ON GALOIS OBJECTS WHICH ARE STRONGLY RADICIAL OVER ITS BASIC RING

Dedicated to Professor M. Takahashi on his 60th birthday

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In [3], Chase and Sweedler introduced a notion of Galois object and extended, in this case, the fundamental theorem of Galois theory for fields. Furthermore, they showed that it contains, as a special case, the fundamental theorem of Galois theory on separable algebras developed by Chase, Harrison and Rosenberg. However they mentioned in [3] that they had, in general, no good characterization of the subalgebras which arose in the Galois correspondence. A purpose of this paper is to show what those subalgebras are in the case of strongly radicial extensions. On the other hand, it is well-known that for a finite purely inseparable extension $K$ of a field $k$, there exists a chain of subfields of $K$: $K = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_r = k$ such that $K_i$ is of exponent one over $K_{i+1}$ for $i=0, 1, 2, \ldots, r-1$. We shall study this analogy in the case of Galois objects over a field which are strongly radicial over their basic field.

Let $H$ be a finite cocommutative split Hopf algebra over a commutative ring $A$ and $C$ a Galois $H^{*\otimes}$-object over $A$ which is strongly radicial over $A$.

In our first section, we shall study a coalgebra structure of $H$. Moreover we shall show that there exists a bijection between the set of admissible Hopf subalgebras of $H$ and the set of distinguished intermediate rings between $A$ and $C$.

In our second section, we shall exhibit an existence of a sequence of subrings of $C$

$$C = C_0 \supset C_1 \supset \cdots \supset C_i \supset \cdots \supset C_n = A$$

satisfying the followings for each $i=0, 1, \ldots, n-1$:

1) $C \# K_i \cong \text{Hom}_{C_i}(C, C)$ via a canonical map where $K_i$ is a Hopf subalgebra of $H$.

2) $d(C_i) \subseteq C_i$ for $d \in H$.

3) $C_i[\Delta \circ \varphi(C_i[C_{i+1}])] = \text{Hom}_{C_{i+1}}(C_i, C_i)$ for $i=0, 1, \ldots, n-1$.

1) For the definition, see §1.
2) $H^*$ denotes a dual Hopf algebra of $H$.
3) For the definition, see [3, Def. 7.1].
4) For the definition, see a following part of the proof of Proposition 3 below.
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(4) \( C_i \) is finitely generated projective as a \( C_{i+1} \)-module.

Throughout the following discussion, all rings are commutative with an identity, and all homomorphisms are unitary. Unadorned \( \otimes \) will mean \( \otimes_A \). If \( A \) is a subring of a ring \( C \), both \( A \) and \( C \) are assumed to have a common identity. In this paper, \( A \) will denote a commutative ring such that \( A/p \) is of characteristic non-zero for each \( p \in \text{Spec}(A) \). We will use the definitions and terminology in [6] with respect to coalgebras and Hopf algebras, and in [4] and [7] with respect to high order derivations and strongly radicial extensions, respectively. The author likes to express his thanks to the referee for comments on Proposition 4.

1. Galois correspondence theorem

Let \( C \) be a commutative algebra over a ring \( A \). Let \( H \) be a finite commutative Hopf-algebra over \( A \). Then \( C \) is called a Galois \( H \)-object if \( C \) is a finitely generated and faithful projective \( A \)-module and there is a map \( \alpha : H^* \otimes C \to C \) which measures \( C \) to \( C \) such that a map \( \varphi : C \otimes H^* \to \text{Hom}_A(C, C) \) by \( \varphi(x \otimes u)(y) = \alpha(u \otimes y) \) is an algebra-isomorphism (c.f. 3).

Let \((H, \Delta, \varepsilon)\) be a coalgebra over a commutative ring \( A \) where \( \Delta \) is its diagonal map and \( \varepsilon \) is its augmentation map. For \( g \in H \), \( g \) is called a grouplike element in \( H \) if \( \Delta(g) = g \otimes g \) and \( \varepsilon(g) = 1 \). Let \( G(H) \) be denoted the set of grouplike elements in \( H \). \( H \) is called a split coalgebra in case \( H = \bigoplus U_g \) as \( A \)-modules where each \( U_g \) is a subcoalgebra of \( H \) in which \( g \) is an only grouplike element and \( U_g = Ag + (U_g \cap \text{Ker} \varepsilon) \).

**Lemma 1.** Let \( C \) be a strongly radicial extension of a ring \( A \). Then so is \( C \otimes C \).

**Proof.** It is obvious from the definition.

**Lemma 2.** Let \( H \) be a finite commutative Hopf-algebra over a local ring \( A \) such that there is a Galois \( H \)-object \( C \) over \( A \) which is strongly radicial over \( A \). Then \( H \) is a local ring.

**Proof.** From Lemma 1, \( C \otimes C \) is a strongly radicial extension of a local ring \( A \) and so is local [c.f., 7, Theorem 5]. On the other hand, we have \( C \otimes C \cong C \otimes H \) as algebras [c.f., 3, Theorem 9.3]. So \( H \) is local.

**Proposition 3.** Let \( H \) be a finite commutative Hopf algebra over a commutative ring \( A \), whose dual coalgebra \( H^* \) is split. Let \( C \) be a Galois \( H \)-object over \( A \). If \( C \) is strongly radicial over \( A \), then \( H^* = A 1 \oplus \text{Ker}(\varepsilon_*) \) as \( A \)-module where \( \varepsilon_* \) is an augmentation map of \( H^* \).

5) For the definition, see [3, Def. 7.1].
Proof. Since $H^*$ is split, $H^*$ has a decomposition $H^* = U_1 \oplus (\bigoplus_{g \in \mathfrak{g}(H)} U_g)$ where each $U_g$ is a subcoalgebra of $H^*$ in which $g$ is an only grouplike element. We need to show that $U_g$ vanishes if $g \neq 1$. Put $\bar{A} = A(p)$, $\bar{C} = \bar{A} \otimes C$ and $\bar{H} = \bar{A} \otimes H$ for $p \in \text{Spec}(A)$. Then $\bar{C}$ is a Galois $\bar{H}$-object. Since $\bar{C}$ is strongly radical over $\bar{A}$, $\bar{H}$ is local. So $(\bar{H})^*$ is irreducible. Then we have $\bar{A} \otimes U_g = 0$ for any $p \in \text{Spec}(A)$ and so $U_g = 0$.

For a coalgebra $H$, let $H^+$ denote $\text{Ker}(\varepsilon)$ where $\varepsilon$ is an augmentation map of $H$. Moreover, assume $H$ is a finite cocommutative split Hopf algebra over a ring $A$ and $C$ a Galois $H^*$-object over $A$ which is strongly radical over $A$. Then $H$ may be considered to be a subalgebra of $\text{Hom}_{A}(C, C)$. Since $C$ is strongly radicial over $A$, $H$ is local. So $(H)^*$ is irreducible. Then we have $A \otimes U_g = 0$ for any $p \in \text{Spec}(A)$ and so $U_g = 0$.

Proposition 4. Let $A, C, H$ be as above. Let $B$ be a distinguished intermediate ring between $A$ and $C$. Then there exists a subcoalgebra $U$ of $H$ such that $U$ is an $A$-module direct summand of $H$ and $C \otimes U \cong \text{Hom}_{B}(C, C)$ via a canonical map.

Proof. Set $K = \text{Proj}_{M}(H^*)$ for a projection above $\text{Proj}_{M}$. Then we have a split exact sequence of $A$-module $0 \to U^+ \to H^+ \to K \to 0$ where $U^+ = \text{Ker}(\text{Proj}_{M} | H^*)$ and the third arrow denotes $\text{Proj}_{M}$. So, $H^+ = U^+ \oplus K$ where $K$ is an $A$-submodule of $H^*$ which is isomorphic to $K_0$. Now we shall show that $C \otimes U^+$ can be identified with $\text{Des}(C/B)$. Since $U^+$ is obviously contained in $\text{Des}(C/B)$, $C \otimes U^+$ may be regarded to be contained in $\text{Des}(C/B)$. For any $p \in \text{Spec}(A)$, put $\bar{A} = A(p)$ and $\bar{C} = \bar{A} \otimes C$. Then we have $\dim_{A}[\bar{C} \otimes U^+] = \dim_{A}[\bar{C} \otimes H^+] - \dim_{A}[\bar{A} \otimes \text{Des}(C/B)] = \dim_{A}[\bar{A} \otimes M]$, because $\bar{C} \otimes K \cong \bar{C} \otimes K_0 \cong \bar{A} \otimes M$. So $C \otimes U^+ \cong \text{Des}(C/B)$, using Nakayama's lemma. Put $U = A1 + U^+$. Then we shall show that $\Delta(d)$ belongs to $U \otimes X$ for $d \in U^+$. We can assume, without loss of generality, that $A$ is local. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ be $A$-module bases for $U^+, K$, respectively. Since $H^+ \otimes H^* = U^+ \otimes U^* + U^+ \otimes K + K \otimes U^* + K \otimes K$, we have $\Delta(d) - 1 \otimes d - d \otimes 1 = \sum a_i u_i \otimes v_j + \sum b_{i,j} v_j \otimes u_i + \sum c_{i,j} v_j \otimes v_j$ for $X \in U^+ \otimes U^*$, $a_{i,j}, b_{i,j}, c_{i,j} \in A$. Since $[D, x]^n$ belongs to $\text{Des}(C/B) \subset \text{Hom}_{B}(C, C)$ for $D \in \text{Des}(C/B)$, $x \in C$, we obtain $\mu(X(x \otimes y)) + \sum a_i u_i(x) v_j(y) + \sum b_{i,j} v_j(x) u_i(y) + \sum c_{i,j} v_j(x) v_j(y) = b \mu(X(x \otimes y)) + b \sum a_i u_i(x) v_j(y) + b \sum b_{i,j} v_j(x) u_i(y) + b \sum c_{i,j} v_j(x) v_j(y)$ for $x$.

6) $A(p)$ denotes the residue field $A/pA_p$.
7) C.f., 4, chap. 1, §1.
where \( \mu \) is a contraction map \( : C \otimes C \to C \). Using the fact that any element of \( U^+ \) commutes with each element of \( B \), we have \( \sum a_{i,j}(v_j(by) - bv_j(y))u_i - \sum c_{i,j}(v_j(by) - bv_j(y))v_i = 0 \) and so \( \sum a_{i,j}(v_j(by) - bv_j(y)) = 0 \). Hence \( \sum a_{i,j}v_j \) and \( \sum c_{i,j}v_j \) belong to \( \Delta \) by \( 2^e(C/B) \), showing \( a_{i,j} = 0 \) and \( c_{i,j} = 0 \) for all \( i, j \). Moreover, we have \( b_{i,j} = 0 \), because \( H \) is cocommutative. This shows that \( U \) is a subcoalgebra of \( H \). Since \( U \) is obviously a subalgebra of \( H \), \( U \) is a subbialgebra of \( H \). This completes the proof.

**Remark.** Let \( A, C, H \) be as above. Let \( B \) be an intermediate ring between \( A \) and \( C \), over which \( C \) is projective. Then \( B \) is distinguished if and only if \( C.(H^+ \cap \Delta \otimes (C/B)) = \Delta \otimes (C/B) \).

**Proof.** The "only if" part follows from the proof of Proposition 4. Since \( C \) is \( B \)-projective, we can write \( \Delta \otimes (C/A) = \Delta \otimes (C/B) \oplus M \) for a submodule \( M \) of \( \Delta \otimes (C/A) \). Now, \( C.(H^+ \cap \Delta \otimes (C/B)) = \Delta \otimes (C/B) \). Then we may regard \( C \otimes (H^+ \cap \Delta \otimes (C/B)) = \Delta \otimes (C/B) \), by identification \( C \otimes H^+ = \Delta \otimes (C/A) \). So we have a canonical isomorphism \( C \otimes \text{Proj}_M(H^+) \cong M \). This shows the "if" part.

**Theorem 5.** Let \( H \) be a cocommutative split Hopf algebra over a commutative ring \( A \). Let \( C \) be a Galois \( H^* \)-object over \( A \) which is strongly radicial over \( A \). Then there exists a bijection between the set \( \mathcal{F} \) of subbialgebras of \( H \) which are \( A \)-module direct summands of \( H \) and the set \( \mathcal{G} \) of distinguished intermediate rings between \( A \) and \( C \). Its correspondence is given by associating \( U \in \mathcal{F} \) with \( Ker(U^+)^{\perp} = \{ x \in C \mid d(x) = 0 \} \) for \( d \in U^+ \).

**Proof.** For \( B \in \mathcal{G} \), take \( U \) as the proposition above. Then \( U \) belongs to \( \mathcal{F} \). Moreover, it is obvious that \( U^+ = \{ d \in H \mid d(by) = bd(x) \} \) for \( x \in C, b \in B \). Conversely, put \( B = Ker(U^+) \) for \( U \in \mathcal{F} \). Then we have \( B = Ker(C \otimes U^+) \) and so \( C \otimes U^+ = \Delta \otimes (C/B) \) [c.f., 7, Theorem 15]. Hence \( B \) belongs to \( \mathcal{G} \), because \( U^+ \) is an \( A \)-module direct summand of \( H^+ \). So, using again [7, Theorem 15], a correspondence : \( U \to Ker(U^+) \) gives a bijection between \( \mathcal{F} \) and \( \mathcal{G} \).

2. Galois objects over a field which are strongly radicial over their basic field

Throughout the following discussion, we shall assume that \( H \) is a cocommutative pointed\(^9\) Hopf algebra over a field \( A \) of characteristic \( p \neq 0 \) and \( C \) is a strongly radicial extension of \( A \) which is a Galois \( H^* \)-object over \( A \). In this case, both \( H \) and \( C \otimes H \) may be regarded to be contained in \( \text{Hom}_A(C, C) \). Since \( H \) measures \( C \) to \( C \), we have \( d(1) = \varepsilon(d)1 \) for \( d \in H \) where \( \varepsilon \) is an augmentation map for \( H \) and \( 1 \) denotes an identity in \( C \). So \( d(1) = 0 \) for \( d \in H^+ = Ker(\varepsilon) \).

\(^8\) This remark was advised by the referee.
\(^9\) For the definition, see [6].
This shows \( C \otimes H^+ = \otimes H(C|A) \), because \( C^*H = \mathcal{D}(C|A) \). Hence we obtain \( \text{Hom}_C(\otimes H(C|A), C) \cong \text{Hom}_C(C \otimes H^+, C) \cong C \otimes \text{Hom}_A(H^+, A) \cong C \otimes (H^+)^* \) and so \( C \otimes (H^+)^* \cong J_{C/A} \) as left C-modules where \( J_{C/A} \) is a kernel of a contraction map: \( C \otimes C \to C \).

**Lemma 6.** \( J_{C/A} \cong C \otimes (H^+)^* \) as rings.

Proof. \( \text{Hom}_C(C \otimes H^+, C) \) forms a ring by a multiplication \( F \ast G : 1 \otimes d \to \sum_{\alpha \gamma} F(1 \otimes d_{(\alpha)}) G(1 \otimes d_{(\gamma)}) \) for \( F, G \in \text{Hom}_C(C \otimes H^+, C) \) where \( \sum d_{(\alpha)} \otimes d_{(\gamma)} = \Delta(d) - 1 \otimes d - d \otimes 1 \) for a diagonal map \( \Delta \) of \( H \). Then \( \text{Hom}_C(C \otimes H^+, C) \) is isomorphic to \( C \otimes (H^+)^* \) as rings. Thus, in order to complete the proof, it suffices to show that \( J_{C/A} \cong \text{Hom}_C(C \otimes H^+, C) \) as rings. A C-module map \( \alpha : J_{C/A} \to \text{Hom}_C(C \otimes H^+, C) \) by \( \alpha(1 \otimes x - x \otimes 1)(c \otimes d) = cd(x) \) for \( c, x \in C, d \in H^+ \) is an isomorphism. We shall show that \( \alpha \) is a ring-homomorphism. Since \( (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1) = 1 \otimes xy - xy \otimes 1 - x(1 \otimes y - y \otimes 1) - y(1 \otimes x - x \otimes 1) \), we have \( \alpha((1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1))(1 \otimes d) = d(xy) - xd(y) - d(x)y = \sum d_{(\alpha)}(x) d_{(\gamma)}(y) \). On the other hand, we obtain \( \{\alpha(1 \otimes x - x \otimes 1) \ast \alpha(1 \otimes y - y \otimes 1)\} \) \( \otimes d = \sum_{\gamma \delta} \alpha(1 \otimes x - x \otimes 1)(1 \otimes d_{(\alpha)}) \alpha(1 \otimes y - y \otimes 1)(1 \otimes d_{(\gamma)}) = \sum_{\alpha \gamma} d_{(\alpha)}(x) d_{(\gamma)}(y) \), showing our requirement.

Since \( H \) is irreducible as a coalgebra, \( A \) is a coradical of \( H \). Let \( P(H) \) denote \( \{d \in H | \Delta(d) = 1 \otimes d + d \otimes 1\} \). Set \( H_i = \wedge_{^i}(A1)^{\otimes} \) for \( i = 0, 1, 2, \ldots \). Then the set \( \{H_i\} \) gives a filtration for \( H \) satisfying the followings:

1. \( H = \bigcup_i H_i \).
2. \( H_0 = A1 \).
3. \( H_i = P(H) \).
4. \( \Delta(H_i) \subseteq \sum_{i=0}^\infty H_i \otimes H_{n-i} \).
5. \( \lambda(H_i) \subseteq H_i \) for \( i = 0, 1, 2, \ldots \) where \( \lambda \) is an antipode of \( H [\text{c.f., 6, Chap. 9}] \).

**Lemma 7.** \( \otimes H(C|A) \cong C \otimes H^+ \) as left C-modules for \( i = 0, 1, 2, \ldots \).

Proof. Since \( (H^+)^* = (H^+)^* \), we have \( J = \text{Hom}_C(J)^{\otimes} \) as rings where \( J = J_{C/A} \), and so \( J/J^{i+1} \cong C \otimes (H^*)^{\otimes} / ((H^*)^+)^{i+1} \). Hence we obtain

\[
\otimes H_i \cong J_{C/A} \cong \text{Hom}_C(J/J^{i+1}, C) \cong \text{Hom}_C(C \otimes (H^*)^{\otimes} / ((H^*)^+)^{i+1}, C) \\
\cong C \otimes \text{Hom}_A(\otimes (H^*)^{\otimes} / ((H^*)^+)^{i+1}, A).
\]

On the other hand, we have \( H_i = \wedge_{^i}(A1)^{\otimes} = [(A1)^{i+1}]^{\otimes} = [(H^*)^{i+1}]^{\otimes} \). This completes the proof.

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10) For the definition, see [6, §9].
Theorem 8. Let $C_i = \text{Ker}(H^*_i)$ for $i = 1, 2, 3, \ldots$. Then the followings hold:

1. $C$ is a strongly radical extension of $C_i$.
2. $C_i = \text{Ker}(\vartheta_{\varphi_1}(C/A))$.
3. $C \# A[H^*_i] = \text{Hom}_C(C, C)$.
4. $C_{i+1} = \text{Ker}(\vartheta_{\varphi_1}(C_i/C_{i+1}))$.

Proof. By an identification $\mathcal{D}(C/A) = C \not\equiv H^*$ from Lemma 8 and so $C[\vartheta_{\varphi_1}(C/A)] = C[C \otimes H^*_i] = C \# A[H^*_i]$. Since $\{C \# A[H^*_i]\}^+$ is a $C$-module direct summand of $C[\vartheta_{\varphi_1}(C/A)]$ where $X^+$ denotes a set $\{d \in X | d(1) = 0\}$ for a subset $X$ of $\text{Hom}_A(C, C)$. Moreover, $\{C[\vartheta_{\varphi_1}(C/A)]\}^+$ is closed by the the multiplication and the operator $[D, x]$ for $D \in \{C[\vartheta_{\varphi_1}(C/A)]\}^+, x \in C$. So (1) and (3) follow easily from [7, Theorem 15]. (2) is obvious. It remains only to show (4). It is trivial that $C_{i+1}$ is contained in $\text{Ker}(\vartheta_{\varphi_1}(C_i/C_{i+1}))$. Assume there is an element $x$ in $\text{Ker}(\vartheta_{\varphi_1}(C_i/C_{i+1}))$ with $x \not\equiv C_{i+1}$. Then we have $d(x) \not\equiv 0$ for some $d \in H^*_i$. Since $C$ is a free $C_i$-module, there is a projection $\rho: C \to C_i$ with $(\rho d)(x) \not\equiv 0$. Since $d$ is an ordinary $C_{i+1}$-derivation, $C \not\equiv C_i$ with $(\rho d)(x) \not\equiv 0$. So we define a map $\varphi: L \to C$ by $\varphi(c_1v_1 + c_2v_2 + \cdots + c_rv_r) = (c_1v_1 + c_2v_2 + \cdots + c_rv_r)$ where $a$ is a non-zero element in $Q^{e-1}$ for a positive integer $e$ with $Q^e = 0$ and $Q^{e-1} \neq 0$. Then $\varphi$ is a non-zero element in $\text{Hom}_C(L, C)$, which is absurd.

Lemma 9. Let $J = J_{C/A}$ under the same situation as above. Then $J/J^2$ is free over $C$.

Proof. Since $C$ admits a $p$-basis over $C_i$ from [8, Theorem 10], $J_{C/A}(J_{C/A})^2$ is free over $C$. So it suffices to show that $J/J^2 \approx J_i/J_i^2$ as $C$-modules where $J_i = J_{C_i}$. Now we have a $C$-split exact sequence of canonical maps

$$0 \to L \to J/J^2 \to J_i/J_i^2 \to 0$$

where $L = \{(C \otimes C)J_{C/A} + J^2\}/J^2$. We have to prove $L = 0$. Since $\text{Hom}_C(J/J^2, C) \approx \vartheta_{\varphi_1}(C/A) = \vartheta_{\varphi_1}(C_i/C_i) = \text{Hom}_C(J_i/J_i^2, C)$, we obtain $\text{Hom}_C(L, C) = 0$. This shows $L = 0$. In fact, assume $L \neq 0$. Let us write $L/QL = (C/Q)v_1 + (C/Q)v_2 + \cdots + (C/Q)v_r (v_i \in L)$ for a unique maximal ideal $Q$ in $C$. Then we have $L = Cv_1 + Cv_2 + \cdots + Cv_r$, because $L$ is finitely generated as a $C$-module. Since $C = F \oplus Q$ as vector spaces over $F$ where $F$ is a subfield of $C$ [c.f., 7], any element $c$ in $C$ can be written as $c^{(0)} + c^{(1)}$ for $c^{(0)} \in F, c^{(1)} \in Q$. Then $\{c^{(0)}, c^{(0)} + \cdots, c^{(0)}\}$ are uniquely determined for $c_1v_1 + c_2v_2 + \cdots + c_rv_r$. For let $c_1v_1 + c_2v_2 + \cdots + c_rv_r = b_1v_1 + b_2v_2 + \cdots + b_rv_r$. Then we have $(c_1^{(0)} - b_1^{(0)})v_1 + \cdots + (c_r^{(0)} - b_r^{(0)})v_r \in QL$. Since $v_1, v_2, \ldots, v_r$ are free mod $Q$, we obtain $c_1^{(0)} - b_1^{(0)} = c_2^{(0)} - b_2^{(0)}, \ldots, c_r^{(0)} - b_r^{(0)}$. So we define a map $\varphi: L \to C$ by $\varphi(c_1v_1 + c_2v_2 + \cdots + c_rv_r) = (c_1^{(0)} + c_2^{(0)} + \cdots + c_r^{(0)})a$ where $a$ is a non-zero element in $Q^{e-1}$ for a positive integer $e$ with $Q^e = 0$ and $Q^{e-1} \neq 0$. Then $\varphi$ is a non-zero element in $\text{Hom}_C(L, C)$, which is absurd.
Lemma 10. Let \(\{t_1, t_2, \ldots, t_n\}\) be a system of generators for an \(A\)-algebra \(C\) such that \(\{I_1, I_2, \ldots, I_n\}\) forms a system of \(p\)-generators for \(C \otimes C = C \otimes 1 + J\) [c.f., 9] where \(I_i = 1 \otimes t_i - t_i \otimes 1 (i = 1, 2, \ldots, n)\). Then we have \(C_i = A((t_i)^p, (t_i)^p, \ldots, (t_i)^p)\).

Proof. Since \(J/J^p\) is free over \(C \otimes 1\) and \(C | Q \otimes C/J^p \cong J/(J+Q)J\) as \(C | Q\)-spaces where \(Q\) is a unique maximal ideal of \(C\), the images of \(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n\) by any canonical map: \(J \rightarrow J/J^p\) form a \(C \otimes 1\)-module basis for \(J/J^p\) [c.f., 2, Chap. II, §3, Prop. 5]. So there are \(d_1, d_2, \ldots, d_n\) in \(\mathcal{E}_{\mathcal{E}}(C/A)\) with \(d_i(t_j) = \delta_{i,j}\) for \(i, j = 1, 2, \ldots, n\). Since \(C = A[t_1, t_2, \ldots, t_n]\), any element \(x\) in \(C_1\) can be written as \(\sum c_{(o)} t_1^{e_1} t_2^{e_2} \cdots t_n^{e_n}\). Then let us write \(x = \sum c_{(o)} t_1^{e_1} t_2^{e_2} \cdots t_n^{e_n}\). Let us write \(a_{(o)} \in C_1\). Then \(c_{(o)} = a_{(o)} t_1^{e_1} \cdots t_n^{e_n}\) with \(p \mid e_i (k = 1, 2, \ldots, s)\) and \(c_{(o)} = a_{(o)} t_1^{e_1} \cdots t_n^{e_n}\) with \(p \nmid e_i (l = 1, 2, \ldots, t)\). Assume there is a term \(c_{(o)} t_1^{e_1} \cdots t_n^{e_n}\) with \(t_1^{e_1} \cdots t_n^{e_n} \neq 1\). Then \(d_i t_1^{e_1} \cdots t_n^{e_n} = \prod (e_{(i)})\) is a unit in \(C_1\). So, if \(c_{(o)} t_1^{e_1} \cdots t_n^{e_n}\) is a non-zero term such that \(e_1 + \cdots + e_n\) is maximal, we have \(d_i t_1^{e_1} \cdots t_n^{e_n} (x) = \prod (e_{(i)})\) which is a contradiction to \(x \in C_1 = \text{Ker} (\mathcal{E}_{\mathcal{E}}(C/A))\). This shows that \(x\) is equal to \(\sum c_{(o)} t_1^{e_1} \cdots t_n^{e_n}\) belonging to \(A((t_i)^p, \ldots, (t_i)^p)\).

Lemma 11. Let \(d, x\) be any element in \(H^+, C\), respectively. Then \(d(x^p)\) belongs to \(A \cdot C^p\) for \(t = 0, 1, 2, \ldots\).

Proof. Let \(\{d_0 = (1), d_1, \ldots, d_I\}\) be an \(A\)-basis for \(H\). Then we have \(\Delta_n(d) = \sum a_{(o)} d_1 \otimes d_2 \otimes \cdots \otimes d_n\) for \(a_{(o)} \in A\) where \(n = p^t\), \(\Delta_n = (1 \otimes \cdots \otimes 1 \otimes \Delta) \cdot (1 \otimes \Delta)\) and \((i) = (i_1, i_2, \ldots, i_n)\) for any permutation \((j_1, j_2, \ldots, j_n)\) of \((i_1, i_2, \ldots, i_n)\) and \(a_{(i_1, i_2, \ldots, i_n)} = a_{(j_1, j_2, \ldots, j_n)}\). Since \(H\) is cocommutative, we have \(\Delta_n(d_1 \otimes d_2 \otimes \cdots \otimes d_n) = \sum a_{(i_1, i_2, \ldots, i_n)} d_1 \otimes d_2 \otimes \cdots \otimes d_n\) for any element \(x \in \text{Ker} (\mathcal{E}_{\mathcal{E}}(C/A))\). This completes the proof, since \(\sum a_{(i_1, i_2, \ldots, i_n)} d_1 \otimes d_2 \otimes \cdots \otimes d_n(x) = 0\) mod \(p\) unless \(k_1 = k_2 = \cdots = k_n\).

Theorem 12. Let \(H\) be a cocommutative pointed Hopf algebra over a field \(A\) and \(C\) is a Galois \(H^+\)-object over an \(A\) which is strongly radical over \(A\). Then there exists a sequence of subrings of \(C: C = C_0 \supset C_1 \supset \cdots \supset C_n \supset A\) satisfying, for each \(i = 1, 2, \ldots, n - 1\),

1. \(C_i\) is finitely generated projective as a \(C_{i+1}\)-module.
2. \(d(C_i) \subseteq C_i\) for all \(d \in H\).
3. A left \(C_i\)-module \(\text{Hom}_A(C_i, C_i)\) is generated by the endomorphisms of \(C_i\) induced by each element in \(H\).
(4) \( C_i[\cong \otimes_1 (C_i/C_{i+1})] = \text{Hom}_{C_{i+1}}(C_i, C_i) \).

Proof. Let \( C_i, H_i \) be as above. Then (1) is obvious. (2) for \( i=1 \) follows from Lem. 10 and 11. Since \( C_i \) is a \( C_i \)-module direct summand of \( C \), \( \text{Hom}_A(C_i, C_i) \) may be considered to be contained in \( \text{Hom}_A(C, C) \). So any homomorphism in \( \text{Hom}_A(C_i, C_i) \) is induced by an element \( \sum c_i \otimes d_i \) in \( C \otimes H \) for \( c_i \in C_i \), \( d_i \in H \). Let us write \( C = C_i \oplus C'_i \) for a \( C_i \)-submodule \( C'_i \) of \( C \). Let \( e_i = c_i^0 + c'_i \) for \( c_i \in C_i, c'_i \in C'_i \). Then we have \( \sum c_i d_i(x) = \sum c_i d_i(x) + \sum c'_i d_i(x) \) for \( x \in C_i \) and so \( \sum c'_i d_i(x) \in C_i \cap C'_i (= 0) \). Hence \( \sum c'_i \otimes d_i \) induces the same homomorphism in \( \text{Hom}_A(C_i, C_i) \). This shows the statement (3) for \( i=1 \). It follows from Theorem 8 that \( C \otimes A[H_2] = \text{Hom}_{C_2}(C, C) \). Then, by the same argument above, a \( C_i \)-module \( \text{Hom}_{C_2}(C_i, C_i) \) is generated by the endomorphisms of \( C_i \) induced by each element in \( A[H_2] \). Since each element of \( H_2 \) induces an ordinary derivation on \( C_i \), we obtain \( C_i[\cong \otimes_1 (C_i/C_{i+1})] = \text{Hom}_{C_2}(C_i, C_i) \) and \( C_2 = \text{Ker}(H_2) = \text{Ker}(\otimes_2 (C_i/C_2)) \). Hence, using again Lemma 10, we have \( C_2 = A[t^0, t^1, \cdots, t^r] \) for \( t_i \in C_i \), and so \( C_2 = A \cdot C^{p^0} \). Repeating the argument above, we complete the proof.

Corollary. Under the situation above, moreover, let \( K \) be a \( C \)-algebra which is finitely generated projective as a \( C \)-module. Then \( H^*(K/A) = H^*(K/C) \) for \( n > 2 \) and there is an exact sequence
\[
0 \to H^0(K/A) \to H^0(K/C) \to H^2(K/C) \to 0
\]
where \( H^*(K/A) \) denotes a Amitsur cohomology group for a extension ring \( K/A \).

Proof. By [5, Theorem 4.3], we have an exact sequence
\[
\cdots \to H^{n-1}(K/C) \to H^n(K/A) \to H^n(K/C) \to H^n(K/C) \to \cdots
\]
So it suffices to show \( H^1(K/C) = 0 \) and \( H^0(K/A) = 0 \) for \( n > 2 \). The first follows from [1, Theorem 3.8]. It follows from [10, Theorem 6] that \( H^*(C_i/C_{i+1}) \) vanish for \( n > 2 \) where the \( C_i \)'s are as above. Hence, using again [4, Theorem 4.3], we obtain \( H^*(C/A) = 0 \) for \( n > 2 \).

References

Galois Objects


