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ON SYMMETRIC STRUCTURE OF A FINITE SET

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1. Introduction

A symmetric structure of a finite set A is defined to be a mapping S of A into the group of permutations on A (the image of an element a in A by S is denoted by S_a or by $S[a]$ and the image of an element b in A by a permutation S_a is denoted by bS_a) such that (i) $aS_a=a$, (ii) $S_a^2=I$ (the identity permutation) and (iii) $S[bS_a]=S_aS_bS_a$ for a and b in A. A set with a symmteric structure is called a symmetric set (with a given symmetric structure). Every group G has a symmetric structure S defined by $bS_a=ab^{-1}a$ for a and b in G, and when we regard a group as a symmetric set we always take this symmetric structure. Generally a symmetric set has a more complicate structure than a group and to develop a structure theory of a symmetric set seems to be an open problem. In this note, we first investigate the following two conditions.

(E) $S_a \neq S_b$ if $a \neq b$.

(H) For any elements a and b, there exists an element c such that $aS_c=b$.

Symmetric sets which satisfy (E) (or (H)) are called *effective* (or *homogeneous*).

Proposition 1. (H) *implies* (E).

Proof. Suppose that (H) is satisfied. Fix an element a and consider a correspondence $b \to b'$ defined by $aS_b=b'$. The correspondence is a surjective mapping of A to A due to (H). Since A is a finite set, it is a bijection. Therefore, if $b \neq c$, then $aS_b \neq aS_c$. Naturally $S_b \neq S_c$.

Actually (H) is stronger than (E).

EXAMPLE 1. Let $A=\{1, 2, 3, 4, 5, 6\}$. Consider S defined by $S_1=(24)(36)$, $S_2=(14)(35)$, $S_3=(25)(16)$, $S_4=(56)(12)$, $S_5=(23)(46)$ and $S_6=(45)(13)$. S is a symmetric structure of A. (E) is satisfied but not (H), since 1 is not mapped to 4 by any S_i.

Next, we consider the group of displacements of A, which is defined to be a subgroup of the group of permutations on A generated by S_aS_b for all a and b in A. Denote it by $G(A)$.

Proposition 2. *Fix an element e in A and consider a mapping of A to G(A) defined by* $a \to S_e S_a$. *Then the mapping is a homomorphism of a symmetric set A to a symmetric set G(A).*

Proof. Let S' be the symmetric structure of a group $G(A)$. We have to show that aS_b is mapped to $(S_eS_a)S'[S_eS_b]$. Now aS_b is mapped to $S_eS[aS_b]$ which is equal to $S_eS_bS_aS_b = S_eS_bS_aS_eS_eS_b = S_eS_b(S_eS_a)^{-1}S_bS_e = (S_eS_a)S'[S_eS_b]$ as we claimed.

If A is effective, then the homomorphism in Proposition 2 is an isomorphism of A into $G(A)$, and hence in this case a symmetric set A is regarded as a subset of a group closed under the operation $ab^{-1}a$. Note also that $G(A)$ is generated by S_eS_a (a in A) as $S_aS_b = S_aS_eS_eS_b$ and $S_aS_e = (S_eS_a)^{-1}$. In **3**, it will be proved that an effective symmetric set is isomorphic with $G(A)$ if and only if $G(A)$ is abelian. (cf. Proposition 2.5. p. 137 [2]) One of the basic concepts in studying the structure of a symmetric set is a cycle which will be defined in **2** as a generalization of a cyclic subgroup of a group. The structure of a cycle will be completely determined in **2**. In **4**, we shall show that a homogeneous symmetric set of p^2 elements where p is an odd prime is isomorphic with an abelian group, but in **5** we shall show that there is a homogeneous symmetric set of 27 elements which is not isomorphic with a group. In **6**, we shall give a complete table of symmetric structures of a set of 5 elements. It would be a rather complicate work to find a complete table of symmetric structures of a set of more than 5 elements.

2. Cycles

Fix an element e in A. For an element a in A, we define

$$a^k = \begin{cases} e(S_eS_a)^i & \text{if } k = 2i \\ a(S_eS_a)^i & \text{if } k = 2i+1 . \end{cases}$$

From now on, we shall denote S_eS_a by U_a. Clearly, $U_a^{-1} = S_aS_e$ and $S[bU_a] = U_a^{-1}S_bU_a$.

Proposition 3. $$S[a^k] = S_eU_a^k .$$

Proof. First suppose $k=2i$. Then $S[a^k] = S[eU_a^i] = U_a^{-i}S_eU_a^i = (S_aS_e)^iS_eU_a^i = S_eS_e(S_aS_e)^iS_eU_a^i = S_eU_a^iS_eS_eU_a^i = S_eU_a^{2i} = S_eU_a^k$. Next, suppose $k=2i+1$. Then $S[a^k] = S[aU_a^i] = U_a^{-i}S_aU_a^i = (S_aS_e)^iS_aU_a^i = S_eS_e(S_aS_e)^iS_aU_a^i = S_eU_a^{i+1+i} = S_eU_a^k$.

Proposition 4. $a^jS[a^k] = a^{-j+2k}$. *Especially* $a^jS[a^{j+1}] = a^{j+2}$.

Proof. $a^jS[a^k] = a^jS_eU_a^k$ by Proposition 3. Suppose $j=2i$. Then $a^jS_eU_a^k = e(S_eS_a)^iS_eU_a^k = eS_e(S_aS_e)^i(S_eS_a)^k = eU_a^{-i+k} = a^{-2i+2k} = a^{-j+2k}$. Suppose $j=2i+1$.

Then $a^j S[a^k] = a(S_e S_a)^i S_e U_a^k = aS_a(S_e S_a)^i S_e U_a^k = aU_a^{-i-1} U_a^k = aU_a^{-i-1+k} = a^{2(-i-1+k)+1} = a^{-j+2k}$.

Now consider a sequence $e, a, a^2, a^3, \cdots$. The latter part of Proposition 4 implies that in the sequence the succeeding element of an element, say, b in the sequence is an image of the preceding element by S_b. We call such a sequence a cycle (generated by a with a base element e). Later we shall consider a set of all distinct elements in a cycle and call it also a cycle. Let $\text{ord}_e a$ (or simply ord a if the base element e is implicitly pregiven) be the least positive integer n such that $a^n = e$, the existence of which is given in the following proposition.

Proposition 5. *There exists ord a, and if we denote it by n and* ord U_a *(the order of a permutation U_a) by m, then $n=m$ or $2m$. If (E) holds, then $n=m$.*

Proof. $a^{2m} = eU_a^m = e$, and so $n \leq 2m$. On the other hand, by Proposition 3, $U_a^n = S_e S[a^n] = S_e S_e = I$. So m divides n. Therefore $n=m$ or $2m$. We have $I = U_a^m = S_e S[a^m]$, which implies that $S[a^m] = S_e$. Therefore, $a^m = e$ or $n=m$ if (E) holds.

From now on, we shall denote $n = \text{ord } a$ and $m = \text{ord } U_a$.

Theorem 1. *If $i \equiv j \bmod 2m$, then $a^i = a^j$. Conversely if $a^i = a^j$, then $i \equiv j \bmod m$.*

Proof. If $i \equiv j \bmod 2m$, then $a^i = a^j$ by definition of a^k. Suppose that $a^i = a^j$. Then $U_a^i = U_a^j$ by Proposition 3, whence $i \equiv j \bmod m$.

Corollary. *$a^k = e$ if and only if $k \equiv 0 \bmod n$.*

Proof. By Theorem 1, a cycle $e, a, \cdots$ consists of repetitions of $e, a, \cdots, a^{2m-1}$. So if $n=2m$, Corollary is clear. Suppose $n=m$. We have to show that if $a^i = e$ for $0 < i < 2m$ then $i=n$. But, by Theorem 1, if $a^i = e$ then $i \equiv 0 \bmod m$ $(=n)$. Therefore $i=n$.

So far we have seen that a cycle $e, a, \cdots$ consists of repetitions of $e, a, \cdots, a^{n-1}$ or of repetitions of $e, a, \cdots, a^{2n-1}$. When we have the former case, we call the cycle *regular*.

Theorem 2. *If n is odd or if $n=2m$, then a cycle $e, a, \cdots$ is regular. If (E) holds, then every cycle is regular.*

Proof. The last statement is clear because $a^i = a^j$ if and only if $S[a^i] = S[a^j]$ when (E) holds, i.e., if and only if $i \equiv j \bmod m$ $(=n)$. Next suppose $n = 2k+1$. To show the regularity of the cycle, it is sufficient to show that $a^{n+1} = a$. Now $a^{n+1} = a^{2k+2} = a^{2(k+1)} = eU_a^{k+1}$. Since $e = a^n = aU_a^k$, we have that $eU_a^{k+1} = aU_a^k U_a^{k+1} = aU_a^{2k+1} = aU_a^n = a$. Here note that in this case $n=m$ because n is odd. If $n=2m$, then the cycle is clearly regular.

Corollary. $a^{n+2k}=a^{2k}$.

Proof. If the cycle is regular, there is nothing to prove. So we may suppose by Theorem 2 that n is even and $n=m$. Then $a^{n+2}=a^{n}S[a^{n+1}]=eS_{e}U_{a}^{n+1}=eU_{a}=a^{2}$. Now consider a cycle $e, a^{2}, a^{4}, \cdots$ It consists of repetitions of $e, a^{2}, \cdots, a^{n-2}$. This completes the proof of Corollary.

EXAMPLE 2. Let $A=\{1,2, \cdots, 6\}$. Define $S_{1}=(26)(45)$, $S_{2}=(13)(46)$, $S_{3}=(24)(56)$, $S_{4}=(13)(25)$, $S_{5}=S_{2}$ and $S_{6}=S_{4}$. S is a symmetric structure of A. We have a cycle 1, 2, 3, 4, 1, 5, 3, 6, 1, 2, $\cdots$ The cycle is not regular. A is not effective and $n=m=4$.

The following proposition will be used in **3**.

Proposition 6. *A symmetric set A is homogeneous if and only if* $\mathrm{ord}_{e}\, a$ *is odd for any e and a in A.*

Proof. Let C be a subset of A consisting of all distict elements of $e, a, \cdots$. C is also called a cycle. C is a symmetric set with a symmetric structure induced from that of A. Generally we call such a subset as a symmetric subset of A. If A is homogeneous, then every symmetric subset B of A is also homogeneous as is seen from the proof of Proposition 1. So if A is homogeneous, then C is so. Then ord a must be odd. Otherwise, $n=2k$ and $S[a^{k}]=S_{e}$ since $a^{t}S[a^{k}]=a^{-t+2k}=a^{-t}=a^{t}S_{e}$ but then $a^{k}=e$ (a contradiction). Conversely suppose that ord a is odd for any e and a. Put ord $a=2k+1$. Consider an element $b=a^{k+1}$, and we see that $aS_{b}=a^{-1+2(k+1)}=a^{2k+1}=e$ by Proposition 4. Thus a is mapped to e. But a and e are taken arbitrarily in A. So (H) is satisfied.

3. Abelian symmetric sets

A is called abelian if $G(A)$ is abelian.

Lemma. *Let e, a and d be elements in an abelian symmetric set A. Put $d^{(k)}=dU_{a}^{k}$. Then $d, d^{(1)}, d^{(2)}, \cdots$ is a cycle. If $m(=\mathrm{ord}\, U_{a})=2j$, then* $\mathrm{ord}\, S_{d}S[d^{(1)}]=j$.

Proof. $S_{d}S[d^{(1)}]=S_{d}S[dU_{a}]=S_{d}S_{a}S_{e}S_{d}S_{e}S_{a}$. But $S_{a}S_{e}S_{d}=S_{d}S_{e}S_{a}$ since $S_{e}S_{a}S_{e}S_{d}=S_{e}S_{d}S_{e}S_{a}$ for $G(A)$ is abelian. Therefore, $S_{d}S[d^{(1)}]=S_{d}S_{d}S_{e}S_{a}S_{e}S_{a}=U_{a}^{2}$, and hence ord $S_{d}S[d^{(1)}]=j$ if ord $U_{a}=2j$. Now if $k=2i$, then $d^{(k)}=dU_{a}^{2i}=d(S_{d}S[d^{(1)}])^{i}$, and if $k=2i+1$, then $d^{(k)}=dU_{a}^{2i+1}=d^{(1)}U_{a}^{2i}=d^{(1)}(S_{d}S[d^{(1)}])^{i}$. This shows that $d, d^{(1)}, d^{(2)}, \cdots$ is a cycle.

Theorem 3. *An effective abelian symmetric set is homogeneous.*

Proof. Suppose that A is abelian and effective. By Proposition 6, we have to show that ord a is odd. Assume on the contrary that ord $a=2j$. Due to (E), $m(=\text{ord } U_a)=n=2j$. Therefore, $j<m$ or $U_a^j\neq I$. Then there exists an element d such that $dU_a^j\neq d$. On the other hand, if we apply the above lemma on d, we have a cycle $d, d^{(1)}, \cdots$ such that ord $S_dS[d^{(1)}]=j$. Due to (E), ord $S_dS[d^{(1)}]=\text{ord}_d\, d^{(1)}$. Thus $d^{(j)}=d$. This is a contradiction.

Theorem 4. *Let A be an effective symmetric set. Then A is abelian if and only if $G(A)=\{S_eS_a \mid a \text{ in } A\}$ for an element e in A.*

Proof. First suppose that A is abelian. By the proof of Theorem 3, ord $a=2k+1$ (odd). Then $e=a^{2k+1}=aU_a^k$, and so $eU_a^{k+1}=aU_a^kU_a^{k+1}=aU_a^{2k+1}=a$. Therefore, $a^{2k+2}=a$, or $a^{2t}=a$ with $t=k+1$. Then $S_bS_eS_a=S_bS_eS[a^{2t}]=S_bS_eS[eU_k^t]=S_bS_e(S_aS_e)^tS_e(S_eS_a)^t=(S_aS_e)^tS_bS_eS_e(S_eS_a)^t=(S_aS_e)^tS_b(S_eS_a)^t=S_c$ with $c=bU_a^t$. This implies that $S_eS_bS_eS_a=S_eS_c$. Also we have that $(S_eS_a)^{-1}=(S_eS_a)^{m-1}=S_eS_d$ with $d=a^{m-1}$. Every element of $G(A)$ is a product of S_eS_a (a in A). Then the above result shows that every element of $G(A)$ is expressed as S_eS_a with an element a in A. As to the converse, note that $G(A)$ has an automorphism (as a group) defined by $T\to S_eTS_e$ with a fixed element e. If $G(A)=\{S_eS_a \mid a \text{ in } A\}$, then the automorphism maps every element of $G(A)$ to its inverse. In such a case, a group must be abelian. (The converse part of Theorem 4 is pointed out by Prof. H. Nagao.)

4. Homogeneous symmetric sets of p^2 elements

Let A be a symmetric set and C a symmetric subset of A. Moreover, suppose that C is a cycle $\{e, a, \cdots, a^{t-1}\}$ where $t=\text{ord } a$. We denote $\{S_eS[a^i] \mid i=0, 1, \cdots, t-1\}$ by $G'(C)$. $G'(C)$ is a cyclic subgroup of $G(A)$. Now suppose that A is homogeneous. For an element b in A, $bG'(C)$ consists of t elements because $bS_eS[a^i]=bS_eS[a^j]$ implies $a^i=a^j$ by the proof of Proposition 1. If d is an element in A, then $bG'(C)$ and $dG'(C)$ are either identical or disjoint as $G'(C)$ is a group. Thus A is a set-theoretical union of disjoint subsets $bG'(C)$, $b'G'(C), \cdots$. This proves the following.

Proposition 7. *Let A be a homogeneous symmetric set of k elements and C a symmetric subset of t elements which is a cycle. Then t divides k.*

Now let A be a homogeneous symmetric set of p^2 elements where p is an odd prime. If A is a cycle, it is naturally abelian and is isomorphic with a cyclic group. So, assume that A is not a cycle. By Proposition 7, every non-trivial cycle consists of p elements. From now on, we are going to use some geometric terms. Call an element in A a point. A cycle is said to be passing through a point if it contains the point. Then we can show that there is one and only one cycle passing through given two points as p is a prime. Two cycles are said to be parallel if they have no point in common. Next we shall show that, if a point a is not contained in a cycle C, then there is one and only one cycle passing through

a and parallel to C. To see it, we first note that the number of cycles passing through a point is $(p^2-1)/(p-1)=p+1$. Now there are p cycles passing through a and points in C. Thus we have the above fact. Then, if C_1 is parallel to C_2 and C_2 to C_3 (C_i are all different cycles), C_1 is then parallel to C_3. By counting the number again, we conclude that there are exactly p cycles which are parallel each other. Now fix a point e in A. Let D_0 be a cycle $\{e, a, \cdots, a^{p-1}\}$. Let C_i be cycles passing through a^i and parallel to C_0 ($i=0, 1, \cdots, p-1$). Let C_0 be $\{e, b, \cdots, b^{p-1}\}$, and D_j cycles passing through b^j and parallel to D_0 ($j=0, 1, \cdots, p-1$). We shall show that $C_iS_d=C_k$ for $i\neq k$ if and only if d is in C_j where $k\equiv 2j-i \bmod p$. First, we have that $C_iS[a^j]=C_k$ since $C_iS[a^j]$ contains a^k and is parallel to C_i. (If $C_iS[a^j]$ and C_i intersect at a point c, then $c=c'S[a^j]$ with a point c' in C_i which implies that a^j is in C_i.) Now consider a set $F=\{u \text{ in } A \mid C_iS_u=C_k\}$. It is not hard to show that F is a symmetric subset of A and is parallel to C_i. Since F contains a^j, $F=C_j$. Similarly $D_iS_d=D_k$ for $i\neq k$ if and only if d is in D_j where $k\equiv 2j-i \bmod p$. Now every point in A is determined as an intersection point of C_i and D_j for some i and j. Denote the point by $u(i, j)$. Then we have by the above result that $u(i, i')S[u(j, j')]=u(k, k')$ where $k\equiv 2j-i$ and $k'\equiv 2j'-i' \bmod p$. Thus A is isomorphic with a group which is a direct product of two cyclic groups of order p.

5. A homogeneous set of 27 elements

Let $A=\{1, 2, \cdots, 9, 1', 2', \cdots, 9', 1'', 2'', \cdots, 9''\}$. Define S as follows. $iS_k=2k-i$, $i'S_k=(i+k)''$, $i''S_k=(i-k)'$; $iS_{k'}=(i+k)''$, $i'S_{k'}=(2k-i)'$, $i''S_{k'}=i-k$; $iS_{k''}=(k-i)'$, $i'S_{k''}=k-i$, $i''S_{k''}=(2k-i)''$, Here all integers are considered mod 9. By routine computations we can verify that S is a symmetric structure of A satisfying (H). For example, we have to check that $S_{k''}S_tS_{k''}=S[tS_{k''}]=S[(k-t)']$. But the both left and right sides of the above will map i to $(k-t+i)''$, i' to $(2k-2t-i)'$, and i'' to $-k+t+i$, and hence we have the identity. A is not isomorphic with a group, because there is one and only one cycle of order 9 passing through a point, say, 1; namely, $\{1, 2, \cdots, 9\}$. On the other hand, in a group of order 27, taking the group identity e, we can see that either there is no cycle (in this case cyclic subgroup) of order 9 passing through e or else there are more than one cycle of order 9 passing through e. (See p. 52 [1].)

6. A table of symmetric structures of a set of 5 elements

The following is a complete table of symmetric structures of a set of 5 elements 1, 2, $\cdots$, 5. There are 14 types including a trivial case.

Type	S_1	S_2	S_3	S_4	S_5
1	(25) (34)	(13) (45)	(24) (15)	(35) (12)	(14) (23)
2	(24)	(13)	(24)	(13)	I
3	(24)	(13)	(24)	(13)	(13)
4	(24)	(13)	(24)	(13)	(13) (24)
5	(23)	(13)	(12)	I	I
6	(23) (45)	(13)	(12)	I	I
7	(23) (45)	(13) (45)	(12) (45)	I	I
8	(23)	I	I	I	I
9	(23) (45)	I	I	I	I
10	(23)	I	I	(23)	I
11	(23)	(45)	(45)	I	I
12	(23)	I	I	(23)	(23)
13	(23) (45)	(45)	(45)	I	I
14	I	I	I	I	I

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[2] O. Loos: Symmetric Spaces; Vol. 1, Benjamin, New York, 1969.