (6.9) and (6.10) we have
We shall calculate the value $\mathbf{C}_0$. Let $SO(n-1)/SO(n-3) \to Q_{n-3}(C)$ be the Hopf fibering. For arbitrary fixed pair $(C_2, C_3)$ of $SO(n-1)/SO(n-3)$ we have

$$\mathbf{C}_0 = \int_{Q_{n-3}(C)} \left| \left< C_2, A_2 \right> + i A_3, \left< C_3, A_2 \right> + i A_3 \right|^2 \Omega^{n-3}. \quad (6.12)$$

We take an orthonormal pair $(D_4, D_5)$ of $SO(n-1)/SO(n-3)$ such that $\left< C_i, D_j \right> = 0 \ (2 \leq i \leq 3, \ 4 \leq j \leq 5)$ and set real orthonormal vectors $A_2, A_3, A_4$ and $A_5$ by

$$A_2 = \sin \phi (\sin \theta \ C_2 - \cos \theta \ C_3) + \cos \phi (\sin \alpha \ D_4 - \cos \alpha \ D_5),$$
$$A_3 = \sin \eta (\cos \theta \ C_2 + \sin \theta \ C_3) + \cos \eta (\cos \alpha \ D_4 + \sin \alpha \ D_5),$$
$$A_4 = -\cos \phi (\sin \theta \ C_2 - \cos \theta \ C_3) + \sin \phi (\sin \alpha \ D_4 - \cos \alpha \ D_5),$$
$$A_5 = -\cos \eta (\cos \theta \ C_2 + \sin \theta \ C_3) + \sin \eta (\cos \alpha \ D_4 + \sin \alpha \ D_5), \quad (6.13)$$

where $0 < \theta, \alpha < \pi, -\pi/2 < \phi, \eta < \pi/2$. By extending $A_2, A_3, A_4$ and $A_5$ to an ordered real orthonormal basis $A_2, A_3, \ldots, A_n$ in $\mathbb{C}^{n-1}$ we get $(A_2, A_3, \ldots, A_n) \in SO(n-1)$. Take an open set $U \subset Q_{n-3}(C)$, where $Q_{n-3}(C)$ is a set $\{\beta \in Q_{n-3}(C): \|\beta, \Pi''((C_2, C_3))\|^2 + \|\beta, \Pi''((C_2, -C_3))\|^2 = 0\}$ in $Q_{n-3}(C)$, and a local cross-section $\sigma = (D_4, D_5)$ of $U$ into $SO(n-3)/SO(n-5) \to Q_{n-3}(C)$. Then we see easily the set $\{(A_2, A_3) \in SO(n-1)/SO(n-3): (A_2, A_3) \text{ is defined at } (6.13) \text{ for } \sigma = (D_4, D_5)\}$ is a double covering of an open set in $Q_{n-3}(C)$. We have

$$\left< dA_2, A_2 \right> = -d\phi, \quad \left< dA_3, A_3 \right> = -d\gamma,$$
$$\left< dA_2, A_3 \right> = -\sin \phi \cos \phi d\theta + \sin \phi \cos \phi d\alpha + \cos \phi \sin \phi \left< dD_4, D_3 \right>,$$
$$\left< dA_3, A_3 \right> = \sin \eta \cos \phi d\theta - \sin \phi \cos \eta d\alpha - \cos \eta \sin \phi \left< dD_4, D_5 \right>,$$
$$\left< dA_4, A_4 \right> = \cos \phi \sin \alpha \left< dD_4, A_i \right> - \cos \phi \left< dD_5, A_i \right>,$$
$$\left< dA_5, A_5 \right> = \cos \eta \left< \cos \alpha \left< dD_4, A_i \right> + \sin \alpha \left< dD_5, A_i \right>\right> \quad (i \geq 6). \quad (6.14)$$

By (6.14) we get

$$\left< dA_2, A_2 \right> \wedge \left< dA_3, A_3 \right> \wedge \cdots \wedge \left< dA_4, A_4 \right> \wedge \left< dA_5, A_5 \right> = \left( \sin \gamma \cos \phi - \sin \phi \cos \gamma \right) \left( \cos \phi \cos \eta \right)^{n-5} 
\times d\phi \wedge d\theta \wedge d\alpha \wedge d\gamma \wedge \prod_{i \geq 6} \left< dD_i, A_i \right> \wedge \left< dD_5, A_i \right>,$$

and

$$\left| \left< C_2, A_2 + i A_3 \right> \right|^2 = 4 \left| \sin \phi \sin \eta \right|^2$$
$$\left| \left< C_2, A_2 - i A_3 \right> \right|^2 = 4 \left| \sin \phi \sin \eta \right|^2$$

Thus we obtain
\[(6.12)' \quad C_0 = (n-3)(n-4)\int |\sin \varphi \sin \eta|^2 |\sin^2 \gamma \cos^2 \varphi - \sin^2 \varphi \cos^2 \eta|
\quad \times |\cos \varphi \cos \eta|^{n-3}d\varphi d\eta \times \int_{Q_{\alpha-(n-1)}} \Omega^{n-3}
\quad = 2(n-3)(n-4)\int |\sin \varphi \sin \eta|^2 |\sin^2 \gamma \cos^2 \varphi - \sin^2 \varphi \cos^2 \eta|
\quad \times |\cos \varphi \cos \eta|^{n-3}d\varphi d\eta
\quad = \frac{16}{(n-1)(n-2)},
\]

because of \(\int_{Q_{\alpha-(n-1)}} \Omega = 2\) and \(\int_E (|\sin \varphi \sin \eta|^2 |\sin^2 \gamma \cos^2 \varphi - \sin^2 \varphi \cos^2 \eta|)
\times (|\cos \varphi \cos \eta|^{n-3}d\varphi d\eta) = \frac{2}{(n-1)(n-2)(n-3)(n-4)},\) where
\(E = \{(\eta, \varphi): 0 \leq \varphi \leq \pi/2\text{ and } 0 \leq \eta \leq \varphi\}. \) Thus we have proved the equation (6.1)
for a sufficiently small \(D. \) Now let \(D\) be an arbitrary open set in \(C^2\) with compact
closure. We take a finite covering \(\{D_s\}_{s=1}^\infty\) of \(D\) such that each \(D_s\) has a differen-
tiable local cross-section of \(f\) into \(SO(n+1)/SO(n-1).\) Let \(\{g_s\}\) be a partition
of unity subordinated to \(\{D_s\}.\) Taking a mapping \(P_s: D_s \times Q_{n-3}(C) \to D_s\) defined
by \(P_s((q, \alpha)) = q\) for \((q, \alpha) \in D_s \times Q_{n-3}(C),\) we put \(n'(D_s, \alpha) = \sum \tau^s(p_s, \alpha)g_s(p_s).\)
Then we obtain
\[(6.17) \quad \int_{Q_{n-3}} \eta(D_s, \alpha)d\alpha = \sum_{s=1}^{\infty} \int_{Q_{n-3}} \eta'(D_s, \alpha)d\alpha
\quad = \sum_{s=1}^{\infty} \int_{D_s \times Q_{n-3}} -g_s(P_s(\alpha')) (t_s'')*d\alpha
\quad = 2 \sum_{s=1}^{\infty} \int_{D_s} g_sf^*\Omega^2
\quad = 2 \int_{D} f^*\Omega^2,
\]
where \(t_s''\) is a mapping of \(D_s \times Q_{n-3}(C)\) onto \(f(D_s)\) defined by (6.6). Q.E.D.

7. Equidistribution theorem

We define the defect \(\delta(\alpha)\) of \(\xi_{\alpha}\) by
\[(7.1) \quad \delta(\alpha) = \lim \inf_{r \to \infty} \frac{m(r, \alpha)}{T(r)},
\]
Since \(m(r, \alpha)\) is non-negative, \(\delta(\alpha)\) is non-negative for any \(\alpha \in Q_{n-1}(C).\) We see
clearly that \(\delta(\alpha) = \delta(\alpha)\) for any \(\alpha \in Q_{n-1}(C).\) By Theorem 2, Lemma 4.5 and the
fact that \(T(r) \to \infty\) if \(r \to \infty,\) we have
\[(7.2) \quad \delta(\alpha) = \lim \inf_{r \to \infty} \left(1 - \frac{N(r, \alpha)}{T(r)}\right).
\]
Theorem 5. \( \delta(\alpha) \) is equal to zero for almost all \( \alpha \in \mathbb{Q}_{n-1}(\mathbb{C}) \) with respect to the volume \( \Omega^{n-1} \).

Proof. By the Fatou's preparation theorem we have

\[
0 \leq \int_{\mathbb{Q}_{n-1}} 0 < \int_{\mathbb{Q}_{n-1}} \left\{ \liminf_{r \to \infty} \left( 1 - \frac{N(r, \alpha)}{T(r)} \right) \right\} d\alpha \\
\leq \liminf_{r \to \infty} \int_{\mathbb{Q}_{n-1}} \left( 1 \right) d\alpha = \liminf_{r \to \infty} \left( 2 - \frac{1}{T(r)} \int_{\mathbb{Q}_{n-1}} N(r, \alpha) d\alpha \right) \\
= \liminf_{r \to \infty} \left( 2 - \frac{1}{T(r)} \int_{\mathbb{Q}_{n-1}} \left\{ \int_0^r n(\Delta(t), \alpha) dt \right\} d\alpha \right) \\
= \liminf_{r \to \infty} \left( 2 - \frac{1}{T(r)} \int_0^r \int_{\mathbb{Q}_{n-1}} n(\Delta(t), \alpha) d\alpha dt \right) \\
= \liminf_{r \to \infty} (2 - 2) = 0 \text{ (by Theorem 4).}
\]

Thus we obtain \( \delta(\alpha) = 0 \) for almost all \( \alpha \in \mathbb{Q}_{n-1}(\mathbb{C}) \). Q.E.D.

If the image \( f(C^2) \) does not intersect with \( \xi_\alpha \), we have \( \delta(\alpha) = 1 \). So we have

Corollary. Let \( f \) be a holomorphic mapping of \( C^2 \) into \( \mathbb{Q}_{n-1}(\mathbb{C}) \) (\( n \geq 3 \)) satisfying Conditions (A) and (B). We put \( W = \{ \alpha \in \mathbb{Q}_{n-1}(\mathbb{C}) : f(C^2) \cap \xi_\alpha = \emptyset \} \). Then the set \( W \) has measure zero with respect to volume \( \Omega^{n-1} \).

Remark 3. In the case of holomorphic curves (\( f : C \to P_n(\mathbb{C}) \) holomorphic mapping), it is known that \( 0 \leq \delta(\xi) \leq 1 \) for each hyperplane \( \xi \) (c.f. [1], [5] and [6]). But in our case we can not prove that \( \delta(\alpha) \leq 1 \).

O S A K A U N I V E R S I T Y

References
