0. Introduction

Let $G$ be a compact connected Lie group and let $T$ be a maximal torus of $G$. Define

$$m(G) = \max \{ \dim H \mid H \text{ is a proper closed subgroup of } G \},$$
$$m_0(G) = \max \{ \dim H \mid H \text{ is a proper closed subgroup of } G \text{ with } \text{rank } H = \text{rank } G \}.$$

Let $M$ be a connected manifold with a non-trivial smooth $G$-action and let $H$ be a closed subgroup of $G$. Denote by $F(H, M)$ the fixed point set of the restricted action of the given $G$-action to the subgroup $H$. Then each connected component $F_a (a \in A)$ of $F(H, M)$ is a regular submanifold of $M$. Define

$$\dim F(H, M) = \max \{ \dim F_a \mid a \in A \}$$
if $F(H, M)$ is non-empty and we put

$$\dim F(H, M) = -1$$
if $F(H, M)$ is empty. Then we have the following results.

Theorem 1.

(a) In general, $\dim M - \dim F(T, M) \geq \dim G - m(G)$.

(b) If $G$ is semi-simple and

$$\dim F(G, M) < \dim F(T, M),$$

then

$$\dim M - \dim F(T, M) \geq \dim G - m_0(G).$$

Theorem 2. If

$$\dim M - \dim F(T, M) = \dim G - m(G),$$
then $G$ is semi-simple, $m(G) = m_s(G)$ and
\[ \dim M - \dim F_a = \dim G - m(G) \]
for each connected component $F_a$ of $F(T, M)$. Moreover
\[ \dim H = m(G) \text{ and } \text{rank } H = \text{rank } G \]
for a principal isotropy group $H$.

1. Preliminary lemmas
In this section we prepare several lemmas.

**Lemma 1.1.** Let $H$ be a closed subgroup of $G$ and assume $T \subseteq H$. Then
\[ F(T, G/H) = N(T)H/H. \]
In particular, $F(T, G/H)$ is a non-empty finite set.

**Proof.** It is clear that
\[ F(T, G/H) = \{ gH \mid g^{-1}Tg \subseteq H \}. \]
If $g^{-1}Tg \subseteq H$, then there is $h \in H$ such that
\[ g^{-1}Tg = hTh^{-1}, \]
since $T$ is a maximal torus of $H^0$, the identity component of $H$. Thus
\[ gh \in N(T) \text{: the normalizer of } T \text{ in } G. \]
Hence we obtain
\[ F(T, G/H) = N(T)H/H. \]
Next, there is a natural surjection $N(T)/T \to N(T)H/H$, where $N(T)/T$ is the Weyl group of $G$ which is a finite group. Therefore $F(T,G/H)$ is a non-empty finite set.

In the following, we assume that $M$ is a connected manifold with a non-trivial smooth $G$-action. It is clear
\[ (1.2) \quad \dim M \geq \dim G - m(G). \]

**Lemma 1.3.** $\dim M - \dim F(G, M) > \dim G - m(G)$.

**Proof.** If $F(G, M)$ is empty, then the inequality is clear from (1.2). If $F(G, M)$ is non-empty, let $n = \dim F(G, M)$ and let $F_a$ be an $n$-dimensional connected component of $F(G, M)$. For $x \in F_a$,
\[ T_xM = T_x(F_a) \oplus N_x \]

as \( G \)-vector spaces, where \( N_x \) is a normal space of \( F_a \) in \( M \). Then there is a non-zero vector \( v \in N_x \) with \( G_v \parallel G \). Thus

\[ \dim G - m(G) \leq \dim G/G_v < \dim N_x = \dim M - n. \]

q.e.d.

**Lemma 1.4.** If

\[ \dim M - \dim F(T, M) \leq \dim G - m_\theta(G) \]

and

\[ \dim F(G, M) < \dim F(T, M), \]

then

\[ M = G \cdot F(H, M). \]

Here \( H \) is a compact connected subgroup of \( G \) such that

\[ \dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G. \]

Proof. Let \( k = \dim F(T, M) \) and denote by \( F^k \) the union of \( k \)-dimensional connected components of \( F(T, M) \). Then

\[ F^k - F(G, M) \]

is non-empty by the assumption. For \( x \in F^k - F(G, M) \),

\[ T_xM = T_x(G \cdot x) \oplus N_x \]

as \( G_x \)-vector spaces, where \( N_x \) is a normal space of the orbit \( G \cdot x \) in \( M \). Since \( T \subset G_x \), \( F(T, G \cdot x) \) is a non-empty finite set by Lemma 1.1. Thus

\[ k = \dim F(T, T_xM) = \dim F(T, N_x) \leq \dim N_x = \dim M - \dim G/G_x \leq \dim M - \dim G + m_0(G). \]

On the other hand,

\[ k \geq \dim M - \dim G + m_0(G) \]

by the assumption. Therefore

\[ (1) \quad \dim G_x = m_\theta(G), \]

\[ (2) \quad F(T, N_x) = N_x. \]

Since the action of \( G_x \) on \( N_x \) is a slice representation at \( x \), a principal isotropy group \( H' \) contains \( T \) by (2), and hence

\[ \dim H' = m_\theta(G) \]
by (1). Let $H$ be the identity component of the principal isotropy group $H'$. Then we have

$$M = G \cdot F(H, M) = \{g \cdot x | g \in G, x \in F(H, M)\}.$$ q.e.d.

**Lemma 1.5.** If

$$\dim M - \dim F(T, M) \leq \dim G - m(G),$$

then $m(G) = m_0(G)$ and

$$M = G \cdot F(H, M).$$

Here $H$ is a compact connected subgroup of $G$ such that

$$\dim H = m(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$ Proof. Taking account of Lemma 1.3 and using similar arguments as in the proof of Lemma 1.4, we can prove this lemma.

**Lemma 1.6.** Let $G$ be a compact connected Lie group and let $H$ be a closed subgroup of $G$ such that

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H^0 = \text{rank } G.$$ Then $N(H)^0 = H^0$, where $H^0$ is the identity component of $H$ and $N(H)$ is the normalizer of $H$ in $G$.

Proof. Assume $N(H)^0 \neq H^0$. Then the assumption on $H$ implies $N(H) = G$. Thus $H$ is a normal subgroup of $G$, and hence

$$\text{rank } G = \text{rank } H^0 + \text{rank } G/H.$$ Then the assumption on $H$ implies $\text{rank } G/H = 0$ and hence $G = H$. But this is a contradiction to

$$\dim H = m_0(G) < \dim G.$$ q.e.d.

**Lemma 1.7.** Let $G$ be a compact connected semi-simple Lie group and let $H$ be a closed connected subgroup of $G$ such that

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$ Let $V$ be a real $G$-vector space such that

$$V = G \cdot F(H, V) \quad \text{and} \quad F(G, V) = \{0\}.$$
Then \( S(V) = G/H \) as \( G \)-manifolds and \( N(H)/H = Z_2 \). Here \( S(V) \) is a \( G \)-invariant unit sphere of \( V \).

Proof. By the assumption on \( H \) and \( V \), the identity component of an isotropy subgroup at each point of \( S(V) \) is conjugate to \( H \) in \( G \). Hence there is an equivariant diffeomorphism

\[
S(V) = G/H \times \frac{F(H, S(V))}{\pi_{\mathfrak{t}}(S(V))}
\]
as \( G \)-manifolds. Here \( F(H, S(V)) \) is a unit sphere of \( F(H, V) \). Since \( N(H)/H \) is a finite group by Lemma 1.6, the natural projection

\[
G/H \times F(H, S(V)) \to S(V)
\]
is a finite covering as \( G \)-manifolds. On the other hand, \( S(V) \) is simply connected, because \( G \) is semi-simple. Therefore

\[
S(V) = G/H
\]
as \( G \)-manifolds and \( F(H, S(V)) \) is a zero-sphere \( S^0 \). Finally,

\[
N(H)/H = F(H, G/H) = F(H, S(V)) = S^0.
\]
Thus \( N(H)/H = Z_2 \), the cyclic group of order 2. q.e.d.

2. Proof of theorems

Let \( G \) be a compact connected Lie group and let \( T \) be a maximal torus of \( G \). Let \( M \) be a connected manifold with a non-trivial smooth \( G \)-action. It is easy to see that

\[
F(T, M) = M \quad \text{implies} \quad F(G, M) = M.
\]
Thus

\[
\dim M - \dim F(T, M) \geq 2,
\]
because

\[
\dim M \equiv \dim F_a \pmod{2}
\]
for each connected component \( F_a \) of \( F(T, M) \).

If \( G \) is not semi-simple, then

\[
\dim G - m(G) = 1
\]
and hence there is nothing to prove. In particular, if

\[
\dim M - \dim F(T, M) = \dim G - m(G),
\]
then \( G \) is semi-simple, and \( m(G) = m_0(G) \) by Lemma 1.5.
Now we assume that $G$ is semi-simple and there is a closed connected subgroup $H$ of $G$ such that

\[ (*) \quad M = G \cdot F(H, M), \quad \dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G. \]

Moreover, (i) first suppose that $F(G, M)$ is empty. Then by the assumption (*), the identity component of an isotropy subgroup at each point of $M$ is conjugate to $H$ in $G$. Hence there is an equivariant diffeomorphism

\[ M = \frac{G|H}{N(H)/H} \times F(H, M) \]

as $G$-manifolds. Since $N(H)/H$ is a finite group by Lemma 1.6, the natural projection

\[ p: \frac{G|H}{N(H)/H} \times F(H, M) \to M \]

is a finite covering as $G$-manifolds. Hence we obtain

\[ F(T, M) = p(F(T, G/H) \times F(H, M)) \cdot \]

Here $F(T, G/H)$ is a non-empty finite set by Lemma 1.1. Therefore

\[ \dim M - \dim F_a = \dim M - \dim F(H, M) \]

\[ = \dim \frac{G|H}{N(H)/H} = \dim G - m_0(G), \]

for each connected component $F_a$ of $F(T, M)$.

(ii) Next suppose that $F(G, M)$ is non-empty. Then each fibre $N_x$ of the normal $G$-vector bundle of $F(G, M)$ in $M$ satisfies the hypothesis of Lemma 1.7, and hence

\[ N(H)/H = Z_2 \quad \text{and} \quad S(N_a) = G/H. \]

Let $U$ be a $G$-invariant closed tubular neighborhood of $F(G, M)$ in $M$. Then there is an equivariant diffeomorphism

\[ M = \partial(D(V) \times F(H, M - \text{int } U))/Z_2 \]

as $G$-manifolds. Here $V$ is a real $G$-vector space (unique up to $G$-isomorphism) with $S(V) = G/H$, $Z_2$ acts on the unit disk $D(V)$ as antipodal involution, and $G$ acts naturally on $D(V)$ and trivially on $F(H, M - \text{int } U)$. Hence we obtain

\[ F(T, M) = \partial(F(T, D(V)) \times F(H, M - \text{int } U))/Z_2 \]

\[ = \partial([-1, 1] \times F(H, M - \text{int } U))/Z_2. \]

Therefore

\[ \dim M - \dim F_a = \dim M - \dim F(H, M - \text{int } U) \]

\[ = \dim D(V) - 1 \]

\[ = \dim G/H \]

\[ = \dim G - m_0(G). \]
for each connected component \( F_a \) of \( F(T, M) \).

Now the proofs of Theorem 1 and Theorem 2 are completed by Lemma 1.4 and Lemma 1.5.

3. Integers \( m(G) \) and \( m_0(G) \)

In this section we show certain properties of \( m(G) \) and \( m_0(G) \). It is easy to see that

\[
(3.1) \quad m(G_1 \times G_2) \geq \max (m(G_1) + \dim G_2, \dim G_1 + m(G_2)),
\]

and

\[
(3.2) \quad m(G) \geq 1, \quad \text{if} \quad G \neq S^1.
\]

**Lemma 3.3.** Let \( G_1 \) and \( G_2 \) be compact connected Lie groups. Suppose that \( G_1 \) is simple and \( G_1 \neq S^1 \). Let \( H \) be a closed connected subgroup of \( G_1 \times G_2 \) with \( \dim H = m(G_1 \times G_2) \). Then

\[
H = H_1 \times G_2 \quad \text{or} \quad H = G_1 \times H_2
\]

where \( H_a \) is a closed subgroup of \( G_a \) (\( a = 1, 2 \)) with \( \dim H_a = m(G_a) \).

**Proof.** Let \( p_a : G_1 \times G_2 \to G_a \) (\( a = 1, 2 \)) be natural projections, and let \( i_a : G_a \to G_1 \times G_2 \) be natural injections defined by

\[
i_1(g) = (g, e_2), g \in G_1, \quad i_2(g) = (e_1, g), g \in G_2
\]

where \( e_a \) is the identity element of \( G_a \) (\( a = 1, 2 \)). Define

\[
H_a = p_a(H) \quad \text{and} \quad H_a' = i_a^{-1}(H).
\]

Then \( H_a' \) is a normal subgroup of \( H_a \) (\( a = 1, 2 \)) and \( H_a' \times H_a' \) is a normal subgroup of \( H \), and \( H \subset H_1 \times H_2 \). Moreover the projection \( p_a \) induces an isomorphism

\[
p_a' : H/H_1' \times H_2' \to H_a/H_a' \quad (a = 1, 2).
\]

(i) First suppose \( H_1 \neq G_1 \). Then

\[
H \subset p_1^{-1}(H_1) = H_1 \times G_2 \neq G_1 \times G_2.
\]

Hence we obtain

\[
H = H_1 \times G_2 \quad \text{and} \quad \dim H_1 = m(G_1)
\]

from the assumption \( \dim H = m(G_1 \times G_2) \).

(ii) Next suppose \( H_1 = G_1 \). Then \( H_1' \) is a normal subgroup of the simple Lie group \( G_1 \) and hence \( H_1' = G_1 \) or \( H_1' \) is a finite group. Since \( m(G_1) \geq 1 \) and
there is an isomorphism
\[ H|_{i_1}(H') = H_2, \]
we obtain
\[ m(G_1 \times G_2) = \dim H = \dim H_1' + \dim H_2 < \dim H_1' + m(G_1) + \dim G_2 \leq \dim H_1 + m(G_1 \times G_2). \]
Thus \( \dim H_1' \neq 0 \), and hence
\[ H_1' = H_1 = G_1. \]
Therefore
\[ H = G_1 \times H_2 \quad \text{and} \quad \dim H_2 = m(G_2) \]
from the assumption \( \dim H = m(G_1 \times G_2) \).

Corollary 3.4. Let \( G_1 \) and \( G_2 \) be compact connected Lie groups. Suppose that \( G_1 \) is simple. Then
\[ \dim (G_1 \times G_2) - m(G_1 \times G_2) = \min \left( \dim G_1 - m(G_1), \dim G_2 - m(G_2) \right). \]
Proof. If \( G_1 \neq S^1 \), then the equation follows from Lemma 3.3. If \( G_1 = S^1 \), then \( m(G_1 \times G_2) = \dim G_2 \) and hence the equation holds. q.e.d.

Theorem 3.5. Let \( G_1 \) and \( G_2 \) be compact connected Lie groups. Then
\[ \dim (G_1 \times G_2) - m(G_1 \times G_2) = \min \left( \dim G_1 - m(G_1), \dim G_2 - m(G_2) \right). \]
Proof. Let \( G^* \) be a compact connected covering group of \( G \). Then it is easy to see that
\[ m(G^*) = m(G). \]
There are covering groups \( G^*_a \) of \( G_a \) \((a=1, 2)\) such that
\[ G_1^* = H_1 \times \cdots \times H_r \times T^m \]
\[ G_2^* = K_1 \times \cdots \times K_s \times T^n \]
where \( H_i, K_j \) are compact connected non-abelian simple Lie groups, and \( T^m, T^n \) are tori. If \( m \) or \( n \) is non-zero, then
\[ \dim (G_1 \times G_2) - m(G_1 \times G_2) = 1 \]
\[ \min \left( \dim G_1 - m(G_1), \dim G_2 - m(G_2) \right) = 1. \]
Next, if \( m=n=0 \), then
\[ \dim (G_1 \times G_2) - m(G_1 \times G_2) = \min \left( \dim H_i - m(H_i), \dim K_j - m(K_j) \right) \]
\[ = \min \left( \dim G_1 - m(G_1), \dim G_2 - m(G_2) \right) \]
be Corollary 3.4.

**Remark 3.6.** The integer $m_0(G)$ can be defined only when $G$ is non-abelian (i.e. $G$ does not coincide with its maximal torus).

**Theorem 3.7.** Let $G_1$ and $G_2$ be compact connected non-abelian Lie groups. Then

$$\dim (G_1 \times G_2) - m_0(G_1 \times G_2) = \min (\dim G_1 - m_0(G_1), \dim G_2 - m_0(G_2)).$$

**Proof.** Let $H$ be a closed connected subgroup of $G_1 \times G_2$ such that

$$\dim H = m_0(G_1 \times G_2) \quad \text{and} \quad \operatorname{rank} H = \operatorname{rank} (G_1 \times G_2).$$

Then there are closed connected subgroups $H_a$ of $G_a (a=1, 2)$ such that

$$H = H_1 \times H_2 \quad \text{and} \quad \operatorname{rank} H_a = \operatorname{rank} G_a (a = 1, 2)$$

from the assumption $\operatorname{rank} H = \operatorname{rank} (G_1 \times G_2)$. Moreover

$$\dim H = m_0(G_1 \times G_2)$$

implies that

$$H_1 = G_1 \quad \text{and} \quad \dim H_2 = m_0(G_2)$$

or

$$H_2 = G_2 \quad \text{and} \quad \dim H_1 = m_0(G_1).$$

q.e.d.

**Table of $m(G)$ and $m_0(G)$ for simple Lie group $G$** (cf. [1], [2])

<table>
<thead>
<tr>
<th>$G$</th>
<th>dim $G$</th>
<th>$m(G)$</th>
<th>$H$</th>
<th>$m_0(G)$</th>
<th>$U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(n), n \geq 4$</td>
<td>$n^2 - 1$</td>
<td>$(n-1)^2$</td>
<td>$SU(n-1) \times U(1)$</td>
<td>$(n-1)^2$</td>
<td>$SU(n-1) \times U(1)$</td>
</tr>
<tr>
<td>$SU(4)$</td>
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<td>10</td>
<td>$Sp(2)$</td>
<td>9</td>
<td>$SO(3) \times U(1)$</td>
</tr>
<tr>
<td>$SO(2n+1)$</td>
<td>$2n^2 + n$</td>
<td>$2n^2 - n$</td>
<td>$SO(2n)$</td>
<td>$2n^2 - n$</td>
<td>$SO(2n)$</td>
</tr>
<tr>
<td>$Sp(n)$</td>
<td>$2n^2 + n$</td>
<td>$2n^2 - 3n + 4$</td>
<td>$Sp(n-1) \times Sp(1)$</td>
<td>$2n^2 - 3n + 4$</td>
<td>$Sp(n-1) \times Sp(1)$</td>
</tr>
<tr>
<td>$SO(2n), n &gt; 3$</td>
<td>$2n^2 - n$</td>
<td>$2n^2 - 3n + 1$</td>
<td>$SO(2n-1)$</td>
<td>$2n^2 - 5n + 4$</td>
<td>$SO(2n-2) \times SO(2)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>14</td>
<td>8</td>
<td>$SU(3)$</td>
<td>8</td>
<td>$SU(3)$</td>
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<tr>
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<td>36</td>
<td>$Spin(9)$</td>
<td>36</td>
<td>$Spin(9)$</td>
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<tr>
<td>$E_8$</td>
<td>248</td>
<td>136</td>
<td></td>
<td>136</td>
<td></td>
</tr>
</tbody>
</table>

Here $H$, $U$ are closed connected subgroups of $G$ with $\dim H = m(G)$, $\dim U = m_0(G)$ and $\operatorname{rank} U = \operatorname{rank} G$. 

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References
