A commutative integral domain \( D \) is said to be a Krull domain provided there is a family \( \{ V_i \}_{i \in I} \) of discrete rank one valuation overrings of \( D \) such that

1. \( D = \cap_{i \in I} V_i \).
2. Each \( V_i \) is essential for \( D \).
3. Given \( x \in D, x \neq 0 \), there is at most a finite number of \( i \) in \( I \) such that \( x \) is a non unit in \( V_i \).

Using noetherian local Asano orders and noetherian simple rings instead of discrete rank one valuation rings we will introduce non commutative Krull rings and generalize some elementary results on commutative Krull domains to the case of non commutative Krull rings.

In §1, we will define non commutative Krull rings and study the relations between a prime Goldie ring \( R \) and noetherian local Asano orders containing \( R \). We will introduce, in §2, the concept of divisor classes on bounded Krull rings and show that the divisor class of a non commutative Krull ring becomes an abelian group under some conditions. In §3, we will study orders over a commutative Krull domain \( o \). Maximal \( o \)-orders are bounded Krull rings. Furthermore we will generalize the approximation theorem for commutative Krull domains to the case of maximal \( o \)-orders (Theorem 3.5). In §4 we will define the \( w \) and \( v \)-operations on one-sided \( R \)-ideals of prime Goldie rings in the same fashion as for commutative domains. We will show that these operations coincide on noetherian bounded Krull rings and maximal \( o \)-orders. Further we will show that every class of right \( v \)-ideals of maximal \( o \)-orders contains a right ideal generated by two regular elements. Several examples of non commutative Krull rings will be given in the final section.

Throughout this paper \( R \) will denote a prime Goldie ring\(^1\) with identity element which is not artinian, and \( Q \) will denote the simple artinian quotient ring of \( R \).

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\(^1\) Conditions assumed on rings will always be assumed to hold on both sides; for example, a Goldie ring always means a right and left Goldie ring.
1. Definitions and localizations

Let \( R \) be a prime Goldie ring with quotient ring \( Q \). A right \( R \)-submodule \( I \) of \( Q \) is called a right ideal (fractional) provided \( I \) contains a regular element of \( Q \) and there is a regular element \( b \) of \( Q \) such that \( bI \subseteq R \). If \( I \subseteq R \), then we say that \( I \) is integral. We call a ring \( R \) an Asano order if its \( R \)-ideals form a group under multiplication. \( R \) is said to be local if its Jacobson radical \( J \) is the unique maximal ideal and \( R/J \) is artinian. Let \( R \) be a noetherian local Asano order. Then, by Proposition 1.3 of [5], \( R \) is hereditary, a principal right and left ideal ring. Let \( F \) be a right additive topology. We denote by \( R_F \) the ring of quotients of \( R \) with respect to \( F \) (cf. [10]). A subring of \( Q \) containing \( R \) is called an overring of \( R \). An overring \( R' \) of \( R \) is said to be right essential if there is a perfect right additive topology \( F \) such that \( R'=R_F \) (cf. p. 74 of [10] for the definition of perfect topologies). By the results of §13 of [10], \( R_F \) is a right essential overring of \( R \) if and only if the inclusion map \( R \to R_F \) is an epimorphism and \( R_F \) is a flat left \( R \)-module. Further, if \( R_F \) is a right essential overring of \( R \), then \( F \) consists of all right ideals \( I \) such that \( IR_F=R_F \). In a similar way, we define the concept of left essential overrings of \( R \). An overring \( R' \) of \( R \) is said to be essential if it satisfies the following two conditions:

(1) \( R' \) is a right and left essential overring of \( R \), that is, \( R'=R_F=R_F \), where \( F(F) \) is a perfect right (left) additive topology.

(2) If \( I \in F(J \in F) \), then \( R'I=R'(JR'=R') \).

If \( A \) is an ideal of \( R \), then we denote by \( C(A) \) those elements of \( R \) which are regular in \( R/A \). If \( R \) satisfies the Ore condition with respect to \( C(P) \), where \( P \) is a prime ideal of \( R \), then we denote by \( R_P \) the local ring of \( R \) with respect to \( P \). Let \( A, B \) be subsets of \( Q \). We use the notation: \( (A : B)_r = \{ q \in Q \mid Bq \subseteq A \} \), \( (A : B)_l = \{ q \in Q : qB \subseteq A \} \). We denote by \( F_r(R) \) the set of right \( R \)-ideals of \( R \), and by \( F_l(R) \) the set of left \( R \)-ideals of \( R \). We set \( F(R) = F_r(R) \cap F_l(R) \).

A prime Goldie ring \( R \) is said to be a Krull ring if there are families \( \{ R_i \}_{i \in I} \) and \( \{ S_j \}_{j \in J} \) of essential overrings of \( R \) such that

(K1) \( R = \bigcap_i R_i = \bigcap_j S_j \) (\( i \in I, j \in J \)).

(K2) Each \( R_i \) is a noetherian local Asano order, each \( S_j \) is a noetherian simple ring and the cardinal number of \( J \) is finite.

(K3) For every regular element \( c \) in \( R \) we have \( cR_i = R_i(R_ic \subseteq R_i) \) for finitely many \( i \) only.

If \( J = \phi \), then we say that \( R \) is bounded.

Proposition 1.1. Let \( R' \) be an essential overring of a prime Goldie ring \( R \) and let \( R' \) be a noetherian local Asano order with unique maximal ideal \( P' \). Then

(1) \( P'=PR'=R'P \), where \( P=R \cap P' \).

(2) \( P \) is a prime ideal of \( R \).

(3) \( R'/P' \) is the quotient ring of \( R/P \).
Proof. There is a perfect right (left) additive topology $F(F_j)$ such that $R' = R_{F_j} = R_{F_i}$. For any $I \in F$, we have $IR' = R'$ so that $IQ = Q$. This implies that $I$ is an essential right ideal of $R$ by Theorem 3.9 of [4]. Hence we easily obtain that $R' = \cup (R : I) (I \in F)$. Similarly $R' = \cup (R : J) (J \in F_i)$.

(1) The containment $PR' \subseteq P'$ is clear. Let $q$ be any element of $P'$. Then there is an element $I \subseteq F$ such that $qI \subseteq P' \cap R = P$. So we have $q \in qR' = qIR' \subseteq PR'$. Hence $P' = PR'$. Similarly $P' = R'P$.

(2) Assume that $AB \subseteq P$ and $B \subseteq P$, where $A$, $B$, are ideals of $R$. Then $R'BR' \subseteq P'$. Since the ideals of $R'$ are only the powers of $P'$, we have $R'BR' = R'$. Write $1 = \sum t_i b_i s_i$, where $t_i$, $s_i \in R'$ and $b_i \in B$. Then there are $I \subseteq F$ and $J \subseteq F_i$ such that $s_i I \subseteq R$ and $Jt_i \subseteq R$. So we have $JI \subseteq B$. Hence $A \subseteq AR' = AJR' \subseteq P'$ and so $A \subseteq P$.

(3) Let $q$ be any element of $R'$ such that $q \not\in P'$. Then $qI \subseteq R$ for some $I \in F$. If $qI \subseteq P$, then $q \in qR' = qIR' \subseteq PR' = P'$, a contradiction. Hence $qI \not\subseteq P$. This implies that $R'/P'$ is an essential extension of $R/P$ as right $R/P$-modules. Since $R'/P'$ is a simple artinian ring and $R/P$ is a prime ring, we obtain that $R'/P'$ is the right quotient ring of $R/P$. Similarly $R'/P'$ is the left quotient ring of $R/P$.

(4) First we shall prove that each element of $C(P)$ is a unit in $R'$. Since $P'$ is the Jacobson radical of $R'$ and $R'/P'$ is a simple artinian ring, an element $q$ of $R'$ is a unit in $R'$ if and only if it is an element in $C(P')$. Hence it suffices to prove that $C(P) \subseteq C(P')$. To prove this, we assume that $cq \in P'$, where $c \in C(P)$ and $q \in R'$, then $cqI \subseteq P$ and $qI \subseteq R$ for some $I \in F$. Hence $qI \subseteq P$ so that $q \in P'$. Consequently $c \in C(P')$. Let $F_p$ be the set of right ideals $I$ of $R$ such that $r^{-1} = \{ x \in R | rx \in I \}$ meets $C(P)$ for all $r \in R$. Then we shall prove that $F = F_p$. Since any element of $C(P)$ is a unit in $R'$, the containment $F_p \subseteq F$ is clear. To prove the converse inclusion let $I$ be any element of $F$ and let $r$ be any element of $R$. Since $r^{-1}I \subseteq F$, we have $(r^{-1}I)R' = R'$. This implies that $[(r^{-1}I + P)/P] (R'/P') = R'/P'$. Hence, by (3) and Theorem 3.9. of [4], $(r^{-1}I + P)/P$ is an essential right ideal of $R/P$ so that $(r^{-1}I + P)/P \cap C(P) = \phi$. Write $c = d + p$, where $c \in C(P)$, $d \in r^{-1}I$ and $p \in P$. Then $d \in r^{-1}I \cap C(P)$ and thus $I \subseteq F_p$. Hence $F_p = F$, as desired. Consequently for any element $q$ of $R'$ we have $qc \in R$ for some $c \in C(P)$. Now take any $r \in R$ and $c \in C(P)$. Then $c^{-1}r \in R'$ so that $c^{-1}rd = s \in R$ for some $d \in C(P)$, that is, $rd = cs$. Therefore $R$ satisfies the Ore condition with respect to $C(P)$ and $R' = R_P$.

2. The divisor classes

Throughout this section we assume that $R$ is a bounded Krull ring, that is, $R = \cap R_i (i \in I)$ and each $R_i$ is a noetherian local Asano order with unique maximal
ideal $P_i'$. We let $P_i=P_i' \cap R$. Then $P_i$ is a prime ideal of $R$ by Proposition 1.1. Further we assume, in this section, the following condition:

(K4) For any $i, j \in I, i \neq j, P_i \supseteq P_j$ and $P_i \nsubseteq P_j$.

This condition is equivalent to the following one.

(K4)' For any $i, j \in I, i \neq j, P_iR_j=R_j=R_jP_i$.

**Lemma 2.1.** Let $R$ be a bounded Krull ring which satisfies (K4). Let $I_i'$ be any element of $F_i(S_i)$ such that $I_i'=R_i$ for almost all $i \in I$. Then $I=\cap I_i' \in F_i(S)$ and $IR_i=I_i'$ for all $i \in I$.

**Proof.** It is clear that $I \in F_i(S)$. First assume that $I_i' \subseteq R_i$ for all $i \in I$. We put $I_i=I_i' \cap R$. Then $I=I_1 \cap \cdots \cap I_k$. Since $R_i$ is bounded, there are natural numbers $n_i$ such that $I_i' \supseteq P_i'^{n_i}$, and so $I_i \supseteq P_i^{n_i}$. Since $R_i$ is flat, we have $I_iR_i=I_i'=I_i'$ for all $i \in I$ by (K4)'... In general case there is a regular element $c$ of $R$ such that $cI_i' \subseteq R_i$ for all $i \in I$. Hence $cIR_i=cI_i'$ so that $IR_i=I_i'$ for all $i \in I$.

**Lemma 2.2** (Robson [7]). Let $S$ be a prime Goldie ring with quotient ring $Q(S)$. Then any right $S$-ideal of $S$ is generated by the units in $Q(S)$ which it contains.

**Proof.** Let $I$ be any element of $F_i(S)$. Then there is a regular element $c$ of $S$ such that $cI \subseteq S$. By Theorem 5.5 of [7], $I=\sum b_iS$, where the set $\{b_i\}$ is the regular elements of $S$ contained in $cI$. Hence $I=\sum (c^k b_i)S$.

For any $A \in F(R)$ we denote by $A_d$ the $(R, R)$-bimodule $\cap R_iAR_i(i \in I)$. There are only finitely many $R_i(1 \leq i \leq k)$ such that $R_iAR_i \neq R_i$. Since $A \in F(R)$, we have $cA \subseteq R$ for some regular element $c$ of $R$. Thus $R_iAR_i \subseteq R_i,c^{-1}R_i$. Since $R_i$ is a bounded Aasano order, $R_iAR_i \in F(R_i)$ so that $R_iAR_i \in F(R_i)$. Therefore $R_iAR_i=P_i^{n_i}(1 \leq i \leq k)$ and $R_iAR_i=R_j$ for $j \in I, j \in \{1, \cdots, k\}$. Hence we get $A_d=P_i^{n_i} \cap \cdots \cap P_k^{n_k} \cap jR_j$ and $A_d \in F(R)$.

We obtain immediately

(i) $A \subseteq A_d$.
(ii) If $A \subseteq B$, then $A_d \subseteq B_d$.
(iii) $A_d=\bar{A}_d$.

If $A=A_d$, then it is said to be a $d$-ideal. If we define an equivalence relation on $F(R)$ by saying that $A \sim B$ if and only if $A_d=B_d$. For any $A \in F(R)$, we denote by $A$ the equivalence class determined by $A$. Each such equivalence class $A$ contains a unique $d$-ideal $A_d$. The set $D(R)$ of all such equivalence classes forms a semi-group under the multiplication "$\ast$" defined by $A \ast B=(A_dB_d)$.

**Theorem 2.3.** If a bounded Krull ring $R$ satisfies (K4), then $D(R)$ forms an abelian group and it is a direct product of infinite cyclic subgroups $\{(P_i)\}_{i=1}$. 

**Proof.** If $A$ is $d$-ideal, then $A=\cap P_i^{n_i}$, where $n_i$ are integers and $n_i=0$
for almost all $i \in I$. We let $B = \cap P_i^{-n_i}$. Then we get the following;

(i) $B_d = B$.
(ii) $B = (R : A)_f = (R : A)_r$, and so we denote $B$ by $A^{-1}$.
(iii) $(A^{-1})^{-1} = A$.

By Lemma 2.1, (i) immediately follows. (ii); we have $BA \subseteq P_i^{-n_i}P_i^{-n_i} = R_i$ for all $i \in I$. So $BA \subseteq R$, and thus $B \subseteq (R : A)_f$. To prove the converse let $c$ be a unit in $Q$ contained in $(R : A)_f$, that is, $cA \subseteq R$ and so $cAR_i = cP_i^{-n_i} \subseteq R_i$ by Lemma 2.1. Hence $c \in P_i^{-n_i}$ for all $i$ and therefore $c \in B$. Consequently $B \supseteq (R : A)_f$ by Lemma 2.2 and the equality holds. Similarly $B = (R : A)_r$. (iii) is clear from (ii). By Lemma 2.1 we get $(AA^{-1})R_i = AP_i^{-n_i}P_i^{-n_i} = R_i$ for all $i \in I$. Hence $R = (AA^{-1})R_i \subseteq (AA^{-1})_d \subseteq R$ so that $\bar{R} = \bar{A} \bar{A}^{-1}$. Therefore $D(R)$ forms a group. Let $A = P_i^{-n_i} \cap \cdots \cap P_i'^{-n_i} \cap_{j \in J} R_j$ be any $d$-ideal of $R$. Then we obtain immediately that $A = (P_i^{-n_i} \cdots P_i'^{-n_i})_d = (P_i^{-n_i} \cdots P_i'^{-n_i})_d$. Hence $\bar{A} = \bar{P}_i^{-n_i} \cdots \bar{P}_i'^{-n_i} = \prod_{i=1}^d \bar{P}_i^{-n_i}$ so that $D(R)$ is an abelian group generated by $\{\bar{P}_i\}_{i \in I}$.

If $\prod_{i=1}^d \bar{P}_i^{-n_i} = 1$, then $(\prod_{i=1}^d \bar{P}_i^{-n_i})_d = R$. This implies that $R_i = P_i'^{-n_i}$ and so $n_i = 0$ for all $i \in I$. Therefore $D(R)$ is a direct product of infinite cyclic subgroups $\{\bar{P}_i\}_{i \in I}$.

3. Maximal orders over commutative Krull domains

In the remainder of this paper, $\delta$ denotes a commutative Krull domains, $K$ denotes the quotient field of $\delta$, and $\Sigma$ a fixed central simple $K$-algebra with finite dimension over $K$. Let $P$ be the set of all minimal prime ideals of $\delta$. Then $\delta = \cap P_i(\delta \in P)$ and $\delta_2$ is a discrete rank one valuation overring of $\delta$.

Following [2], $A$ subring $\Lambda$ of $\Sigma$ is said to be $\delta$-order if the following conditions are satisfied:

(i) $\delta \subseteq \Lambda$.
(ii) $K\Lambda = \Sigma$.
(iii) Each element of $\Lambda$ is integral over $\delta$.

If $\Lambda$ is a $\delta$-order, then, by Proposition 1.1 of [2], there is a finitely generated $\delta$-free submodule $F$ of $\Sigma$ such that $F \supseteq \Lambda$. Further if $\Lambda$ is maximal $\delta$-order, then $\Lambda \otimes \delta_2$ is also a maximal $\delta_2$-order by Proposition 1.3 of [2]. Therefore if $\Lambda$ is a maximal $\delta$-order, then $\Lambda \otimes \delta_2$ is a noetherian local Asano order and $\Lambda = \cap \Lambda \otimes \delta_2(\delta \in P)$. For any $\delta \in P$, we denote by $P'$ the unique maximal ideal of $\Lambda \otimes \delta_2$, and denote by $P$ the contracted prime ideal $P' \cap \Lambda$.

**Proposition 3.1.** Let $\Lambda$ be a maximal $\delta$-order. Then

1. $\Lambda$ is a bounded Krull ring and satisfies (K4).
2. $D(\Lambda) \cong D(\delta)$.

Proof. For any $\delta \in P$, we denote by $F(F_\delta)$ the set of all right (left) ideals
$I(f)$ of $\Lambda$ such that $I(f)$ meets $0-\bar{z}$. Then it follows that $F(F_i)$ is a perfect right (left) additive topology and that $\Lambda_F=\Lambda\otimes\omega_0=\Lambda_{F_i}$. Hence $\Lambda\otimes\omega_0$ is an essential overring of $\Lambda$. Let $x$ be any regular element of $\Lambda$. Since $x$ is algebraic over $K$, we get; $a_0x^k+\cdots+a_0=0$ for some $a_i\in\omega_0$. We assume that $a_0\neq0$. Put $y=-a_0x^{k-1}-\cdots-a_i$. Then we have $a_0y=xy=y\in\Lambda$. So $(xy)\omega_0=\omega_0$ for almost all $\bar{z}\in P$. Therefore $\Lambda(\Lambda\otimes\omega_0)=\Lambda\otimes\omega_0$ for almost all $\bar{z}\in P$. It is evident that $P\cap\omega_0=\bar{z}$ for any $\bar{z}\in P$ and that $\omega_0$ satisfies (K4). Therefore $\Lambda$ also satisfies (K4). (2) is clear from (1) and Theorem 2.3.

**Proposition 3.2.** Let $\Lambda$ be a maximal $\omega$-order and let $\bar{z}\in P$. Then the set of prime ideals of $\Lambda$ lying over $\bar{z}$ is only $\{P\}$. In particular, $P$ is a minimal prime ideal of $\Lambda$.

Proof. Assume that $P_\omega$ is a non zero prime ideal of $\Lambda$ such that $P_\omega\cap\omega=\bar{z}$ and $P_\omega+P$. We set $\bar{K}=\Lambda/P_\omega$, $\bar{\omega}=/\omega_0\omega_1$ and $\bar{\Lambda}=\Lambda\otimes\bar{\omega}$. First we shall prove that $\bar{\Lambda}$ is a prime Goldie ring with quotient ring $\Lambda_\omega=\Lambda\otimes\bar{\omega}$. It is evident that $\bar{\Lambda}$ is $\bar{\omega}$-torsion-free. Hence the natural map $\Lambda\to\bar{\Lambda}$ is a morphism and $\bar{\Lambda}$ is an essential extension of $\bar{\Lambda}$ as $\bar{\Lambda}$-modules. Since $\Lambda\otimes\omega_0$ is finitely generated as $\omega_1$-modules, $\Lambda\otimes\bar{\omega}$ is an $\bar{\omega}$-algebra with finite dimension over $\bar{\omega}$, so that $\Lambda\otimes\bar{\omega}$ is a simple artinian ring. Therefore $\Lambda_\omega$ is a quotient ring of $\Lambda$, and so $\Lambda$ is a prime Goldie ring.

(i) In case $P\subsetneq P_\omega$. Let $P_\omega(\Lambda\otimes\omega_0)=P_\omega^n$ for some natural number $n$. We shall prove that $C(P_\omega)\subseteq C(P')$. By Lemma 2.3 of [5], $C(P'^n)=C(P')$, and so it suffices to prove that $C(P_\omega)\subseteq C(P'^n)$. If $c\bar{x}\in P'^n$, where $c\in C(P_\omega)$, $x\in\Lambda\otimes\omega_0$, then there exists $m\in\omega-\bar{z}$ such that $c\bar{x}m\in P_\omega$ and $xm\in\Lambda$. Hence $x\in P_\omega$ and so $x\in P'^n$. Therefore each element of $C(P_\omega)$ is a unit in $\Lambda\otimes\omega_1$. Since $P_\omega\cap\omega=\bar{z}$ and $\Lambda$ is a prime Goldie ring, we have $P\cap C(P_\omega)=\phi$. This implies that $P(\Lambda\otimes\omega_0)=\Lambda\otimes\omega_0$, a contradiction.

(ii) In case $P\supseteq P_\omega$. The family $F=\{I| x^{-1}I\cap C(P_\omega)=\phi$ for any $x\in\Lambda, I$: right ideal} is a topology (cf. Exer. 4 of [10, p. 18]). The $F$-torsion submodule $t(\Lambda)$ of $\Lambda$ is an ideal and $t(\Lambda)\subseteq P_\omega$. We denote by $\bar{\Lambda}$ the factor ring $\Lambda/t(\Lambda)$ and by $\Lambda_F$ the ring of quotients of $\Lambda$, that is, $\Lambda_F=\lim\hom(I, \bar{\Lambda})(I\in F)$. Then $\bar{\Lambda}$ is a subring of $\Lambda_F$ (cf. Chap. 2 of [10]). We shall prove that $\bar{P}_\omega\Lambda_F\subseteq\Lambda_F$, where $\bar{P}_\omega=P_\omega/t(\Lambda)$. It suffices to prove that $\bar{P}_\omega\Lambda_F\cap \bar{\Lambda}=\bar{P}_\omega$. Assume that $\bar{T}=\bar{P}_\omega\Lambda_F\cap \bar{\Lambda}=\bar{P}_\omega$, where $T$ is an ideal of $\Lambda$. Then $T\supseteq P_\omega$ and so $T\cap C(P_\omega)+\phi$. Let $c$ be any element of $T\cap C(P_\omega)$. Write $c=\sum p_iq_i$, where $p_i\in P_\omega$ and $q_i\in \Lambda_F$. There exists $I\in F$ such that $q_i\in I$. Hence $cI\subseteq P_\omega$ and thus $I\subseteq P_\omega$. This contradicts to $C(P_\omega)\cap I+\phi$. Therefore $\bar{P}_\omega\Lambda_F\cap \bar{\Lambda}=\bar{P}_\omega$. It is evident that $m\Lambda\subseteq F$ for every $m\in\omega-\bar{z}$. Hence we may assume that $\bar{\Lambda}\otimes\omega_1\subseteq\Lambda_F$. Since $P_\omega\subseteq P$, $C(P)\cap P_\omega+\phi$. This implies that $P_\omega(\Lambda\otimes\omega_0)=\Lambda\otimes\omega_0$, a contradiction.

Finally we shall prove that $P$ is a minimal prime ideal of $\Lambda$. If $P\supseteq P_\omega+\phi$
and \( P_0 \) is a prime ideal of \( \Lambda \), then \( P_0 \cap \mathfrak{a} = 0 \) is a prime ideal of \( \mathfrak{a} \) and \( \mathfrak{a} \supseteq P_0 \cap \mathfrak{a} \). Hence \( \mathfrak{a} = P_0 \cap \mathfrak{a} \). Therefore \( P_0 = P \) by (i).

**Lemma 3.3.** Let \( \mathfrak{p}_i \) (1 \( \leq i \leq k \)) be any elements in \( P \). Then \( \Lambda_i = \bigcap_{i=1}^{k} \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i} \) is a bounded Dedekind prime ring, and is a right and left principal ideal ring.

**Proof.** The set \( M = \mathfrak{a} - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k) \) is a multiplicative closed set and \( \mathfrak{a} M = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k \) is a Dedekind domain. Further it is a principal ideal ring. By Corollary 1.9 of [2], \( \Lambda \otimes \mathfrak{o}_M = \bigcap_{i=1}^{k} \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i} \) is a maximal \( \mathfrak{o}_M \)-order. Hence it is a bounded Dedekind prime ring. Since the set of prime ideals of \( \Lambda \otimes \mathfrak{o}_M \) is \( \{P_i(\Lambda \otimes \mathfrak{o}_M) | 1 \leq i \leq k\} \), \( \Lambda \otimes \mathfrak{o}_M \) is a right and left principal ideal ring.

**Lemma 3.4.** Let \( \mathfrak{p}_i \) (1 \( \leq i \leq k \)) be any elements in \( P \) and let \( n \) be any positive integer. Then there is a regular element \( \lambda \in \Lambda \) such that \( \lambda(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i} \) and \( \lambda(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}(2 \leq i \leq k) \).

**Proof.** We put \( M = \mathfrak{a} - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k) \) and \( P_i' = (\Lambda \otimes \mathfrak{o}_M) \cap P_i \). By Lemma 3.3, \( P_i' = y(\Lambda \otimes \mathfrak{o}_M) = (\Lambda \otimes \mathfrak{o}_M)y \) for some regular element \( y \in \Lambda \otimes \mathfrak{o}_M \). Since \( y = zm^{-1}(z \in \Lambda \otimes \mathfrak{o}_M) \), we may assume that \( y \in \Lambda \cap P_i' = P_i \). We shall prove that \( y \in C(P_i) \). By Proposition 3.1, \( \Lambda \) satisfies (K4) and so \( P_i \cap C(P_i) = P_i \). Let \( \epsilon = yw \in (\Lambda \otimes \mathfrak{o}_M) \) be any element of \( P_i \cap C(P_i) \). If \( zy \in P_i \), where \( z \in \Lambda \), then \( zy = z \in P_i' \cap \Lambda = P_i \) so that \( z \in P_i \). Hence \( y \) is a unit in \( \Lambda \otimes \mathfrak{o}_M(2 \leq i \leq k) \). We put \( x = y^n \). Then \( x \) satisfies the assertion of the lemma.

**Theorem 3.5.** Let \( \mathfrak{p}_1, \ldots , \mathfrak{p}_k \in P \) and let \( n_1, \ldots , n_k \) be any integers. Then there is a unit \( x \in \Sigma \) such that \( x(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}) = P_{i,n} \) for all \( \mathfrak{p}_i \in P \) with \( \mathfrak{p}_i \neq \mathfrak{p}_j \).

**Proof.** It is enough to show that for any \( i \), there exists a unit \( x_i \in \Sigma \) such that \( x_i(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}) = P_{i,n_i} \) and \( x_i(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}(i \neq j, 1 \leq j \leq k) \) and \( x_i \in \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j} \) for all \( \mathfrak{p}_j \in P \). We will exhibit a unit \( x_i \in \Sigma \) such that \( x_i(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}) = P_{i,n_i} \) and \( x_i(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}(2 \leq i \leq k) \) and \( x_i \in (\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) \) for all \( \mathfrak{p}_j \in P \) with \( \mathfrak{p}_j \neq \mathfrak{p}_i \).

(i) If \( n_i > 0 \), then the assertion follows from Lemma 3.4.

(ii) If \( n_i < 0 \), then, by Lemma 3.4, there is a regular element \( y \in \Lambda \) such that \( y(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i}) = (\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_i})y = P_{i,n_i} \) and \( y(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}(2 \leq i \leq k) \). There are only finitely many elements \( \mathfrak{p}_j \) such that \( y(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) \). For any \( j(k+1 \leq j \leq l) \), there is a positive integer \( h_j \) such that \( y(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) = \Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j} \). There are only finitely many elements \( \mathfrak{p}_j \) such that \( y(\Lambda \otimes \mathfrak{o}_{\mathfrak{p}_j}) \).
4. *-operations

Let $R$ be a prime Goldie ring with quotient ring $Q$. Following [3], a mapping $*: I \rightarrow I^*$ of $F_r(R)$ into $F_r(R)$ is called a *-operation on $R$ if the following conditions hold for any unit $a$ in $Q$ and any $I, J \in F_r(R)$:

(i) $(aR)^* = aR$, $(aI)^* = aI^*$.

(ii) $I \subseteq I^*$, if $I \subseteq J$, then $I^* \subseteq J^*$.

(iii) $(I^*)^* = I^*$.

If $I = I^*(I \in F_r(R))$, then we say that $I$ is a right $*$-ideal. In this section we shall define *-operations of two kinds and give sufficient conditions that these two *-operations coincide. Let $I$ be any element of $F_r(R)$. We define the $v$-operation on $R$ by

$I_v = (R : (R : I)_v)$.

If $R$ is a Krull ring, that is, $R = \cap R_i \cap S_j$, then we define the $w$-operation on $R$ by

$I_w = \cap IR_i \cap IS_j$.

If $\mathfrak{o}$ is a commutative Krull domain and if $\Lambda$ is a maximal $\mathfrak{o}$-order, that is, $\Lambda = \cap \Lambda \otimes \mathfrak{o}_\mathfrak{a} (\mathfrak{a} \subseteq \mathfrak{P})$, then $A_w = A_d$ for any $A \in F(\Lambda)$. It is evident that the $w$-operation is a $*$-operation.

By using Lemma 2.2, the following proposition is proved by the same way as in commutative domains (cf. Theorem 28.1 of [3]).

**Proposition 4.1.** Let $R$ be a prime Goldie ring with quotient ring $Q$ and let $I$ be any element in $F_r(R)$. Then $I_v$ is the intersection of the set of principal right $R$-ideals of $R$ containing $I$.

**Corollary 4.2.** (1) The $v$-operation on $R$ is a $*$-operation.

(2) If $I \in F_r(R)$, then $I^* \subseteq I_v$ for any $*$-operation on $R$. In particular, any right $v$-ideal is a right $*$-ideal.

(3) If $J \in F_r(R)$, then $(R : J)_v$ is a right $v$-ideal.

**Proof.** (1) and (2) are evident from Proposition 4.1 and the definitions. (3): An element $x$ in $Q$ is an element in $(R : J)_v$ if and only if $Jx \subseteq R$. This condition is equivalent to $cx \in R$ for any unit $c$ in $Q$ which is contained in $J$. Hence $(R : J)_v = \cap c^{-1}R = \cap (c^{-1}R)_v = (\cap c^{-1}R)_v$, where $c$ ranges over all units in $Q$ which are contained in $J$. Therefore $(R : J)_v$ is a right $v$-ideal.

**Proposition 4.3.** Let $R$ be a noetherian Krull ring, that is, $R = \cap R_i \cap S_j$ ($i \in I, j \in J$) and let each $S_j$ be hereditary. Then $I_w = I_v$ for any $I \in F_r(R)$.

**Proof.** For a convenience, we denote $R_i$ or $S_j$ by $T_k$. We let $J = (R : I)_v$. First we shall prove that $(T_k : IT_k)_v = T_k J$. Let $x$ be any element in $(T_k : IT_k)_v$. Then $xI \subseteq T_k$. Since $R$ is noetherian, $xI$ is finitely generated so that $J x \subseteq R$ for some $J x \subseteq F_k$, where $F_k$ is the left additive topology on $R$ such that $T_k = R_{F_k}$. Thus $J x \subseteq T_k J$. Consequently we have $x \in T_k x = T_k J x \subseteq T_k J$ and so $(T_k : IT_k)_v$
The converse containment is evident. Similarly we get \((T_k : T_kJ)_r = (R : J)_r \). By Corollary 4.2, \(J\) is a left \(w\)-ideal so that \(J = \cap (T_k : J)\). Similarly \((R : J)_r = \cap (T_k : T_kJ)_r\). By Corollary 4.6 of [7], simple rings are also maximal orders. So \((T_k : T_kJ)_r = (T_k : (T_k : J))_r = IT_k\) for all \(k\). Hence we get \(I_v = (R : J)_r = \cap (T_k : T_kJ)_r = IT_k = I_w\), as desired.

**Theorem 4.4.** Let \(o\) be a commutative Krull domain and let \(\Lambda\) be a maximal \(o\)-order. Then \(I_v = I_w\) for any \(I \in F_r(\Lambda)\).

**Proof.** (i) First we assume that \(I\) is integral. By Corollary 4.2 \(I_w \subseteq I_v\). To prove the converse inclusion it suffices to prove that any regualar element \(c\) in \(\Lambda - I_w\) is not contained in \(I_v\) by Lemma 2.2. We set \(I_w = \cap I'_i\), where \(I'_i = I(\Lambda \otimes o_{b_i})\) and \(b_i \in \mathcal{P}\). There are only finitely many \(I'_1, \ldots, I'_l\) such that \(I'_i \subseteq \Lambda \otimes o_{b_i}\) \((1 \leq i \leq k)\). Put \(M = o - (b_1 \cup \cdots \cup b_k)\). By Lemma 3.3 there is a regular element \(x\) in \(\Lambda\) such that \(x(\Lambda \otimes o_M) = I'_i \cap \cdots \cap \Lambda \otimes o_{\Lambda - I'_i} = I'_i\) for \(i(1 \leq i \leq k)\). There are only finitely many \(k+1, \ldots, l\) such that \(x(\Lambda \otimes o_{b_i}) \subseteq \Lambda \otimes o_{b_j}\) \((k+1 \leq j \leq l)\). Since \(\Lambda \otimes o_{b_j}\) is bounded, there are positive integers \(n_{b_1}, \ldots, n_l\) such that \(z = xy^{-1}\). Then we get: \(z(\Lambda \otimes o_{b_j}) \supseteq P_j^{n_j}\). By Theorem 3.5, there is a unit \(y\) in \(\Sigma\) such that \(y(\Lambda \otimes o_{b_j}) \subseteq \Lambda \otimes o_{b_j}\) \((1 \leq i \leq k)\), \(y(\Lambda \otimes o_{\Lambda - I'_i}) = y = P_j^{n_j}(k+1 \leq j \leq l)\) and \(y(\Lambda \otimes o_{b_j}) \subseteq \Lambda \otimes o_{b_j}\) \((k+1 \leq j \leq l)\). Set \(M_2 = o - (b_{k+1} \cup \cdots \cup b_k)\) and \(N = M_1 \cap M_2\). Now for any \(I'_i\) we get \(I'_i \supseteq P_i^{n_i}\) for some positive number \(n_i\) and so \(I'_i \subseteq \Lambda \supseteq P_i^{n_i}\). We put \(A = P_i^{n_1} \cap \cdots \cap P_i^{n_l}\). Then \(I_v \supseteq A\) and \(A(\Lambda \otimes o_N) \supseteq d(\Lambda \otimes o_N)\) for some regular element \(d\) in \(\Lambda\). Since \(\Lambda\) satisfies (K4) and \(\Lambda \otimes o_{b_j}\) is \(\Lambda\)-flat, we have \(d(\Lambda \otimes o_{b_j}) \subseteq \Lambda \otimes o_{b_j}\) \((k+1 \leq j \leq l)\). Therefore we get: \((c\Lambda + d\Lambda)(\Lambda \otimes o_{b_j}) = I'_i + P_i^{n_i} = I'_i(1 \leq i \leq k)\) and \((c\Lambda + d\Lambda)(\Lambda \otimes o_{b_j}) = \Lambda \otimes o_{b_j}\) \((k+1 \leq j \leq l)\). Thus we obtain \((c\Lambda + d\Lambda)_v = I_v\), as
desired.

5. Examples

In this section we shall give some examples of Krull rings.

(i) A commutative Krull domain \( \mathfrak{o} \) and a maximal \( \mathfrak{o} \)-order are both bounded Krull rings (cf. §3).

(ii) Noetherian Asano orders are Krull rings (cf. [5]). In particular, bounded noetherian Asano orders are bounded Krull rings.

(iii) If \( R \) is Krull ring, then the complete matrix ring \( (R)_n \) is also Krull. In particular, if \( R \) is bounded, then \( (R)_n \) is also bounded.

(iv) If \( R \) is a Krull ring, then the polynomial ring \( R[x] \) over \( R \) is also Krull.

In the remainder of this section, we shall give the proof of (iv).

Lemma 5.1. Let \( R \) be a noetherian local Asano order with unique maximal ideal \( P \), and let \( Q \) be the quotient ring of \( R \). Then

1) \( R[x] \) satisfies the Ore condition with respect to \( C(P[x]) \) and \( R[x]_{P[x]} \) is also a noetherian local Asano order.

2) \( Q[x] \cap R[x]_{P[x]} = R[x] \).

Proof. It is well known that \( R[x] \) is a noetherian prime ring. Since \( P[x] \) is invertible, the proof of (1) is similar to one in §2 of [5].

(2) It is evident that \( Q[x] \cap R[x]_{P[x]} \subseteq R[x] \). Let \( f(x) = a_n x^n + \cdots + a_0 \) be any element of \( Q[x] \cap R[x]_{P[x]} \), where \( a_i \in Q \) and let \( f(x) = g(x) h(x)^{-1} \), where \( g(x) \), \( h(x) \in R[x] \) and \( h(x) \in C(P[x]) \). Since \( h(x) = [h(x) + P[x]] \) is a regular element in \( R[x]/P[x] \cong \bar{R}[x] \), where \( \bar{R} = R/P \), there is an element \( r(x) \) in \( R[x] \) such that \( h(x) \bar{r}(x) = \bar{e}_m x^m + \cdots + \bar{e}_0 \) by Lemma 2 of [9], where \( e_m \) is a unit in \( R \). Hence \( e_m \) is a unit in \( R \). So \( f(x) h(x) r(x) = g(x) r(x) \in \bar{R}[x] \) and \( f(x) \in Q/P[x] \). From this we have \( a_n e_m \in \bar{R} \). Thus \( a_n \in \bar{R} \), that is, \( a_n \in R \). By induction, we get \( a_i \in R \) for all \( i \) and thus \( f(x) \in R[x] \).

Lemma 5.2. Let \( R \) be a prime Goldie ring with quotient ring \( Q \) and let \( S \subseteq T \) be overrings of \( R \). If \( S \) is an essential overring of \( R \) and if \( T \) is an essential overring of \( S \), then \( T \) is an essential overring of \( R \).

Proof. It is evident that \( T \) is a flat \( R \)-module. Because

\[
T \cong T \otimes S T \cong T \otimes S (S \otimes S T) \cong T \otimes S (S \otimes R S) \otimes S T \cong
T \otimes S (S \otimes R T) \cong T \otimes R T,
\]

the inclusion map: \( R \rightarrow T \) is an eqimorphism. Therefore there is a perfect right (left) additive topology \( F(F_t) \) such that \( R_F = T = R_{F_t} \) by Theorem 13.10 of [10].
From the assumption, there are topologies \( F_i, F_j (i = 1, 2) \) such that \( R_{F_i} = S = R_{F_j} \) and \( S_{F_2} = T = S_{F_1} \). Let \( I \) be any element of \( F \). Then \( T = IT = IST \) so that \( IS \subseteq F_2 \). Hence \( TIS = T \). Write \( 1 = \sum t_i a_{s_i} \), where \( t_i \in T, a_i \in I \) and \( s_i \in S \). There is an element \( A \) of \( F_1 \) such that \( s_i A \subseteq R \). Therefore we get \( A = (\sum t_i a_{s_i}) A \subseteq \Sigma (t_{\alpha} a_{\gamma}) (s, \alpha) \subseteq TI \). Since \( SA = S \), we have \( T \supseteq TI \supseteq TA = TSA = TS = T \) so that \( T = TI \). Similarly, we have \( JT = T \) for any \( J \in F_j \). Hence \( T \) is an essential overring of \( R \).

**Lemma 5.3.** Let \( S \) be a (right) essential overring of \( R \). Then \( S[x] \) is a (right) essential overring of \( R[x] \).

**Proof.** There is a perfect right additive topology \( F \) such that \( S = R_F \). We shall prove that \( F \) is a right essential overring of \( R[x] \) and that \( S[x] = R[x]_F \). Let \( I \) be any element of \( F \) and let \( f(x) = a_0 x^n + \cdots + a_0 \) be any element of \( R[x] \). (i) We shall prove that \( f(x)^{-1} I \subseteq F^* \) by induction on degree \( f(x) = n \). From the assumption there is an element \( I \) in \( F \) such that \( I \subseteq F_1 \). If \( n = 0 \), then \( a_0 I \subseteq a_0^{-1} I^* \) and \( a_0 I \subseteq F \) so that \( f(x)^{-1} I \subseteq F^* \). By the induction hypothesis, \( f_1(x)^{-1} I \supseteq I_1 \), where \( f_1(x) = a_{n-1} x^{n-1} + \cdots + a_0 \) and \( I_1 \subseteq F \). Since \( a_0 x^n (a_0^{-1} I) = x^n a_0 (a_0^{-1} I) \subseteq x^n I = I^* \), we get \( a_0 I \subseteq (a_0 x^n)^{-1} I^* \). Hence we obtain that \( f(x)^{-1} I \subseteq (a_0 x^n)^{-1} I^* \cap f_1(x)^{-1} I \subseteq I_1 \cap a_0^{-1} I \). This implies that \( f(x)^{-1} I \subseteq F^* \). (ii) We shall prove that if \( J \) is a right ideal of \( R[x] \) and if there is \( a \in J^* \) such that \( a^{-1} J \subseteq F^* \) for every \( a \in I \), then \( J^* \subseteq F^* \). There exists \( I \subseteq F \) such that \( I \supseteq J \). For any \( a \in I \), there exists \( J_\alpha \subseteq F \) with \( a \subseteq J_\alpha \supseteq I_\alpha \). Set \( J = \sum a_\alpha J_\alpha (a \in I) \). Since \( a^{-1} J \supseteq J_\alpha \) for any \( a \in I \), we have \( J \subseteq F \). It is evident that \( J \supseteq F^* \) so that \( J^* \subseteq F^* \), as desired. Hence \( F^* \) is a right additive topology. (iii) We shall prove that \( F^* \) is perfect and \( S[x] = R[x]_{F^*} \). Let \( I \) be any element of \( F^* \). Then it is clear that \( I^* S[x] = S[x] \). Conversely, let \( I \) be a right ideal of \( R[x] \) with \( I^* S[x] = S[x] \). Write \( 1 = \sum t_i a_{s_i} \), where \( a_i \in I, q_i \in S[x] \). There exists \( I \subseteq F \) such that \( q_i I \subseteq R[x] \). Hence \( I = (\sum q_i a_{s_i} I) \subseteq I^* \) so that \( I^* \subseteq F^* \). Thus, by Theorem 13.10 of [10], \( F^* \) is perfect and \( S[x] = R[x]_{F^*} \). Consequently \( S[x] \) is a right essential overring of \( R[x] \). If \( S \) is an essential overring of \( R \), then it is evident from the definition of \( F^* \) that \( S[x] \) is an essential overring of \( R[x] \).

**Theorem 5.4.** If \( R \) is a Krull ring, then \( R[x] \) is a Krull ring.

**Proof.** Let \( R = \cap R_i \cap S_j \), where \( i \in I, j \in J \), the cardinal number of \( J \) is finite, each \( R_i \) is a noetherian local Asano order with unique maximal ideal \( P_i \) and each \( S_j \) is a noetherian simple ring. By Lemma 5.1 \( R_i[x]_{P_i(x)} \) is a noetherian local Asano order, and \( R_i[x]_{P_i(x)} \cap Q[x] = R_i[x] \). By Example 6.1 of [8], \( S_j[x] \) is a noetherian Asano order. Hence \( S_j[x] = S_{j0} \cap S_{j0}^{jk} (jk \in J_j) \), where \( S_{j0}^{jk} \) is a noetherian simple ring and \( S_{j0}^{jk} \) are noetherian local Asano orders (cf. [5]). \( Q[x] \) is a Dedekind prime ring by Example 6.3 of [8]. Hence \( Q[x] = T_0 \cap T_i (i \in L) \),
where $T_0$ is a simple hereditary ring and $T_i$ are noetherian local Asano orders (cf. [6]). By Lemmas 5.2 and 5.3, all $R_i[x]_{P_i(x)}$, $S^*_P$, $S^*_j$, $T_0$, and $T_i$ are essential overrings of $R[x]$. Further we get:

$$R[x] = \bigcap R_i[x] \cap S_j[x] = \bigcap \{Q[x] \cap R_i[x]_{P_i(x)} \cap \{S^*_P \cap S^*_j\} \cap [\bigcap j, S^*_P \cap T_i],$$

and the simple overrings are only $T_0$, $S^*_j(j \in J)$ so that the number of these is finite.

Finally we shall prove that $R[x]$ satisfies the axiom (K3). Let $I^*$ be any essential right ideal of $R[x]$. Then there exists $f(x)=a_n x^n + \cdots + a_0$ in $I^*$ such that $a_n$ is a regular element in $R$ by Lemma 2 of [9]. Hence $a_n R_i = R_i$ for almost all $i \in I$, that is, $a_n \in C(P_i)$ and so $f(x) \in C(P_i[x])$. This implies that $I^* R_i[x]_{P_i(x)} = R_i[x]_{P_i(x)}$ for almost all $i \in I$. Since $I^* Q[x]$ is an essential right ideal of $Q[x]$, we have $T_i = I^* Q[x] T_i = I^* T_i$ for almost all $i \in I$. Similarly $S^*_j = I S^*_j$ for almost all $j \in J$. Therefore $R[x]$ is a Krull ring.

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