LINEAR SU(n)-ACTIONS ON COMPLEX PROJECTIVE SPACES

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0. Introduction

Let $U_*$ be the bordism ring of weakly complex manifolds and let $G$ be a compact Lie group. Denote by $SF(G)$, an ideal in $U_*$ of those bordism classes represented by a weakly complex manifold on which the group $G$ acts smoothly without stationary points and the action preserves a weakly complex structure.

For a compact abelian Lie group $G$ the ideal $SF(G)$ was computed by tom Dieck [8]. Such ideals are similarly defined in the bordism ring $\Omega_*$ of oriented manifolds and those were computed for certain abelian groups by Floyd [3] and Stong [7]. But it seems that there is no useful method to compute the ideal $SF(G)$ for a non-abelian Lie group $G$.

First we give an upper bound and a lower bound of $SF(G)$ for any compact Lie group $G$. To state our result precisely we introduce some notations as follows. Denote by $I(G)$, a set of positive integers such that $n \in I(G)$ if and only if there is an $n$-dimensional complex $G$-vector space without $G$-invariant one-dimensional subspaces, by $m(G)$ the maximum dimension of proper closed subgroups of $G$, and put

$$n(G) = \dim G - m(G).$$

It is known that the bordism ring $U_* = \sum_{k \geq 0} U_{2k}$ is generated by a set of bordism classes

$$\{[P_n(C)], [H_{p,q}(C)]; n \geq 0, p \geq q > 0\}$$

as a ring. Now we define ideals $L(G), M(G)$ in $U_*$ as follows. Let $L(G)$ be an ideal in $U_*$ generated by a set

$$\{[P_n(C)], [H_{m+n,n}(C)]; n+1 \in I(G), m \geq 0\}$$

and let

$$M(G) = \sum_{2k \in n(G)} U_{2k}.$$ 

Then we have following results.
Theorem 0.1. For any compact Lie group $G$,

$$L(G) \subset SF(G) \subset M(G).$$

Corollary. $SF(SU(2)) = SF(U(2)) = \sum_{n \geq 0} U_{2n}$. 

For each positive integer $n$, $P_n(C)$ admits a linear $SU(2)$-action without stationary points, but for example $P_3(C)$ does not admit a linear $SU(3)$-action without stationary points. Thus we next consider $SU(3)$-actions on $P_3(C)$ and we have a following result. Denote by $hP_3(C)$, a compact smooth 6-dimensional manifold homotopy equivalent to $P_3(C)$.

Theorem 0.2. (a) Any smooth $SU(3)$-action on $hP_3(C)$ has at least one stationary point. (b) Any non-trivial smooth $SU(3)$-action on $hP_3(C)$ is equivariantly diffeomorphic to a linear $SU(3)$-action on $P_3(C)$.

1. Weakly complex $G$-manifolds without stationary points

Let $G$ be a Lie group and $V$ be an $n$-dimensional complex $G$-vector space. Denote by $P(V)$ the complex projective space $P_n(C)$ with an induced $G$-action. We call such a $G$-action on $P_n(C)$ a linear $G$-action. Then $P(V)$ is a weakly complex $G$-manifold in the sense of Conner-Floyd [1]. Denote by $[v]$ a point of $P(V)$ represented by a non-zero vector $v$ of $V$. Then

Lemma 1.1. A point $[v]$ of $G$-manifold $P(V)$ is a stationary point if and only if the vector $v$ spans a $G$-invariant one-dimensional subspace of $V$.

Lemma 1.2. Any smooth $G$-action on a manifold $M$ is trivial, if $\dim M \leq n(G) = \dim G - m(G)$. Here $m(G)$ is the maximum dimension of proper closed subgroups of $G$.

Proof. If $x \in M$ is not a stationary point, then the isotropy subgroup $G_x$ at $x$ is a proper closed subgroup of $G$, the orbit $G \cdot x$ is a submanifold of $M$, and $G \cdot x$ is diffeomorphic to the homogeneous space $G/G_x$. Then

$$\dim M \geq \dim (G \cdot x) = \dim G - \dim G_x \geq \dim G - m(G).$$

Remark. The integer $m(G)$ was calculated by Mann [5] for compact connected simple Lie groups $G$, by making use of Dynkin's work [2].

Proof of Theorem 0.1. Let $V$ be an $n$-dimensional complex $G$-vector space and $W$ be an $m$-dimensional complex $G$-vector space. The canonical $G$-action on the dual space $V^* = \text{Hom}_C(V, C)$ is defined by

$$(g \cdot u)(v) = u(g^{-1} \cdot v); g \in G, u \in V^*, v \in V.$$
Define
\[ H(V \oplus W, V^*) = \{ ([v+w], [u]) \in P(V \oplus W) \times P(V^*): u(v) = 0 \} , \]
then \( H(V \oplus W, V^*) \) is a manifold \( H_{m+n-1, n-1}(\mathbb{C}) \) with a weakly complex \( G \)-action. If \( V \) has no \( G \)-invariant one-dimensional subspaces, then the \( G \)-action on \( H(V \oplus W, V^*) \) has no stationary points by Lemma 1.1. Therefore the inclusion \( L(G) \subset SF(G) \) is proved. Next the inclusion \( SF(G) \subset M(G) \) follows from Lemma 1.2. This completes the proof of Theorem 0.1.

Next we consider the case for \( G = SU(n) \), the special unitary group. Let \( I(G) \) be the set of positive integers defined in the introduction. Then by definition
\[ n_1, n_2 \in I(G) \text{ implies } n_1 + n_2 \in I(G) . \]

**Lemma 1.4.** Any binomial coefficient \( \binom{n+k-1}{k} \) is contained in \( I(SU(n)) \) for \( n \geq 2 \) and \( k \geq 1 \).

Proof. Denote by \( V_n \) the complex vector space \( \mathbb{C}^n \) with the standard \( SU(n) \)-action. Then the \( k \)-th symmetric product \( S_k(V_n) \) is irreducible as a complex \( SU(n) \)-vector space for each \( k \geq 1 \) and
\[ \dim cS_k(V_n) = \binom{n+k-1}{k} . \]

**Corollary 1.5.** \( SF(SU(2)) = SF(U(2)) = \sum_{n>0} U_{2n} \).

Proof. Since \( I(SU(2)) = I(U(2)) \) consists all positive integers \( n \geq 2 \) by Lemma 1.4,
\[ L(SU(2)) = L(U(2)) = \sum_{n>0} U_{2n} . \]

On the other hand,
\[ M(SU(2)) = M(U(2)) = \sum_{n>0} U_{2n} \]
by the connectivity of \( SU(2) \) and \( U(2) \).

2. \( SU(3) \)-actions on \( \mathbb{P}(\mathbb{C}) \)

Let us first recall some basic facts in differentiable transformation groups.

(i) Let \( G \) be a compact Lie group acting on a manifold \( M \). Then by averaging an arbitrary given Riemannian metric on \( M \), we may have a \( G \)-invariant Riemannian metric on \( M \).

(ii) Let \( x \in M \), then the isotropy subgroup \( G_x \) acts on a normal vector space \( N_x \) of the orbit \( G \cdot x \) at \( x \) orthogonally; we call it the normal representation of \( G_x \) at \( x \) and denote by \( \rho_x \).
(iii) (The differentiable slice theorem) Let $E(\nu)$ be the normal bundle of the orbit $G \cdot x = G/G_x$. Then

$$E(\nu) = G \times_{G_x} N_x$$

where $G_x$ acts on $N_x$ via $\rho_x$. We note that $G$ acts naturally on $E(\nu)$ as bundle mappings and we may choose small positive real number $\epsilon$ such that the exponential mapping gives an equivariant diffeomorphism of the $\epsilon$-disk bundle of $E(\nu)$ onto an invariant tubular neighborhood of $G \cdot x$. ([6], Lemma 3.1)

(iv) Let $H \subseteq G$ be a closed subgroup. Denote by $(H)$, the set of all subgroups of $G$ which are conjugate to $H$ in $G$. We introduce the following partial ordering relation "<" by defining $(H_1) < (H_2)$ if and only if there exist $H_1 \subseteq (H_1)$ and $H_2 \subseteq (H_2)$ such that $H_1 \subset H_2$. If $M$ is connected, then there exists an absolute minimal $(H)$ among the conjugate classes in $\{G_x | x \in M\}$, moreover the set

$$M_{\text{min}} = \{x \in M | G_x \in (H)\}$$

is a dense open submanifold. The conjugate class $(H)$ is called the type of principal isotropy subgroups. ([6], (2.2) and (2.4))

Combining (iii) and (iv), we have a following lemma.

**Lemma 2.1.** If $M$ is connected, then the normal representation of $G_x$ at $x \in M$ is trivial if and only if $G_x$ is a principal isotropy subgroup.

Now we consider $SU(3)$-actions. Let $H$ be a closed subgroup of $SU(3)$. Denote by $N(H)$ the normalizer of $H$ in $SU(3)$.

**Lemma 2.2.** (a) Let $H$ be a closed connected proper subgroup of $SU(3)$ with $\dim H \geq 3$, then $H$ is conjugate to $SU(2)$, $SO(3)$ or $N(SU(2))$. (b) There are isomorphisms, $N(SU(2))/SU(2) \simeq S^1$, the circle group; $N(SO(3))/SO(3) \simeq Z_3$, the cyclic group of order 3; $N(N(SU(2))) = N(SU(2))$, as the subgroups of $SU(3)$. (c) $N(SU(2))$ does not contain any subgroup which is conjugate to $SO(3)$.

Proof. (a) is proved by considering the structure of Lie algebra of $SU(3)$ and the 3-dimensional unitary representations of $SU(2)$. (b) is proved by direct calculation. (c) is true since $N(SU(2)) \subset SU(3)$ is not irreducible but $SO(3) \subset SU(3)$ is irreducible.

**Remark.** $\dim SU(3) = 8$ and $\dim SU(2) = \dim SO(3) = 3$.

**Lemma 2.3.** Let $M$ be an orientable connected 6-dimensional manifold with smooth $SU(3)$-action. If an isotropy subgroup $SU(3)_x$ is of 3-dimensional, then $SU(3)_x$ is a principal isotropy subgroup.

Proof. First we may prove that the homogeneous space $SU(3)/SU(3)_x$ is an orientable 5-manifold by Lemma 2.2. Thus the normal bundle $E(\nu)$ of
SU(3)/SU(3)_x is a trivial line bundle, since M and SU(3)/SU(3)_x are orientable. But if the normal representation of SU(3)_x at x∈M is non-trivial, then the normal bundle E(ν) is non-orientable. This is a contradiction. Therefore the result follows from Lemma 2.1.

Now we consider non-trivial smooth SU(3)-actions on hP_3(C), a compact 6-dimensional manifold with the homotopy type of P_3(C).

**Lemma 2.4.** (a) Any isotropy subgroup is of dimension ≥3. (b) hP_3(C) does not admit only one type (H) of isotropy subgroups for any proper subgroup H of SU(3).

**Proof.** If dim SU(3)_x≤1, then the 6-dimensional manifold hP_3(C) contains a submanifold SU(3)/SU(3)_x of dimension ≥7. This is a contradiction. Next if dim SU(3)_x=2, then SU(3)/SU(3)_x is an open and closed submanifold of hP_3(C). Therefore

\[ hP_3(C) = SU(3)/SU(3)_x. \]

By an exact sequence of homotopy groups

\[ \pi_2(SU(3)) \to \pi_2(SU(3)/SU(3)_x) \to \pi_1(SU(3)_x) \to \pi_1(SU(3)), \]

we obtain \( \pi_1(SU(3)_x)=\mathbb{Z}, \) an infinite cyclic group, since SU(3) is 2-connected. On the other hand, since dim SU(3)_x=2, the identity component of SU(3)_x is isomorphic to a 2-dimensional toral group, and hence \( \pi_1(SU(3)_x)=\mathbb{Z} \oplus \mathbb{Z}. \) This is a contradiction. Next we prove (b). It is sufficient to consider the case

\[ \dim H = 3 \text{ or } 4, \]

by (a) and Lemma 2.2. If hP_3(C) admits only one type (H) of isotropy subgroups, then there is a differentiable fibering

\[ SU(3)/H \to h_3P(C) \overset{P}{\longrightarrow} h_3P(C)/SU(3), \]

and the orbit space hP_3(C)/SU(3) is a compact manifold without boundary, by the differentiable slice theorem (iii). First if dim H=3, then the orbit space is of one-dimensional and hence

\[ hP_3(C)/SU(3) = S. \]

By exact sequences

\[ \pi_3(S') \to \pi_3(SU(3)/H) \to \pi_3(hP_3(C)) \overset{P^*}{\longrightarrow} \pi_3(S'), \]

\[ \pi_4(SU(3)) \to \pi_4(SU(3)/H) \to \pi_4(H) \to \pi_4(SU(3)), \]

we obtain \( \pi_4(H)=\mathbb{Z}. \) On the other hand \( \pi_4(H)=0 \) or \( \mathbb{Z}_2, \) since \( \pi_4(SU(2))=0 \)
and \( \pi_1(SO(3)) = \mathbb{Z}_2 \). This is a contradiction. Next if \( \dim H = 4 \), then \( H \) is conjugate to \( N(SU(2)) \) and the orbit space \( hP_3(C)/SU(3) \) is of 2-dimensional. Since

\[
SU(3)/N(SU(2)) = P_2(C),
\]

there is an exact sequence

\[
\pi_1(hP_3(C)) \to \pi_1(hP_3(C)/SU(3)) \to \pi_1(P_3(C)).
\]

Thus the orbit space is a simply connected 2-dimensional compact manifold without boundary. Therefore

\[
hP_3(C)/SU(3) = S^2.
\]

Then there is a contradiction in the following exact sequence

\[
\pi_4(hP_3(C)) \to \pi_4(hP_3(C)/SU(3)) \to \pi_4(P_3(C)),
\]

since \( \pi_4(S^n) = \mathbb{Z}_2 \).

**Remark 2.5.** By the above consideration, if there is a smooth \( SU(3) \)-action on \( hP_3(C) \) without stationary points, then \( hP_3(C) \) admits just two types \((H)\) and \((N(SU(2)))\) of isotropy subgroups, where the identity component of \( H \) is \( SU(2) \).

Proof of Theorem 0.2 (a). If there is a smooth \( SU(3) \)-action on \( hP_3(C) \) with just two types \((H)\) and \((N(SU(2)))\) of isotropy subgroups, where the identity component of \( H \) is \( SU(2) \), then \( hP_3(C) \) is a special \( SU(3) \)-manifold in the sense of Hirzebruch-Mayer [4]. Therefore the orbit space \( hP_3(C)/SU(3) \) is a compact smooth manifold with boundary, and hence

\[
hP_3(C)/SU(3) = [0, 1].
\]

Let \( p: hP_3(C) \to [0, 1] \) be a projection and

\[
X_0 = p^{-1} \left( \left[ 0, \frac{1}{2} \right] \right), \quad X_1 = p^{-1} \left( \left[ \frac{1}{2}, 1 \right] \right).
\]

Then \( X_0 \) and \( X_1 \) are diffeomorphic to the disk bundle of \( n \)-fold tensor product of the canonical complex line bundle over \( P_3(C) \) for certain positive integer \( n \), by the differentiable slice theorem (iii). Therefore \( X_0 \cap X_1 \) is a 5-dimensional rational homology sphere. Then there is a contradiction in the following exact sequence of cohomology groups with rational coefficients,

\[
H^4(X_0 \cap X_1) \to H^4(hP_3(C)) \to H^4(X_0) \oplus H^4(X_1) \to H^4(X_0 \cap X_1).
\]

Therefore any smooth \( SU(3) \)-action on \( hP_3(C) \) has at least one stationary point, by Remark 2.5.
Lemma 2.6. Consider a non-trivial smooth $SU(3)$-action on a connected 6-dimensional manifold $M$. Let $x \in M$ be a stationary point. Then the normal representation $SU(3) \rightarrow O(6)$ is equivalent to the standard inclusion $SU(3) \subset O(6)$, and $SU(2)$ is a principal isotropy subgroup.

Proof. This follows from the fact that non-trivial 6-dimensional real representation of $SU(3)$ is isomorphic to the real restriction of the standard 3-dimensional complex representation.

Remark 2.7. Denote by $V_3$, the 3-dimensional complex vector space $\mathbb{C}^3$ with the standard $SU(3)$-action. Then $P(C^3 \oplus V_3)$ is the complex projective space $P_3(C)$ with a non-trivial linear $SU(3)$-action, where the $SU(3)$-action on $C^3$ is trivial. Denote by $D^6$ the unit disk in $V_3$. Then there is an equivariant decomposition

$$P(C^3 \oplus V_3) = (SU(3) \times_{N(SU(2))} D^3) \cup D^6,$$

where the $N(SU(2))$-action on $D^2$ is induced from the standard action of $N(SU(2))/SU(2) = S^1$ on $D^2$ and $h$ is an equivariant diffeomorphism on boundaries.

Lemma 2.8. Any equivariant diffeomorphism on $\partial D^6$ is extendable to an equivariant diffeomorphism on $D^6$.

Proof. Since the $SU(3)$-action on $\partial D^6$ is transitive, it is easy to prove that any equivariant diffeomorphism on $\partial D^6$ is given by a scalar multiplication

$$(z_1, z_2, z_3) \rightarrow (uz_1, uz_2, uz_3),$$

where $(z_1, z_2, z_3) \in \partial D^6$, $u \in \mathbb{C}$ and $|u| = 1$. Such a diffeomorphism is canonically extended to an equivariant diffeomorphism on $D^6$.

Proof of Theorem 0.2 (b). Let $hP_3(C)$ admit a non-trivial smooth $SU(3)$-action. Then we can use Lemma 2.6, via Theorem 0.2 (a). Thus $SU(2)$ is a principal isotropy subgroup, and hence the possible types of isotropy subgroups are

$$(SU(2)), (N(SU(2))) \text{ and } (SU(3)),$$

by Lemma 2.2 and Lemma 2.3. In any case, $hP_3(C)$ becomes a special $SU(3)$-mainfold with the orbit space $[0,1]$. If the type $(N(SU(2)))$ does not appear, then $hP_3(C)$ is diffeomorphic to $D^6 \cup D^2$. This is a contradiction. Therefore $hP_3(C)$ has isotropy subgroups of type $(N(SU(2)))$ and of type $(SU(3))$. Hence, by the differentiable slice theorem (iii), there is an equivariant decomposition

$$hP_3(C) = (SU(3) \times_{N(SU(2))} D^3) \cup D^6,$$
where $k$ is an equivariant diffeomorphism on boundaries. Moreover, there is an equivariant diffeomorphism from $hP_3(C)$ to $P(C' \oplus V_3)$, by making use of Lemma 2.8 and Remark 2.7.

3. Concluding remarks

3.1. If $G = T^n$, the $n$-dimensional toral group, then it is known that for any smooth $G$-action on an oriented compact manifold $M$ without boundary, each connected component of the stationary point set $M^G$ is canonically oriented and the index formula

$$I(M) = I(M^G)$$

holds. Thus, we ask whether the above is true or not when $G$ is a compact connected Lie group. The answer is no as follows. Denote by $S_k(V_n)$ the $k$-th symmetric product of $V_n$ which is $C^n$ with the standard $SU(n)$-action. If $n \geq 2$ and $n-1 < 2^s$, then

$$t = \dim c S_k(V_n)$$

is odd, and there is a linear $SU(n)$-action on $P_{s+t}(C)$ with $P_s(C)$ as the stationary point set for each integer $s$. This example shows that the index formula is false for $SU(n)$-actions in general. Similarly, we can construct linear $SO(n)$-actions on $P_{s+t}(R)$ with $P_s(R)$ as the stationary point set. This example shows that there are smooth $SO(n)$-actions for which the stationary point sets are not orientable.

3.2. Let $V_n$ be as above, then $SU(n)$-manifold $P(C' \oplus V_n)$ has only one stationary point for each $n \geq 2$. Such a phenomenon does not appear for compact $G$-manifold without boundary when $G$ is an abelian group such as a toral group or a finite cyclic group of prime order.

3.3. Let $G$ be a compact Lie group. Denote by $F_A$ the family of all closed subgroups of $G$, and by $F_P$ the family of all closed proper subgroups of $G$. Then there is an exact sequence of bordism modules of weakly complex $G$-manifolds,

$$\cdots \to U_*(G; F_p) \xrightarrow{i_*} U_*(G; F_A) \xrightarrow{j_*} U_*(G; F_A, F_p) \xrightarrow{\partial_*} U_*(G; F_P) \to \cdots$$

It is known that $i_*$ is trivial for $G = T^n$ and almost trivial for $G$ a finite cyclic group of prime order. On the other hand, we can prove that $i_*$ is injective when $G$ is a compact connected semi-simple Lie group, by making use of projective space bundles associated to complex $G$-vector bundles.
References


