Throughout \( R \) will represent a ring with unit element 1, and all modules will be unitary \( R \)-modules. We call a module \( M \) a **completely indecomposable module** if the endomorphism ring of \( M \) is a local ring. Let \( \mathcal{M} = \{M_\alpha\} \) be a set of completely indecomposable right \( R \)-modules, and \( \mathcal{A} \) the full subadditive category of the category of all right \( R \)-modules, whose objects consist of all \( R \)-modules which are isomorphic to direct sums of \( M_\alpha \)'s in \( \mathcal{M} \). We define the subclass \( \mathcal{Y} \) of the morphisms in \( \mathcal{A} \) as follows: for any objects \( M = \sum_{\alpha \in \mathcal{K}} \oplus M_\alpha', \ N = \sum_{\beta \in \mathcal{L}} N_\beta \) in \( \mathcal{A} \), \( \mathcal{Y} \mathcal{T} \mathcal{I} \) \( \text{Hom}_R(M, N) = \{ f \in \text{Hom}_R(M, N) \mid p_\beta f i_\alpha \text{ is not isomorphic, for all } \alpha \in \mathcal{K}, \beta \in \mathcal{L}, \text{ where } i_\alpha : M_\alpha' \rightarrow M \text{ is the inclusion and } p_\beta : N \rightarrow N_\beta \text{ is the projection} \} \). Then, \( \mathcal{Y} \) does not depend on the decompositions of \( M \) and \( N \) (see Corollary to Lemma 5 in [5]).

M. Harada and Y. Sai [4], [5] gave several equivalent conditions for \( S_M \cap \mathcal{Y} \) to be equal to the Jacobson radical \( J(S_M) \) of \( S_M \), where \( M \in \mathcal{A} \) and \( S_M = \text{Hom}_R(M, N) \). Among those conditions, they made great use of structures of the factor category \( \mathcal{A}/\mathcal{Y} \) in order to show the following fact: if \( J(S_M) = S_M \cap \mathcal{Y} \), then for any two decompositions \( M = \sum_{\alpha \in \mathcal{K}} \oplus M_\alpha = \sum_{\beta \in \mathcal{L}} \oplus N_\beta \) and any subset \( \mathcal{K}' \) of \( \mathcal{K} \), there exists a one-to-one mapping \( \varphi \) of \( \mathcal{K}' \) into \( \mathcal{L} \) such that \( M_\alpha \approx N_{\varphi(\alpha)} \) for all \( \alpha \in \mathcal{K}' \) and \( M = \sum_{\alpha \in \mathcal{K}'} \oplus N_{\varphi(\alpha)} \oplus \sum_{\alpha \notin \mathcal{K}'} \oplus M_\alpha' \).

The purpose of this note is to give a ring-theoretical proof of the above fact by using a few structure of \( \mathcal{A}/\mathcal{Y} \). We shall define a concept of locally direct summands of \( M \) in \( \mathcal{A} \) for this purpose. Let \( N = \sum_{\gamma \in \mathcal{L}'} \oplus N_\gamma \) be a submodule of \( M \) in \( \mathcal{A} \). If \( \sum_{\gamma \in \mathcal{L}'} \oplus N_\gamma \) is a direct summand of \( M \) for every finite subset \( \mathcal{L}' \) of \( \mathcal{L} \), we call a **locally direct summand** of \( M \) (with respect to the decomposition \( N = \sum_{\gamma \in \mathcal{L}'} \oplus N_\gamma \)). We shall give a relation between some locally direct summands of \( M \) and dense submodules of \( M \) defined in [4], and using this relation we shall give a proof of the statement above.

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We begin with preliminary definitions and results on $\mathcal{Y}$ and $S_M$. From now, we understand that a module $M$ is in $\mathfrak{A}$ and that $M_a$'s are completely indecomposable, if there are no confusions.

Let $M, N$ be in $\mathfrak{A}$, and $f \in \text{Hom}_R(M, N)$. $f$ is said to be left regular modulo $\mathcal{Y}$ if, for any homomorphism $g$ of any $L$ in $\mathfrak{A}$ to $M$, $fg$ in $\mathcal{Y}$ implies $g$ in $\mathcal{Y}$. The right regularity of $f$ modulo $\mathcal{Y}$ is defined similarly. $f$ is said to be an isomorphism modulo $\mathcal{Y}$ if there exists some $g: N \rightarrow M$ such that $gf = 1_M \text{mod. } \mathcal{Y}$ and $fg = 1_N \text{mod. } \mathcal{Y}$.

**Remark 1.** Let $M = \sum_{\rho \in K} \bigoplus M_{\rho}$, $N = \sum_{\rho' \in K'} \bigoplus M_{\rho'}$ be in $\mathfrak{A}$ where $K'$ is a subset of $K$, $i$ the inclusion of $N$ to $M$ and $p$ the projection of $M$ onto $N$. Then, $i$ is left regular mod. $\mathcal{Y}$ and $p$ is right regular mod. $\mathcal{Y}$.

**Lemma 1.** For any morphism $f$ in $\mathfrak{A}$ and any $g$ in $\mathcal{Y}$, $fg$ and $gf$ are in $\mathcal{Y}$.

**Lemma 2.** Let $M = \sum_{\rho \in K} \bigoplus M_{\rho}$ be in $\mathfrak{A}$, and $S_M$ the endomorphism ring of $M$. Then,

1. $S_M/S_M \cap \mathcal{Y}$ is a regular ring (in the sense of von Neumann), moreover
2. for any $f \in S_M$ with $f = f' \text{ modulo } \mathcal{Y} \cap S_M$, there exist some elements $a$ and $e$ in $S_M$ such that $a$ is regular in $S_M/S_M \cap \mathcal{Y}$, $e$ is a projection of $M$ to $\sum_{\rho' \in K'} \bigoplus M_{\rho'}$ for some subset $K'$ of $K$ and $f = aed$ modulo $\mathcal{Y}$, where $aa' = a'a = 1$ modulo $\mathcal{Y}$ and $a' \in S_M$.

See [1], Lemma 6 and Theorem 7 in [5] and [6].

**Corollary 1.** Let $M, N$ be in $\mathfrak{A}$, and $f: M \rightarrow N$. Then,

1. $f$ is left (resp. right) regular mod. $\mathcal{Y}$ if and only if there exists some $g: N \rightarrow M$ such that $gf = 1_M \text{mod. } \mathcal{Y}$, and
2. $f$ is an isomorphism mod. $\mathcal{Y}$ if and only if $f$ is left and right regular mod. $\mathcal{Y}$.

**Proof.** (1) "If" part is trivial. Conversely, we assume that $f$ is left regular mod. $\mathcal{Y}$. Since $S_M/S_M \cap \mathcal{Y}$ is a regular ring by the lemma, there exists some $g: N \rightarrow M$ such that $gf = 1_M \text{mod. } \mathcal{Y}$. The left regularity of $f$ mod. $\mathcal{Y}$ implies that $gf = 1_M \text{mod. } \mathcal{Y}$. The right regularity is similar. (2) is clear.

**Corollary 2.** If $f: M \rightarrow N$ is left regular mod. $\mathcal{Y}$ for $M, N$ in $\mathfrak{A}$ and $S_M \cap \mathcal{Y}$ is equal to the Jacobson radical $J(S_M)$ of $S_M$, then $f$ is an $R$-monomorphism and $M$ is $R$-isomorphic to a direct summand of $N$.

**Proof.** By Corollary 1(1), there exists some $g: N \rightarrow M$ such that $gf = 1_M \text{mod. } \mathcal{Y}$, since $f$ is left regular mod. $\mathcal{Y}$. Hence, $1_M - gf \in S_M \cap \mathcal{Y} = J(S_M)$ and so $gf$ is an $R$-isomorphism. Therefore, $f$ is an $R$-monomorphism and $M$ is $R$-isomorphic to a direct summand of $N$. 
Let $U, V$ be right $R$-modules, $/ : U \rightarrow V$, and $U = \sum_{\gamma \in K} 0 U_\gamma$ a direct sum of right $R$-submodules of $U$. Then, we consider the following condition:

/ is an \textbf{R-monomorphism} and

\textit{(*)} for any finite subset $K'$ of $K$, $f(\sum_{\gamma \in K'} \Phi U_\gamma)$ is a direct summand of $V$.

If $f$ satisfies the above \textit{(*)}-condition, we call $f$ a \textbf{(*)-monomorphism} (with respect to this decomposition of $U$).

For example, let $f, U$, and $V$ be as above. If $f$ is an $R$-monomorphism and each $U_\gamma$ is injective, then $/ \cdot f$ is a \textbf{(*)-monomorphism} (with respect to the decomposition $U = \sum_{\gamma \in K} 0 U_\gamma$).

From now on, \textbf{(*)-monomorphisms} will be considered in $\mathfrak{H}$.

The following lemma on \textbf{(*)-monomorphisms} is essential in this note.

\textbf{Lemma 3.} Let $M = \sum_{\alpha \in K} \oplus M_\alpha, N$ be in $\mathfrak{H}$ and $f : M \rightarrow N$. Then, $f$ is left regular mod. $\mathfrak{Y}$ if and only if $f$ is a \textbf{(*)-monomorphism} (w.r.t. the decomposition $M = \sum_{\alpha \in K} \oplus M_\alpha$).

\textbf{Proof.} First, we assume that $f$ is left regular mod. $\mathfrak{Y}$. Put $M_0 = \sum_{\alpha \in K} \oplus M_\alpha$ for any finite subset $K'$ of $K$. Let $i$ be the inclusion of $M_0$ to $M$. Then, $f \cdot i$ is left regular mod. $\mathfrak{Y}$ and $S_{M_0} \cap \mathfrak{Y} = (S_{M_0})$ by Lemma 8 in [5], because $K'$ is a finite set. Hence, $f \cdot i$ is an $R$-monomorphism and $f \cdot i(M_0)$ is a direct summand of $N$ by Corollary 2 to Lemma 2. Therefore, $/ \cdot f$ is an $R$-monomorphism and $f(M_0)$ is a direct summand of $N$, i.e. $/ \cdot f$ is a \textbf{(*)-monomorphism} (w.r.t. the decomposition $M = \sum_{\alpha \in K} \oplus M_\alpha$). Conversely, let $g \in \text{Hom}_R(T, M)$ for any module $T = \sum_{\gamma \in K} 0 T_\gamma$ in $\mathfrak{H}$ and assume that $fg$ in $\mathfrak{Y}$. Put $g_\gamma = g_i \gamma$, where $i_\gamma$ is the inclusion of $T_\gamma$ to $T$ for all $\gamma \in L$. Then, we can express $g_\gamma$ as a column-summable matrix for all $\gamma \in L$. Hence, $g_\gamma$ is a column-matrix whose finite components are isomorphic and the others are all non-isomorphisms. We can rearrange $g_\gamma$ as follows: the first $n$ components are isomorphisms. Put $M_0 = \sum_{i = 1}^n \oplus M_i$. Let $i$ be the inclusion of $M_0$ to $M$, and $p$ the projection of $M$ onto $M_0$. Then, $f \cdot p g_\gamma = f g_\gamma = f g_i \gamma \equiv 0 \mod. \mathfrak{Y}$. Since $f \cdot i$ is left regular mod. $\mathfrak{Y}$, $f g_\gamma$ is in $\mathfrak{Y}$. Hence, $g_\gamma$ and so $g$ are in $\mathfrak{Y}$, because $\alpha p g_\gamma + (1 - \alpha) g_\gamma = g_\gamma \mod. \mathfrak{Y}$. Therefore, $/ \cdot f$ is left regular mod. $\mathfrak{Y}$.

We note that a \textbf{(*)-monomorphism} does not depend on the decomposition of $M$ from Lemma 3.

\textbf{Corollary 1} (cf. Lemma 3.2.3 in [3]) (1) // $/ : M \rightarrow N$ is left regular mod. $\mathfrak{Y}$, then $f$ is an $R$-monomorphism. (2) For any $f$ in $S_M \cap \mathfrak{Y}$, $1_M - f$ is an $R$-monomorphism.

\textbf{Proof.} (1) is clear by the lemma. (2) Since $/ \cdot f$ is in $S_M \cap \mathfrak{Y}$, $1_M - f$ is left
regular mod. $\mathfrak{X}$ and hence an $R$-monomorphism by (1).

**Corollary 2.** Let $M, N$ be in $\mathfrak{A}$, and $f: M \to N$ an isomorphism mod. $\mathfrak{Y}$. Then $f$ is an $R$-isomorphism provided either $S_M \cap \mathfrak{Y} = J(S_M)$ or $S_N \cap \mathfrak{Y} = J(S_N)$. Especially, if $M$ is a finite direct sum of $M_\alpha$'s in $\mathfrak{M}$, then an isomorphism mod. $\mathfrak{Y}$ means an $R$-isomorphism.

**Proof.** Since $f$ is isomorphic mod. $\mathfrak{Y}$, there exists some $g: N \to M$ such that $gf = 1_M$ mod. $\mathfrak{Y}$ and $fg = 1_N$ mod. $\mathfrak{Y}$. Hence, $f$ and $g$ are left regular mod. $\mathfrak{Y}$, that is, both are $R$-monomorphisms by Corollary 1. In case $S_N \cap \mathfrak{Y} = J(S_N)$, $1_N - fg \in J(S_N)$ Hence, $f$ is an $R$-isomorphism and so is $f$. On the other hand, if $S_M \cap \mathfrak{Y}$ is equal to $J(S_M)$, then $1_M - gf \in J(S_M)$ and hence $gf$ is an $R$-isomorphism. Therefore, $f$ is an $R$-isomorphism. The latter assertion is clear by Lemma 8 in [5].

We define here an important concept as follows (see [3]): let $M, N$ be in $\mathfrak{A}$, and $N = \sum_{\beta \in K} N_\beta$ a submodule of $M$. Then, $N$ is said to be a **locally direct summand** of $M$ (with respect to the decomposition $N = \sum_{\beta \in K} N_\beta$) if the inclusion $i: N' \to M$ is a $(\ast)$-monomorphism (with respect to this decomposition of $N$).

In the following lemma, we consider the existence of locally direct summands of a module $M$ in $\mathfrak{A}$.

**Lemma 4.** Let $M, N$ be in $\mathfrak{A}$, and $f: M \to N$. Then, there exist a locally direct summand $N'$ of $N$ in $\mathfrak{A}$ via the inclusion $i: N' \to N$ and some $f': M \to N'$ such that $f = if'$ mod. $\mathfrak{Y}$, $i$ is left regular mod. $\mathfrak{X}$ and $f$ is right regular mod. $\mathfrak{Y}$.

**Proof.** We begin with the case $M = N$ and $f = f^2$ mod. $\mathfrak{Y}$. There exist a projection $e$ of $M = \sum_{a \in K} \oplus M_a$ onto $\sum_{a' \in K'} \oplus M_{a'}$ for some subset $K'$ of $K$ and elements $a, a'$ in $S_M$ such that $f = aea'$ mod. $\mathfrak{Y}$ and $aa' = a'a = 1$ mod. $\mathfrak{Y}$, by Lemma 2(2). Put $N' = aeM, N'' = eM$, and consider the inclusions $i: N' \to M, i': N'' \to M$. Then, by Lemma 3, $N'$ is a locally direct summand of $M$ and $i$ is left regular mod. $\mathfrak{Y}$, since $N'$ is isomorphic to $N''$ under $ai'$ that is left regular mod. $\mathfrak{Y}$. Moreover, $ea'$ is right regular mod. $\mathfrak{Y}$, and hence so is $f' = aea': M \to N'$. Thus, our lemma holds. In the general case, for $f: M \to N$, there exist some homomorphisms $g: N \to N$ and $k: N \to M$ such that $g = g^2$ mod. $\mathfrak{Y}, f = gf$ mod. $\mathfrak{Y}$ and $g = f k$ mod. $\mathfrak{Y}$, by Lemma 2. For $g: N \to N$ with $g = g^2$ mod. $\mathfrak{Y}$, there exist a locally direct summand $N'$ of $N$ in $\mathfrak{S}$, some $g': N \to N'$ and the inclusion $i: N' \to N$ such that $g = ig'$ mod. $\mathfrak{Y}, g'$ and $i$ are right and left regular mod. $\mathfrak{Y}$, respectively, by the above argument. We can easily show that $g'f$ is
right regular mod. $\mathcal{Y}'$ since $g'i=1_{N'}$ mod. $\mathcal{Y}'$, and $f=ig'$ mod. $\mathcal{Y}'$.

**Lemma 5.** For $M$ and $N$ in $\mathfrak{A}$, a homomorphism $f: M \to N$ is right regular mod. $\mathcal{Y}'$ if and only if there exist a locally direct summand $M'$ of $M$ in $\mathfrak{A}$ and some $g: N \to M'$ such that $f(1_{N'}$ mod. $\mathcal{Y}')$ and $g$ is an isomorphism mod. $\mathcal{Y}'$, where $i$ is the inclusion of $M'$ to $M$.

Proof. "If" part is trivial. Conversely, suppose that $f$ is right regular mod. $\mathcal{Y}'$. Then there exists some $g': N \to M$ such that $fg'=1_{N}$ mod. $\mathcal{Y}'$, by Corollary 1 to Lemma 2. Since $g'$ is left regular mod. $\mathcal{Y}'$, there exists a locally direct summand $M'$ of $M$ in $\mathfrak{A}$ such that $g'='i$ mod. $\mathcal{Y}'$, where $g: N \to M'$ is right regular mod. $\mathcal{Y}'$ and the inclusion $i: M' \to M$ is left regular mod. $\mathcal{Y}'$, by Lemma 4. Therefore, $fg=1_{N}$ mod. $\mathcal{Y}'$ and $g$ is an isomorphism mod. $\mathcal{Y}'$.

**Lemma 6.** Let $M, N$ be in $\mathfrak{A}$, $e$ an idempotent element in $S_M$ where $N$ is contained in $eM$, and let the inclusion $i: N \to M$ be left regular mod. $\mathcal{Y}'$. Then, there exists a locally direct summand $N'$ of $M$ in $\mathfrak{A}$ such that $e=ip+ip'$ mod. $\mathcal{Y}'$, $pi=1_{N'}$ mod. $\mathcal{Y}'$, $p'i'=1_{N'}$ mod. $\mathcal{Y}'$ and $p'i'=p'ip=0$ mod. $\mathcal{Y}'$, where $i'$ is the inclusion of $N'$ to $M$ and $p, p'$ are homomorphisms of $M$ to $N$, $N'$ respectively. Furthermore, the formal direct sum $N \oplus N'$ is $R$-isomorphic to a locally direct summand of $eM$.

Proof. For $f: N \to M$, there exists some $p_i: M \to N$ such that $ip_i=i$ mod. $\mathcal{Y}'$. Since $e=ip+ip'$ mod. $\mathcal{Y}'$, $pi=1_{N'}$ mod. $\mathcal{Y}'$. Now, we put $f=ip$ and $g=e-f$. Then, $ef=fefind ge=ge=g$. For $g: M \to M$ there exist a locally direct summand $N'$ of $M$ in $\mathfrak{A}$ and some $p': M \to N'$ such that $g=i'p'$ mod. $\mathcal{Y}'$, the inclusion $i': N' \to M$ is left regular mod. $\mathcal{Y}'$ and $p, p'$ are right regular mod. $\mathcal{Y}'$. Moreover, $g=g'$ mod. $\mathcal{Y}'$ implies $p'i'=p'ip=0$ mod. $\mathcal{Y}'$, because $i$ and $i'$ are left regular mod. $\mathcal{Y}'$ and $p, p'$ are right regular mod. $\mathcal{Y}'$. Finally, we show that the formal direct sum $N \oplus N'$ is $R$-isomorphic to a locally direct summand of $eM$. Let $I=(0, i')$: $N \oplus N' \to M$ and $t=\{t_s\}: T \to N \oplus N'$ for any $T$ in $\mathfrak{A}$. Suppose that $It=it_1+it_2$ is in $\mathcal{Y}'$. Then, $p_1t_1+p_2t_2$ is in $\mathcal{Y}'$. Since $p_2t_2$ is in $\mathcal{Y}'$, $t_2$ is in $\mathcal{Y}'$. Hence, $t$ is in $\mathcal{Y}'$. It follows that $/ is a (*)-monomorphism. Therefore, $N \oplus N'$ is a locally direct summand of $M$ in $\mathfrak{A}$. On the other hand, $g=eg$ and $g=t'p'$ mod. $\mathcal{Y}'$ imply $e'=i'p'$ mod. $\mathcal{Y}'$, and so we may assume $e'=i'$ in the above. Since $i'$ is a (*)-monomorphism, Im($i'$) is contained in $eM$. Hence, Im($I$) is contained in $eM$, whence $N \oplus N'$ is $R$-isomorphic to a locally direct summand of $eM$.

**Corollary.** Let $N \subset M$ be in $\mathfrak{A}$. If the inclusion $i: N \to M$ is left regular mod. $\mathcal{Y}'$, then there exist a locally direct summand $N'$ of $M$ in $\mathfrak{A}$, the inclusion $i'$:
\( N' \to M \) and some \( p, p' \) of \( M \) to \( N, N' \) respectively such that \( 1_M = ip + ip' \mod \mathcal{Y}, \) \( pi = 1_N \mod \mathcal{Y}, p'i' = 1_{N'} \mod \mathcal{Y}' \) and \( p'i = p'i' = 0 \mod \mathcal{Y}' \).

Proof. Put \( e = 1_M \) in the lemma.

Let \( N \) be an \( R \)-module, and \( \{N_j\} = \bigoplus_{i \in J} N_{i}^{\to} \) the set of submodules of \( N \) in \( \mathfrak{A} \) which are locally direct summands of \( N \). We define an order \( > \) in the set \( \{N_j\} \) as follows:

for each locally direct summand \( N_j \) of \( N \),

\( N_j > N_{if} \) if and only if \( \{N_i^{(f)}\} \supset \{N_i^{(k)}\} \) for any \( f \neq k \) in \( J \).

Then, there exists a maximal submodule of \( N \) among the set \( \{N_j\} \) with respect to this order \( > \), by Zorn's lemma. We call it a \textit{maximal} locally direct summand of \( N \).

**Proposition 7.** Assume that all \( M_a \) in \( \mathfrak{A} \) are injective. Let \( N \subseteq M \) be in \( \mathfrak{A} \). Then, \( N \) is essential in \( M \) if and only if \( N \) is a maximal locally direct summand of \( M \).

Proof. "Only if" part is trivial. Conversely, if \( N \) is not essential, there exists a cyclic submodule \( N' \) of \( M \) with \( N \cap N' = (0) \). Then, the injective hull \( E(N') \) in \( M \) is a direct summand of \( M \). On the other hand, \( N \cap E(N') = (0) \).

Since \( E(N') \) contains an injective submodule \( M_{\beta} \) for some \( \beta \), this contradicts the maximality of \( N \). Hence, \( N \) is an essential submodule of \( M \).

Next, we show that a dense submodule of a module in \( \mathfrak{A} \) defined in [4] is equal to a maximal locally direct summand of the module.

**Lemma 8.** Let \( N \subseteq M \) be in \( \mathfrak{A} \). Then, \( N \) is a maximal locally direct summand of \( M \) if and only if the inclusion \( i \) of \( N \) to \( M \) is an isomorphism \( \mod \mathcal{Y}' \).

Proof. First, we assume that \( N \) is a maximal locally direct summand of \( M \). If the inclusion \( i: N \to M \) is not isomorphic modulo \( \mathcal{Y}' \), there exists a locally direct summand \( N' \) of \( M \) in \( \mathfrak{A} \) such that \( 1_M = ip + ip' \mod \mathcal{Y}' \), where \( i' \) is the inclusion of \( N' \) to \( M \), \( p: M \to N \) and \( p': M \to N' \), by Corollary to Lemma 6. Then, \( I = (i, i') : N \oplus N' \to M \) is a (*)-monomorphism. Hence, the image of \( / \) is equal to a locally direct summand \( N \oplus \operatorname{Im}(i') \) of \( M \) in \( \mathfrak{A} \) which contains \( N \); this contradicts the maximality of \( N \). Hence, \( N' = 0 \). Therefore, \( 1_M = ip \mod \mathcal{Y}' \) and so \( i \) is an isomorphism \( \mod \mathcal{Y}' \). Conversely, suppose that \( i \) is an isomorphism \( \mod \mathcal{Y}' \). Then, there exists some \( p: M \to N \) such that \( pi = 1_N \mod \mathcal{Y} \) and \( ip = 1_M \mod \mathcal{Y} \), and also \( N \) is a locally direct summand of \( M \).

If \( N \) is not maximal in \( M \), there exists a locally direct summand \( N' \) of \( M \) in \( \mathfrak{A} \) such that \( N \oplus N' \) is a locally direct summand of \( M \) in \( \mathfrak{A} \). Hence, the inclusion \( I = (i, i') : N \oplus N' \to M \) is left regular \( \mod \mathcal{Y}' \), where \( i' \) is the inclusion of \( N' \) to \( M \). Therefore, there exists some \( g: M \to N \oplus N' \) such that \( gI = 1_{N \oplus N'} \mod \mathcal{Y}' \),
by Corollary 1(1) to Lemma 2. Let \( p_i \) be the projection of \( N \oplus N' \) onto \( N \). Then, \( p_i g_i = 1_N = p_i \mod \mathcal{Y}' \) and so \( p_i g = p_i \mod \mathcal{Y}' \), which implies that \( p_i = 0 \mod \mathcal{Y}' \). Hence, \( N' = 0 \); a contradiction. It follows that \( N \) is a maximal locally direct summand of \( M \).

**REMARK 3.** The submodule \( N \) in the lemma is called a dense submodule of \( M \), in [4]. We note that \( N \oplus N' \mod \mathcal{Y}' \) is a dense submodule of \( M \).

**Corollary 1.** Let \( M, N \) be in \( \mathfrak{A} \), and \( f: M \to N \). Then, there exist locally direct summands \( M' \) and \( N' \) of \( M \) and \( N \) in \( \mathfrak{A} \), respectively, such that the restriction \( f \mid_{M'} \to N' \) is an \( R \)-isomorphism. Especially, \( f \) is isomorphic mod. \( \mathcal{Y}' \) if and only if \( M' \) and \( N' \) are dense in \( M \) and \( N \), respectively.

**Proof.** For \( f: M \to N \), there exist a locally direct summand \( N'' \) of \( N \) in \( \mathfrak{A} \), the inclusion \( i': N'' \to N \) and some \( f': M \to N'' \) such that \( f = i' f' \mod \mathcal{Y}' \), \( i' \) is left regular mod. \( \mathcal{Y}' \) and \( f' \) is right regular mod. \( \mathcal{Y}' \), by Lemma 4. Since \( f' \) is right regular mod. \( \mathcal{Y}' \), there exist a locally direct summand \( M' \) of \( M \) in \( \mathfrak{A} \) and some \( g: N'' \to M' \) such that \( f' i/g = 1_N \mod \mathcal{Y}' \) and \( g \) is isomorphic mod. \( \mathcal{Y}' \), where \( i \) is the inclusion of \( M' \) to \( M \), by Lemma 5. Since \( f g = \) left regular mod. \( \mathcal{Y}' \) and \( g \) is isomorphic mod. \( \mathcal{Y}' \), \( f i \) is left regular mod. \( \mathcal{Y}' \) and so \( f i \) is monomorphic. Let \( N' \) be the image of \( f i \) in \( N \). Then, \( f i: M' \to N' \) is an \( R \)-isomorphism, whence it follows that \( N' \) is a locally direct summand of \( N \). Particularly, in case \( f \) is isomorphic mod. \( \mathcal{Y}' \), \( i' \) and \( f' \) are isomorphic mod. \( \mathcal{Y}' \) by Corollary 1(2) to Lemma 2 and \( f i g = 1_N \mod \mathcal{Y}' \). Since \( f h = \) left regular mod. \( \mathcal{Y}' \), \( f i \) is left regular mod. \( \mathcal{Y}' \) and so \( f i \) is monomorphic. Let \( N' \) be the image of \( f i \) in \( N \). Then, \( f i: M' \to N' \) is an \( R \)-isomorphism, whence it follows that \( N' \) is a locally direct summand of \( N \). Particularly, in case \( f \) is isomorphic mod. \( \mathcal{Y}' \), \( i' \) and \( f' \) are isomorphic mod. \( \mathcal{Y}' \), and hence \( f \) is isomorphic mod. \( \mathcal{Y}' \) by the lemma.

**Corollary 2.** \( S_M \cap \mathcal{Y}' = f(S_M) \) for a module \( M \) in \( \mathfrak{A} \), then \( M \) is the only one dense submodule in \( M \).

**Proof.** Let \( N \) be a dense submodule of \( M \). Then, the inclusion \( i: N \to M \) is isomorphic mod. \( \mathcal{Y}' \) by the lemma. Hence, \( i \) is an \( R \)-isomorphism by Corollary 2 to Lemma 3 and so \( N = M \).

**Lemma 9.** Let \( e \) be an idempotent element in \( S_M \) for module \( M = \sum_{a \in K} M_a \) in \( \mathfrak{A} \). Then, there exist a submodule \( N \) of \( eM \) in \( \mathfrak{A} \) and \( p: M \to N \) such that \( e = ip \mod \mathcal{Y}' \) and \( pi = 1_N \mod \mathcal{Y}' \), where \( i: N \to M \) is the inclusion.

**Proof.** Since \( eM \) is a direct summand of \( M \), \( eM \) contains some \( M_a \) by [2]. Hence, there exists a maximal locally direct summand of \( eM \) in \( \mathfrak{A} \). Let \( N \) be the maximal one, and \( i \) the inclusion of \( N \) to \( M \). Since \( i \) is left regular
mod. \mathcal{Y}', there exists a locally direct summand \( N' \) of \( M \) in \( \mathfrak{B} \) such that \( e = ip + i'p' \mod. \mathcal{Y}' \), \( pi = 1_N' \mod. \mathcal{Y}' \) and \( N \oplus N' \) is \( R \)-isomorphic to a locally direct summand of \( eM \), where \( p: M \to N, p': M \to N' \) and \( i' \) is the inclusion of \( N' \) to \( M \), by Lemma 6. Since \( N \) is maximal in \( eM \), \( N' = 0 \) and hence \( e = ip \mod. \mathcal{Y}' \).

**Corollary 1** (cf. Theorem 1 in [4]). Let \( P = \bigoplus_{\alpha \in L} P_\alpha \) in \( \mathfrak{B} \) (not necessarily each \( P_\alpha \) is in \( \mathfrak{M} \)). Then, there exists a submodule \( N_\alpha \) of \( P_\alpha \) in \( \mathfrak{A} \) such that \( e_\alpha = i_\alpha p_\alpha \mod. \mathcal{Y}' \), where \( p_\alpha: P \to N_\alpha, i_\alpha: N_\alpha \to P \) is the inclusion and \( e_\alpha: P \to P_\alpha \) is the projection, for each \( \alpha \in L \). Moreover, \( \bigoplus_{\alpha \in L} N_\alpha \) is a maximal locally direct summand of \( P \) in \( \mathfrak{S} \). (Such \( N_\alpha \) is called a dense submodule of \( P_\alpha \), in [4].)

**Proof.** We can find a maximal locally direct summand \( N_\alpha \) of \( e_\alpha P = P_\alpha \) such that \( e_\alpha = i_\alpha p_\alpha \mod. \mathcal{Y}' \), where \( p_\alpha: P \to N_\alpha, i_\alpha: N_\alpha \to P \) is the inclusion and \( e_\alpha: P \to P_\alpha \) is the projection, for every \( \alpha \in L \), by the lemma. Since a finite direct sum \( \sum_{\alpha \in L} N_\alpha \) is a direct summand of \( P \), \( \sum_{\alpha \in L} N_\alpha \) is a locally direct summand of \( P \). Hence, the inclusion \( /: \sum_{\alpha \in L} N_\alpha \to P \) is left regular mod. \( \mathcal{Y}' \). In order to see that \( \sum_{\alpha \in L} N_\alpha \) is dense in \( P \), we have only to prove that \( I \) is right regular mod. \( \mathcal{Y}' \). Let \( t \) be a homomorphism of \( P \) to any module \( T \) in \( \mathfrak{B} \) and assume that \( tI \) is in \( \mathcal{Y}' \). If \( t \) is not in \( \mathcal{Y}' \), there exists some direct summand \( P_\beta \) in \( P \) such that the restriction \( t|_{P_\beta} \) is not in \( \mathcal{Y}' \). Therefore, \( ti \) is in \( \mathcal{Y}' \) and so \( I \) is right regular mod. \( \mathcal{Y}' \).

**Corollary 2.** Let \( M \) be in \( \mathfrak{B} \), and \( N \) a direct summand of \( M \). If \( S_M \cap \mathcal{Y}' \) is equal to \( J(S_M) \), then \( N \) is in \( \mathfrak{B} \).

**Proof.** Since \( N \) is a direct summand of \( M \), there exists a submodule \( N' \) of \( M \) such that \( M = N \oplus N' \). Hence, there exist dense submodules \( N_0 \) and \( N'_0 \) of \( N \) and \( N' \) in \( \mathfrak{B} \), respectively such that \( N_0 \oplus N'_0 \) is dense in \( M \), by the above corollary. Hence, \( N_0 \oplus N'_0 = M \) by Corollary 2 to Lemma 8, which implies that \( N \) is isomorphic to a direct sum of completely indecomposable modules \( M_\alpha \)'s in \( \mathfrak{M} \).

**Proposition 10.** Let \( M, N \) be in \( \mathfrak{B} \), and \( f: M \to N \). If either \( S_M \cap \mathcal{Y}' = J(S_M) \) or \( S_N \cap \mathcal{Y}' = J(S_N) \), then there exist submodules \( M_1 \) and \( M_2 \) of \( M \) in \( \mathfrak{S} \) such that \( M = M_1 \oplus M_2 \) and the restrictions off to \( M_1 \) and \( M_2 \) are a zero homomorphism mod. \( \mathcal{Y}' \) and an \( R \)-monomorphism, respectively.
Proof. By Corollary 1(1) to Lemma 2 and Lemma 4, there exist a locally
direct summand \( N' \) of \( N \) in \( \mathfrak{A}, f': M \rightarrow N', g': N' \rightarrow M \) and the inclusion \( i: N' \rightarrow N \) such that \( f = \text{im}(f') \mod. \mathfrak{Y}, \) \( f'g' = 1_{N'} \mod. \mathfrak{Y}, \) \( i \) is left regular \( \mod. \mathfrak{Y}, \) and \( f' \) is
right regular \( \mod. \mathfrak{Y}. \) In case \( S_N \cap \mathfrak{Y} = J(S_N), S_N \cap \mathfrak{Y} = J(S_N') \) and hence \( f'g' \)
is an \( R \)-isomorphism. Therefore, \( M = \text{im}(g') \oplus \ker(f') \). We put \( M_1 = \ker(f') \)
and \( M_2 = \text{im}(g') \). Then, the restriction \( f|_{M_1} \) is a zero homomorphism \( \mod. \mathfrak{Y}. \) Since
\( f|_{M_2} \) is an \( \mathfrak{Y}-\)isomorphism, \( f|_{M_2} \) is an \( \mathfrak{Y}-\)monomorphism. On the other hand, if \( S_M \cap \mathfrak{Y} = J(S_M), S_M \cap \mathfrak{Y} = J(S_M') \) where
\( M' \) is a locally direct summand of \( M \) in \( \mathfrak{A} \) such that some \( g: N' \rightarrow M' \) is isomorphic
\( \mod. \mathfrak{Y} \) (cf. Lemma 5). Since \( g \) is an \( \mathfrak{Y}-\)isomorphism by Corollary 2 to
Lemma 3, \( S_{M'} \cap \mathfrak{Y} = J(S_{M'}) \) and so \( M = \text{im}(g') \oplus \ker(f') \) as above. We put
\( M_3 = \text{im}(g') \). Then, \( M_1 \) and \( M_2 \) satisfy the proposition.

Now, we shall show ring-theoretically the main theorem in this note by
\( M_1 = \ker(f') \) and only using the concept "modulo \( \mathfrak{Y}' \)."

\textbf{Theorem 11.} Let \( M = \bigoplus_{\alpha \in K} M_\alpha = \bigoplus_{\beta \in J} N_\beta \) be any two direct sum decompositions of a module \( M \) in \( \mathfrak{A} \) into completely indecomposable modules \( M_\alpha \)'s and \( N_\beta \)'s, respectively and assume that \( S_M \cap \mathfrak{Y} = J(S_M) \). Then, for any subset \( K' \) of \( K \), there exists a one-to-one mapping \( \varphi \) of \( K' \) into \( J \) such that \( M = \bigoplus_{\alpha \in K'} M_\alpha \) and \( M = \bigoplus_{\beta \in J'} N_\beta \) for \( \alpha \in K' \).

Proof. For any subset \( K' \) of \( K \), we put \( M_0 = \bigoplus_{\alpha \in K \setminus K'} M_\alpha \). Then, there exists a maximal member \( M^* \) in the set \( \{ M_0 \oplus \bigoplus_{\alpha \in K \setminus K'} N_\alpha \} \) of locally direct summands of \( M \) with each subset \( J_1 \) of \( J \), by Zorn's lemma. Since \( M \) is the only one dense submodule of \( M \) by Corollary 2 to Lemma 8, \( M^* \) is a direct summand of \( M \), say, \( M = M^* \oplus M' \) for some submodule \( M' \) of \( M \). By Corollary 2 to
Lemma 9, \( M' \) is in \( \mathfrak{A} \) if \( M' \neq 0 \). And so by [2] there exists some \( N_\beta \) such that
\( M^* \oplus N_\beta \) is a direct summand of \( M \). This contradiction shows that \( M^* = M \).

Since \( \sum_{\alpha \in K'} M_\alpha \approx M \cap M_0 \approx \bigoplus_{\gamma \in J'} N_\gamma \) with some subset \( J' \) of \( J \), by [2] we can
find a one-to-one mapping \( 99 \) of \( K' \) onto \( J' \) such that \( M_\alpha \approx N_{\varphi(\alpha)} \) for \( \alpha \in K' \).

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\textbf{References}


