Introduction. Let $P_N(c)$ denote the $N$-dimensional complex projective space $P_N(C)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature $c>0$. For an irreducible symmetric Kahlerian manifold $M$ of compact type, Nakagawa-Takagi [5] constructed a series of full equivariant Kahlerian imbeddings

$$f_p: (M, g_p) \to P_{N_p}(c),$$

parametrized by positive integers $p$, and observed that the degree $d(f_p)$ of $f_p$ (See §1 for the definition) is given by

$$d(f_p) = rp,$$

where $r = \text{rank} M$,

in the case where $p=1$ or $M$ is a complex quadric or a complex Grassmann manifold.

In this note we shall prove the above equality for general symmetric Kahlerian submanifolds of $P_N(c)$: Let

$$f_i: (M_i, g_i) \to P_{N_i}(c) \quad (1 \leq i \leq s)$$

be the $p_i$-th full Kahlerian imbedding of an irreducible symmetric Kahlerian manifold $M_i$ of rank $r_i$ ($1 \leq i \leq s$). Take the tensor product (See §2 for the definition)

$$f = f_1 \otimes \cdots \otimes f_s: (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \to P_N(c)$$

of the $f_i$ ($1 \leq i \leq s$). Then (Theorem 2) the degree $d(f)$ is given by

$$d(f) = \sum_{i=1}^s r_i p_i.$$

It should be noted that any full Kahlerian immersion $f$ into $P_N(c)$ of a symmetric Kahlerian manifold of compact type is obtained in this way.

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1. Degree of Kählerian immersions

Let $V$ be a real vector space of dimension $2n$, equipped with an almost complex structure $J$ and an inner product $g$ satisfying

$$g(Jx, Jy) = g(x, y) \quad \text{for } x, y \in V.$$  

Such a pair $(J, g)$ will be called a hermitian structure on $V$. Denoting complex linear extensions of $J$ and $g$ to the complexification $V^c$ of $V$ by the same $J$ and $g$ respectively, we define subspaces $V^\pm$ of $V^c$ and a hermitian inner product $\langle , \rangle$ on $V^c$ by

$$V^\pm = \{x \in V^c; Jx = \pm \sqrt{-1}x\},$$

$$\langle x, y \rangle = g(x, \bar{y}) \quad \text{for } x, y \in V^c,$$

where $x \mapsto \bar{x}$ denotes the complex conjugation of $V^c$ with respect to $V$. Then we have $V^\mp = V^\pm$ and

$$V^c = V^+ \oplus V^- \quad \text{(orthogonal direct sum with respect to $\langle , \rangle$).}$$

A basis $u = (u_1, \cdots, u_n)$ of $V^+$ satisfying $\langle u_i, u_j \rangle = \delta_{ij}$ ($1 \leq i, j \leq n$) is called a unitary frame of $V$.

Let $E$ be a smooth real vector bundle over a smooth manifold $M^*$ with a smooth assignment $(J, g): p \mapsto (J^p, g^p)$ of hermitian structures on fibres $E^p$. $(J, g)$ is called a hermitian structure on $E$. Then, getting together the constructions on fibres $E^p$, we have a hermitian inner product $\langle , \rangle$ on the complexification $E^c$ of $E$, and subbundles

$$E^\pm = \bigcup_{p \in M} E^\pm_p$$

of $E^c$ satisfying

$$E^c = E^+ \oplus E^- \quad \text{(orthogonal Whitney sum),}$$

and the complex conjugation $E^c \to E^\mp$. The map on the space of smooth sections induced from the complex conjugation will be also denoted by

$$C^\infty(E^\pm) \to C^\infty(E^\mp).$$

Let $(M, g)$ be a Kählerian manifold of $\dim_{\mathbb{C}} M = n$. Then the almost complex structure tensor $J$ and the Kählerian metric $g$ give a hermitian structure on the tangent bundle $T(M)$ of $M$. Thus we get a hermitian inner product $\langle , \rangle$ on the complexification $T(M)^c$ of $T(M)$ and subbundles $T(M)^\pm$ of $T(M)^c$ such that

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*) In this note, a manifold will be always assumed to be connected.
\[ T(M)^c = T(M)^+ \oplus T(M)^- \] (orthogonal Whitney sum).

Denote by \( U_{\rho}(M) \) the totality of unitary frames of \( T_{\rho}(M) \). Then the union

\[ U(M, g) = \bigcup_{\rho \in \mathbb{R}} U_{\rho}(M) \]

has a structure of smooth principal bundle over \( M \) with the structure group \( U(n) \). The Levi-Civita’s connection form \( \omega \) and the canonical form \( \theta \) of \((M, g)\) will be considered as a \( \mathfrak{u}(n) \)-valued 1-form and \( \mathbb{C}^n \)-valued 1-form on \( U(M, g) \) respectively. \( \omega_A^B \) \((1 \leq A, B \leq n)\) and \( \theta^A \) \((1 \leq A \leq n)\) denote the components of \( \omega \) and \( \theta \) respectively.

Now let \((M, g)\) and \((M', g')\) be Kahlerian manifolds of complex dimensions \( n \) and \( N \) respectively and \( f: (M, g) \rightarrow (M', g') \) be a Kahlerian immersion, i.e., a holomorphic isometric immersion of \((M, g)\) into \((M', g')\). The almost complex structure tensors of \( M \) and \( M' \) will be denoted by \( J \) and \( J' \) respectively. The Levi-Civita connections of \( T(M) \) and \( T(M') \) are denoted by \( \nabla \) and \( \nabla' \) respectively. The induced bundle \( f^*T(M') \) over \( M \) has a hermitian structure \((J', g')\) induced from the one on \( T(M') \). Also it has a connection induced from the Levi-Civita connection on \( T(M') \), which will be also denoted by \( \nabla' \). If we denote the orthogonal complement of \( f_*T_{\rho}(M) \) in \( T_{f(\rho)}(M') \) with respect to \( g'_{f(\rho)} \) by \( N_{\rho}(M) \), the union

\[ N(M) = \bigcup_{\rho \in \mathbb{R}} N_{\rho}(M) \]

is a subbundle of \( f^*T(M') \), having a hermitian structure \((J', g')\) induced from the one on \( f^*T(M') \). The hermitian inner products on \( T(M)^c, T(M')^c, f^*T(M')^c \) and \( N(M)^c \) will be denoted by the same \( \langle, \rangle \). We have the following orthogonal Whitney sum decompositions:

\[ f^*T(M') = f_*T(M) \oplus N(M), \]
\[ f^*T(M')^c = f_*T(M)^c \oplus N(M)^c, \]
\[ f^*T(M')^\perp = f_*T(M)^\perp \oplus N(M)^\perp, \]

where the complex linear extension of the differential \( f_* \) is denoted by the same \( f_* \). The injections \( f_*: T(M) \rightarrow f^*T(M'), f_*: T(M)^c \rightarrow f^*T(M')^c \) and \( f_*: T(M)^\perp \rightarrow f^*T(M')^\perp \) preserve the respective inner products. So we shall often identify \( T(M) \) etc. with a subbundle of \( f^*T(M') \) etc. through the injections \( f_* \). The orthogonal projection \( f^*T(M') \rightarrow N(M) \) will be denoted by \( x \mapsto x^\perp \) and the induced projection \( \mathbb{C}^\infty(f^*T(M')) \rightarrow \mathbb{C}^\infty(N(M)) \) will be also denoted by \( \xi \mapsto \xi^\perp \). Then the normal connection \( D \) on \( N(M) \) satisfies

\[ D_x \xi = (\nabla_x \xi)^\perp \quad \text{for } X \in \mathbb{C}^\infty(T(M)), \xi \in \mathbb{C}^\infty(N(M)). \]
Now we shall define the higher fundamental form $H^m$ of $f$ as a smooth section of the complex vector bundle $\text{Hom}(\otimes^n T(M)^+, N(M)^+)$. In the sequel, for a real linear object, its complex linear extension will be denoted by the same notation. For vector spaces $V$ and $W$, the space $\text{Hom}(\otimes^n V, W)$ of linear maps from the $m$-fold tensor product $\otimes^m V$ of $V$ into $W$ will be identified with the space of $m$-multilinear maps on $V$ into $W$. Let $h^2 \in C^\infty(\text{Hom}(\otimes^2 T(M), N(M)))$ be the second fundamental form of $f$, i.e.,

$$h^2(x, y) = (\nabla_x Y)^\perp$$

for $x, y \in T_p(M)$,

where $Y$ is a local smooth vector field on $M$ around $p$ such that $Y_p = y$. It is known (cf. Kobayashi-Nomizu [3]) that

$$h^2(x, y) = h^2(y, x), \quad h^2(Jx, y) = J'h^2(x, y) \quad \text{for } x, y \in T_p(M),$$

and hence

$$h^2(T_p(M)^+, T_p(M)^-) = \{0\}, \quad h^2(T_p(M)^\mp, T_p(M)^\pm) \subset N_p(M)^\pm.$$

We define $h^m \in C^\infty(\text{Hom}(\otimes^m T(M), N(M)))$ ($m \geq 3$) inductively as follows:

$$(1.2) \quad h^{m+1}(x_1, \ldots, x_m, x_{m+1}) = D_{x_{m+1}} h^m(x_1, \ldots, x_m) - \sum_{i=1}^m h^m(x_1, \ldots, \nabla_{x_{m+1}} x_i, \ldots, x_m) \quad \text{for } x_i \in T_p(M),$$

where the $X_i$ are smooth local vector fields on $M$ around $p$ such that $(X_i)_p = x_i$.

Note that (1.1) and (1.2) imply

$$h^m(x_1, \ldots, x_m) \in N_p(M)^+ \quad \text{for } x_1, x_2 \in T_p(M)^+ \text{ and } x_3, \ldots, x_m \in T_p(M)^-.$$

Now $H^m \in C^\infty(\text{Hom}(\otimes^m T(M)^+, N(M)^+))$ ($m \geq 2$) is defined by

$$H^m(x_1, \ldots, x_m) = h^m(x_1, \ldots, x_m) \quad \text{for } x_i \in T_p(M)^+.$$

We write

$$\sum_{m \geq 2} h^m \in C^\infty(\text{Hom}(\sum_{m \geq 2} \otimes^m T(M), N(M)))$$

and

$$\sum_{m \geq 2} H^m \in C^\infty(\text{Hom}(\sum_{m \geq 2} \otimes^m (T(M)^+, N(M)^+))$$

by $h$ and $H$ respectively. Note that then we have

$$(1.3) \quad \left\{ \begin{array}{l}
H(X_1, X_2) = \nabla_{x_2}X_1 - \nabla_{x_2}X_1, \\
H(X_1, \ldots, X_m, X_{m+1}) = D_{x_{m+1}} H(X_1, \ldots, X_m) - \sum_{i=1}^m H(X_1, \ldots, \nabla_{x_{m+1}} X_i, \ldots, X_m) \quad (m \geq 2) \\
\quad \text{for } X_i \in C^\infty(T(M)^+). \end{array} \right.$$
Making use of the higher fundamental form $H$ we shall define the degree $d(f)$ of the Kählerian immersion $f$. Let $p \in M$. For a positive integer $m$, we define a subspace $\mathcal{H}_p^m(M)$ of $T_{f(p)}(M')^+$ to be the subspace spanned by $T_p(M)^+$ and $H\left(\sum_{2 \leq k \leq m} \otimes^k T_p(M)^+\right)$. Then we get a series

$$\mathcal{H}_p^0(M) \subset \mathcal{H}_p^1(M) \subset \cdots \subset \mathcal{H}_p^m(M) \subset \mathcal{H}_p^{m+1}(M) \subset \cdots \subset T_{f(p)}(M')^+$$

of increasing subspaces of $T_{f(p)}(M')^+$. We define $O_p^m(M)$ to be the orthogonal complement of $\mathcal{H}_p^{m-1}(M)$ in $\mathcal{H}_p^m(M)$ with respect to $\langle \cdot, \cdot \rangle$, where $\mathcal{H}_p^0(M)$ is understood to be $\{0\}$. Thus we have an orthogonal direct sum:

$$\mathcal{H}_p^m(M) = O_p^1(M) \oplus O_p^2(M) \oplus \cdots \oplus O_p^m(M).$$

For each positive integer $m$, we define the set $R_m$ of $m$-regular points of $M$ inductively as follows. Define $R_1 = M$. For $m \geq 2$, assume $R_{m-1}$ is already defined. Then we define

$$R_m = \{p \in R_{m-1}; \dim \mathcal{H}_p^m(M) = \max_{p' \in R_{m-1}} \dim \mathcal{H}_p^{m-1}(M)\}.$$

We have inclusions: $R_1 \supset R_2 \supset \cdots \supset R_m \supset R_{m+1} \supset \cdots$. Note that each $R_m$ is an open non-empty subset of $M$ and that

$$\mathcal{H}_p^m(M) = \bigcup_{p \in R_m} \mathcal{H}_p^m(M)$$

is a smooth complex vector bundle over $R_m$ which is a subbundle of $f^*T(M')^+ | R_m$ for each $m$.

**Lemma 1.** Let $p \in R_m$, $m \geq 1$.

1) For each $x \in T_p(M)^+$ and each local smooth section $Y$ of $\mathcal{H}_p^m(M)$ around $p$ we have

$$\nabla^*_x Y \in \mathcal{H}_p^{m+1}(M).$$

2) $O_p^{m+1}(M) = \{0\}$ if and only if for each $x \in T_p(M)^+$ and each local smooth section $Y$ of $\mathcal{H}_p^m(M)$ around $p$ we have

$$\nabla^*_x Y \in \mathcal{H}_p^m(M).$$

Proof. Induction on $m$. Let $x \in T_p(M)^+$ and $Y$ a local smooth section of $\mathcal{H}_p^1(M) = T(M)^+$ around $p$. Then by (1.3)

$$\nabla^*_x Y \equiv H(Y_x, x) \mod \mathcal{H}_p^1(M),$$

which implies the Lemma for $m=1$. Let $m \geq 2$ and $x \in T_p(M)^+$. Each local smooth section $Y$ of $\mathcal{H}_p^m(M)$ around $p$ is written as

$$Y = Z + \sum H(X_1, \ldots, X_m)$$

by a local smooth section $Z$ of $\mathcal{H}_p^{m-1}(M) | R_m$ and local smooth sections $X_i$ of
From the assumption of the induction, we have $\nabla'_i Z \in \mathcal{A}_p^*(M)$. Further (1.3) implies

$$\nabla'_i H(X_1, \ldots, X_m) \equiv D_\xi H(X_1, \ldots, X_m) \equiv H((X_1)_p, \ldots, (X_m)_p, x) \mod \mathcal{A}_p^*(M),$$

and hence

$$\nabla'_i Y \equiv \sum H((X_1)_p, \ldots, (X_m)_p, x) \mod \mathcal{A}_p^*(M).$$

This implies the Lemma for $m$. q.e.d.

It follows from Lemma 1, 2) that there exists uniquely a positive integer $d$ such that

$$\begin{cases} O_p^d(M) \neq \{0\} & \text{for some } p \in \mathcal{R}_d, \\ O_p^{d+1}(M) = \{0\} & \text{for each } p \in \mathcal{R}_d. \end{cases}$$

Such integer $d$ is called the degree of the Kählerian immersion $f$ and denoted by $d=d(f)$. We have

$$\mathcal{R}_d = \mathcal{R}_{d+1} = \cdots.$$

This open subset $\mathcal{R}_d$ of $M$ will be denoted by $\mathcal{R}$ and called the set of regular points of $M$.

**Lemma 2** (Nakagawa-Takagi [5]). If $(M', g')=P_N(c)$, then:

1) $H^m$ is symmetric multilinear for each $m\geq 2$;
2) For each $u=(u_1, \ldots, u_n) \in U(M, g)$, we have
   (a) $h(u_i, u_j, u_k)=0$,
   (b) $h(u_{i_1}, \ldots, u_{i_m}, u_j)=\frac{m-2}{2} \sum_{r=1}^{\frac{m}{2}} (-1)^{r(m-r)} \sum_{r=1}^{\frac{m}{2}} (-1)^{r(m-r)} \sum_{\sigma} \langle H(u_{i_{\sigma(r+1)}}, \ldots, u_{i_{\sigma(m)}}), H(u_{i_1}, u_j) \rangle \times H(u_i, u_{i_{\sigma(1)}}, \ldots, u_{i_{\sigma(m)}})$ $(m\geq 3)$,

where $\sigma$ runs through the permutations of $\{1, 2, \ldots, m\}$.

**Lemma 3** (Nakagawa [4]). Let $M$ be a smooth manifold, $p_0 \in M$ and $f: M \to P_N(C)$ a smooth immersion. Let $\pi: U(P_N(c)) \to P_N(C)$ be the bundle of unitary frames of $P_N(c)$, $A^A$ $(1 \leq A \leq N)$ and $\omega^A_B$ $(1 \leq A, B \leq N)$ be canonical forms and Levi-Civita's connection forms of $P_N(c)$ respectively. Then, $f(M)$ is contained in an $N'$-dimensional linear subvariety of $P_N(C)$ if and only if we can find $u_0 \in U(P_N(c))$ with $\pi(u_0)=f(p_0)$ such that for each smooth curve $\{p_t\}$ of $M$ through $p_0$ there exists a smooth curve $\{u_t\}$ of $U(P_N(c))$ through $u_0$ with $\pi(u_t)=f(p_t)$ satisfying
Now we prove the following theorem, giving a geometric interpretation of the degree $d(f)$.

**Theorem 1.** Let $(M', g') = P_N(c)$ and

$$f: (M, g) \rightarrow P_N(c)$$

be a Kählerian immersion. Then the dimension $N'(f)$ of the smallest linear subvariety of $P_N(c)$ containing $f(M)$ is given by

$$N'(f) = \text{rank}_c \mathcal{H}^{d(f)}(M).$$

**Proof.** First we show that for each $x \in T_p(M)$, $p \in R_m$ ($m \geq 1$) and for each local smooth section $Y$ of $\mathcal{H}^m(M)$ around $p$, we have

$$\nabla_X Y \in \mathcal{H}_p^{m+1}(M).$$

By virtue of Lemma 1, it suffices to show (1.4) for $x \in T_p(M)^-$. It follows from (1.1) and (1.2) that for each local smooth sections $X, X_i$ of $T(M)^+$ around $p$ we have

$$\begin{align*}
\nabla_X Y &= \sum_{i=1}^n h(X, X_i) X_i + \sum_{i=1}^n H(X, \ldots, \nabla_X X_i) X_i \quad (m \geq 2). 
\end{align*}$$

Here we know that $h(X, \ldots, X_m, X)$ is a local smooth section of $\mathcal{H}^m(M)$ in view of Lemma 2,2), and hence we can prove (1.4) for $x \in T_p(M)^-$ in the same way as Lemma 1.

Take a connected component $M_0$ of the set $\mathcal{R}$ of regular points and take $p_0 \in M_0$. (1.4) implies

$$\nabla_X Y \in \mathcal{H}_p^{d(f)}(M)$$

for each $x \in T_p(M)$, $p \in M_0$, and for each local smooth section $Y$ of $\mathcal{H}^{d(f)}(M)|M_0$ around $p$. Using the notation in Lemma 3, we choose a unitary frame $u_0 = (u_1(0), \ldots, u_N(0)) \in U(P_N(c))$ with $\pi(u_0) = f(p_0)$ such that $\{u_1(0), \ldots, u_N(0)\}$ spans $\mathcal{H}_p^{d(f)}(M)$, where $N' = \text{rank}_c \mathcal{H}^{d(f)}(M)$. For each smooth curve $\{t \mapsto (u(t), \ldots, u_N(t))\}$ of $U(P_N(c))$ through $u_0$ with $\pi(u(t)) = f(p_t)$ such that $\{u_1(t), \ldots, u_N(t)\}$ spans $\mathcal{H}_p^{d(f)}(M)$. This is possible since $\mathcal{H}^{d(f)}(M)|M_0$ is a subbundle of $f^*T(M)^+|M_0$. Then (1.6) implies

$$\begin{align*}
\theta^A(u_t) &= \langle f_*(\hat{p}_t), u_A(t) \rangle = 0 \quad (N' + 1 \leq A \leq N), \\
\omega^B(u_t) &= \langle \nabla_{f_*(\hat{p}_t)} u_B(t), u_A(t) \rangle = 0 \quad (N' + 1 \leq A \leq N, 1 \leq B \leq N').
\end{align*}$$
Thus, by Lemma 3, \( f(M_0) \) is contained in an \( N' \)-dimensional linear subvariety \( P \) of \( P_N(\mathbb{C}) \). From the analyticity of the immersion \( f \), we conclude \( f(M) \subset P \), and hence \( N'(f) \leq N' \).

Assume that \( f(M) \) is contained in a linear subvariety \( P' \) of \( P_N(\mathbb{C}) \). Since \( P' \) is a totally geodesic complex submanifold of \( P_N(\mathbb{C}) \), we have

\[ \mathcal{H}^p(f)(M) \subset T_{f(p)}(P')^+ \quad \text{for} \quad p \in \mathcal{R}. \]

This implies \( N' \leq N'(f) \) and hence \( N' = N'(f) \).

q.e.d.

2. Symmetric Kählerian submanifolds of \( P_N(\mathbb{C}) \)

A holomorphic immersion \( f \) of a complex manifold \( M \) into \( P_N(\mathbb{C}) \) is said to be full if \( f(M) \) is not contained in any proper linear subvariety of \( P_N(\mathbb{C}) \). In this section we recall the construction of full Kählerian imbeddings into \( P_N(\mathbb{C}) \) of a symmetric Kählerian manifold of compact type. (cf. Borel [1], Takeuchi [6], Nakagawa-Takagi [5])

Let \( \Pi^\alpha = \{ \alpha_1, \ldots, \alpha_l \} \) be an irreducible Dynkin diagram and \( \Sigma \) the root system with the fundamental root system \( \Pi \). Take a lexicographic order \( \succ \) on \( \Sigma \) such that the set of simple roots in \( \Sigma \) with respect to \( \succ \) coincides with \( \Pi \). Assume that the highest (with respect to \( \succ \)) root \( \gamma_1 \) of \( \Sigma \) has the following expression:

\[ \gamma_1 = \alpha_1 + \sum_{i=2}^{l} m_i \alpha_i. \]

Put \( \Pi_0 = \{ \alpha_2, \ldots, \alpha_l \} \) and fix a positive integer \( p \). To the triple \( (\Pi, \Pi_0; p) \) we can associate a full Kählerian imbedding of an irreducible symmetric Kählerian manifold into \( P_N(\mathbb{C}) \) as follows.

Take a compact simple Lie algebra \( g \) with the Dynkin diagram \( \Pi \). Let \( t \) be a maximal abelian subalgebra of \( g \) and denote by \( g^c \) and \( t^c \) the complexifications of \( g \) and \( t \) respectively. We identify a weight of \( g^c \) relative to the Cartan subalgebra \( t^c \) with an element of \( \mathcal{J}^\alpha - \Pi_0^\alpha \) by means of the duality defined by the Killing form \((\ , \) \) of \( g^c \). Thus the root system \( \Sigma \) of \( g^c \) relative to \( t^c \) is a subset of \( \sqrt{-1} t \). Let \( \{ \Lambda_i, \ldots, \Lambda_l \}, \{ \varepsilon_i, \ldots, \varepsilon_l \} \subset \sqrt{-1} t \) be the fundamental weights of \( g^c \) and the dual basis for \( \Pi \) respectively:

\[ 2(\Lambda_i, \alpha_j) = \delta_{ij}, \quad (\alpha_i, \varepsilon_j) = \delta_{ij}, \quad (1 \leq i, j \leq l). \]

Put \( \Sigma^+ = \{ \alpha \in \Sigma; \alpha > 0 \} \), \( \Sigma^0 = \Sigma \cap \{ \Pi_0 \}^\alpha \) and \( \Sigma^0_+ = \Sigma^+ - \Sigma^0 \), where \( \{ \Pi_0 \} \) denotes the subgroup of \( \sqrt{-1} t \) generated by \( \Pi_0 \). Define subalgebras \( t^c, m^c \) and \( u \) of \( g^c \) by
\[ \mathfrak{k}^c = \mathfrak{t}^c + \sum_{\alpha \in \Sigma_0} \mathfrak{g}_\alpha^c, \quad m^+ = \sum_{\alpha \in \Sigma_1^{\text{nt}}} \mathfrak{g}_\alpha^c, \]
\[ u = \mathfrak{t}^c + \sum_{\alpha \in \Sigma_0 \cup \Sigma_1^{\text{nt}}} \mathfrak{g}_\alpha^c, \]

where \( \mathfrak{g}_\alpha^c \) denotes the root space of \( \mathfrak{g}^c \) for \( \alpha \in \Sigma \). Let \( \mathfrak{k} = \mathfrak{k}^c \cap \mathfrak{g} \), which is a real form of \( \mathfrak{k}^c \), and \( m \) be the orthogonal complement of \( \mathfrak{k} \) in \( \mathfrak{g} \) with respect to \(( , )\). Then the automorphism \( \theta = \exp \text{ad} \sqrt{-1} \xi \) of \( \mathfrak{g} \) is involutive and gives the decomposition \( \mathfrak{g} = \mathfrak{k} + m \) with
\[ \mathfrak{k} = \{ X \in \mathfrak{g}; \theta X = X \}, \quad m = \{ X \in \mathfrak{g}; \theta X = -X \}. \]

\( G \) and \( G^c \) denote the adjoint groups of \( \mathfrak{g} \) and \( \mathfrak{g}^c \) respectively, \( G \) and \( G^c \) the universal covering groups of \( G \) and \( G^c \) respectively. We may identify as \( G \subseteq G^c \) and \( G \subseteq G^c \). Let \( K \) and \( U \) denote the (closed) connected subgroups of \( G^c \) generated by \( \mathfrak{k} \) and \( \mathfrak{u} \) respectively. We define a complex manifold \( M \) by
\[ M = G^c / U. \]

Then the natural map \( G / K \to G^c / U \) induces the identification \( M = G / K \) as smooth manifolds. The tangent space \( T_o(M) \) of \( M \) at the origin \( o = U \) is identified with \( m \) and \( T_o(M)^+ \) with \( m^+ \) in the natural way.

Let
\[ \rho: G \to SU(N+1) \]
be an irreducible unitary representation of \( G \) with the highest weight \( \rho \Lambda_1 \). By virtue of the irreducibility it induces a homomorphism
\[ \rho: G \to PU(N+1) = SU(N+1) / \{ \xi_1^{N+1}; \xi^{N+1} = 1 \} \]
such that the diagram
\[ \begin{CD}
G @> \bar{\rho} >> SU(N+1) \\
@V \pi VV \quad @V \pi VV \\
G @>> \rho >> PU(N+1)
\end{CD} \]
is commutative, where the \( \pi \) are respective covering homomorphisms. They are extended holomorphically to \( G^c \) and \( G^c \) in such a way that the diagram
\[ \begin{CD}
G^c @> \bar{\rho} >> SL(N+1, \mathcal{C}) \\
@V \pi VV \quad @V \pi VV \\
G^c @>> \rho >> PL(N+1, \mathcal{C}) = SL(N+1, \mathcal{C}) / \{ \xi_1^{N+1}; \xi^{N+1} = 1 \}
\end{CD} \]
is commutative, where we have used the same letters for extensions. Let
$P_N(C) = \mathbb{C}^{N+1} - \{0\}/\mathbb{C}^*$ be the complex projective space associated to the representation space $\mathbb{C}^{N+1}$ of $\check{\rho}$. For $v \in \mathbb{C}^{N+1} - \{0\}$, the equivalence class of $v$ will be denoted by $[v]$. Taking a highest weight vector $v_0 \in \mathbb{C}^{N+1} - \{0\}$, we can define a full holomorphic imbedding $f: M \to P_N(C)$ by

$$f(x) = \rho(x)[v_0]$$

for $x \in G^C$.

We take the $SU(N+1)$-invariant Fubini-Study metric on $P_N(C)$ of constant holomorphic sectional curvature $c$ and introduce a Kählerian metric $g$ on $M$ in such a way that

$$f: (M, g) \to P_N(c)$$

becomes a Kählerian imbedding. Then $(M, g)$ is an irreducible symmetric Kählerian manifold of compact type. If we denote the group of Kählerian automorphisms of $(M, g)$ and the one of holomorphisms of $M$ by $Aut(M, g)$ and $Aut(M)$ respectively, the identity-components $Aut^0(M, g)$ and $Aut^0(M)$ are identified with $G$ and $G^C$ respectively. Further $f$ is $G^C$-equivariant by the homomorphism $\rho$:

$$f(xp) = \rho(x)f(p)$$

for $x \in G^C, p \in M$.

where $\rho(G) \subset PU(N+1) = Aut(P_N(c))$.

Put

$$\kappa(M) = \# \{ \alpha \in \sum_+; \alpha - \alpha_i \in \sum \} + 2.$$ 

Then (Nakagawa-Takagi [5]) the scalar curvature $k$ of $(M, g)$ is given by

$$k = \frac{(\dim_c M)c \kappa(M)}{\rho},$$

which gives a geometric characterization of the positive integer $\rho$. It is also characterized (Nakagawa-Takagi [5]) by

$$g = \frac{\rho(\alpha_1, \alpha_i)}{c} g_0,$$

where $g_0$ is a $G$-invariant Kählerian metric on $M$ defined from the inner product $-(\, , \,)$ on $g$. The imbedding $f$ will be called the $\rho$-th full Kählerian imbedding of $M$.

Now we shall construct a full Kählerian imbedding of a general (not necessarily irreducible) symmetric Kählerian manifold into $P_N(c)$. For complex projective spaces $P_{N_1}(C)$ and $P_{N_2}(C)$ associated to $\mathbb{C}^{N_1+1}$ and $\mathbb{C}^{N_2+1}$ respectively, we define a holomorphic imbedding $\iota$ of $P_{N_1}(C) \times P_{N_2}(C)$ into the complex projective space $P_{N_1N_2+N_1+N_2}(C)$ associated to the tensor product $\mathbb{C}^{N_1+1} \otimes \mathbb{C}^{N_2+1}$ by
where \([*]\) denotes the point of the projective space with homogeneous coordinates \(*\). Then it defines a full Kählerian imbedding

\[
\iota: P_{N_1}(c) \times P_{N_2}(c) \to P_{N_1N_2+N_1+N_2}(c).
\]

Let

\[ f_i: (M_i, g_i) \to P_{N_i}(c) \quad (i = 1, 2) \]

be two Kählerian immersions. Then the composite

\[
f_1 \boxtimes f_2 = \iota(f_1 \times f_2): (M_1 \times M_2, g_1 \times g_2) \to P_{N_1N_2+N_1+N_2}(c)
\]

is also a Kählerian immersion, which will be called the tensor product of \(f_1\) and \(f_2\).

One can easily check the associativity of the tensor product, and so the multi-fold tensor product \(f_1 \boxtimes \cdots \boxtimes f_s\) is well-defined.

Now let

\[ f_i: (M_i, g_i) \to P_{N_i}(c) \quad (1 \leq i \leq s) \]

be full Kählerian imbeddings of irreducible symmetric Kählerian manifolds of compact type constructed as before. Then the tensor product

\[
f = f_1 \boxtimes \cdots \boxtimes f_s: (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s) \to P_N(c),
\]

where \(N = \prod_{i=1}^{s} (N_i + 1) - 1\), is a full Kählerian imbedding of the symmetric Kählerian manifold \((M, g) = (M_1 \times \cdots \times M_s, g_1 \times \cdots \times g_s)\). Note that

\[
G^c = G_1^c \times \cdots \times G_s^c, \quad G = G_1 \times \cdots \times G_s,
\]

where \(G^c = \text{Aut}^c(M)\), \(G = \text{Aut}(M, g)\), \(G_i^c = \text{Aut}^c(M_i)\), \(G_i = \text{Aut}(M_i, g_i)\), and that \(f\) is \(G^c\)-equivariant by the homomorphism \(\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_s\) induced from the external tensor product \(\hat{\rho}_1 \boxtimes \cdots \boxtimes \hat{\rho}_s\) of respective representations \(\hat{\rho}_i\). The tangent space \(T_o(M)\) of \(M\) at the origin \(o = o_1 \times \cdots \times o_s\) of \(M\), where \(o_i\) is the origin of \(M_i\), is identified with the direct sum

\[
m = m_1 \oplus \cdots \oplus m_s
\]

of respective complements \(m_i\), and hence \(T_o(M)^+\) with

\[
m^+ = m_1^+ \oplus \cdots \oplus m_s^+.
\]

Further the stabilizer \(K\) of the origin \(o\) in \(G\) is the direct product
of respective stabilizers \( K_i \).

It is known (Nakagawa-Takagi [5]. See also Takeuchi [8]) that any full Kahlerian immersion into \( P_N(c) \) of a symmetric Kahlerian manifold of compact type is obtained in this way.

3. Degree of symmetric Kahlerian submanifolds of \( P_N(c) \)

Let

\[
f: (M, g) \rightarrow P_N(c)
\]

be the \( p \)-th full Kahlerian imbedding of an irreducible symmetric Kahlerian manifold \( (M, g) \) constructed in § 2. We recall first the construction of the Hermann map for \( M \) (cf. Takeuchi [7]). Choose root vector \( E_{\alpha} \in g_+ \) for \( \alpha \in \Sigma \) in such a way that

\[
[E_{\alpha}, E_{-\alpha}] = -\alpha, \quad (E_{\alpha}, E_{-\alpha}) = -1.
\]

Then the complex conjugation \( X \mapsto \bar{X} \) of \( g^c \) with respect to \( g \) satisfies \( E_{\alpha} = E_{-\alpha} \) for each \( \alpha \in \Sigma \). We put

\[
X_{\alpha} = \sqrt{-2} \left( \frac{\alpha}{\alpha, \alpha} \right) E_{\alpha}, \quad H_{\alpha} = \left( \frac{\alpha}{\alpha, \alpha} \right) \alpha \quad \text{for} \quad \alpha \in \Sigma.
\]

Then we have

\[
[X_{\alpha}, X_{-\alpha}] = -H_{\alpha}, \quad (X_{\alpha}, X_{-\alpha}) = -\left( \frac{2}{\alpha, \alpha} \right), \quad X_{\alpha} = X_{-\alpha}.
\]

Let \( \{\gamma_1, \ldots, \gamma_r\} \subset \sum_i \) be a maximal system of strongly orthogonal roots containing the highest root \( \gamma_1 \) such that \( r = \text{rank} M \) and \( (\gamma_j, \gamma_i) = (\alpha_i, \alpha_i) \) for each \( j \) (cf. Helgason [2]). An injective homomorphism \( \phi_j: \mathfrak{s}l(2, \mathbb{C}) \rightarrow g^c \) is defined by

\[
X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_{\gamma_j}, \quad X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \mapsto X_{-\gamma_j},
\]

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -H_{\gamma_j}.
\]

Since \( \phi_j(-X) = \bar{\phi}_j(X) \) for \( X \in \mathfrak{s}l(2, \mathbb{C}) \), we have \( \phi_j(\mathfrak{s}u(2)) \subset \mathfrak{g} \). If we define a map \( \phi \) from the \( r \)-fold direct sum \( \mathfrak{s}l(2, \mathbb{C})^r \) of \( \mathfrak{s}l(2, \mathbb{C}) \) into \( g^c \) by

\[
\phi(X_1, \ldots, X_r) = \sum_{j=1}^r \phi_j(X_j) \quad \text{for} \quad X_j \in \mathfrak{s}l(2, \mathbb{C}),
\]

then it is also an injective homomorphism such that \( \phi(\mathfrak{s}u(2)^r) \subset \mathfrak{g} \). The extension of \( \phi \) to the \( r \)-fold direct product \( SL(2, \mathbb{C})^r \) of \( SL(2, \mathbb{C}) \) is also denoted by
It satisfies $\phi(SU(2)^r) \subset G$. Putting

$$SL(1, 1; C) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C); \ c = 0 \right\},$$

we identify the $r$-fold direct product $P_1(C)^r$ of $P_1(C)$ with $SL(2, C)/(SL(1, 1; C)^r$. Then the map

$$xSL(1, 1; C)^r \mapsto \phi(x)p \quad \text{for } x \in SL(2, C)^r$$

defines a holomorphic imbedding

$$\phi: P_1(C)^r \to M,$$

which is $SL(2, C)^r$-equivariant:

$$\phi(xp) = \phi(x)\phi(p) \quad \text{for } x \in SL(2, C)^r, \ p \in P_1(C)^r.$$

The imbedding $\phi$ is called the Hermann map. The Kählerian metric $h$ on $P_1(C)^r$ induced from $(M, g)$ is the direct product $h_1 \times \cdots \times h_r$ of Kählerian metrics $h_j$ on $P_1(C)$ of constant holomorphic sectional curvatures, since $SU(2)^r$ acts transitively on $P_1(C)^r$ as Kählerian automorphisms of $(P_1(C), h)$. The tangent space $T_x(\phi(P_1(C)^r))$ will be identified with a subspace $\mathfrak{p}$ of $\mathfrak{m}$, and hence $T_x(\phi(P_1(C)^r))^\perp$ with a subspace $\mathfrak{p}^\perp$ of $\mathfrak{m}^\perp$.

**Lemma 4.** Let

$$\phi: (P_1(C)^r, h_1 \times \cdots \times h_r) \to (M, g)$$

be the Hermann map as above. Then:

1) $\mathfrak{m}^+ = K\mathfrak{p}^+;$

2) $\phi$ is totally geodesic;

3) Each $h_j$ has the holomorphic sectional curvature $\frac{c}{\mathfrak{p}}$.

**Proof.** 1) If we put

$$U_{\gamma_j} = E_{\gamma_j} + E_{-\gamma_j}, \quad V_{\gamma_j} = \sqrt{-1}(E_{\gamma_j} - E_{-\gamma_j}) \quad (1 \leq j \leq r),$$

$\mathfrak{p}$ is spanned over $\mathfrak{R}$ by the $U_{\gamma_j}, V_{\gamma_j} (1 \leq j \leq r)$. The subspace $\alpha$ of $\mathfrak{m}$ spanned over $\mathfrak{R}$ by the $U_{\gamma_j} (1 \leq j \leq r)$ is a maximal abelian subalgebra in $\mathfrak{m}$, and hence $\mathfrak{m} = K\alpha$. Since the projection $\varpi: \mathfrak{m}^c \to \mathfrak{m}^+$ relative to the decomposition $\mathfrak{m}^c = \mathfrak{m}^+ \oplus \bar{\mathfrak{m}}^+$ is $K$-equivariant, we have $\mathfrak{m}^+ = K\varpi^+(\alpha)$. But $\varpi^+(\alpha)$ is spanned over $\mathfrak{R}$ by the $E_{-\gamma_j} (1 \leq j \leq r)$ and hence is contained in $\mathfrak{p}^+ = \mathfrak{\varpi}^+(\mathfrak{p})$, which is spanned over $\mathfrak{C}$ by the $E_{-\gamma_j} (1 \leq j \leq r)$. Thus we conclude $\mathfrak{m}^+ = K\mathfrak{p}^+$.

2) From the relations
\[ [U_{\gamma_j}, V_{\gamma_j}] = 2\sqrt{-1} \gamma_j, \quad [\sqrt{-1} \gamma_j, U_{\gamma_j}] = V_{\gamma_j}, \quad [\sqrt{-1} \gamma_j, V_{\gamma_j}] = -U_{\gamma_j}, \]

called the Killing vector, and hence \( \phi \) is totally geodesic (cf. Helgason [2]).

3) Identifying \( X^+ + X^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) with a tangent vector of \( P_i(\mathbb{C}) \) at the origin, we have by (2.2)

\[
h_j(X^+ + X^-, X^+ + X^-) = g(X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j})
\]

\[
= -\frac{b(\alpha_1, \alpha_1)}{c} (X_{\gamma_j} + X_{-\gamma_j}, X_{\gamma_j} + X_{-\gamma_j})
\]

\[
= -\frac{2b(\gamma_j, \gamma_j)}{c} (X_{\gamma_j}, X_{-\gamma_j}) = \frac{2b(\gamma_j, \gamma_j)}{c} \frac{2}{\gamma_j, \gamma_j} 
\]

\[
= \frac{4}{c}. 
\]

It follows that \( h_j \) is \( p \) times the Fubini-Study metric of \( P_i(\mathbb{C}) \), which implies the assertion 3). q.e.d.

Now we shall prove the following

**Theorem 2.** Let

\[ f_i: (M_i, g_i) \to P_{N_i}(\mathbb{C}) \quad (1 \leq i \leq s) \]

be the \( p_i \)-th full Kählerian imbedding of an irreducible symmetric Kählerian manifold \((M_i, g_i)\) of compact type, with rank \( M_i - r_i \) \((1 \leq i \leq s)\), and

\[ f: (M, g) \to P_n(\mathbb{C}) \]

be the tensor product of the \( f_i \) \((1 \leq i \leq s)\). Then the degree \( d(f) \) of \( f \) is given by

\[ d(f) = \sum_{i=1}^{s} r_i p_i. \]

For the proof of the Theorem we need the following Lemma.

**Lemma 5** (Nakagawa-Takagi [5]). Let

\[ f: (M, g) \to P_n(\mathbb{C}) \]

be a Kählerian immersion of a locally symmetric Kählerian manifold \((M, g)\). Then:

1) \( \langle H(\otimes^m T_p(M)^\lor), H(\otimes^{m'} T_p(M)^\lor) \rangle = 0 \) for \( m \neq m' \), and hence \( O^r(M) = H(\otimes^m T_p(M)^\lor) \) for each \( m \);

2) For each \( u = (u_1, \ldots, u_n) \in U(M, g) \),

\[
h(u_{i_1}, \ldots, u_{i_m}, u_i) = -\frac{c}{2} \sum_{1 \leq r < s \leq m} \delta_{i_r, i_s} H(u_{i_1}, \ldots, \hat{u}_{i_r}, \ldots, u_{i_m})
\]

\[
+ \sum_{1 \leq k < l \leq n} \langle R(u_{i_k}, u_{i_l}) u_{i_k}, u_{i_l} \rangle H(u_{i_1}, u_{i_1}, \ldots, \hat{u}_{i_k}, \ldots, \hat{u}_{i_l}, \ldots, u_{i_m})
\]

\[ (m \geq 3), \]
where $R$ is the curvature tensor of $(M, g)$.

Proof of Theorem 2. Let $r=r_1+\cdots+r_s$ be the rank of $M$. We use the notation in the end of §2. Taking the direct product of respective homomorphisms $\phi_i: SL(2, \mathbb{C})^{r_i} \to G_i^c$ for $M_i (1 \leq i \leq s)$ and the one of Hermann maps $\phi_i: P_i\left(\frac{\mathbb{C}}{\mathbb{P}_i}\right)^r \to (M_i, g_i) (1 \leq i \leq s)$, we get a homomorphism $\phi: SL(2, \mathbb{C})^r \to G^c$ such that $\phi(SU(2)^r) \subset G$ and a totally geodesic Kählerian imbedding

$$\phi: P = P_1\left(\frac{\mathbb{C}}{\mathbb{P}_1}\right)^{r_1} \times \cdots \times P_s\left(\frac{\mathbb{C}}{\mathbb{P}_s}\right)^{r_s} \to (M, g),$$

which is $SL(2, \mathbb{C})^r$-equivariant:

$$\phi(xp) = \phi(x)\phi(p) \quad \text{for } x \in SL(2, \mathbb{C})^r, p \in P.$$

The tangent space $\mathfrak{p} = T_0(\phi(P))$ of $\phi(P)$ at the origin is the direct sum

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$$

of respective tangent spaces $\mathfrak{p}_i$ of $\phi_i\left(P_i\left(\frac{\mathbb{C}}{\mathbb{P}_i}\right)^{r_i}\right)$ at the origin, and hence

$$\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \cdots \oplus \mathfrak{p}_s^+.$$ 

It follows from Lemma 4 and decompositions (2.3), (2.4) that

$$(3.1) \quad m^+ = K\mathfrak{p}^+.$$ 

Let us consider a Kählerian imbedding

$$f' = f \circ \phi: P \to P_N(c).$$

If we put

$$\rho' = \rho \circ \phi: SL(2, \mathbb{C})^r \to PL(N+1, \mathbb{C}),$$

$f'$ is $SL(2, \mathbb{C})^r$-equivariant by the homomorphism $\rho'$:

$$f'(xp) = \rho'(x)f'(p) \quad \text{for } x \in SL(2, \mathbb{C})^r, p \in P,$$

Note that $\rho'(SU(2)^r) \subset PU(N+1) = Aut(P_N(c))$ and $SU(2)^r$ acts transitively on $P$ as Kählerian automorphisms of $P$. We shall identify as $P \subset (M, g)$ through the imbedding $\phi$. Denote the higher fundamental forms of $f$ and $f'$ by $H$ and $H'$ respectively. We shall prove the following two assertions:

(i) \quad $d(f) = d(f')$.

(ii) \quad $d(f') = \sum_{j=1}^{r_s} r_j p_j$.

But in view of the $\text{Aut}^0(M, g)$-equivariance of $f$ and Lemma 5, 1), we know that each point of $M$ is regular and $d(f)$ is determined by conditions
In the same way, \( d(f') \) is determined by conditions

\[
H^{d(f')}\theta = 0 \quad \text{and} \quad H^{d(f')+1} \theta = 0.
\]

Here \( H^\theta \) and \( H^\theta' \) are understood to be always not 0. Hence the assertion (i) is equivalent to the assertion

\[
(i)' \quad H^\theta_s = 0 \Leftrightarrow H'^\theta_s = 0 \quad (m \geq 2).
\]

Proof of (i)'. Note first that if we denote by \( X \mapsto kX \) the action of \( k \in K \) on \( N_\kappa(M) \) through the differential \( \rho(k)_\bullet \), we have

\[
H(kX_1, \ldots, kX_m) = kH(X_1, \ldots, X_m) \quad \text{for} \quad X_i \in m^+, \quad k \in K,
\]

because of the \( \text{Aut}^\theta(M, g) \)-equivariance of \( f \). Now

\[
\begin{align*}
H^\theta_s &= 0 \\
\Leftrightarrow H(X_1, \ldots, X_m) &= 0 \quad \text{for each} \quad X_i \in m^+, \\
\Leftrightarrow H(X, \ldots, X) &= 0 \quad \text{for each} \quad X \in m^+ \quad \text{by Lemma 2.1}, \\
\Leftrightarrow H(Y, \ldots, Y) &= 0 \quad \text{for each} \quad Y \in p^+ \quad \text{by (3.1), (3.2)}, \\
\Leftrightarrow H'(Y, \ldots, Y) &= 0 \quad \text{for each} \quad Y \in p^+ \quad \text{since} \ \phi \ \text{is totally geodesic}, \\
\Leftrightarrow H'(Y, \ldots, Y) &= 0 \quad \text{for each} \quad Y_i \in p^+ \quad \text{by Lemma 2.1} \\
\Leftrightarrow H'^\theta_s &= 0.
\end{align*}
\]

Proof of (ii). For an index \( j, \ 1 \leq j \leq r \), we define \( \nu(j), \ 1 \leq \nu(j) \leq s \), by

\[
\nu(j) = \nu \quad \text{if} \quad r_1 + \cdots + r_{s-1} + 1 \leq j \leq r_1 + \cdots + r_{s-1} + r_s.
\]

Take a unitary frame \( u=(u_1, \ldots, u_r) \) of \( P \) at the origin \( o \) such that \( u_i \) is tangent to the \( i \)-th factor of \( P \) for each \( i \), and fix it once for all. Then the curvature tensor \( R \) of \( P \) satisfies

\[
\langle R(u_k, u_l)u_j, u_i \rangle = \frac{c}{p_{s(i)}} \delta_{ij} \delta_{jk} \delta_{kl}.
\]

For each \( i_1, \ldots, i_m, j \ (m \geq 2) \), the following equality holds:

\[
h'(u_{i_1}, \ldots, u_{i_m}, u_j, u_j) = \frac{c(a_j+1)}{2p_{s(i)}} (a_j-p_{s(i)}) H'(u_{i_1}, \ldots, u_{i_m}),
\]

where \( a_j \) is an integer given by
Indeed, Lemma 5.2) and (3.3) imply
\[ h'(u_{i_1}, \ldots, u_{i_{m+1}}, u_j) = \frac{c}{2} \sum_{r=1}^{m+1} \delta_{i_j} H'(u_{i_1}, \ldots, \hat{u}_{i_r}, \ldots, u_{i_{m+1}}, u_j) \]
\[ + \frac{c}{p_{s(j)} \cdot \delta_{i_j} \delta_{i_j}} H'(u_{i_1}, \ldots, \hat{u}_{i_j}, \ldots, u_{i_{m+1}}, u_j) \].

Put \( i_{m+1} = j \). Recalling that \( H' \) is symmetric, we have
\[ h'(u_{i_1}, \ldots, u_{i_m}, u_j) \]
\[ = -\frac{c}{2} \sum_{r=1}^{m} \delta_{i_j} H'(u_{i_1}, \ldots, \hat{u}_{i_r}, \ldots, u_{i_m}, u_j) - \frac{c}{2} H'(u_{i_1}, \ldots, u_{i_m}) \]
\[ + \frac{c}{p_{s(j)} \cdot \delta_{i_j} \delta_{i_j}} H'(u_{i_1}, \ldots, \hat{u}_{i_j}, \ldots, u_{i_m}, u_j) \]
\[ + \frac{c}{p_{s(j)} \cdot i_{m+1}} H'(u_{i_1}, \ldots, \hat{u}_{i_j}, \ldots, u_{i_{m+1}}, u_j) \]
\[ = \left\{ \frac{c}{2} a_j - \frac{c}{2} + \frac{c}{p_{s(j)}} a_j(a_j - 1) + \frac{c}{p_{s(j)}} a_j \right\} H'(u_{i_1}, \ldots, u_{i_m}) \]
\[ = \frac{c(a_j + 1)}{2p_{s(j)}} (a_j - p_{s(j)}) H'(u_{i_1}, \ldots, u_{i_m}) \].

Now we are in a position to prove (ii). If \( d' = d(f') = 1 \), then \( f' \) is totally geodesic, and hence \( s = 1 \), \( r_i = 1 \), \( p_i = 1 \). So we may assume \( d' \geq 2 \). Then there exist indices \( i_1, \ldots, i_{d'} \) such that \( H'(u_{i_1}, \ldots, u_{i_{d'}}) = 0 \). It follows from (1.5) and \( H'^{d'+1} = 0 \) that
\[ h'(u_{i_1}, \ldots, u_{i_{d'}}, u_j, u_j) = 0 \quad \text{for each } j, 1 \leq j \leq r. \]

Thus (3.4) implies
\[ \# \{k; 1 \leq k \leq d', i_k = j\} = p_{s(j)} \quad \text{for each } j, 1 \leq j \leq r, \]
and hence
\[ d' = \sum_{j=1}^{r} p_{s(j)} = \sum_{i=1}^{r} r_i p_i. \quad \text{q.e.d.} \]
References


